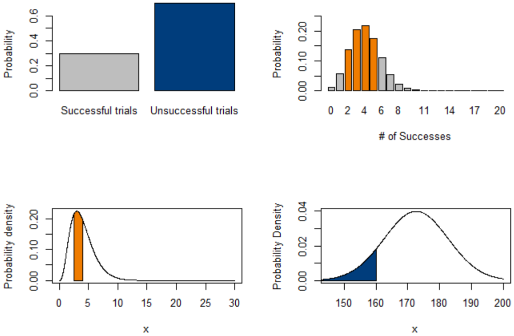
Probability Distributions



Outline

1. Properties of Probability
2. Probability Mass Function
3. Probability Density Function
4. Support of a Random Variable

Learning Objectives

By the end of this video, we hope that you will be able to:

- Understand probability mass function & probability density function and their respective properties.
- Understand the concept of support of a random variable.

Properties of Probability

Properties of Probability

Probability of an Event

A probability is simply a number between 0 and 1, that assigns a likelihood of occurrence to an event. Events are defined in terms of the random variable representing the outcomes.

- Out of next 5 buses, let X be the number of buses that are full.
- Possible events:
  - $X = 0$: No buses that are full.
  - $X = 3$: Exactly 3 buses that are full.
  - $X \leqslant 5$: Observing 5 or less buses that are full.
- Suppose that the probability of $X = 0$ is 0.1, and that the probability of $X = 4$ is 0.3.
  - It is 3 times more likely to observe 4 buses that are full than no buses that are full.

Properties of Probability

cont'd

$P(X = x) = q$

The probability that random variable X takes on the value x is q.
E.g.

- $P(X = 0) = 0.1$
- $P(X = 1) = 0.2$
- $P(X = 2) = 0.2$
- $P(X = 3) = 0.1$
- $P(X = 4) = 0.3$
- $P(X = 5) = 0.1$

Possible values of q

The probability of the event, q always lies between 0 and 1.

$P(\text{some events}) = 0$

The event will never occur. For instance, P(X=6)=0

$P(\text{some events}) = 1$

The event will certainly occur. For instance, $P(X \geqslant 0) = 1$.

Sum of all $P(X = x)$

The sum of the probability of all the individual events must be 1. For instance,
$P(0 \leqslant X \leqslant 5) = P(X = 0) + P(X = 1) + \ldots$
$+P(X = 5) = 1$.

Probability Distributions

Probability Distribution

A mathematical function that describes possible values and likelihoods that a random variable can take within a given range.

Probability Mass Function

- Discrete random variables.

Probability Density Function

- Continuous random variables.

Probability Mass Function

Probability Mass Function (*pmf*)

- Discrete random variables are defined by *probability mass function*.

Probability Mass Function

Give the probabilities of all individual values for a discrete random variable.

- Represent *pmf* as $p(x)$, where x denotes the value of the random variable of a certain event.
- $p(x)$ is a short-form for $P(X = x)$, i.e. The probability that the random variable X takes on the value x.

Probability Mass Function (*pmf*)

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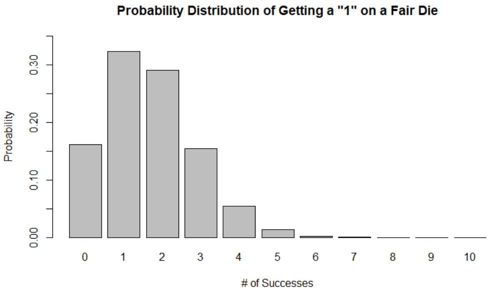
Suppose random variable X denotes the number of times we observe "1", when we roll a fair die 10 times.



- The probability of observing "1" once in 10 rolls gives the highest value, with a value of about $p(1) = 0.323$.
- The x-axis stretches from 0 to 10. These are all the possible values that X can take on in this experiment (Support of X).
- Probability of observing "1" seven or more times in 10 rolls,
  $P(X \geqslant 7) = p(7) + p(8) + p(9) + p(10)$.
- All the probability values are derived from the **binomial distribution**.

Probability Mass Function (*pmf*)

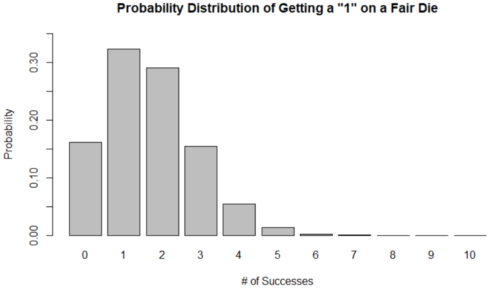
cont'd

Suppose random variable X denotes the number of times we observe "1", when we roll a fair die 10 times.

x	$P(X = x)$
0	0.161
1	0.323
2	0.291
3	0.155
4	0.054
5	0.013
6	0.002
7	2.48×10^{-4}
8	1.86×10^{-5}
9	8.27×10^{-7}
10	1.65×10^{-8}

- The probability of observing "1" once in 10 rolls gives the highest value, with a value of about $p(1) = 0.323$.
- The x-axis stretches from 0 to 10. These are all the possible values that X can take on in this experiment (Support of X).
- Probability of observing "1" 7 or more times in 10 rolls, $P(X \geqslant 7) = p(7) + p(8) + p(9) + p(10)$.
- All the probability values are derived from the **binomial distribution**.

Properties of *pmf*



1. The height of each bar is between 0 and 1. This corresponds to probabilities always being between 0 and 1.
2. The sum of heights of all the bars will be exactly equal to 1. This corresponds to one of these events will occur for certain.

Recap from the properties of probability:

Possible values of q

The probability of the event, q always lies between 0 and 1.

$P(\text{some events}) = 1$

The event will certainly occur. For instance, $P(X \geqslant 0) = 1$.

Sum of all $P(X = x)$

The sum of the probability of all the individual events must be 1.

Properties of *pmf*

cont'd

Summary

- We refer to the *pmf* as the "distribution of the discrete random variable".
- A *pmf* is simply a denumeration of all possible probabilities of events related to that random variable.
- We can represent a *pmf* using either a bar chart or a table.
- If it is clear what *pmf* $p(x)$ corresponds to, then we will write $X \sim p(x)$:
  - the *pmf* of X is $p(x)$, or equivalently that X has distribution $p(x)$.

Probability Density Function

Probability Density Function (*pdf*)

- Since a continuous random variable can take on an infinite number of values, we cannot plot an individual bar/rectangle for each possible value it can take on.

Probability Density Function

A function that returns the relative likelihood (probability density) of any value of the random variable occurring, but not the actual probability

- Note: Probability density is **not the same as** probability.
  - Instead, it is the area under the curve that corresponds to the probability of X being valued in a *range of values*.
- To distinguish from a discrete random variable, we shall use $f(x)$ to represent the *pdf* of a continuous random variable.

Probability Density Function (*pdf*)

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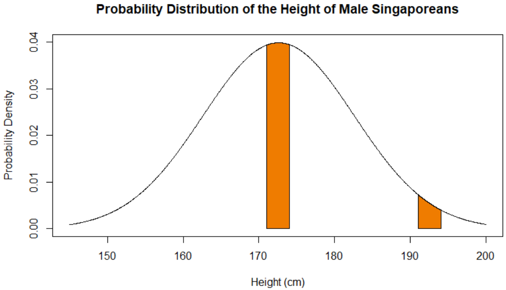
Suppose we measure the heights of some randomly selected male Singaporeans. Then, height can be taken to be a random variable Y that follows a distribution as shown.

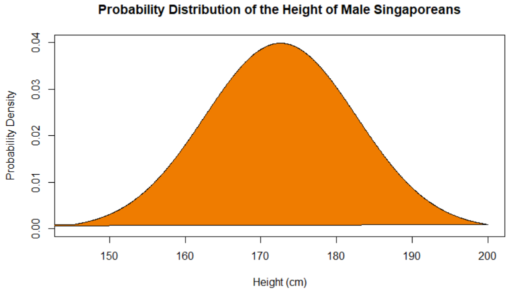


- The probability of this selected person having a height between 171 cm and 174 cm is the highest.
- The area between 171 cm and 174 cm under the curve is bigger than the area between 191 cm and 194 cm under the curve.
- All the probability values are derived from the **normal distribution**.

Properties of *pdf*

Compute probabilities of interest from a *pdf* by **finding areas under it**.

- The probability of someone's height that is between 170 cm and 175 cm.
- The probability of someone's height that is less than 160 cm.
- The *pdf* $f(x)$ is **always non-negative for all values in the random variable X**.
- Note that the area under the graph of the *pdf* is always equal to 1. This corresponds to the fact that one of the possible height values must be observed.



Support of a Random Variable

Support of a Random Variable

The support of a random variable is the set of values that it can possibly take on.

- For values outside the support of a random variable, the probability will be 0.
- E.g. We know the maximum capacity of the bus stop is 50, it means that the support of X will be $0, 1, 2, \ldots, 50$.
  - Therefore, the probability of observing 100 people waiting at the bus stop must be 0.
- E.g. Shortest male Singaporean (140 cm) vs. tallest male Singaporean (210 cm).
  - Therefore, the *pdf* of observing a male with a height of less than 140 cm, or more than 210 cm must be 0.

Support of a Random Variable

cont'd

Why do we need to know the support of a random variable?

- It helps us to decide whether it is discrete or continuous.
- It helps us to choose what *pdf* or *pmf* to use for it.
- It serves as a sanity check when we are simulating or modelling outcomes – for instance, we would not want to use a *pdf* with negative values in its support to represent inter-arrival times.

Summary

Learning Outcomes

- Understand two types of probability distributions: probability mass function & probability density function as well as their respective properties.
- Understand the support of a random variable.

Probability Distribution	**Probability Mass Function**	**Probability Density Function**
Short-form	*pmf*	*pdf*
Types of Random Variables	Discrete	Continuous
Notation	$X \sim p(x)$	$X \sim f(x)$
Function	Give the probabilities of all individual values for a discrete random variable	Specifies the probability of the continuous random variable falling within a particular *range of values*.

References

Diez, D. M., Barr, C. D., and Mine, C.-R. (2019). page 115–123. OpenIntro, Inc., 4th edition.

Ross, S. M. (2020). *A first course in probability*. Pearson Education Limited.

