NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1522 Linear Algebra for Computing

AY2023/24 Semester 1 — Homework 1

(1) Write your answers on A4 size papers. You may choose to write or type your answer digitally.

(2) Write the page number on the top right corner of each page of the answer script.

(3) You may use MATLAB to aid in your calculations, however, you need to show all your workings clearly.

(4) If you have written your answer on an A4 paper, to submit, scan or take pictures of your work (make sure the images can be read clearly). Merge all your images into one PDF file (make sure they are in order of the page).

(5) If you have written or typed your answer digitally, print it into a PDF file.

(6) Name the PDF file by **StudentNo_HW1** (e.g. **A1234567Z_HW1.pdf**).

(7) Submit the PDF file to Canvas assignments, Homework 1.

1. Consider the linear system

$$\begin{array}{rcrcrcl} x & + & y & + & (3-a)z & = & 2 \\ 3x & + & 4y & + & 2z & = & b \\ 2x & + & 3y & - & z & = & 1. \end{array}$$

(i) (2 marks) Write down the augmented matrix of the above linear system.

(ii) (4 marks) Determine the values of a and b in which the above system has no solution.

(iii) (3 marks) Determine the values of a and b in which the above system has infinitely many solutions, and find the general solution.

(iv) (3 marks) Determine the values of a and b in which the above system has a unique solution. What is the value of z for this unique solution?

Show all your elementary row operations.

2. Consider the 3 by 3 matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 6 \end{pmatrix}.$$

(i) (3 marks) We perform the following elementary row operations on A to get the matrix B.

$$A \xrightarrow{R_2-2R_1} \xrightarrow{R_3-4R_1} \xrightarrow{-R_2} \xrightarrow{R_3-R_2} \xrightarrow{-\frac{1}{3}R_3} B.$$

Compute B.

(ii) (3 marks) Starting from B in (i), compute the reduced row echelon form R.

(iii) (4 marks) Using (i) and (ii), write R as a product of elementary matrices E_i and A, i.e.

$$R = E_k E_{k-1} \cdots E_1 A$$

where $E_1, E_2, \ldots, E_k$ are elementary matrices.

(iv) (2 marks) Using (ii) or otherwise, explain why A is an invertible matrix.

(v) (6 marks) Compute the inverse matrix A^{-1}.

3. (10 marks) Let X and Y be two 6 by 6 matrices.

We write

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}$$

where each a_{ij} is a scalar.

We group the entries of the matrix X into 2 by 2 submatrices

$$X = \left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where $A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, A_{12} = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}, \ldots, A_{33} = \begin{pmatrix} a_{55} & a_{56} \\ a_{65} & a_{66} \end{pmatrix}$.

We do this likewise for the matrix Y

$$Y = \left(\begin{array}{cc|cc|cc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ \hline b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{array}\right) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

and the 6 by 6 matrix

$$XY = \left(\begin{array}{cc|cc|cc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ \hline c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ \hline c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{array}\right) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

Show that the 2 by 2 matrix

$$C_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}.$$

Remark. This remark is NOT part of the homework.

The last question points to the formula

$$(*) \qquad \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

where

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j}$$

for $i, j = 1, 2, 3$.

Formula (*) behaves like multiplying two 3 by 3 matrices.

Formula (*) is a special case of a more general phenomena, namely

$$(**)\qquad \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1k} \\ B_{21} & B_{22} & \dots & B_{2k} \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nk} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1k} \\ C_{21} & C_{22} & \dots & C_{2k} \\ \dots & \dots & \dots & \dots \\ C_{m1} & C_{m2} & \dots & C_{mk} \end{pmatrix}$$

where

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}.$$

Here the A_{ij}'s are r by s matrices, the B_{ij}'s are s by t matrices and the C_{ij}'s are r by t matrices. They may not even be square matrices.

As mentioned, this remark is NOT part of the homework but it is a fun exercise to supply proofs to (*) and (**).