Chapter 9

Multiple and logistic regression[1]

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[1] These notes use content from OpenIntro Statistics Slides by Mine Cetinkaya-Rundel.

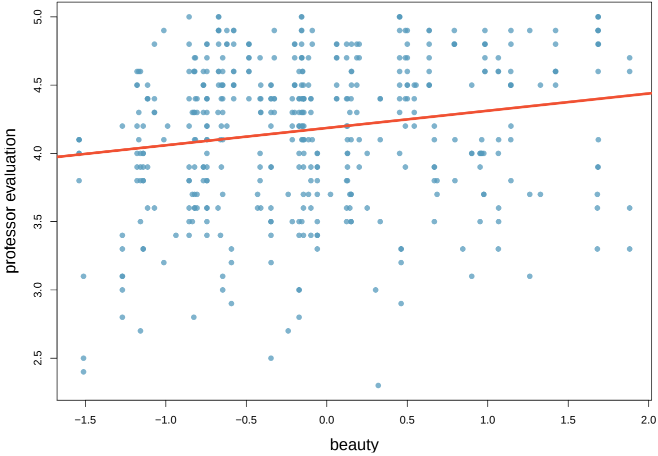
Model selection

Beauty in the classroom

- Data: Student evaluations of instructors' beauty and teaching quality for 463 courses at the University of Texas.
- Evaluations conducted at the end of semester, and the beauty judgements were made later, by six students who had not attended the classes and were not aware of the course evaluations (2 upper level females, 2 upper level males, one lower level female, one lower level male).

Professor rating vs. beauty

Professor evaluation score (higher score means better) vs. beauty score (a score of 0 means average, negative score means below average, and a positive score above average):



Which of the below is correct based on the model output?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.19	0.03	167.24	0.00
beauty	0.13	0.03	4.00	0.00

$R^2 = 0.0336$

(a) Model predicts 3.36% of professor ratings correctly.

(b) Beauty is not a significant predictor of professor evaluation.

(c) Professors who score 1 point above average in their beauty score are tend to also score 0.13 points higher in their evaluation.

(d) 3.36% of variability in beauty scores can be explained by professor evaluation.

(e) The correlation coefficient could be $\sqrt{0.0336} = 0.18$ or -0.18, we can't tell which is correct.

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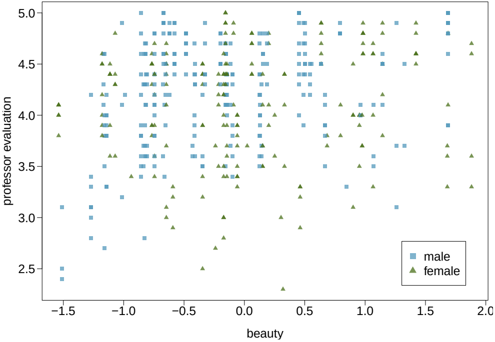
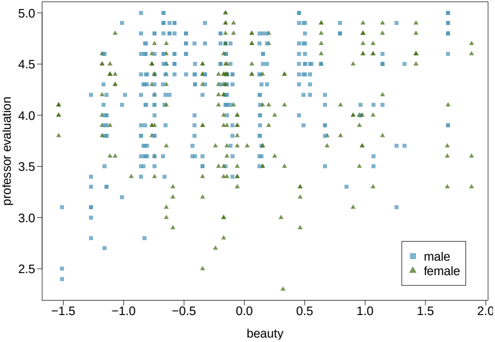
Exploratory analysis



Any interesting features?

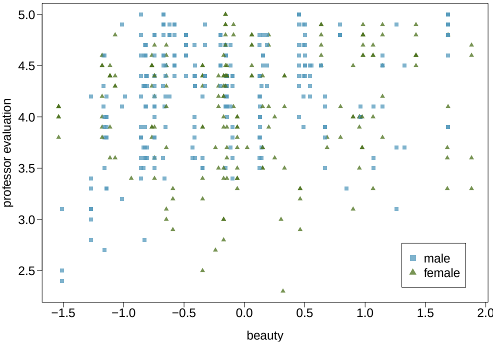
Exploratory analysis



Any interesting features?

Few females with very low beauty scores.

Exploratory analysis

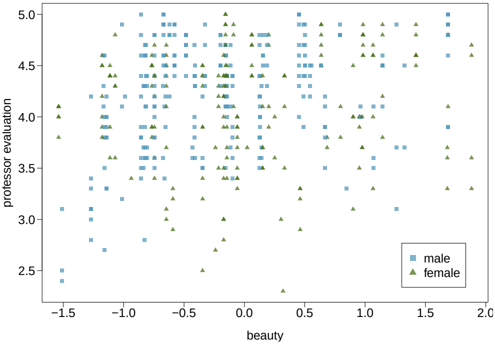


Any interesting features?

Few females with very low beauty scores.

For a given beauty score, are male professors evaluated higher, lower, or about the same as female professors?

Exploratory analysis



Any interesting features?

Few females with very low beauty scores.

For a given beauty score, are male professors evaluated higher, lower, or about the same as female professors?

Difficult to tell from this plot only.

Professor rating vs. beauty + gender

For a given beauty score, are male professors evaluated higher, lower, or about the same as female professors?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.09	0.04	107.85	0.00
beauty	0.14	0.03	4.44	0.00
gender.male	0.17	0.05	3.38	0.00

$R^2_{adj} = 0.057$

A) higher

B) lower

C) about the same

Professor rating vs. beauty + gender

For a given beauty score, are male professors evaluated higher, lower, or about the same as female professors?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.09	0.04	107.85	0.00
beauty	0.14	0.03	4.44	0.00
gender.male	0.17	0.05	3.38	0.00

$R^2_{adj} = 0.057$

A) higher $\rightarrow$ Beauty held constant, male professors are rated 0.17 points higher on average than female professors.

B) lower

C) about the same

Full model

\begin{center}

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.6282	0.1720	26.90	0.00
beauty	0.1080	0.0329	3.28	0.00
gender.male	0.2040	0.0528	3.87	0.00
age	-0.0089	0.0032	-2.75	0.01
formal.yes [2]	0.1511	0.0749	2.02	0.04
lower.yes [3]	0.0582	0.0553	1.05	0.29
native.non english	-0.2158	0.1147	-1.88	0.06
minority.yes	-0.0707	0.0763	-0.93	0.35
students [4]	-0.0004	0.0004	-1.03	0.30
tenure.tenure track [5]	-0.1933	0.0847	-2.28	0.02
tenure.tenured	-0.1574	0.0656	-2.40	0.02

[2] `formal`: picture wearing tie&jacket/blouse, levels: `yes`, `no`

[3] `lower`: lower division course, levels: `yes`, `no`

[4] `students`: number of students

[5] `tenure`: tenure status, levels: `non-tenure track`, `tenure track`, `tenured`

Hypotheses

Just as the interpretation of the slope parameters take into account all other variables in the model, the hypotheses for testing for significance of a predictor also takes into account all other variables.

H_0 : $B_i = 0$ when other explanatory variables are included in the model.

H_A : $B_i \neq 0$ when other explanatory variables are included in the model.

Assessing significance: numerical variables

The p-value for age is 0.01. What does this indicate?

	Estimate	Std. Error	t value	Pr(>|t|)
...				
age	-0.0089	0.0032	-2.75	0.01
...				

A) Since p-value is positive, higher the professor's age, the higher we would expect them to be rated.

B) If we keep all other variables in the model, there is strong evidence that professor's age is associated with their rating.

C) Probability that the true slope parameter for age is 0 is 0.01.

D) There is about 1% chance that the true slope parameter for age is -0.0089.

Assessing significance: numerical variables

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B) If we keep all other variables in the model, there is strong evidence that professor's age is associated with their rating.

C) Probability that the true slope parameter for age is 0 is 0.01.

D) There is about 1% chance that the true slope parameter for age is -0.0089.

Assessing significance: categorical variables

Tenure is a categorical variable with 3 levels: non tenure track, tenure track, tenured. Based on the model output given, which of the below is false?

	Estimate	Std. Error	t value	Pr(>|t|)
...				
tenure.tenure track	-0.1933	0.0847	-2.28	0.02
tenure.tenured	-0.1574	0.0656	-2.40	0.02

(a) Reference level is non tenure track.

(b) All else being equal, tenure track professors are rated, on average, 0.19 points lower than non-tenure track professors.

(c) All else being equal, tenured professors are rated, on average, 0.16 points lower than non-tenure track professors.

(d) All else being equal, there is a significant difference between the average ratings of tenure track and tenured professors.

Assessing significance: categorical variables

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(a) Reference level is non tenure track.

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(c) All else being equal, tenured professors are rated, on average, 0.16 points lower than non-tenure track professors.

(d) All else being equal, there is a significant difference between the average ratings of tenure track and tenured professors.

Assessing significance

Which predictors do not seem to meaningfully contribute to the model, i.e. may not be significant predictors of professor's rating score?

\begin{center}

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.6282	0.1720	26.90	0.00
beauty	0.1080	0.0329	3.28	0.00
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formal.yes	0.1511	0.0749	2.02	0.04
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students	-0.0004	0.0004	-1.03	0.30
tenure.tenure track	-0.1933	0.0847	-2.28	0.02
tenure.tenured	-0.1574	0.0656	-2.40	0.02

Model selection strategies

Based on what we've learned so far, what are some ways you can think of that can be used to determine which variables to keep in the model and which to leave out?

Backward-elimination

1. Start with the full model
2. Drop one variable at a time and record R^2_{adj} of each smaller model
3. Pick the model with the highest increase in R^2_{adj}
4. Repeat until none of the models yield an increase in R^2_{adj}

Backward-elimination

Full	beauty + gender + age + formal + lower + native + minority + students + tenure	0.0839

Backward-elimination

Full	beauty + gender + age + formal + lower + native + minority + students + tenure	0.0839
Step 1	gender + age + formal + lower + native + minority + students + tenure	0.0642
	beauty + age + formal + lower + native + minority + students + tenure	0.0557
	beauty + gender + formal + lower + native + minority + students + tenure	0.0706
	beauty + gender + age + lower + native + minority + students + tenure	0.0777
	beauty + gender + age + formal + native + minority + students + tenure	0.0837
	beauty + gender + age + formal + lower + minority + students + tenure	0.0788
	beauty + gender + age + formal + lower + native + students + tenure	0.0842
	beauty + gender + age + formal + lower + native + minority + tenure	0.0838
	beauty + gender + age + formal + lower + native + minority + students	0.0733

Backward-elimination

Full	beauty + gender + age + formal + lower + native + minority + students + tenure	0.0839
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	beauty + age + formal + lower + native + minority + students + tenure	0.0557
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	beauty + gender + age + formal + native + minority + students + tenure	0.0837
	beauty + gender + age + formal + lower + minority + students + tenure	0.0788
	beauty + gender + age + formal + lower + native + students + tenure	0.0842
	beauty + gender + age + formal + lower + native + minority + tenure	0.0838
	beauty + gender + age + formal + lower + native + minority + students	0.0733
Step 2	gender + age + formal + lower + native + students + tenure	0.0647
	beauty + age + formal + lower + native + students + tenure	0.0543
	beauty + gender + formal + lower + native + students + tenure	0.0708
	beauty + gender + age + lower + native + students + tenure	0.0776
	beauty + gender + age + formal + native + students + tenure	0.0846
	beauty + gender + age + formal + lower + native + tenure	0.0844
	beauty + gender + age + formal + lower + native + students	0.0725

Backward-elimination

Full	beauty + gender + age + formal + lower + native + minority + students + tenure	0.0839
Step 1	gender + age + formal + lower + native + minority + students + tenure	0.0642
	beauty + age + formal + lower + native + minority + students + tenure	0.0557
	beauty + gender + formal + lower + native + minority + students + tenure	0.0706
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	beauty + age + formal + lower + native + students + tenure	0.0543
	beauty + gender + formal + lower + native + students + tenure	0.0708
	beauty + gender + age + lower + native + students + tenure	0.0776
	beauty + gender + age + formal + native + students + tenure	0.0846
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	beauty + age + formal + native + students + tenure	0.0534
	beauty + gender + formal + native + students + tenure	0.0707
	beauty + gender + age + native + students + tenure	0.0786
	beauty + gender + age + formal + students + tenure	0.0756
	beauty + gender + age + formal + native + tenure	0.0855
	beauty + gender + age + formal + native + students	0.0713

Backward-elimination

Full	beauty + gender + age + formal + lower + native + minority + students + tenure	0.0839
Step 1	gender + age + formal + lower + native + minority + students + tenure	0.0642
	beauty + age + formal + lower + native + minority + students + tenure	0.0557
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	beauty + age + formal + lower + native + students + tenure	0.0543
	beauty + gender + formal + lower + native + students + tenure	0.0708
	beauty + gender + age + lower + native + students + tenure	0.0776
	beauty + gender + age + formal + native + students + tenure	0.0846
	beauty + gender + age + formal + lower + native + tenure	0.0844
	beauty + gender + age + formal + lower + native + students	0.0725
Step 3	gender + age + formal + native + students + tenure	0.0653
	beauty + age + formal + native + students + tenure	0.0534
	beauty + gender + formal + native + students + tenure	0.0707
	beauty + gender + age + native + students + tenure	0.0786
	beauty + gender + age + formal + students + tenure	0.0756
	beauty + gender + age + formal + native + tenure	0.0855
	beauty + gender + age + formal + native + students	0.0713
Step 4	gender + age + formal + native + tenure	0.0667
	beauty + age + formal + native + tenure	0.0553
	beauty + gender + formal + native + tenure	0.0723
	beauty + gender + age + native + tenure	0.0806
	beauty + gender + age + formal + tenure	0.0773
	beauty + gender + age + formal + native	0.0713

step function in R

```
##
## Call:
## lm(formula = profevaluation ~ beauty + gender + age + formal +
##     lower + native + minority + students + tenure, data = d)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.79845 -0.37270  0.09849  0.39052  0.93273
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)         4.6282155  0.1720227  26.905  < 2e-16 ***
## beauty              0.1079530  0.0329357   3.278 0.001127 **
## gendermale          0.2040127  0.0527509   3.867 0.000126 ***
## age                -0.0089405  0.0032458  -2.755 0.006115 **
## formalyes           0.1511348  0.0749453   2.017 0.044328 *
## loweryes            0.0581603  0.0553270   1.051 0.293723
## nativenon english  -0.2157998  0.1146764  -1.882 0.060503 .
## minorityyes        -0.0706677  0.0762621  -0.927 0.354607
## students           -0.0003726  0.0003603  -1.034 0.301536
## tenuretenure track -0.1932547  0.0846549  -2.283 0.022903 *
## tenuretenured      -0.1574315  0.0655919  -2.400 0.016791 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5206 on 452 degrees of freedom
## Multiple R-squared:  0.1037, Adjusted R-squared:  0.0839
## F-statistic: 5.231 on 10 and 452 DF,  p-value: 2.748e-07
```

Best model: beauty + gender + age + formal + native + tenure

Forward selection

1. Start with regressions of response vs. each explanatory variable
2. Pick the model with the highest R^2_{adj}
3. Add the remaining variables one at a time to the existing model, and once again pick the model with the highest R^2_{adj}
4. Repeat until the addition of any of the remaining variables does not result in a higher R^2_{adj}

- Backward elimination with the p-value approach:
  1. Start with the full model
  2. Drop the variable with the highest p-value and refit a smaller model
  3. Repeat until all variables left in the model are significant
- Forward selection with the p-value approach:
  1. Start with regressions of response vs. each explanatory variable
  2. Pick the variable with the lowest significant p-value
  3. Add the remaining variables one at a time to the existing model, and pick the variable with the lowest significant p-value
  4. Repeat until any of the remaining variables does not have a significant p-value

Backward-elimination: $p - value$ approach

Step	**Variables included & p-value**									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.29	0.06	0.35	0.30	0.02	0.02

Backward-elimination: $p-value$ approach

Step	Variables included & p-value									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.29	0.06	0.35	0.30	0.02	0.02
Step 1	beauty	gender male	age	formal yes	lower yes	native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01

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Step	Variables included & p-value									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
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Step 1	beauty	gender male	age	formal yes	lower yes	native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01
Step 2	beauty	gender male	age	formal yes		native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.05		0.02		0.44	0.01	0.01

Backward-elimination: $p - value$ approach

Step	Variables included & p-value									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
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	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01
Step 2	beauty	gender male	age	formal yes		native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.05		0.02		0.44	0.01	0.01
Step 3	beauty	gender male	age	formal yes		native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.06		0.02			0.01	0.01

Backward-elimination: $p - value$ approach

Step	**Variables included & p-value**									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.29	0.06	0.35	0.30	0.02	0.02
Step 1	beauty	gender male	age	formal yes	lower yes	native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01
Step 2	beauty	gender male	age	formal yes		native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.05		0.02		0.44	0.01	0.01
Step 3	beauty	gender male	age	formal yes		native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.06		0.02			0.01	0.01
Step 4	beauty	gender male	age			native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01			0.06			0.01	0.01

Backward-elimination: $p - value$ approach

Step	Variables included & p-value									
Full	beauty	gender	age	formal	lower	native	minority	students	tenure	tenure
		male		yes	yes	nonenglish	yes		tenure track	tenured
	0.00	0.00	0.01	0.04	0.29	0.06	0.35	0.30	0.02	0.02
Step 1	beauty	gender	age	formal	lower	native		students	tenure	tenure
		male		yes	yes	nonenglish			tenure track	tenured
	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01
Step 2	beauty	gender	age	formal		native		students	tenure	tenure
		male		yes		nonenglish			tenure track	tenured
	0.00	0.00	0.01	0.05		0.02		0.44	0.01	0.01
Step 3	beauty	gender	age	formal		native			tenure	tenure
		male		yes		nonenglish			tenure track	tenured
	0.00	0.00	0.01	0.06		0.02			0.01	0.01
Step 4	beauty	gender	age			native			tenure	tenure
		male				nonenglish			tenure track	tenured
	0.00	0.00	0.01			0.06			0.01	0.01
Step 5	beauty	gender	age						tenure	tenure
		male							tenure track	tenured
	0.00	0.00	0.01						0.01	0.01

Backward-elimination: $p - value$ approach

Step	Variables included & p-value									
	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
Full	0.00	0.00	0.01	0.04	0.29	0.06	0.35	0.30	0.02	0.02
Step 1	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01
Step 2	0.00	0.00	0.01	0.05		0.02		0.44	0.01	0.01
Step 3	0.00	0.00	0.01	0.06		0.02			0.01	0.01
Step 4	0.00	0.00	0.01			0.06			0.01	0.01
Step 5	0.00	0.00	0.01						0.01	0.01

Backward-elimination: $p - value$ approach

Step	Variables included & p-value									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.29	0.06	0.35	0.30	0.02	0.02
Step 1	beauty	gender male	age	formal yes	lower yes	native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.38	0.03		0.34	0.02	0.01
Step 2	beauty	gender male	age	formal yes		native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.05		0.02		0.44	0.01	0.01
Step 3	beauty	gender male	age	formal yes		native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.06		0.02			0.01	0.01
Step 4	beauty	gender male	age			native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01			0.06			0.01	0.01
Step 5	beauty	gender male	age						tenure tenure track	tenure tenured
	0.00	0.00	0.01						0.01	0.01

Best model: beauty + gender + age + tenure

Adjusted R^2 vs. p-value approaches

- The two approaches are similar, but they sometimes lead to different models, with the adjusted R^2 approach tending to include more predictors in the final model.
- When the sole goal is to improve prediction accuracy, use R^2 . This is commonly the case in machine learning applications.
- When we care about understanding which variables are statistically significant predictors of the response, or if there is interest in producing a simpler model at the potential cost of a little prediction accuracy, then the p-value approach is preferred.
- Regardless of the approach we use, our job is not done after variable selection – we must still verify the model conditions are reasonable.