



Introduction to geodynamic modelling

Solving equations

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Goals of this lecture

- Give a quick overview of what we need to know to **solve equations**
- Provide a bit of background for why we need to use **numerical solutions** to equations



Finding a solution

- We have now seen the general form of several different equations we will be using to model geodynamic processes, such as the [heat transfer equation](#) below

$$\rho c_P \left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T \right) = \nabla \cdot k \nabla T + A$$

- But how do we go from this form of the equation to a solution in the form

$$T(x, y, z, t) = \dots?$$



A simpler example

- Let's start simpler
 - Steady-state heat conduction in 1D

$$\rho c_P \left(\frac{\partial T}{\partial t} + v_z \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

Time dependence Advection Conduction Production



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Time dependence Advection **Conduction** Production



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Time dependence Advection **Conduction** Production

- In this case we can ignore most terms of this equation and we're left with

$$\frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) = 0$$



A simpler example

- In fact, we can even factor out k and simplify our equation further

$$\frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) = 0$$

$$\frac{\partial}{\partial z} \left(\frac{\partial T}{\partial z} \right) = 0$$

$$\frac{\partial^2 T}{\partial z^2} = 0$$

- Here, we are left with simply the second derivative of temperature T with respect to depth z
- **We can solve this**, we just need to integrate twice with respect to z



A simpler example...?

- We can integrate once to find

$$\int \frac{d^2T}{dz^2} dz = \frac{dT}{dz} + c_1$$

$$\frac{dT}{dz} + c_1 = 0$$

- And a second time to get

$$\begin{aligned} \int \left(\frac{dT}{dz} + c_1 \right) dz &= \int \frac{dT}{dz} dz + \int c_1 dz \\ &= T(z) + c_1 z + c_2 \end{aligned}$$

$$T(z) + c_1 z + c_2 = 0$$

$$T(z) = -c_1 z - c_2$$



OK, now what?

$$T(z) = -c_1 z - c_2$$

- Great, we have our solution, but we can't yet calculate temperatures because we don't know the values of c_1 and c_2 , the constants of integration
- What do we need to be able to find c_1 and c_2 ?



Boundary conditions

- With two integrations we end up with two unknown constants
- In order to find the values of the constants, we need to use **boundary conditions**, known (or assumed) values for certain variables in our equation
- There are two natural choices for our heat transfer problem:
 - Known temperatures and certain depths
 - Known temperature gradients at certain depths



Boundary conditions

$$T(z) = -c_1 z - c_2$$

- Let's make some assumptions
 - We know the surface temperature at $z = 0$ is $T = 0$
 - We know the temperature at some depth $z = L$ is $T = 1000^\circ\text{C}$
- If we substitute in assumption 1, we find $c_2 = 0$
- If we then substitute in assumption 2, we find $c_1 = -1000 / L$



Boundary conditions

- Now, we can finally see out exciting result with $c_1 = -1000 / L$ and $c_2 = 0...$

$$T(z) = -c_1 z - c_2$$

$$T(z) = \frac{1000}{L} z$$

...a straight line.



Other considerations

- So that was an almost embarrassingly simple example
- Heat conducted between two known temperatures at steady state will simply produce a linear temperature increase across the thickness of the material
- What happens, though, if we consider a heat transfer problem that is not at a steady state?
- What other information might we need to know for that case?



Initial conditions

Time-dependent
advection and conduction

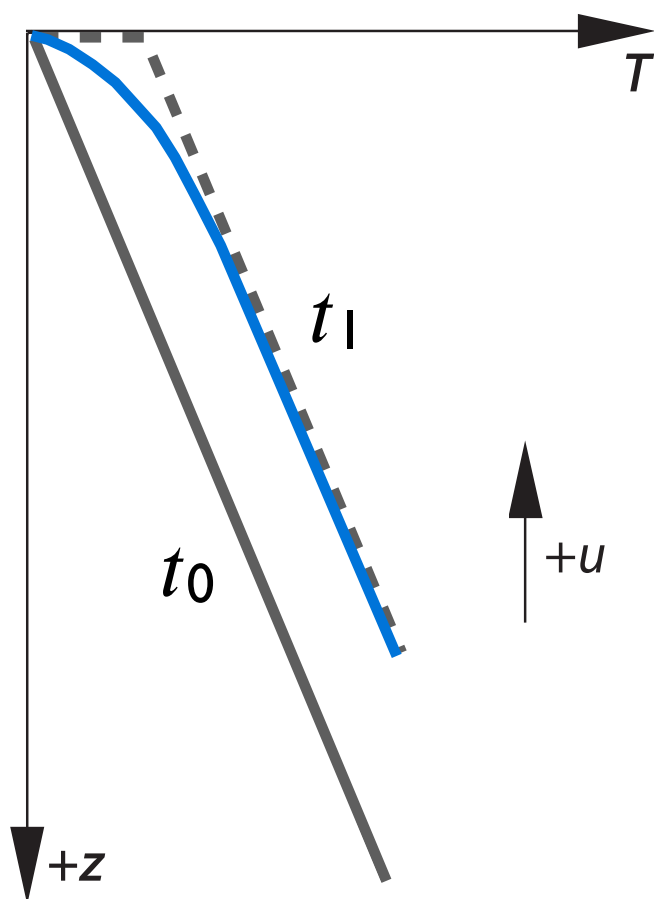


Fig. 3.13, Stüwe, 2007

- In order to consider the time evolution of the temperatures in the Earth, we would also need to know the starting distribution of temperature
- This is known as an **initial condition**
- Clearly, the temperatures we expect after some time can strongly depend on the initial conditions we assume
- Why don't we need initial conditions for the steady-state case?



What about other scenarios

- What if we want to solve slightly more complicated problems?
- 2D heat transfer with surface topography
- Spatially variable material properties
- Temperature-dependent material properties



What about other scenarios

- What if we want to solve slightly more complicated problems?
 - 2D heat transfer with surface topography
 - Spatially variable material properties
 - Temperature-dependent material properties
- In most of these cases **we cannot simply directly integrate our heat transfer equation** because we cannot find the constants of integration
 - Consider the example above with topography
 - We likely do not have a function that can give us the required known values of temperature and elevation at the surface



The need for numerical integration

- In cases where the geometry of the problem or material property distributions/behaviors are more complex, **we need to use numerical methods to integrate our equations** of interest
- Much days 2 and 3 in this course will focus on the use of the **finite difference method** for solving equations, how it can be used, and its limitations
- The finite difference method is one of the most popular approaches used for solving equations in geodynamic models
- We'll start learning the details of the finite difference method tomorrow morning



Summary

- To solve equations used in geodynamic modelling (or anything really) we need to know
 - **Boundary conditions**
 - **Initial conditions** (in some cases)
- We can solve these equations directly in some cases, but most of the time we will need to use numerical methods such as the **finite difference method** to find solutions



References

Stüwe, K. (2007). *Geodynamics of the lithosphere: an introduction*. Springer Science & Business Media.