LECTURE 2

Diversification & the Efficient Frontier of Risky Assets

EC3333 Financial Economics I

Learning Objectives

- Given a portfolio of stocks, including the holdings in each stock, the expected return in each stock, and its variance, compute the following:
  - Expected return on the portfolio;
  - Variance and volatility of the portfolio; and
  - Covariance and correlation of each pair of stocks in the portfolio.
- Define an efficient portfolio and the efficient frontier.
- Describe short sales and how it extends the efficient frontier.
- Define and contrast idiosyncratic (firm specific) and systematic (market) risk.
- Discuss the limits to diversification.

Portfolios of Risky Assets

- Portfolio Weights
  - The fraction of the total investment in the portfolio held in each individual investment in the portfolio.
  - The portfolio weights must add up to 1.

$$x_i = \frac{\text{Value of investment } i}{\text{Total value of portfolio}} = \frac{\text{Value of investment } i}{\sum_i \text{ Value of investment } i}$$

- The return on the portfolio R_p is the weighted average of the returns on the investments in the portfolio.

$$R_P = x_1 R_1 + x_2 R_2 + \ldots + x_n R_n = \sum_i x_i R_i$$

$$E[R_P] = E\left[\sum_i x_i R_i\right] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i]$$

Portfolio Return: An Example

- Portfolio return

$$R_p = x_D R_D + x_E R_E$$

x_D = Portfolio weight on Bond

R_D = Bond return

x_E = Portfolio weight on Equity

R_E = Equity return

- Portfolio's expected return

$$E(R_p) = x_D E(R_D) + x_E E(R_E)$$

Portfolio Risk: An Example

- Portfolio variance

$$\sigma_p^2 = x_D^2\sigma_D^2 + x_E^2\sigma_E^2 + 2x_D x_E \,\mathrm{Cov}(R_D, R_E)$$

$$\sigma_D^2 = \text{ Bond variance}$$

$$\sigma_E^2 = \text{ Equity variance}$$

$$\mathrm{Cov}(R_D, R_E) = \sigma_{DE}$$

$$=\text{Covariance of returns for bond and equity}$$

Covariance

- More generally, covariance is the expected product of the deviations of two returns from their means.
- Covariance between Returns *Ri* and *Rj*

$$\text{Cov}(R_i, R_j) = E[(R - E[R_i])(R_j - E[R_j])]$$

- Estimate of the Covariance from Historical Data

$$\text{Cov}(R_i, R_j) = \frac{1}{T-1}\sum_t \left(R_{it} - \bar{R}_i\right)\left(R_{j,t} - \bar{R}_j\right)$$

  - If the covariance is positive, the two returns tend to move together.
  - If the covariance is negative, the two returns tend to move in opposite directions.
  - But high covariance could be due to the returns being more volatile or the returns moving more closely together

Covariance and Correlation

- Covariance of returns on bond and equity

$$\text{Cov}(r_D r_E) = \rho_{DE}\sigma_D\sigma_E$$

  - ρ_{DE} = Correlation coefficient of returns
    - A measure of the common risk shared by assets that does not depend on their volatility
  - σ_D = Standard deviation of bond returns
  - σ_E = Standard deviation of equity returns

- Range of values for the correlation coefficient ρ

$$-1 \leq \rho \leq 1$$

  - If $\rho = 1$, the securities are perfectly positively correlated
  - If $\rho = 0$, the securities are uncorrelated
  - If $\rho = -1$, the securities are perfectly negatively correlated

The Volatility of a Portfolio of Stocks

- The variance of a portfolio is equal to the weighted average covariance of each stock with the portfolio

$$\mathrm{Var}(R_P) = \mathrm{Cov}(R_P, R_P) = \mathrm{Cov}\left(\sum_i x_i R_i, R_P\right) = \sum_i x_i \,\mathrm{Cov}(R_i, R_P)$$

- The risk of a portfolio depends on how each stock's return moves in relation to it

$$\begin{aligned}\mathrm{Var}(R_P) &= \sum_i x_i \,\mathrm{Cov}(R_i, R_p) = \sum_i x_i \,\mathrm{Cov}(R_i, \Sigma_j x_j R_j) \\ &= \sum_i \sum_j x_i x_j \,\mathrm{Cov}(R_i, R_j)\end{aligned}$$

- The overall variability of the portfolio depends on the total co-movement of the stocks within it

Example: Two-Asset Portfolio

- For a Two-Asset portfolio

$$\begin{aligned}\mathrm{Var}(R_p) &= \mathrm{Cov}(R_p, R_p) \\ &= \mathrm{Cov}(x_1R_1 + x_2R_2, x_1R_1 + x_2R_2) \\ &= x_1x_1\,\mathrm{Cov}(R_1, R_1) + x_1x_2\,\mathrm{Cov}(R_1, R_2) + x_2x_1\,\mathrm{Cov}(R_2, R_1) + x_2x_2\,\mathrm{Cov}(R_2, R_2)\end{aligned}$$

- The Variance of a Two-Asset Portfolio

$$\mathrm{Var}(R_p) = x_1^2\,\mathrm{Var}(R_1) + x_2^2\,\mathrm{Var}(R_2) + 2x_1x_2\,\mathrm{Cov}(R_1, R_2)$$

Correlation Coefficient & Diversification

- When $\rho_{DE} = 1$:

$$\sigma_P = x_E\sigma_E + x_D\sigma_D$$

- When $\rho_{DE} = -1$, a perfect hedge is possible.
  - To find it, set σ_P to 0, and solve for the portfolio weight

$$\sigma_P = x_E\sigma_E - x_D\sigma_D = 0$$

$$x_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - x_D$$

Efficient vs. Inefficient Portfolios

- In an inefficient portfolio, it is possible to find another portfolio that is better in terms of both expected return and volatility.
- The efficient portfolios are the portfolios that offer the highest possible expected return for a given level of volatility
- In other words, in an efficient portfolio there is no way to reduce the volatility of the portfolio without lowering its expected return.

Table 11.4 Expected Returns and Volatility for Different Portfolios of Two Stocks: Intel and Coca-Cola
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.4
0.00	1.00	6.0	25.0

Figure 11.3 Volatility Versus Expected Return for Portfolios of Intel and Coca-Cola Stock (suppose these stocks are *uncorrelated*)
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

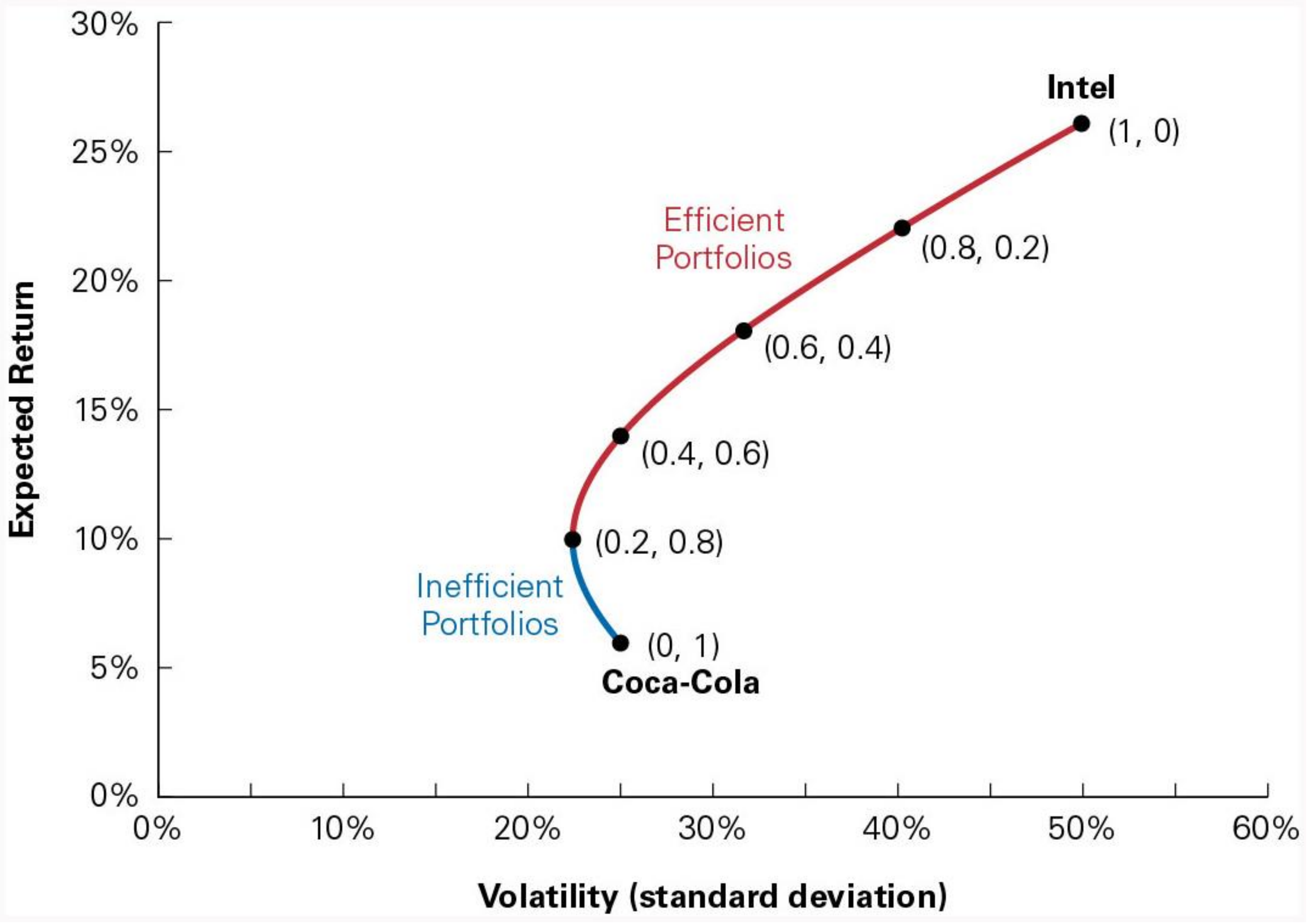
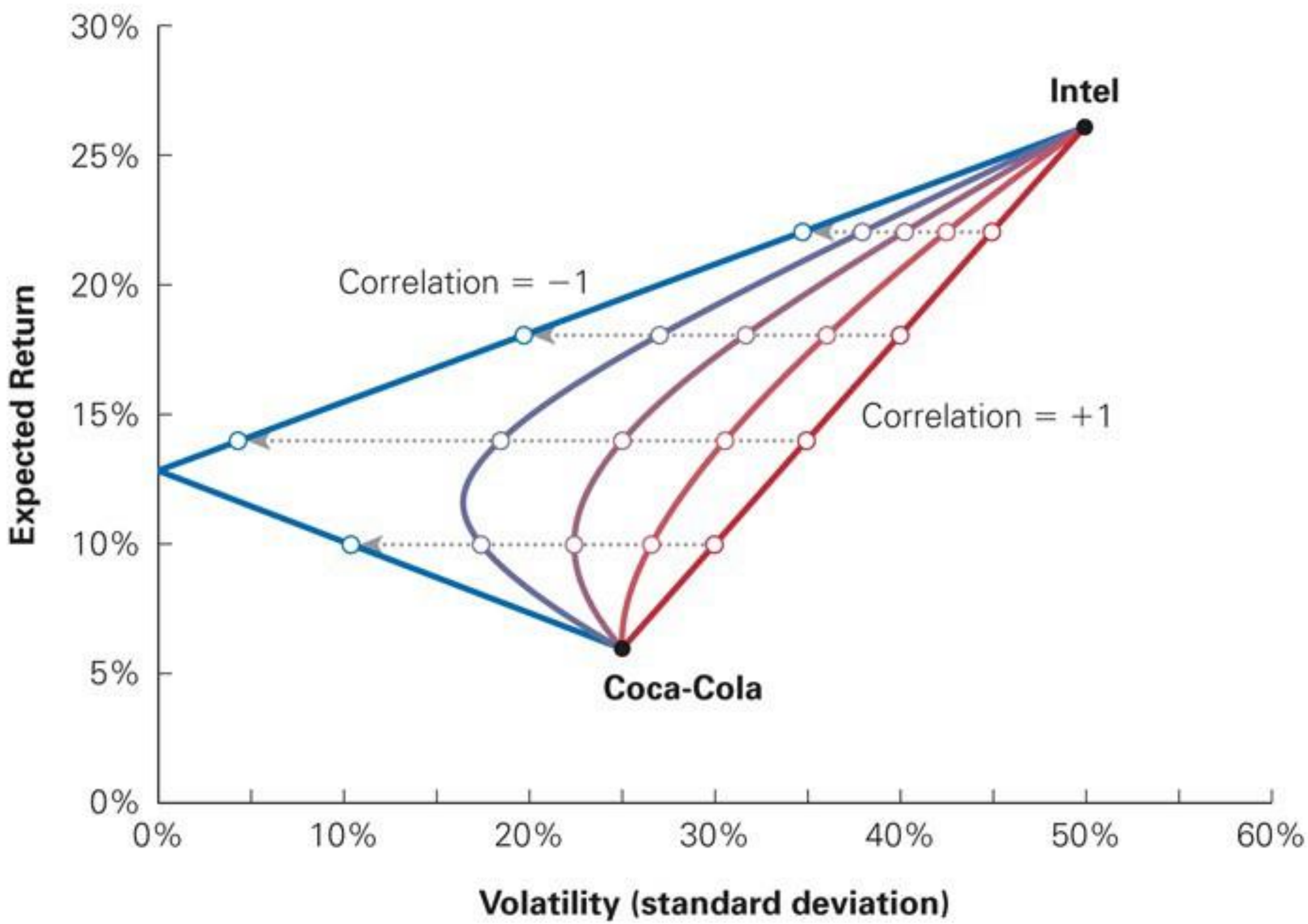


Figure 11.4 Effect on Volatility and Expected Return of Changing the Correlation between Intel and Coca-Cola Stock
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)



The Minimum Variance Portfolio

- The minimum variance portfolio is the portfolio composed of the risky assets that has the smallest standard deviation; i.e. the portfolio with least risk.

- The amount of possible risk reduction through diversification depends on the correlation:
  - If $\rho = 1$, no risk reduction is possible;
  - If $\rho = 0$, σ_P may be less than the standard deviation of either component asset; and
  - If $\rho = -1$, a riskless hedge is possible.

The Minimum Variance Portfolio with Two Risky Assets: Derivation

- To minimize portfolio variance, differentiate wrt x_D

$$\sigma_p^2 = x_D^2 \sigma_D^2 + (1 - x_D)^2 \sigma_E^2 + 2 x_D (1 - x_D) \rho_{DE} \sigma_D \sigma_E$$

- Set the derivative equal to zero and solve for x_D

$$x_D^* = \frac{\sigma_E^2 - \rho_{DE} \sigma_D \sigma_E}{\sigma_D^2 + \sigma_E^2 - 2 \rho_{DE} \sigma_D \sigma_E}$$

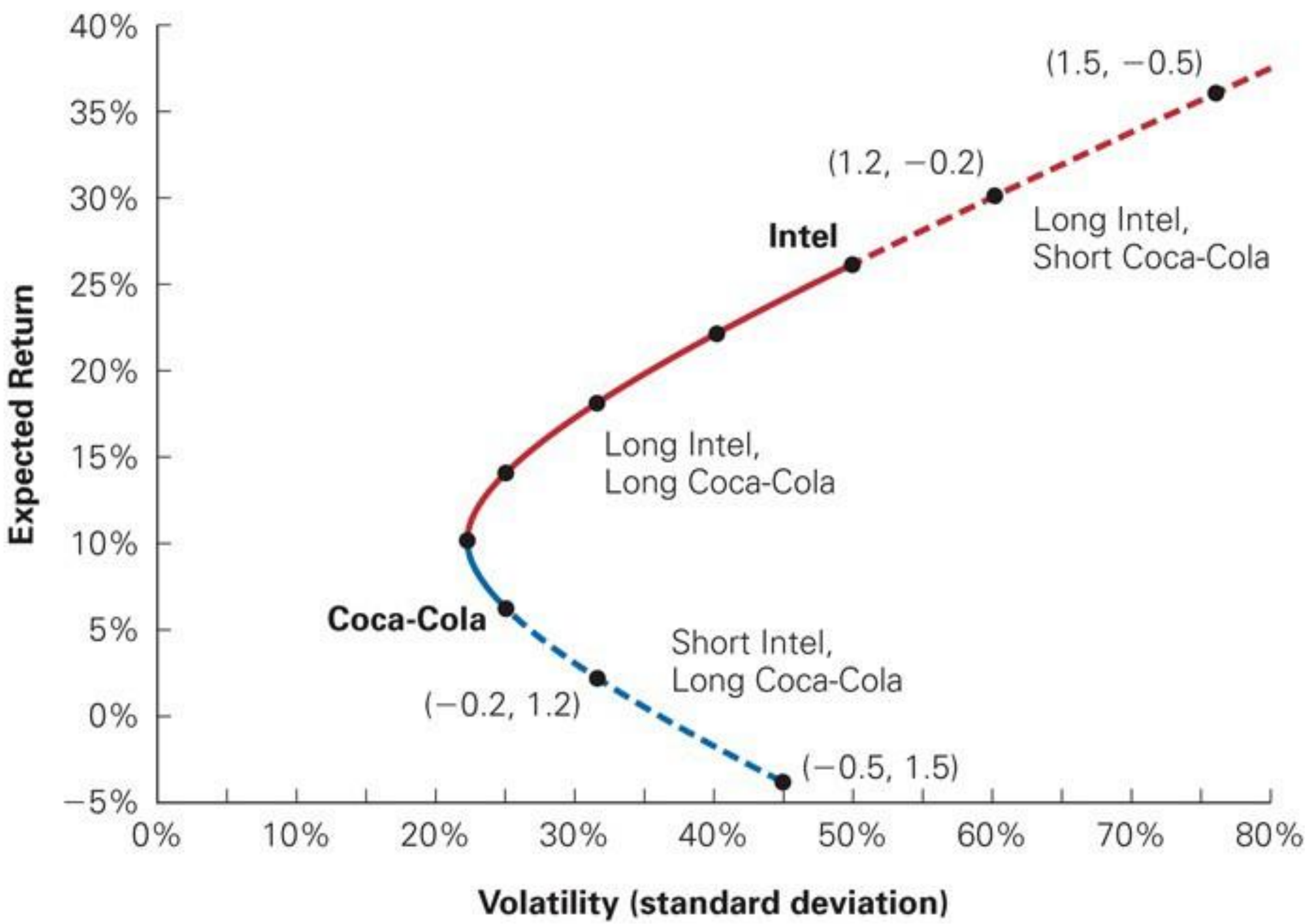
$$x_E^* = 1 - x_D$$

Short Sales

- Long Position
  - A positive investment in a security

- Short Position
  - A negative investment in a security
  - In a short sale, you sell a stock that you do not own and then buy that stock back in the future

Figure 11.5 Figure 11.5 Portfolios of Intel and Coca-Cola Allowing for Short Sales
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)



Efficient Portfolio with more than 2 stocks

- What happens when a third stock is added to the two-stock portfolio?

  (Table from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

				Correlation with	
Stock	**Expected Return**	**Volatility**	**Intel**	**Coca-Cola**	**Bore Ind.**
Intel	26%	50%	1.0	0.0	0.0
Coca-Cola	6%	25%	0.0	1.0	0.0
Bore Industries	2%	25%	0.0	0.0	1.0

Figure 11.6 Expected Return and Volatility for Selected Portfolios of Intel, Coca-Cola, and Bore Industries Stocks
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

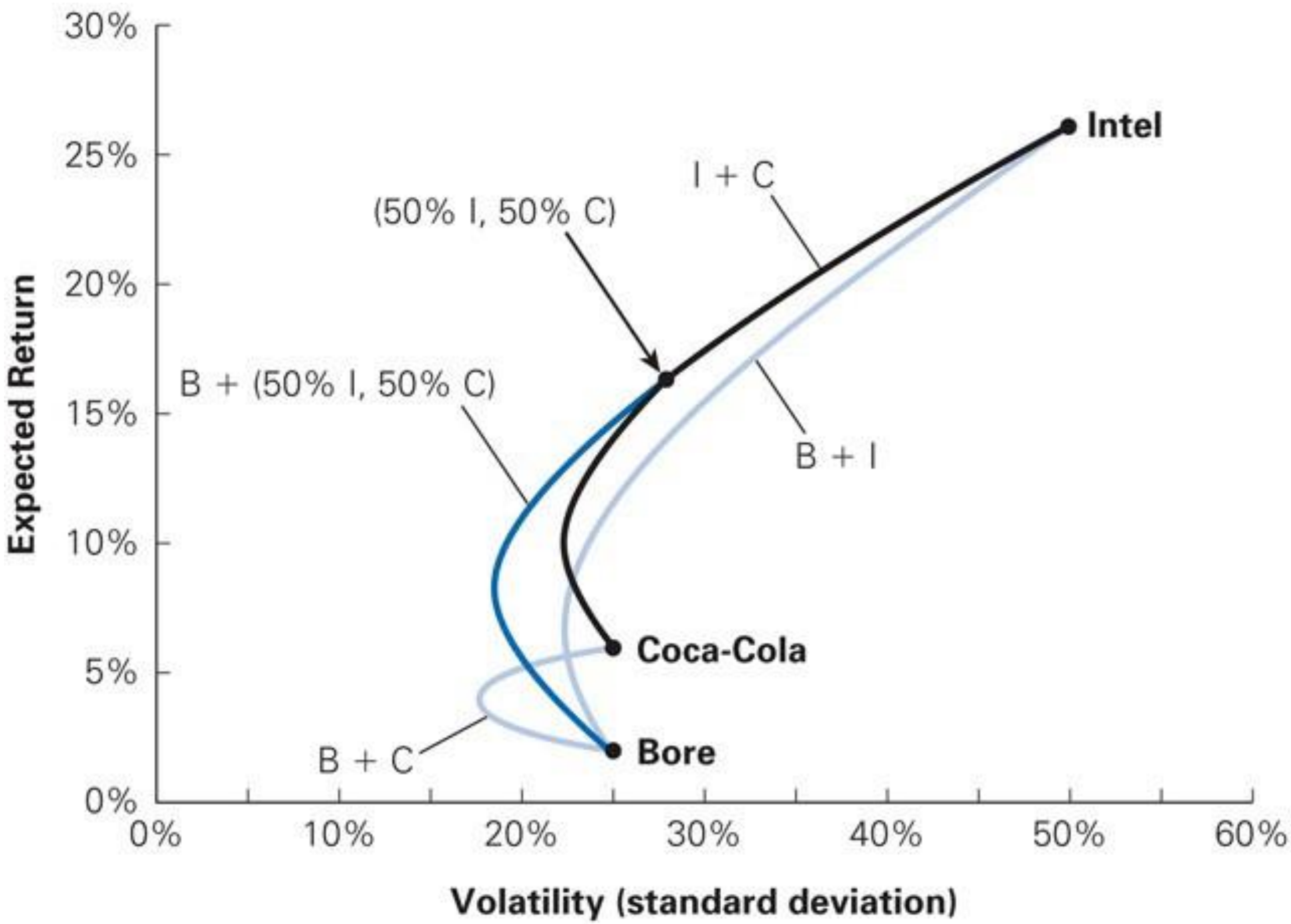


Figure 11.7 The Volatility and Expected Return for All Portfolios of Intel, Coca-Cola, and Bore Stock
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

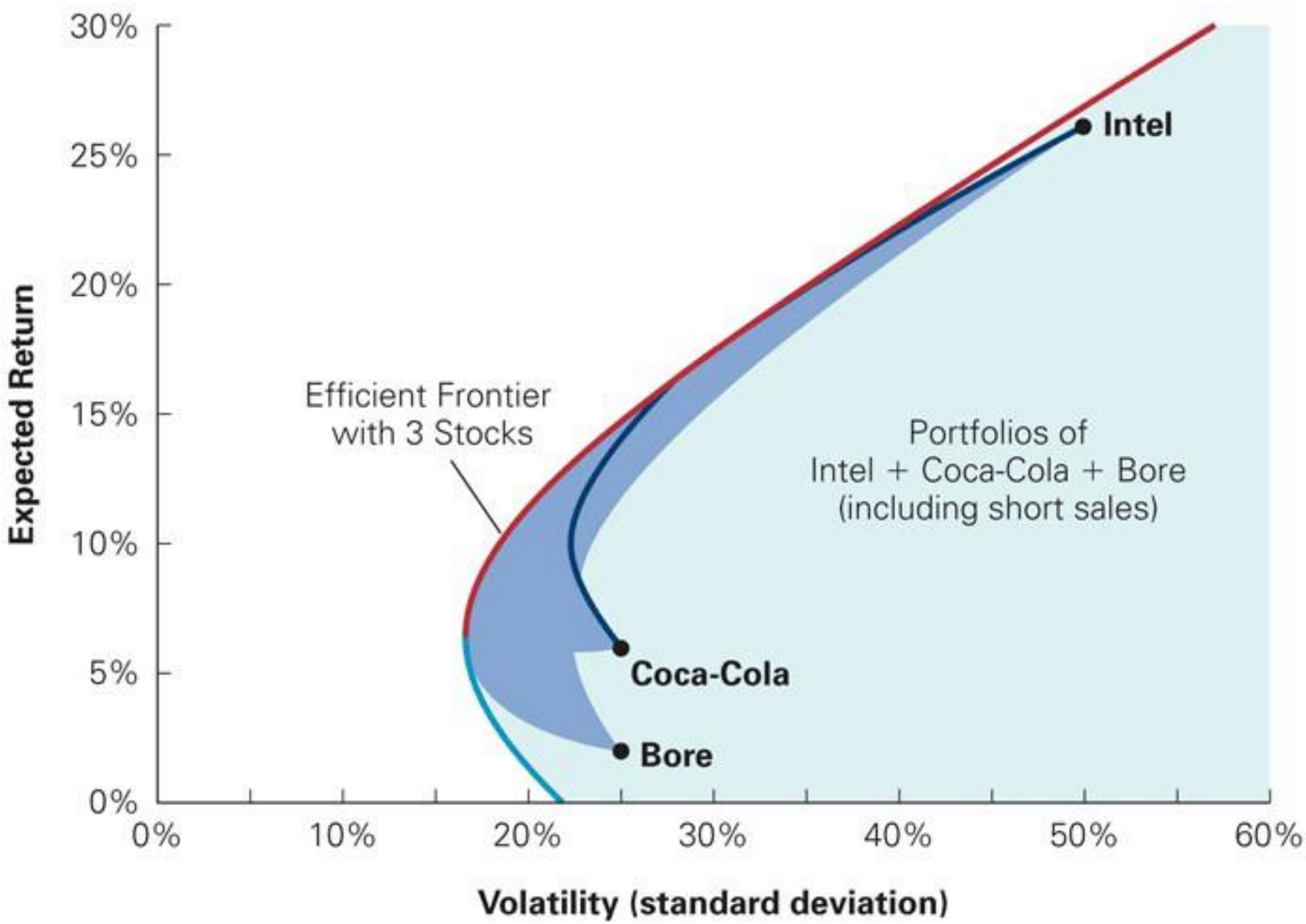
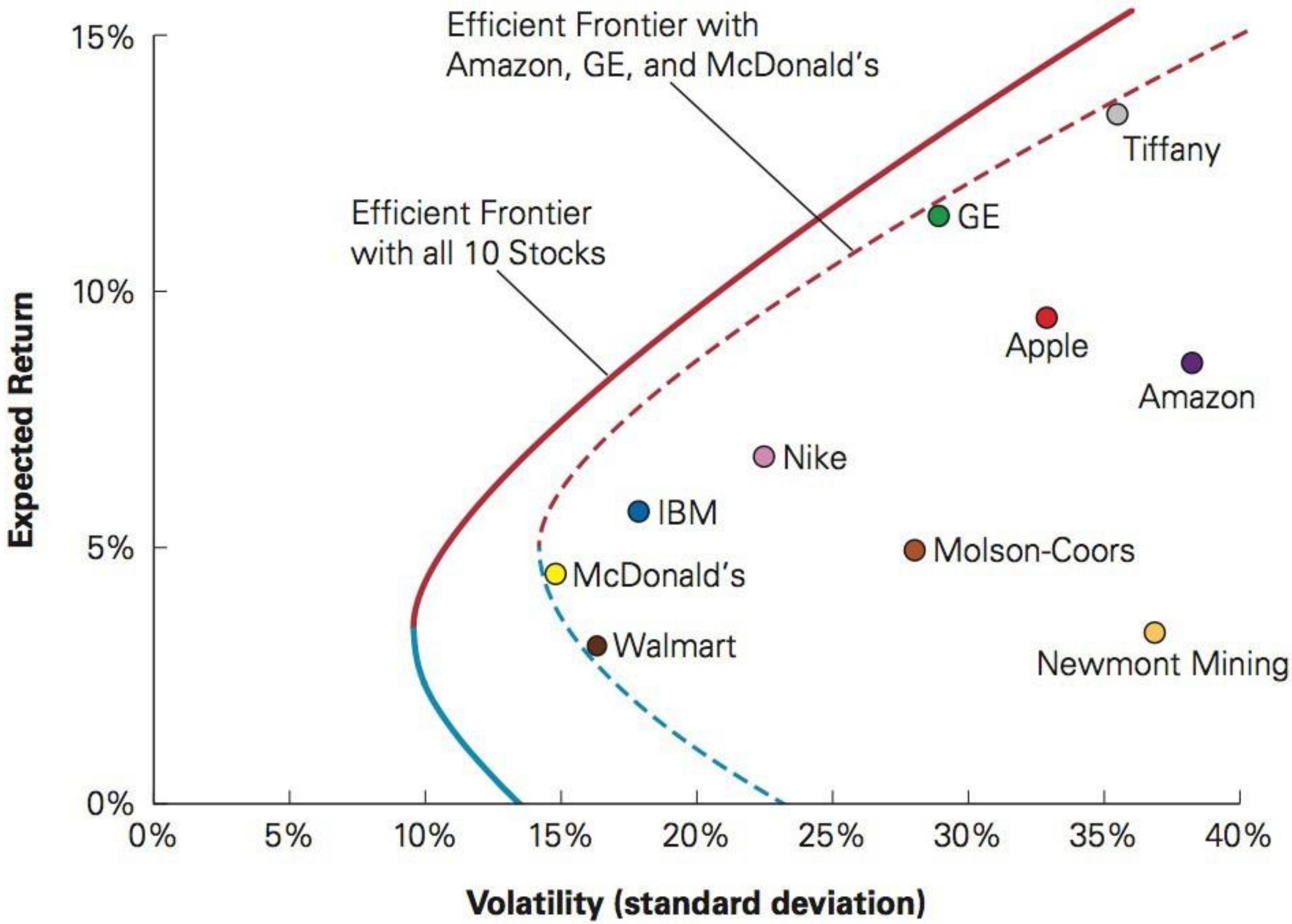


Figure 11.8 Efficient Frontier with 10 Stocks vs. 3 Stocks
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

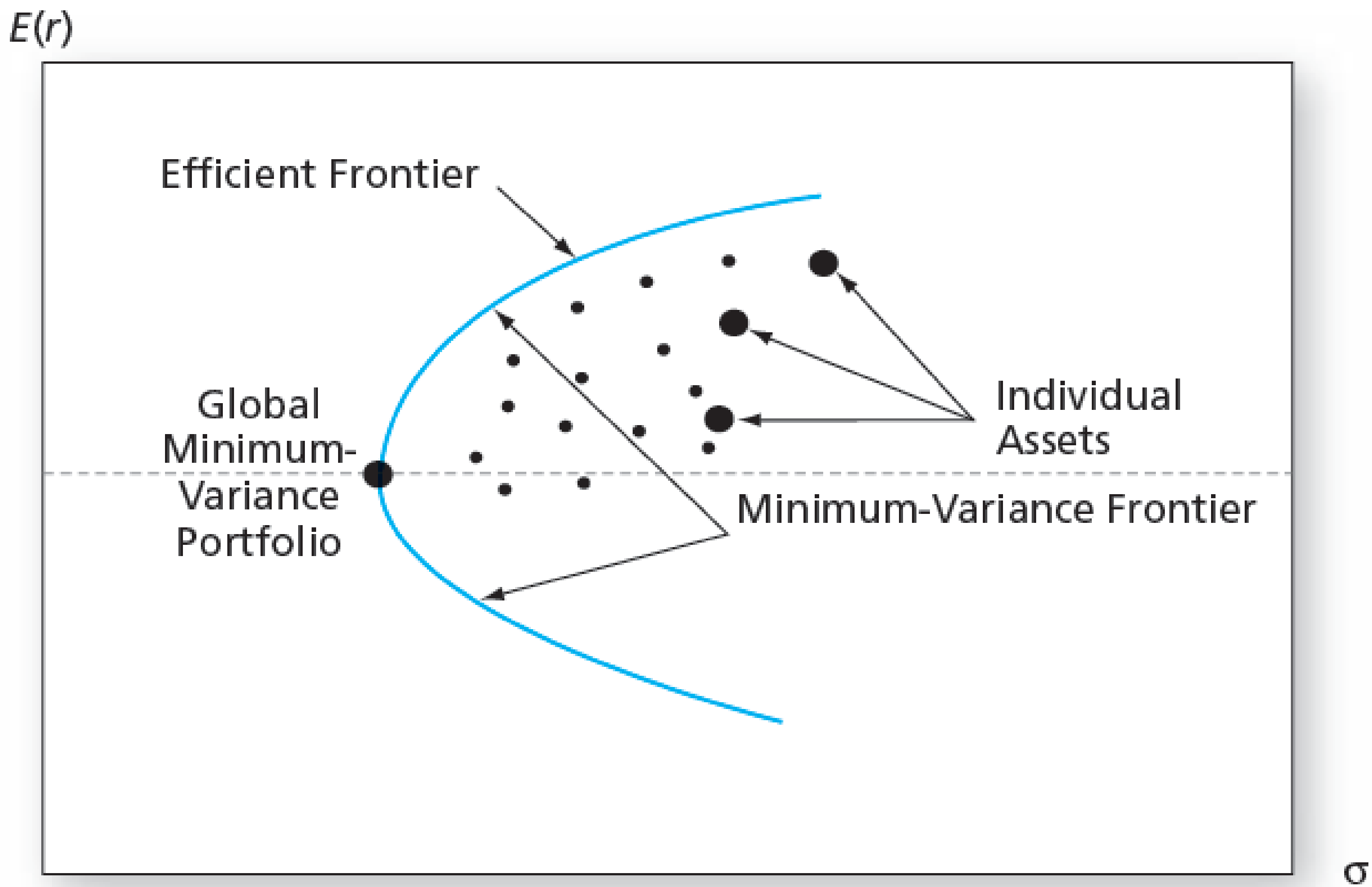


Risk Versus Return: Many Stocks

- The minimum-variance frontier plots the lowest variance that is attainable for every portfolio expected return.

- The efficient portfolios are those on the northwest edge of the minimum-variance frontier, which is called the efficient frontier (of risky assets).
  - It consists of the portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward.

Figure 7.10 The Efficient Frontier with Multiple Risky Assets
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Limits to Diversification

- The variance of a well-diversified portfolio is equal to the average covariance between the risky assets in the portfolio.
- The irreducible risk of a diversified portfolio depends on the covariance of the returns, which is a function of the systematic factors in the economy.
- Even for a very large portfolio, we cannot eliminate all the risk.
- The benefit of diversification is most dramatic initially.

Diversification & Portfolio Risk

- Market risk
  - Risk attributable to marketwide risk sources and remains even after extensive diversification.
  - Also call systematic or non-diversifiable.

- Firm-specific risk
  - Risk that *can* be eliminated by diversification.
  - Also called diversifiable or nonsystematic.

Figure 7.1 Firm-Specific Risk vs. Market Risk
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)