Macroeconomics analysis II, EC3102

Tutorial 8 Solution

Question 1.

Part a.

Let g_X denote the growth rate of a variable X (X can be anything). Given:

$$Y_t = \bar{A}K_t^{\frac{1}{3}}\bar{L}^{\frac{2}{3}}$$
$$\Longrightarrow \frac{Y_t}{\bar{L}} = \bar{A}K_t^{\frac{1}{3}}\frac{\bar{L}^{\frac{2}{3}}}{\bar{L}}$$
$$\Longrightarrow \frac{Y_t}{\bar{L}} = \bar{A}K_t^{\frac{1}{3}}\bar{L}^{-\frac{1}{3}}$$
$$\text{or } \left(\frac{Y_t}{\bar{L}} = \bar{A}K_t^{\alpha}\bar{L}^{-\alpha}\right) \tag{1}$$

(Here note that the production has the Cobb-Douglas form $Y_t = \bar{A}K_t^{\alpha}\,\bar{L}^{1-\alpha}$, where α is specifically $\frac{1}{3}$. So $Y_t = \bar{A}K_t^{\alpha}\,\bar{L}^{1-\alpha}$ is the general form of Cobb-Douglas function. Whereas, $Y_t = \bar{A}K_t^{\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$ is a specific Cobb-Douglas function)

From question 4 of the previous tutorial, we know that:

$$g_{\frac{Y}{L}} = g_Y - g_L \tag{2}$$

From (1) and (2) , we have:

$$\begin{aligned}
\boldsymbol{g}_{\frac{Y}{\bar{L}}} &= \boldsymbol{g}_Y - \boldsymbol{g}_{\bar{L}} = \boldsymbol{g}_{\bar{A}} + \boldsymbol{g}_{K^{\alpha}} + \boldsymbol{g}_{\bar{L}^{-\alpha}} \\
&= \boldsymbol{g}_{\bar{A}} + \alpha\boldsymbol{g}_K + \boldsymbol{g}_{\bar{L}^{-\alpha}} \\
&= 0 \;\; + \alpha\boldsymbol{g}_K + (-\alpha)\boldsymbol{g}_{\bar{L}} \\
&= 0 \;\; + \alpha\boldsymbol{g}_K + (-\alpha)\cdot 0 \\
&= \alpha g_K \\
\Longrightarrow g_{\frac{Y}{L}} &= \alpha g_K = \frac{1}{3}g_K
\end{aligned} \tag{3}$$

Part b.

***Comment**: (this would walk you through the thinking process) We have equation (3), $g_{\frac{Y}{L}} = \frac{1}{3}g_K$, and the question asks us to prove $g_{\frac{Y}{L}} = \frac{1}{3}\bar{s}\frac{Y^{\star}}{K^{\star}}\cdot\left(\frac{K^{\star\frac{2}{3}}}{K^{\frac{2}{3}}} - 1\right)$. This means that we need to express g_K as $\bar{s}\frac{Y^{\star}}{K^{\star}}\cdot\left(\frac{K^{\star\frac{2}{3}}}{K^{\frac{2}{3}}} - 1\right)$. Now, where can we get g_K? Any equation that involves the like of g_K? Growth* *rate of something involves the change of it. So logically, we should think of the motion equation or the accumulation equation of K, that is*

$$\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t \tag{4}$$

$$(\text{or the difference between saving and break} - \text{even investment})$$

From equation (4), divide both sides by K_t, we obtain:

$$\frac{\Delta K_{t+1}}{K_t} = \bar{s}\frac{Y_t}{K_t} - \bar{d} \tag{5}$$

***Comment**: (this would walk you through the thinking process) In one of the previous tutorial, we discussed why $\frac{\frac{\Delta K_{t+\Delta t}}{K_t}}{\Delta t}$ is the growth rate of K_t. So why is $\frac{\Delta K_{t+1}}{K_t}$ the growth rate of K. $\frac{\Delta K_{t+1}}{K_t}$ is the percentage change of capital (Right?) . But wait, if you look carefully, you can notice that this percentage change of K_t is over a period of time. What is this period? It is actually 1 (going from time t to time $t+1$). As such, we can write $\frac{\Delta K_{t+1}}{K_t}/\mathbf{1}$ or $\frac{\frac{\Delta K_{t+\Delta t}}{K_t}}{\Delta t}$ $(\text{where } \Delta t = 1)$.*

From equation (5), we can write:

$$g_K = \bar{s}\frac{Y_t}{K_t} - \bar{d} \tag{6}$$

Substitute (6) into (3), we have:

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot \left(\bar{s}\frac{Y_t}{K_t} - \bar{d}\right) \tag{7}$$

***Comment**: Keep in sight the final target which is $\frac{1}{3} \cdot \bar{s}\frac{Y^\star}{K^\star} \cdot \left(\frac{K^{\star\frac{2}{3}}}{K^{\frac{2}{3}}} - 1\right)$. It does not involve $\bar{d}$. So we have to find some expression to get rid of $\bar{d}$. We realize that the final target involves $Y^\star$ and $K^\star$, which are the steady-state output and capital levels.*

The steady state is the point where:

$$\frac{\Delta K_{t+1}}{K_t} = \bar{s}\frac{Y^\star}{K^\star} - \bar{d} = 0 \quad (\text{no change in capital level})$$

$$\Rightarrow \bar{d} = \bar{s}\frac{Y^\star}{K^\star} \tag{8}$$

Substitute (8) into (7), we have:

$$\begin{aligned} g_{\frac{Y}{L}} &= \frac{1}{3} \cdot \left(\bar{s}\frac{Y_t}{K_t} - \bar{s}\frac{Y^\star}{K^\star}\right) \\ &= \frac{1}{3} \cdot \bar{s} \cdot \left(\frac{Y_t}{K_t} - \frac{Y^\star}{K^\star}\right) \\ &= \frac{1}{3} \cdot \bar{s} \cdot \left(\frac{Y_t}{K_t} \cdot \frac{Y^\star}{K^\star} \cdot \frac{K^\star}{Y^\star} - \frac{Y^\star}{K^\star}\right) \end{aligned}$$

$$\left(\text{factorizing } \frac{Y^\star}{K^\star} \text{ out of the expression}\right)$$

$$= \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\frac{Y_t}{K_t} \cdot \frac{K^\star}{Y^\star} - 1\right) \tag{9}$$

***Comment**: If you get to here, pat yourself on the shoulder. Now, we need to show $\frac{Y_t}{K_t} \cdot \frac{K^\star}{Y^\star} = \frac{K^{\star\frac{2}{3}}}{K^{\frac{2}{3}}}$ and we are done. How? Note that $Y_t = \bar{A}K_t^{\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$ and $Y^\star = \bar{A}K^{\star\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$. Please be very careful here:*

- $Y_t = \bar{A}K_t^{\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$ is the production function

- $Y^\star = \bar{A}K^{\star\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$ is the steady state output which is the value of the production function evaluated at $K^\star$. That is why there is no subscript t.

Substituting $Y_t = \bar{A}K_t^{\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$ and $Y^\star = \bar{A}K^{\star\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}$ into (9), we get:

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\frac{\bar{A}K_t^{\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}}{K_t} \cdot \frac{K^\star}{\bar{A}K^{\star\frac{1}{3}}\,\bar{L}^{\frac{2}{3}}} - 1\right)$$

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\frac{\cancel{\bar{A}}K_t^{\frac{1}{3}}\,\cancel{\bar{L}^{\frac{2}{3}}}}{K_t} \cdot \frac{K^\star}{\cancel{\bar{A}}K^{\star\frac{1}{3}}\,\cancel{\bar{L}^{\frac{2}{3}}}} - 1\right)$$

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\frac{K_t^{\frac{1}{3}}}{K_t} \cdot \frac{K^\star}{K^{\star\frac{1}{3}}} - 1\right)$$

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\frac{K^{\star\frac{2}{3}}}{K_t^{\frac{2}{3}}} - 1\right)$$

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\left(\frac{K^\star}{K_t}\right)^{\frac{2}{3}} - 1\right) \tag{10}$$

What is the point of having this final expression (10)? Note that in the right-hand side, K_t is the only variable; the rest - $Y_t^\star$, $K_t^\star$ and $\bar{s}$ - are constants (note $Y_t^\star$, $K_t^\star$ are steady state values, thus fixed). As such, the expression maps out how the growth rate of Y_t will change with respect to K_t. A few things that we can note here:

- If at time t, $K_t < K^\star$, then $\left(\frac{K^\star}{K_t}\right)^{\frac{2}{3}} > 1 \Longrightarrow g_{\frac{Y}{L}} = \frac{1}{3} \cdot \bar{s} \cdot \frac{Y^\star}{K^\star}\left(\left(\frac{K^\star}{K_t}\right)^{\frac{2}{3}} - 1\right) > 0$. This corresponds well with the graph analysis that if the capital is below the steady state level, there is accumulation of capital which in turn leads to higher output Y - meaning, $g_{\frac{Y}{L}} > 0$. We can easily show that if $K_t > K^\star$, $g_{\frac{Y}{L}} < 0$ (this is because of capital decumulation).

- If $K_t = K^\star$, we reach the steady state. Intuitively, we should expect $g_{\frac{Y}{L}} = 0$. Is it true using expression (10)? $K_t = K^\star \Longrightarrow \left(\frac{K^\star}{K_t}\right)^{\frac{2}{3}} - 1 = 0 \Longrightarrow g_{\frac{Y}{L}} = 0$. It is true.
- If K_t is extremely small (≈ 0), then $\left(\frac{K^\star}{K_t}\right)^{\frac{2}{3}} - 1$ is extremely large or $g_{\frac{Y}{L}}$ is extremely large. That is, poor economies with positive saving rate, ***according to the model***, should grow immensely fast.

Question 2.

Part a.

The law of motion of ideas is:

$$\Delta A_{t+1} = \bar{z} A_t^{\alpha} L_{at}$$

To get the growth rate of A_t, divide both sides by A_t, we obtain:

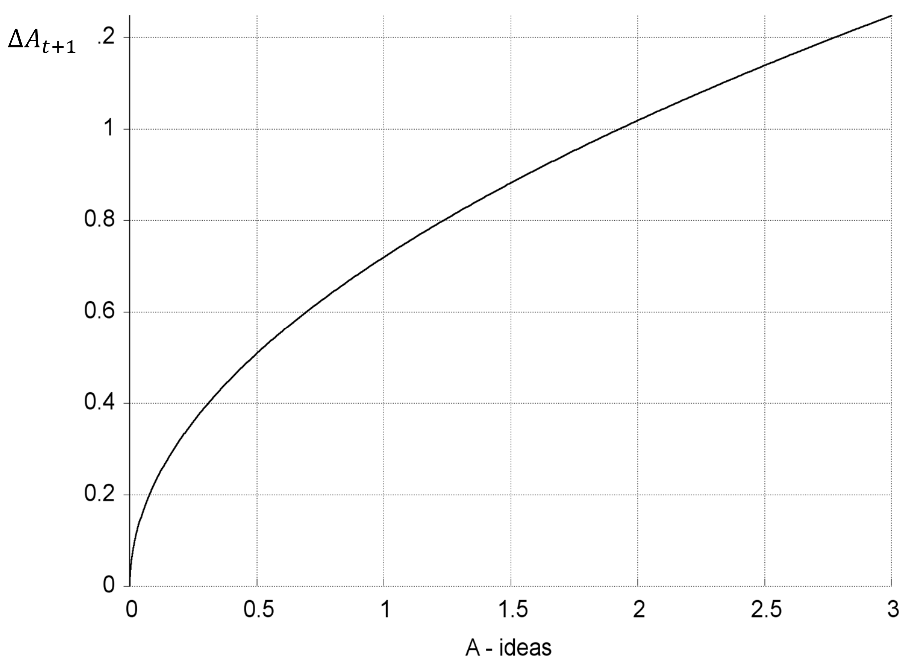
$$\frac{\Delta A_{t+1}}{A_t} = \bar{z} A_t^{\alpha - \mathbf{1}} L_{at}$$

$$g_A = \bar{z} A_t^{\alpha - \mathbf{1}} L_{at} = \frac{\bar{z} L_{at}}{A_t^{1-\alpha}}$$

$$\text{since } \alpha < 1 \Longrightarrow \alpha - 1 < 0,$$

$$\Longrightarrow \frac{\partial g_A}{\partial A_t} = (\alpha - \mathbf{1}) \bar{z} A_t^{\alpha - \mathbf{2}} L_{at} < 0 \quad (\text{or: } A_t \uparrow, g_A \downarrow)$$

Thus, if $A_t \to \infty$, $g_A \to 0$. That is, if A_t gets very large, g_A goes to zero. This means that some point in time, growth of idea is zero. This is due to the diminishing return to ideas (see the graph below).



$$\Delta A_{t+1} = \bar{z} A_t^{\alpha} L_{at}$$

with $\bar{z} = 1.2$, $\alpha = 0.5$,

and $L_{at} = 0.6$

Part b.

With $\alpha = 1$, the law of motion of ideas is:

$$\Delta A_{t+1} = \bar{z} A_t L_{at}$$

Let g_{A_t} and g_{y_t} be the growth rates of knowledge and output per capita respectively. From the law of motion of ideas (idea production function), we have:

$$\begin{aligned}
\Delta A_{t+1} &= \bar{z} A_t L_{at} \\
\frac{A_{t+1} - A_t}{A_t} &= \bar{z} L_{at} \\
\frac{A_{t+1} - A_t}{A_t} &= \bar{z}\,\bar{l}\,\bar{L} \\
\Longrightarrow g_{A_t} &= \bar{z}\,\bar{l}\,\bar{L} \qquad (11)
\end{aligned}$$

From the aggregate production function, and taking the $\ln(\quad)$ and differentiating both sides with respect to time,

$$y_t = \frac{Y_t}{\bar{L}} = A_t \cdot \frac{(1-\bar{l})\bar{L}}{\bar{L}} = A_t \cdot (1 - \bar{l})$$

$$\Longrightarrow g_{y_t} = g_{A_t} = \bar{z}\,\bar{l}\,\bar{L} \qquad (12)$$

Thus, if the research productivity increases from $\bar{z}$ and $\bar{z}'$, the new growth rates of knowledge and output per capita are:

$$g_{y_t} = g_{A_t} = \bar{z}'\,\bar{l}\,\bar{L}$$

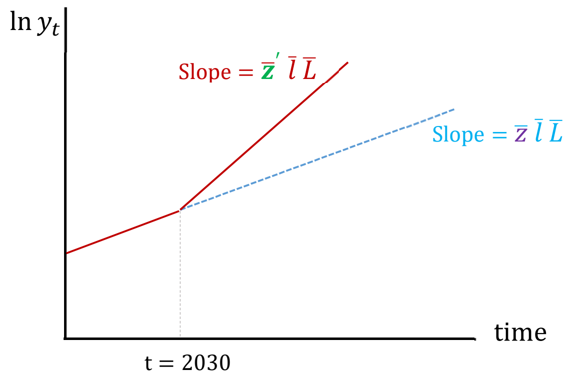
Comment: *The equality $g_{y_t} = g_{A_t}$ in equation (12) is important because the growth rate of output is identity to the growth rate of ideas (**according to the model**). And if the law of motion of ideas is that of part a, then this economy will eventually run into a problem. Let 's figure it out:*

$$\begin{aligned}
g_{y_t} = g_{A_t} &= \bar{z} A_t^{\alpha - 1} L_{at} \\
&= \bar{z}\, A_t^{\alpha-1} \bar{l}\,\bar{L} \\
\Longrightarrow \frac{\partial g_{y_t}}{\partial A_t} &= (\alpha - 1)\bar{z} A_t^{\alpha-2} \bar{l}\,\bar{L} < 0 \quad (\text{or: } A_t \uparrow, g_{y_t} \downarrow)
\end{aligned}$$

*That is eventually, the growth rate of per capita output will reach zero as ideas get very large. In other words, in the long run, the economy will reach **some steady state**, where output and the pool of ideas stop growing.*

Part c.

The growth rate of output per capita with permanent increase in productivity will look like this:



Comment: *Here is why the slope of the graph is the growth rate of y_t. Notice that the y-axis is $\boldsymbol{ln(y_t)}$, not just y_t. In some books, authors can call this **ratio scale** $\boldsymbol{y_t}$.*

Suppose that we have a variable X_t which grows at the rate of g_X. Thus we have

$$X_t = (1 + g_X)^t X_0 \quad where\ X_0\ is\ X\ at\ t = 0$$

Taking $ln(\quad)$ both sides, we have:

$$\begin{aligned} ln\, X_t &= ln(1 + g_X)^t + ln\, X_0 \\ &= \boldsymbol{t} \cdot ln(1 + g_X) + ln\, X_0 \end{aligned}$$

Thus, $ln(X_t)$ is linear with time, ***t****. So, if we draw a graph of $ln\, X_t$ against* time, ***t****, we will have a straight line with the slope of $ln(1 + g_X)$. So how then the slope of the straight line is the growth rate? It is because for small g_X, $ln(1 + g_X) \approx g_X$. Or:*

$$\ln X_t \approx \boldsymbol{t} \cdot ln(1 + g_X) + \ln X_0$$

Applying to this question, we have:

$$\ln y_t \approx \boldsymbol{t} \cdot ln(1 + g_A) + \ln y_0$$

(where y_0 is just some initial value of y (a constant))

$$\text{and} \begin{cases} \ln y_t \approx \boldsymbol{t} \cdot \bar{\boldsymbol{z}}\, \bar{l}\, \bar{L} + \ln y_0 & \text{before 2030} \\ \ln y_t \approx (\boldsymbol{t - 2030}) \cdot \bar{\boldsymbol{z}}'\, \bar{l}\, \bar{L} + \ln y_{2030} & \text{after 2030} \end{cases}$$

For those who are strong with math:

*If X_t is growing **continuously** at the rate of g_X, then we can write:*

$$X_t = X_0 e^{g_X \boldsymbol{t}} \quad \text{where } X_0 \text{ is the initial value of } X$$

Taking $\ln(\quad)$ *on both sides, we have:*

$$\ln X_t = g_X t + \ln X_0$$

In this case, $\ln X_t$ *is clearly linear in* t *with the slope of* g_X

Part d.

Research productivity may increase for a few reasons:

- Education level of the population increases
- New technological applications that speeds up and lower the cost of productions or that help testing and prototyping of new ideas cheaper and faster. For example, we can use computer for simulation to generate different possibilities and filter out the infeasible or erroneous ideas and to choose good ones. (previously, Keppel Lab in NUS has a team working on simulations for off-shore oil rig). The 3-D printing helps engineers to prototype ideas visually. Use of computers can help chemists to design new drugs before testing on animals; or businesses to adopt more efficient inventory management and delivery systems.
- Government could change the law to promote research productivity.
- New management method which can help companies to manage more efficiently.