ASSET PRICING

Using INFINITE-PERIOD FRAMEWORK

CHAPTER 8

OUTLINES

- MODEL Specification:
  - Basics – Explaining terms and concepts
  - Subjective discount factor
  - Utility function (The sum of all periods' utilities)
  - Budget constraints
- SOLVING MODEL using Lagrange (Sequential approach):
- ASSET PRICING (from model result, learn how stocks are priced)
  - How macroeconomic events affect asset prices
  - Understanding how the representative consumer maximize utility in this model
  - Long-run theory of Macroeconomics: From asset pricing, how we can understand relationship between impatience (discount factor) and real interest rate from Long-run perspective
- Steady-state: why are interest rates positive?

MODEL SPECIFICATIONS

BASICS

- Modern workhorse macroeconomic frameworks feature an infinite number of periods
  - A more realistic (?) view of time

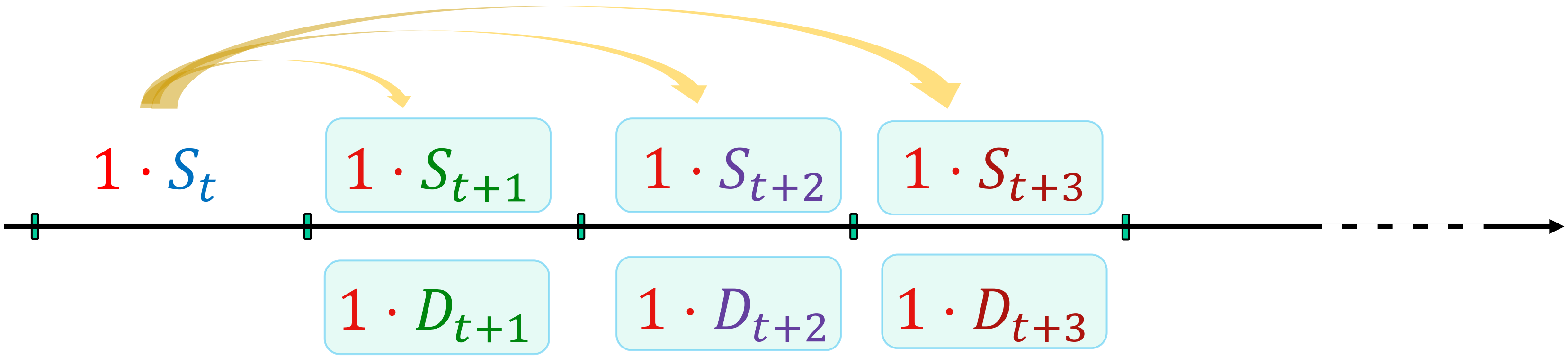
- Especially useful for thinking about asset accumulation and asset pricing
  - The intersection of modern macro theory and modern finance theory

BASICS

- Here, suppose just one real asset
  - Call it a "stock" – i.e., a share in the S&P 500
  - (In monetary analysis, two nominal assets: bonds and money)

- Index time periods by arbitrary indexes t, $t+1$, $t+2$, etc.
  - Important: all analysis conducted from the perspective of the very beginning of period t…
  - …so an "infinite future" (period $t+1$, period, $t+2$, period $t+3$, …) for which to save

Buying 1 stock



Total \$ earned from **1** stock after **1** period: $\frac{D_{t+1} + S_{t+1}}{1+i}$

If continue to hold:

Total \$ earned from **1** stock after **2** period: $\frac{D_{t+1}}{1+i} + \frac{D_{t+2} + S_{t+2}}{(1+i)^2}$

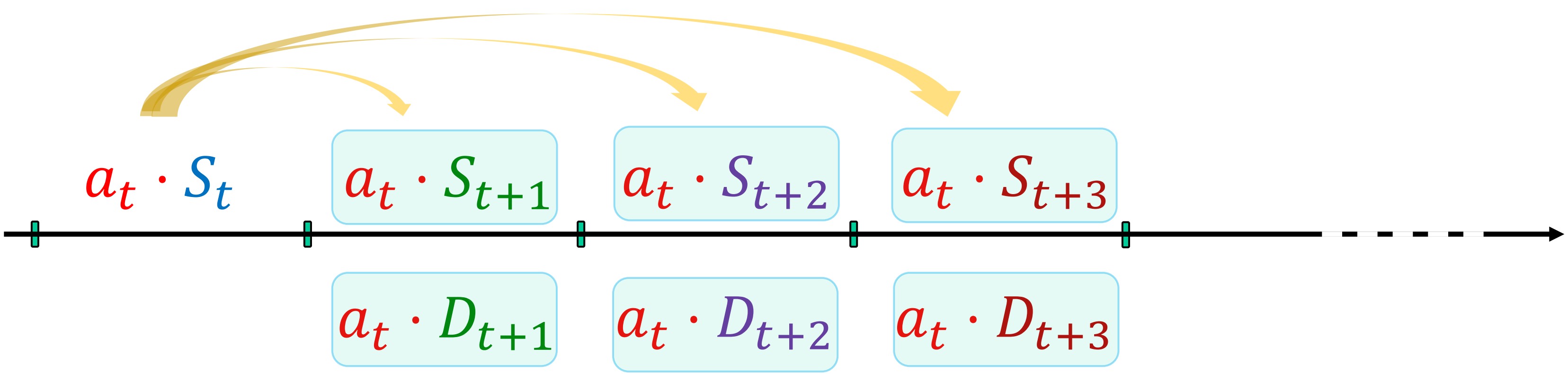
If continue to hold:

Total \$ earned from **1** stock after **3** period: $\frac{D_{t+1}}{1+i} + \frac{D_{t+2}}{(1+i)^2} + \frac{D_{t+3} + S_{t+3}}{(1+i)^{23}}$

Total \$ gotten from **1** stock after **n** periods:

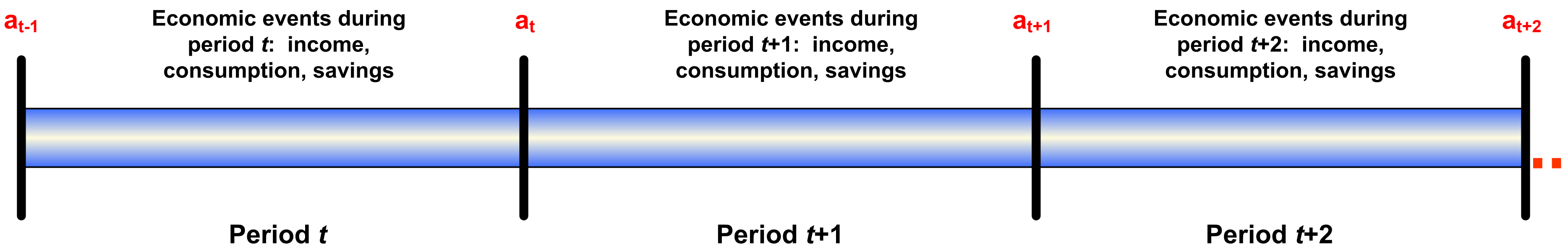
$$\underbrace{\frac{D_{t+1}}{1+i}+\frac{D_{t+2}}{(1+i)^2}+\frac{D_{t+3}}{(1+i)^3}+\cdots+\cdots+\frac{\boldsymbol{D_{t+n}}}{\boldsymbol{(1+i)^n}}}_{\textit{Flow of present-value discounted dividends}}+\frac{\boldsymbol{S_{t+n}}}{\boldsymbol{(1+i)^n}}$$

If buying a_t stock



Basics

- Timeline of events



- Notation
  - c_t: consumption in period t
  - P_t: nominal price of consumption in period t
  - Y_t: nominal income in period t ("falls from the sky")
  - a_{t-1}: number of stocks held at beginning of period t/end of period t-1 (this is the wealth brought to period t)

Basics

- Notation
  - …

The "definining features" of stock

  - S_t: nominal price of a unit of stock in period *t*
  - D_t: nominal dividend paid in period *t* by each unit of stock held at the start of *t*

  - π_{t+1}: inflation rate between period *t* and period *t*+1

$$\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t}$$

  - y_t: real income in period *t* ($= Y_t/P_t$)

Basics

- Notation
  - c_{t+1}: consumption in period t+1
  - P_{t+1}: nominal price of consumption in period t+1
  - Y_{t+1}: nominal income in period t+1 ("falls from the sky")
  - a_t: number of stocks held at beginning of period t+1/end of period t (this is the wealth brought to period t+1)

Basics

- Notation
  - ….
  - S_{t+1}: nominal price of a unit of stock in period t+1
  - D_{t+1}: nominal dividend paid in period t+1 by each unit of stock held at the start of t+1
  - π_{t+2}: net inflation rate between period $t+1$ and $t+2$

$$\pi_{t+2} = \frac{P_{t+2} - P_{t+1}}{P_{t+1}}$$

  - y_{t+1}: real income in period $t+1$ $\left(= \frac{Y_{t+1}}{P_{t+1}}\right)$
  - And so on for period t+2, t+3, etc…

The "definining features" of stock

Subjective Discount Factor

- Infinite number of periods *a more serious view of time*

- Impatience potentially an issue when taking a serious view of time

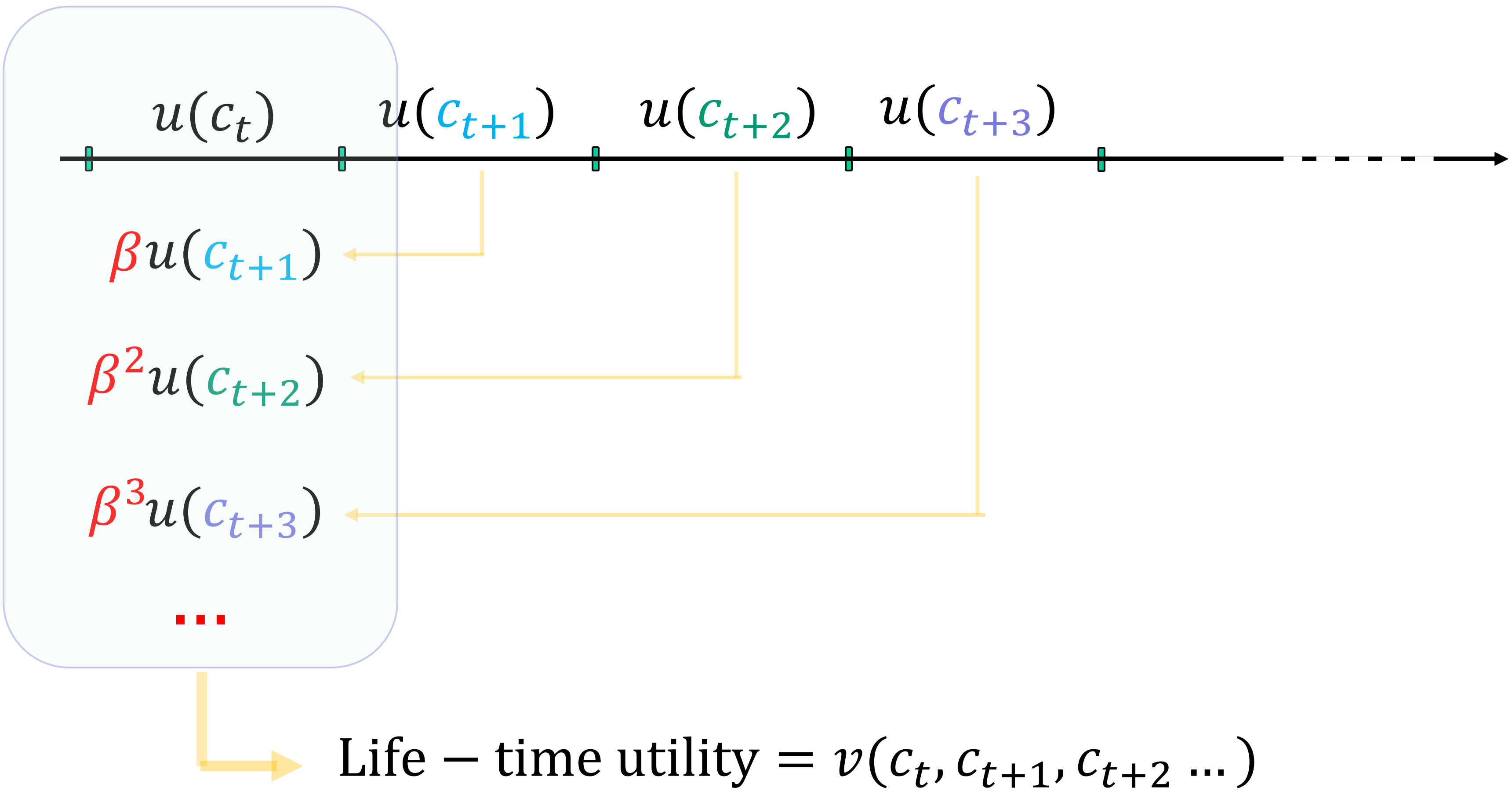
- Individuals (i.e., consumers) are impatient
  - All else equal, would rather have X utils today than identical X utils at some future date
  - An introspective statement about the world
  - An empirical statement about the world

Subjective Discount Factor

- Subjective discount factor
  - A simple model of consumer impatience
  - β (a number between zero and one) measures impatience
    - The lower is β, the less does individual value future utility → More impatient

SUBJECTIVE DISCOUNT FACTOR

- Subjective discount factor
  - ……
  - Simple assumption about how "impatience" builds up over time
    - Multiplicatively: i.e., discount one period ahead by β, discount two periods ahead by β^2, discount three periods ahead by β^3, etc.
    - Do individuals' impatience really build up over time in this way?...limited empirical evidence so really don't know…



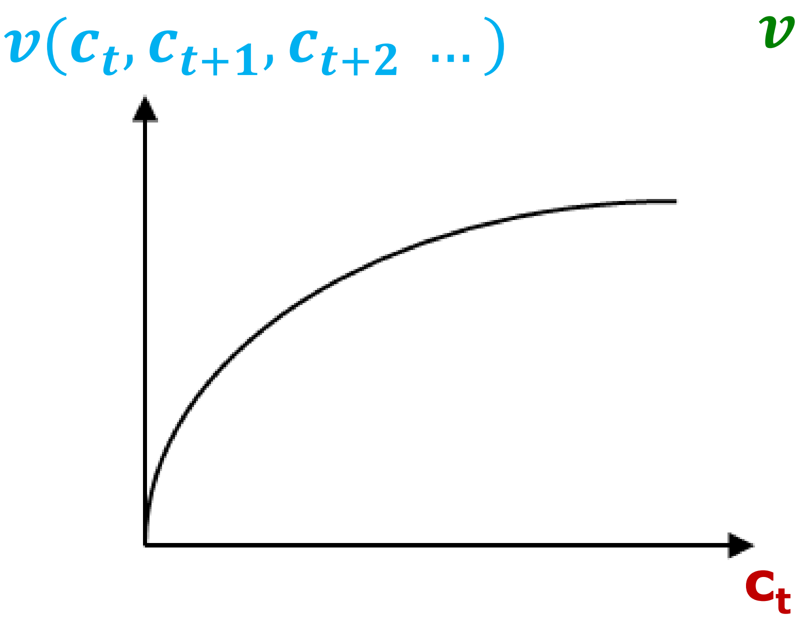
$$\text{Life} - \text{time utility} = v(c_t, c_{t+1}, c_{t+2} \dots)$$

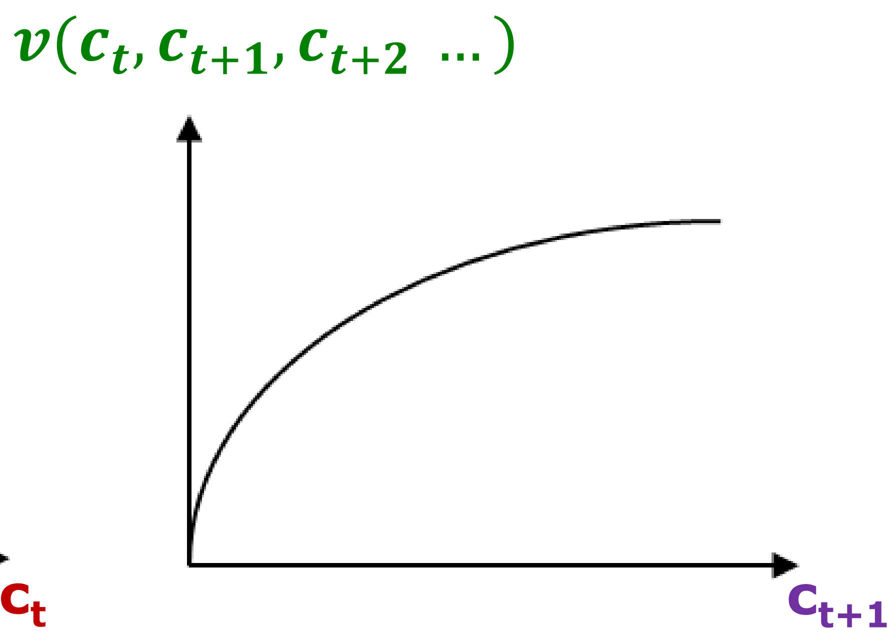
$$u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \cdots$$

UTILITY

present period is denoted as t

- Preferences $v(c_t, c_{t+1}, c_{t+2} \ldots)$ with all the "usual properties"
  - Lifetime utility function
  - Strictly increasing in c_t, c_{t+1}, c_{t+2} … Diminishing marginal utility in c_t, c_{t+1}, c_{t+2} …





…etc.

Utility

- Lifetime utility function additively-separable across time (a simplifying assumption), starting at time *t*

$$v(c_t, c_{t+1}, c_{t+2}, c_{t+3} \dots)$$

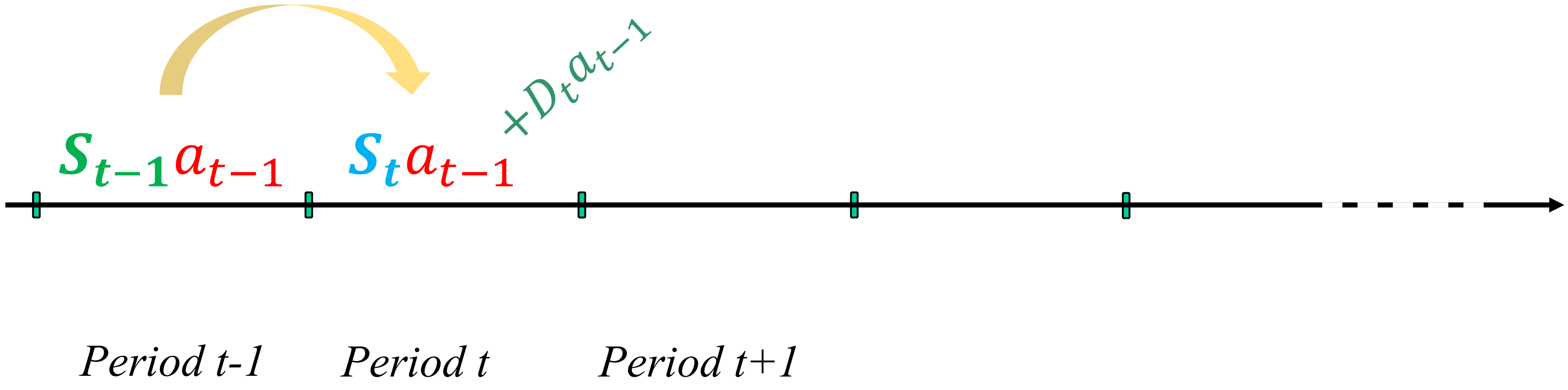
$$= u(c_t) + \boldsymbol{\beta} u(c_{t+1}) + \boldsymbol{\beta}^{\mathbf{2}} u(c_{t+2}) + \boldsymbol{\beta}^{\mathbf{3}} u(c_{t+3}) + \cdots$$

- Utility side of infinite-period framework no different than Chapter 1 model – except no longer possible to represent graphically

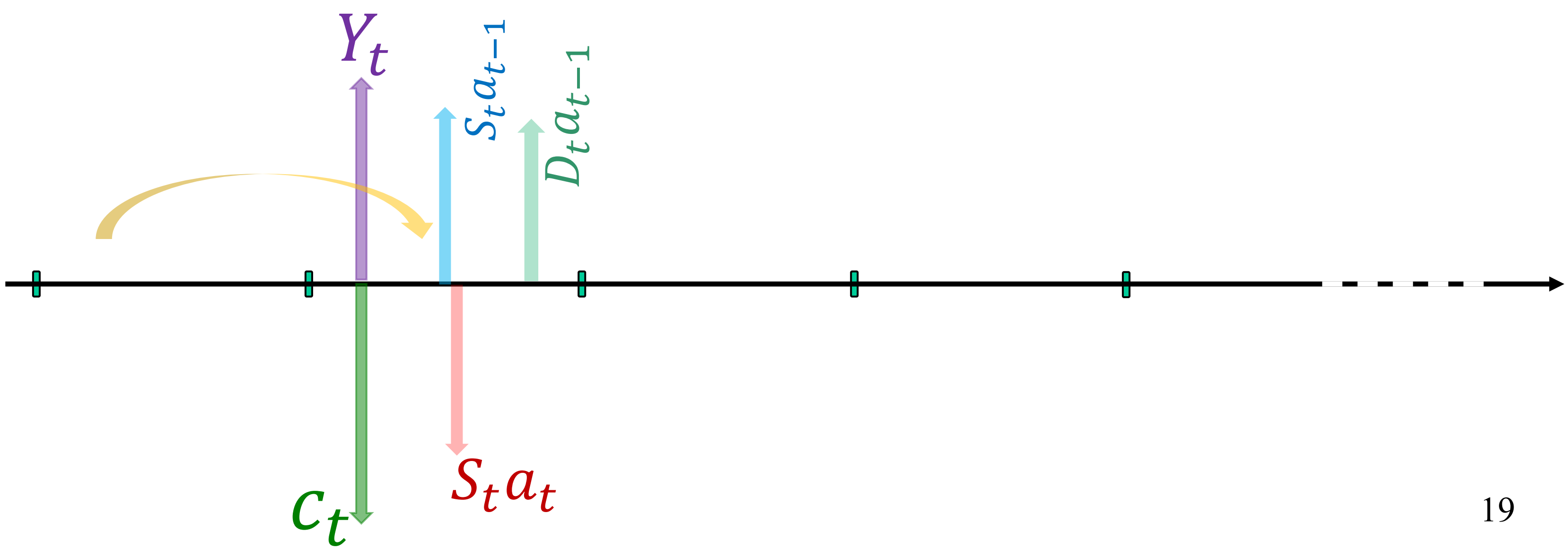
Budget Constraint(s)

- Suppose again Y "falls from the sky"
  - Y_t in period t, Y_{t+1} in period t+1, $Y_{\mathrm{t+2}}$ in period t+2, etc.

- Need infinite budget constraints to describe economic opportunities and possibilities
  - One for each period



$$P_t c_t + S_t a_t = Y_t + S_t a_{t-1} + D_t a_{t-1}$$



Budget Constraint(s)

- Period *t* budget constraint

$$\underbrace{P_t c_t + S_t a_t}_{\text{Total expenditure in period } t} = \underbrace{Y_t + S_t a_{t-1} + D_t a_{t-1}}_{\text{Total income in period } t}$$

Total expenditure in period *t*: period-*t* consumption + wealth to *carry into period t*+1

Total income in period *t*: period-*t* *Y* + income from stock-holdings *carried into period t* (has value S_t and pays dividend D_t)

Budget Constraint(s)

- Period *t+1* budget constraint

$$\underbrace{P_{t+1}c_{t+1} + S_{t+1}a_{t+1}} = \underbrace{Y_{t+1} + S_{t+1}a_t + D_{t+1}a_t}$$

<u>Total expenditure in period t+1:</u> period-t+1 consumption + wealth to *carry into period* t+2

<u>Total income in period t+1:</u> period-t+1 Y + income from stock-holdings *carried into period* t+1 (has value S_{t+1} and pays dividend D_{t+1})

$$P_t c_t + S_t a_t = Y_t + S_t a_{t-1} + D_t a_{t-1}$$

$$P_t c_t + S_t \underbrace{\left(a_t - a_{t-1}\right)}_{\Delta a_t} = Y_t + D_t a_{t-1}$$

$$\boxed{a_{t-1} + \Delta \boldsymbol{a}_t = \boldsymbol{a}_t}$$

Budget Constraint(s)

$$P_t c_t + S_t a_t = Y_t + S_t a_{t-1} + D_t a_{t-1}$$

can rewrite as

$$P_t c_t + \underbrace{S_t \overbrace{(a_t - a_{t-1})}^{\text{Savings during period } t \text{ (a flow)}}}_{S_t \Delta a_t} = Y_t + \overset{\text{Dividend income during period } t \text{ (a flow)}}{D_t a_{t-1}}$$

$$P_{t+1} c_{t+1} + S_{t+1} a_{t+1} = Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t$$

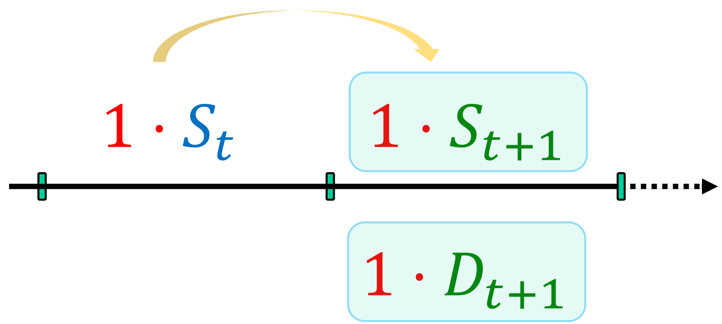
can rewrite as

$$P_{t+1} c_{t+1} + S_{t+1} \overbrace{(a_{t+1} - a_t)}^{\text{Savings during period } t+1 \text{ (a flow)}} = Y_{t+1} + \overset{\text{Dividend income during period } t+1 \text{ (a flow)}}{D_{t+1} a_t}$$

And identical-looking budget constraints for *t*+2, *t*+3, *t*+4, etc…

Budget Constraint(s)

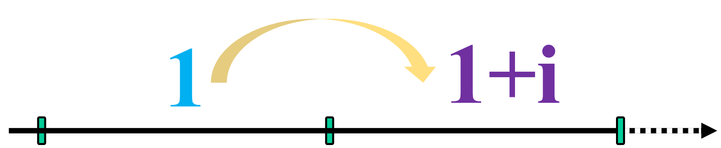
- Where is interest rate in the budget constraint?



S_t is the investment

$S_{t+1} + D_{t+1}$ is what we get back

$$\frac{S_{t+1} + D_{t+1}}{S_t} = 1 + i$$



Budget Constraint(s)

$$P_t c_t + S_t a_t = Y_t + S_t a_{t-1} D_t a_{t-1}$$

$$P_{t+1} c_{t+1} + S_{t+1} a_{t+1} = Y_{t+1} + \underbrace{S_{t+1} a_t + D_{t+1} a_t}_{S_t a_t (\mathbf{1} + \boldsymbol{i})}$$

SOLVING MODEL using Lagrange

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- Sequential formulation *highlights the role of stock holdings* (a_t) between period t and period $t+1$

  - Advantage: allows us to think about interaction between asset prices and macroeconomic events (intersection of finance theory and macro theory)

Lagrange Analysis: Sequential Approach

- Apply Lagrange tools to consumption-savings optimization
- Objective function: $v(c_t, c_{t+1}, c_{t+2} \ldots)$
- Constraints:
  - Period-t budget constraint: $Y_t + S_t a_{t-1} + D_t a_{t-1} - P_t c_t - S_t a_t = 0$
  - Period-t+1 budget constraint:
    $$Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t - P_{t+1} c_{t+1} - S_{t+1} a_{t+1} = 0$$
  - Period-t+2 budget constraint:
    $$Y_{t+2} + S_{t+2} a_{t+1} + D_{t+2} a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2} = 0$$
  - **etc…**

INFINITE constraints

- Seq. Lagrange formulation requires inf. multipliers

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- Step 1: Construct Lagrange function (starting from t)

IMPORTANT:
Discount factor β multiplies *both* future utility *and* future budget constraints

Everything (utility *and* income) about the future is discounted

$$u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \cdots$$

$$+\lambda_t[Y_t + (S_t + D_t)a_{t-1} - P_t c_t - S_t a_t]$$

$$+\beta\lambda_{t+1}[Y_{t+1} + (S_{t+1} + D_{t+1})a_t - P_{t+1}c_{t+1} - S_{t+1}a_{t+1}]$$

$$+\beta^2 \lambda_{t+2}[Y_{t+2} + (S_{t+2} + D_{t+2})a_{t+1} - P_{t+2}c_{t+2} - S_{t+2}a_{t+2}]$$

$$+\beta^3 \lambda_{t+3}[Y_{t+3} + (S_{t+3} + D_{t+3})a_{t+2} - P_{t+3}c_{t+3} - S_{t+3}a_{t+3}]$$

…

$$\partial L(\boldsymbol{c_t}, c_{t+1}, c_{t+2} \dots ; a_t, a_{t+1}, a_{t+2}, \dots ; \lambda_t, \lambda_{t+1}, \dots)/\partial \boldsymbol{c_t} =$$

$$\frac{\partial \left\{ \begin{array}{l} u(\boldsymbol{c_t}) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \cdots \\ +\lambda_t [Y_t + (S_t + D_t) a_{t-1} - P_t \boldsymbol{c_t} - S_t a_t] \\ +\beta \lambda_{t+1} [Y_{t+1} + (S_{t+1} + D_{t+1}) a_t - P_{t+1} c_{t+1} - S_{t+1} a_{t+1}] \\ +\beta^2 \lambda_{t+2} [Y_{t+2} + (S_{t+2} + D_{t+2}) a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2}] \\ \dots \end{array} \right\}}{\partial \boldsymbol{c_t}}$$

$$= 0$$

$$\boxed{u'(c_t) - \lambda_t P_t = 0}$$

$$\partial L(c_t, c_{t+1}, c_{t+2} \ldots ; \boldsymbol{a_t}, a_{t+1}, a_{t+2}, \ldots ; \lambda_t, \lambda_{t+1}, \ldots)/\partial \boldsymbol{a_t} =$$

$$\frac{\partial \left\{ \begin{array}{l} u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \cdots \\ +\lambda_t [Y_t + (S_t + D_t) a_{t-1} - P_t c_t - S_t \boldsymbol{a_t}] \\ +\beta \lambda_{t+1} [Y_{t+1} + (S_{t+1} + D_{t+1}) \boldsymbol{a_t} - P_{t+1} c_{t+1} - S_{t+1} a_{t+1}] \\ +\beta^2 \lambda_{t+2} [Y_{t+2} + (S_{t+2} + D_{t+2}) a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2}] \\ \cdots \end{array} \right\}}{\partial \boldsymbol{a_t}}$$

$$= 0$$

$$\boxed{\boldsymbol{\lambda_t} S_t - \beta \boldsymbol{\lambda_{t+1}} [S_{t+1} + D_{t+1}] = 0}$$

$$\partial L(c_t, \boldsymbol{c_{t+1}}, c_{t+2} \dots; a_t, a_{t+1}, a_{t+2}, \dots; \lambda_t, \lambda_{t+1}, \dots)/\partial \boldsymbol{c_{t+1}} =$$

$$\frac{\partial \left\{ \begin{array}{l} u(c_t) + \beta u(\boldsymbol{c_{t+1}}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \cdots \\ +\lambda_t [Y_t + (S_t + D_t) a_{t-1} - P_t c_t - S_t a_t] \\ +\beta \lambda_{t+1} [Y_{t+1} + (S_{t+1} + D_{t+1}) a_t - P_{t+1} \boldsymbol{c_{t+1}} - S_{t+1} a_{t+1}] \\ +\beta^2 \lambda_{t+2} [Y_{t+2} + (S_{t+2} + D_{t+2}) a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2}] \\ \dots \end{array} \right\}}{\partial \boldsymbol{c_{t+1}}}$$

$$= 0$$

3rd FOC: $\boxed{\cancel{\beta} u'(c_{t+1}) - \cancel{\beta} \boldsymbol{\lambda_{t+1}} P_{t+1} = 0}$

Very similar to: $\boxed{u'(c_t) - \boldsymbol{\lambda_t} P_t = 0}$

Lagrange Analysis: Sequential Approach

- Step 1: Construct Lagrange function (starting from t)
- Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …

Identical except for time subscripts

with respect to c_t: $$u'(c_t) - \lambda_t P_t = 0$$ **Equation 1**

with respect to a_t: $$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ **Equation 2**

with respect to c_{t+1}: $$\cancel{\beta} u'(c_{t+1}) - \cancel{\beta}\lambda_{t+1} P_{t+1} = 0$$ **Equation 3**

ASSET PRICING
(from model result, learn how stocks are priced)

The Basics of Asset Pricing

$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ **Equation 2**

$$\lambda_t S_t = \beta\lambda_{t+1}(S_{t+1} + D_{t+1})$$

$$S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$$

The Basics of Asset Pricing

$$S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$$

Period-*t* stock price = **Pricing kernel** x **Future return**

BASIC ASSET-PRICING EQUATION

finance

Price of financial asset = **discounted value** of **future earnings**.

In **finance**, the discounting factor is just a generic **pricing kernel** (from empirics) found by running regression of $S_{t+1} + D_{t+1}$ on to S_t

Here, we explain this price kernel theoretically in economics.

The Basics of Asset Pricing

- Much of finance theory concerned with pricing kernel
  - Theoretical properties
  - Empirical models of kernels

- Pricing kernel where macro theory and finance theory intersect
  - Allows studying common "macro factors" that affect "all" asset markets/asset prices

- To take more macro-centric view
  - Solve equations 1 and 3 for λ_t and λ_{t+1}
  - Insert in asset-pricing equation

$$u'(c_t) - \lambda_t P_t = 0$$

$$u'(c_t) = \lambda_t P_t$$

Equation 1

$$\beta u'(c_{t+1}) - \beta \lambda_{t+1} P_{t+1} = 0$$

$$u'(c_{t+1}) = \lambda_{t+1} P_{t+1}$$

Equation 3

$$\Rightarrow \frac{u'(c_t)}{u'(c_{t+1})} = \frac{\lambda_t P_t}{\lambda_{t+1} P_{t+1}}$$

$$\Rightarrow \frac{u'(c_t)}{u'(c_{t+1})} \cdot \frac{P_{t+1}}{P_t} = \frac{\lambda_t}{\lambda_{t+1}} \Rightarrow \boxed{\frac{u'(c_{t+1})}{u'(c_t)} \cdot \frac{P_t}{P_{t+1}} = \frac{\lambda_{t+1}}{\lambda_t}}$$

Macroeconomic Events Affect Asset Prices

$$S_t = \left(\frac{\beta \lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$$

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

MRS of consumption across time

Using definition of inflation:

$$1 + \pi_{t+1} = \frac{P_{t+1}}{P_t}$$

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{1}{1 + \pi_{t+1}}\right)$$

MACROECONOMIC EVENTS AFFECT ASSET PRICES

- Consumption across time (c_t and c_{t+1}) affects stock prices
  - Fluctuations over time in aggregate consumption impact S_t

- Inflation affects stock prices
  - Fluctuations over time in inflation impact S_t

- ANY factor (monetary policy, fiscal policy, globalization, etc.) that affects inflation and GDP in principle impacts stock/asset markets

Since consumption is affected by income

Consumer Optimization

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS between period } t \text{ consumption and period } t+1 \text{ consumption}} = \underbrace{\left(\frac{S_{t+1} + D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)}_{\boxed{\text{must be } (1+r_t)}}$$

i.e., ratio of marginal utilities

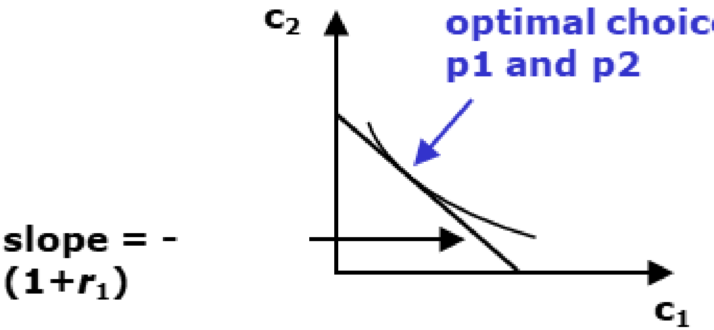
MRS between period t consumption and period t+1 consumption

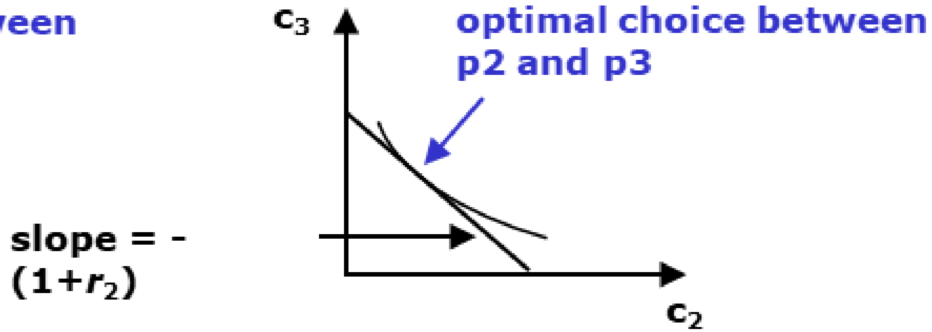
must be $(1+r_t)$

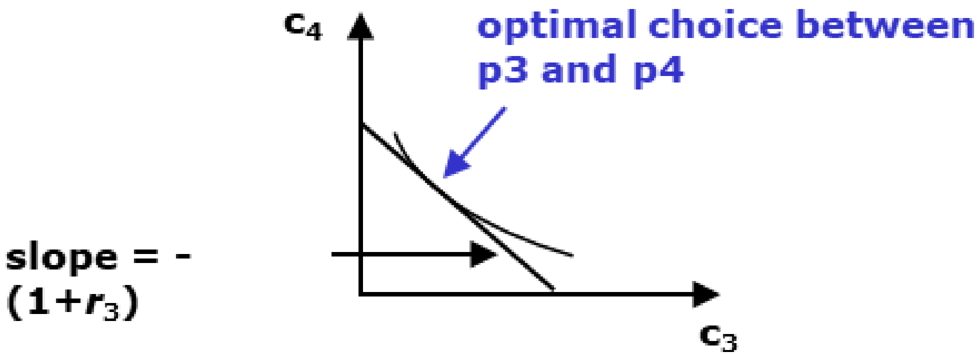
Recall real interest rate is a <u>price</u>

CONSUMER OPTIMIZATION

- Infinite-period framework is sequence of overlapping two-period frameworks







etc
…

A Long-Run Theory of Macro

- Consumption-savings optimality condition at the heart of modern macro theories
  - Emphasize the dynamic nature of aggregate economic events
  - Foundation for understanding the periodic ups and downs ("business cycles") of the economy
  - (Chapter 14: business cycle theories)

$$\boxed{\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t}$$

Back to outline

A Long-Run Theory of Macro

$$\boxed{\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t}$$

$$\frac{1}{\beta} = 1 + r$$

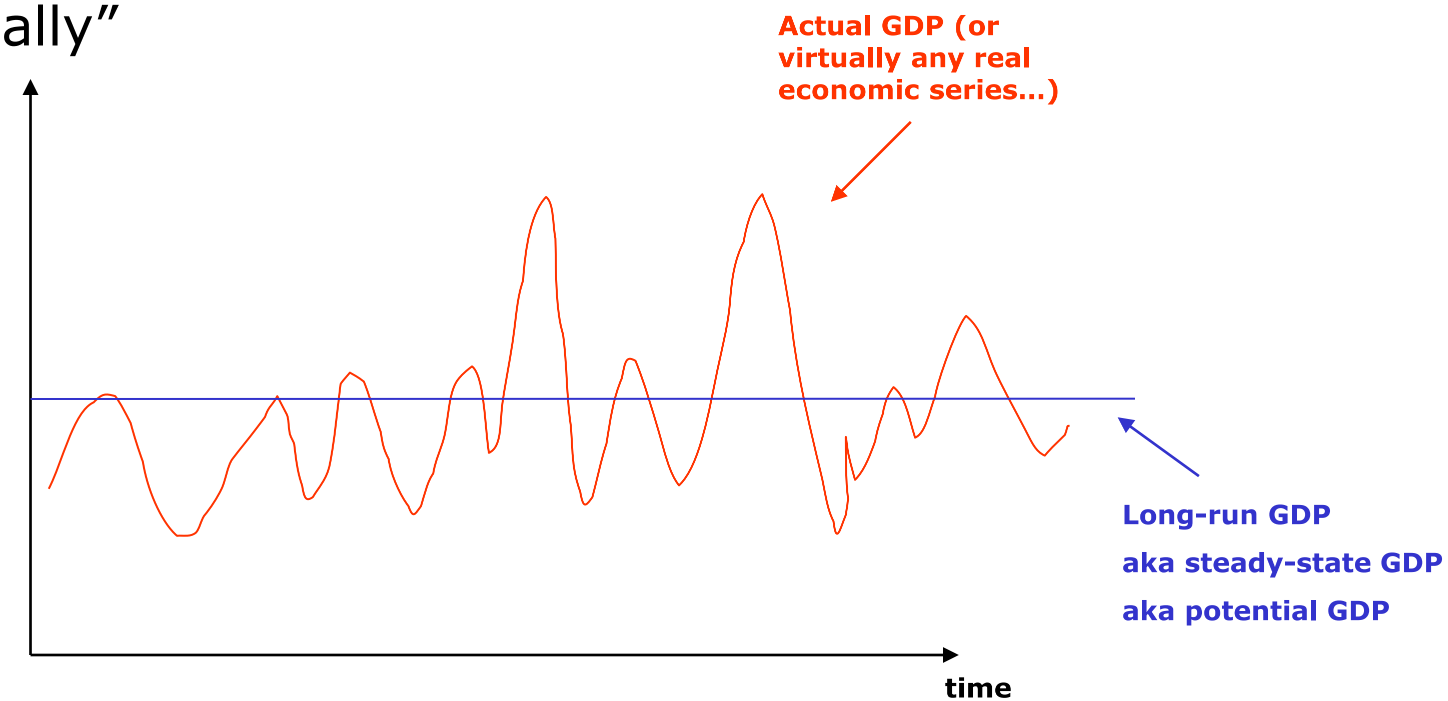
NEXT: Impose "steady state" and examine long-run relationship between interest rates and consumer impatience

Steady-State (Long-Run) of Infinite-Period Economy: Why are Interest Rates Positive?

Back to outline

A LONG-RUN THEORY OF MACRO

- Aggregate economic activity tends to "settle down eventually"



- The "ups and downs" are business cycles
- The "average" is the long-run
  - Technical terminology: steady-state

STEADY STATE

- Steady state
  - Heuristic definition: in a dynamic (mathematical) system, a steady-state is a condition in which the variables that are moving over time settle down to constant values

Steady State

- In dynamic macro models, a steady state is a condition in which all *real* variables *settle down to constant values*
  - But *nominal* variables (i.e., price level) may still be moving over time (will be important in monetary models)
  - Simple example
    - …

STEADY STATE

- …
  - …
  - Simple example
    - Suppose $M_t/P_t = c_t$ is an optimality condition of an economic model (c_t is consumption, P_t is nominal price level, M_t is nominal money stock of economy)
    - Even if $\boldsymbol{c_t}$ eventually becomes constant over time (i.e., reaches a steady-state), it is *possible* for *both* M_t and P_t to continue growing over time (at the same rate of course…)

- Bottom line: in ss, *real* variables do not change over time, nominal variables may change over time (*inflation* is a *real* variable)

A Long-Run Theory of Macro

- Consumption-savings optimality condition at the heart of modern macro analysis

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t$$

Steady-state:

$c_t = c_{t+1} = c$

And $r_t = r_{t+1} = r$

(i.e., just dropping all time subscripts on real variables!)

$$\frac{\cancel{u'(c)}}{\beta \cancel{u'(c)}} = 1 + r_t$$

$$\underbrace{\frac{1}{\beta}}_{\text{Inverse of subjective discount factor}} = \underbrace{1 + r}_{\text{(one plus) real interest rate}}$$

KEY RELATIONSHIP

Inverse of subjective discount factor

(one plus) real interest rate

REAL INTEREST RATE

- Recall earlier interpretation of r
  - Price of consumption in a given period in terms of consumption in the next period
  - (Chapter 3 & 4: *r measures the price of period-1 consumption in terms of period-2 consumption*)

$$\frac{1}{\beta} = 1 + r$$

REAL INTEREST RATE

- Now a second interpretation of r: long-run (i.e., steady state) real interest rate simply a reflection of degree of impatience of individuals in an economy

  *Comment: **in short run**, it is possible to have negative r*

  - The lower is β, the higher is r
  - The more impatient a populace is, the higher are interest rates

- Which came first, β or r?
  - Modern macro view: β < 1 causes r > 0, not the other way around
  - A deep view of <u>why</u> positive interest rates exist in the world