LECTURE EIGHT

The Solow Growth Model (Part 2)

Chapter 5 of Charles Jones' text book

Outline for Lecture 5 part 2

- Looking at Data through the Lens of the Solow Model
  - Observing from data *(To see how far our model matches the reality)*
  - Differences in Y/L
    - Mathematical derivation: *(to see how TFP is important in explaining output differences)*
- Understanding the Steady State
- Economic Growth in the Solow Model
  - Economic growth in the Solow model
  - Case study: Population growth
- Some economic experiments
  - An increase in the investment rate (s)
  - A rise in the depreciation rate (d)
- Principle of Transition Dynamics
  - Principle of Transition Dynamics
  - Understanding differences in Growth rates
  - Growth rates in the OECD, 1960-2014
  - Case study: South Korea and Phillippines
- Strength and weakness of Solow model
- The Saving Rate and Consumption
  - Dynamics of steady state consumption and saving rate
  - Saving rate and consumption (Policy)

5.5 Looking at Data through the Lens of the Solow Model

5.5 Looking at Data through the Lens of the Solow Model

- Recall the steady state:

$$\bar{s}Y^\star = \bar{d}K^\star$$

- The capital to output ratio is the ratio of the investment rate to the depreciation rate:

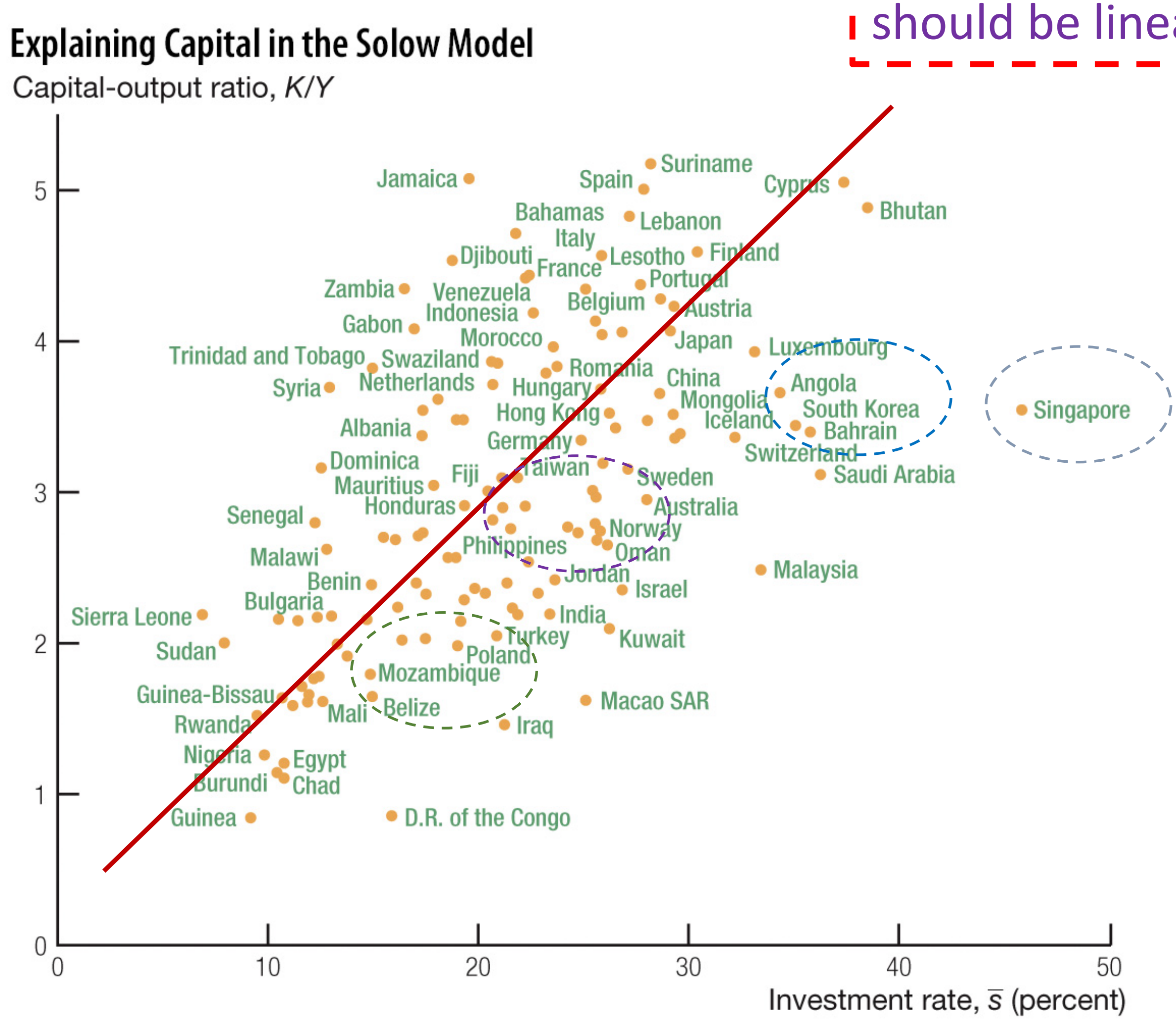
$$\frac{K^\star}{Y^\star} = \frac{\bar{s}}{\bar{d}}$$

Please note: This result is based on the Solow model

- Investment rates vary across countries
- It is assumed that the depreciation rate is relatively constant

Explaining Capital in the Solow Model (Fig 5.3)

$\frac{K^*}{Y^*} = \frac{\bar{s}}{\bar{d}}$ ➔ means that $\frac{K^*}{Y^*}$ should be linear with $\bar{s}$



Observing from the data (previous slide) —1

- A key determinant of a country's capital–output ratio is its investment/saving rate.
  - As predicted by the Solow model, these variables should be positively related.
  - This prediction holds up remarkably well in the data.
- Singapore and South Korea have high average investment rates and high capital–output ratios.

Observing from the data (previous slide) —2

- US lies somewhere in the middle, with $\bar{s} = 25\%$ and a capital–output ratio of below $\frac{K^*}{Y^*} = 3$.
- Zimbabwe and Haiti have low investment rates and low capital–output ratios.
- This shows that the saving rate, $\bar{s}$, does account for some elements of the differences in capital-output ratio $\frac{K^*}{Y^*}$ among economies

Differences in *Y/L*

- The Solow model shows TFP is more important in explaining per capita output
  - Can be used to understand differences in *y*
- Take the ratio of *y** for two countries, assuming the depreciation rate is the same:

$$\underbrace{\frac{y^*_{\text{rich}}}{y^*_{\text{poor}}}}_{70} = \underbrace{\left(\frac{\bar{A}_{\text{rich}}}{\bar{A}_{\text{poor}}}\right)^{3/2}}_{35} \times \underbrace{\left(\frac{\bar{s}_{\text{rich}}}{\bar{s}_{\text{poor}}}\right)^{1/2}}_{2} \qquad (\star)$$

See next slide

Mathematical derivation for $(\star)$ —1:

Recall:

$$\begin{cases} y^*_{rich} = \bar{A}_{rich}^{\frac{1}{1-\alpha}} \left(\dfrac{\bar{s}_{rich}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}} \\ y^*_{poor} = \bar{A}_{poor}^{\frac{1}{1-\alpha}} \left(\dfrac{\bar{s}_{poor}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}} \end{cases}$$

$$\Longrightarrow \frac{y^*_{rich}}{y^*_{poor}} = \frac{\bar{A}_{rich}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}_{rich}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}}{\bar{A}_{poor}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}_{poor}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}}$$

$$\boxed{\Longrightarrow \frac{y^*_{rich}}{y^*_{poor}} = \left(\frac{\bar{A}_{rich}}{\bar{A}_{poor}}\right)^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}_{rich}}{\bar{s}_{poor}}\right)^{\frac{\alpha}{1-\alpha}}}$$

Mathematical derivation for $(\star)$—2:

$$\Longrightarrow \frac{y^*_{rich}}{y^*_{poor}} = \left(\frac{\bar{A}_{rich}}{\bar{A}_{poor}}\right)^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}_{rich}}{\bar{s}_{poor}}\right)^{\frac{\alpha}{1-\alpha}}$$

If we assume $\alpha = \frac{1}{3}$, then $\frac{1}{1-\alpha} = \frac{3}{2}$ and $\frac{\alpha}{1-\alpha} = \frac{1}{2}$

And thus, we have:

$$\Longrightarrow \frac{y^*_{rich}}{y^*_{poor}} = \left(\frac{\bar{A}_{rich}}{\bar{A}_{poor}}\right)^{\frac{1}{3}} \left(\frac{\bar{s}_{rich}}{\bar{s}_{poor}}\right)^{\frac{1}{2}}$$

5.6 Understanding the Steady State

5.6 Understanding the Steady State

- The economy reaches a steady state because:
  - investment has diminishing returns
  - the rate at which production and investment rise is smaller as the capital stock is larger
- Also, a constant fraction of the capital stock depreciates every period.
  - Depreciation is not diminishing as capital increases.
- Eventually, *net investment* is zero.
  - The economy rests in steady state.

5.7 Economic Growth in the Solow Model

5.7 Economic Growth in the Solow Model —1

- Important result:
  - There is ***no long-run growth*** in the Solow model.
- In the steady state, growth stops, and
  - output, $Y^* = \bar{A}K_t^{*\alpha}\,\bar{L}_t^{1-\alpha}$
  - capital, K_t^*
  - output per person $(\frac{Y^*}{\bar{L}} = y^*)$, and
  - consumption per person

are **constant**.

Economic Growth in the Solow Model —2

- Empirically, however, economies appear to continue to grow over time.
  - Per capita GDP in the United States grew at an average annual rate **of 2 percent per year** for more than a century.
  - Economic growth shows no signs of disappearing (but this is what happens in Solow).
  - Thus, we see a drawback of the model.

Economic Growth in the Solow Model — 3

- According to the model:
  - Capital accumulation is not the engine of long-run economic growth.
  - After we reach the steady state, there is no long-run growth in output.
  - Saving and investment
    - are beneficial in the short run
    - do not sustain long-run growth due to diminishing returns (depreciation and new investment offset each other)

Case Study: Population Growth in the Solow Model—1

- Can **growth in the labor force** lead to overall economic growth?
  - It ***can*** in the ***aggregate***
  - It ***cannot*** in ***output per person***
- Diminishing returns lead k and y to approach the steady state.
  - This occurs even with more workers.

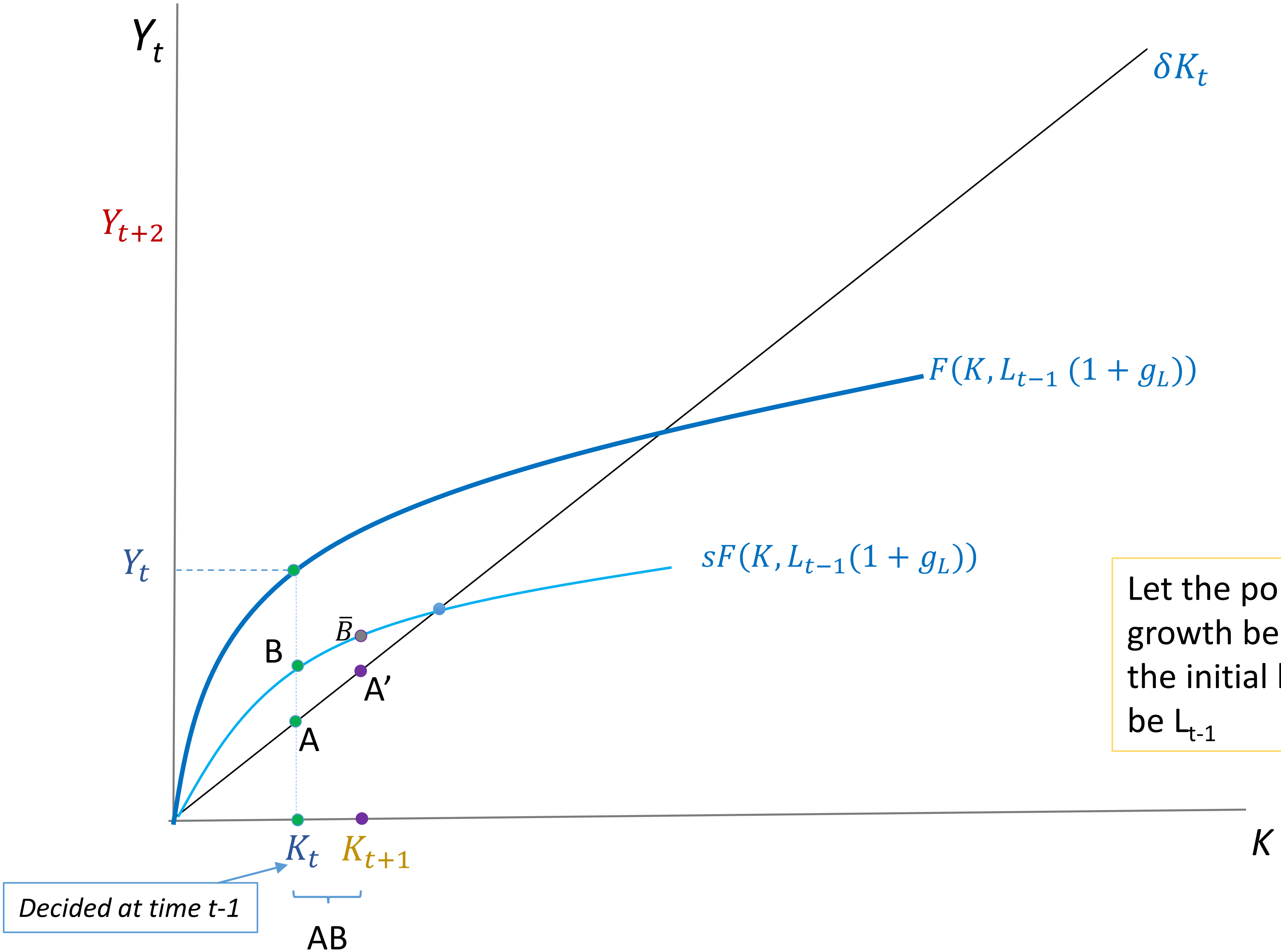
Case Study: Population Growth …—2

- If we wish to solve Solow Model **graphically**, we need to convert all 3 graphs – output, investment (aka saving) curve and break-even curve to “per-capita” form.

- The reason is if with population growth, the graph of Y_t against K_t shifts up every period and so is the saving graph. This makes the graphical analysis extremely difficult because we have a continually shifting graphs and cannot pin down the steady state.

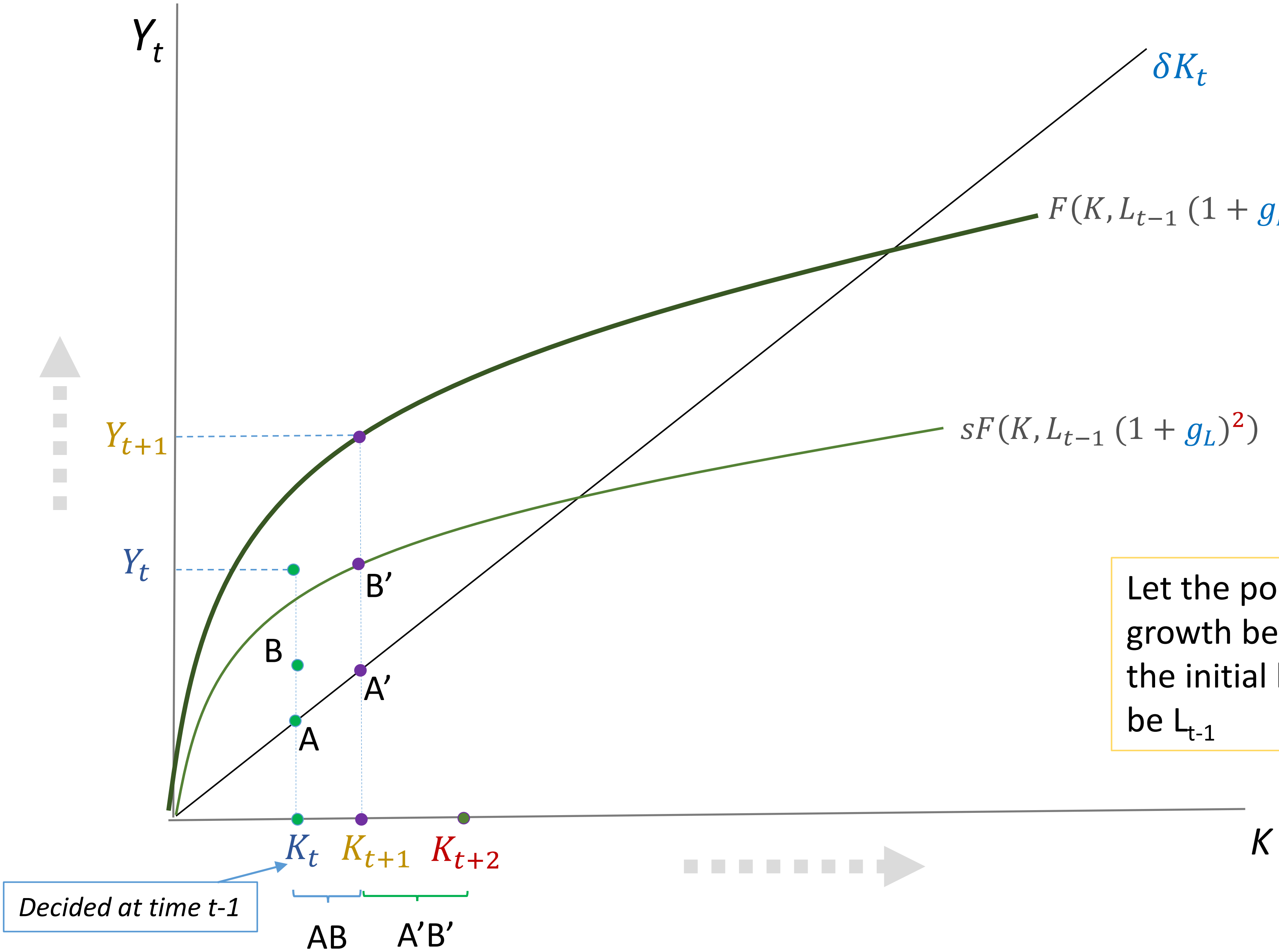
recall: $Y_t = \bar{A}L_t^{1-\alpha}K_t^{\alpha}$

This diagram is not tested. It is created for deeper understanding



Let the population growth be g_L and the initial labour be L_{t-1}

This diagram is not tested. It is created for deeper understanding



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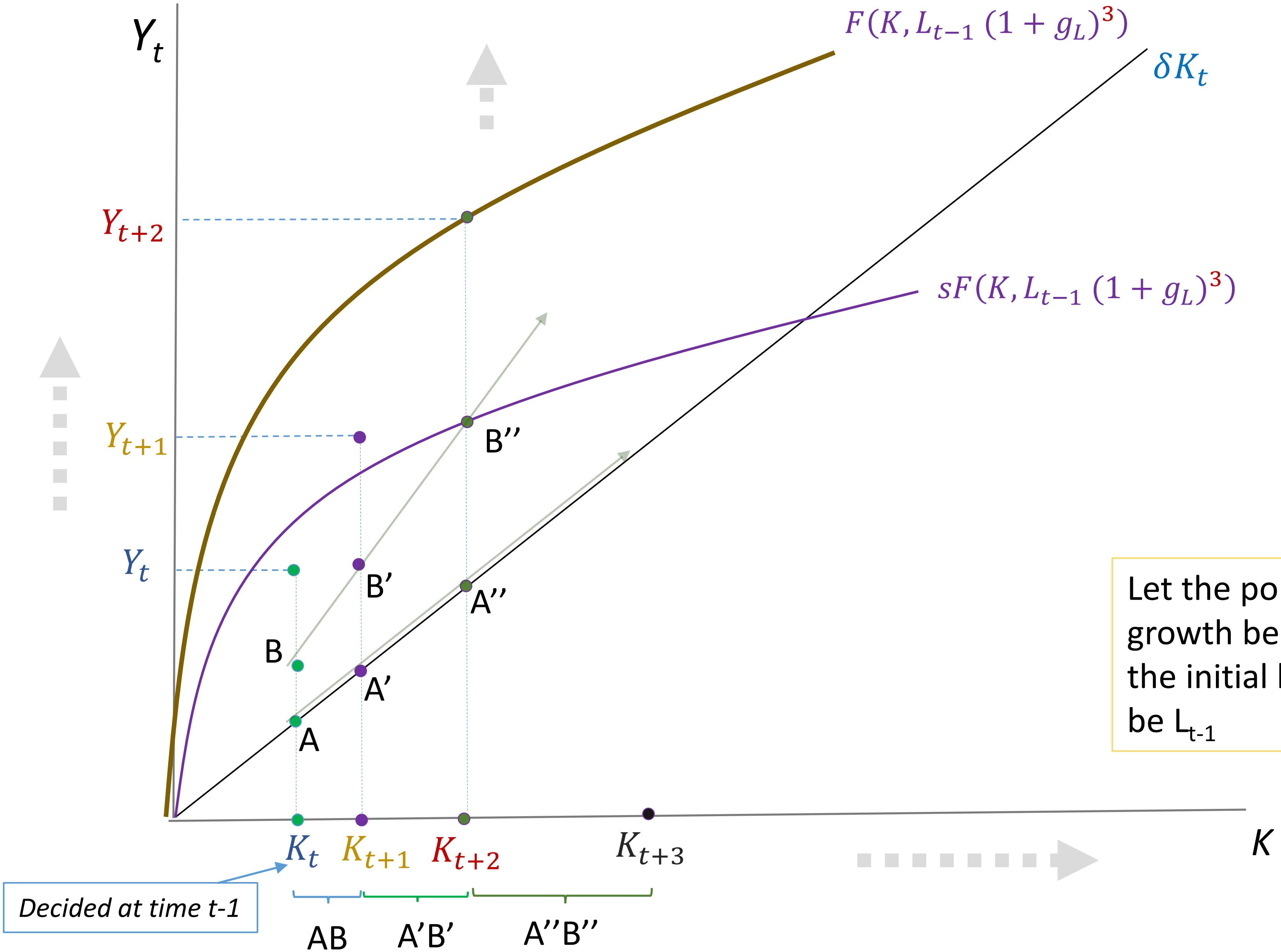


Let the population growth be g_L and the initial labour be L_{t-1}

Case Study: Population Growth …—3

- The rationale for converting the graphs into "per-capita" form is because:
  - The "per-capita" graphs do not shift up every period
  - And in the steady state, $\frac{Y_t}{L_t}$ will be a constant since:

$$g_{\frac{Y}{L}} = g_y = g_Y - g_L \qquad (tutorial\ 5)$$

and in the long run, we can show that g_Y is equal to $g_L \Longrightarrow g_y = 0$. This means that $y^\star$ (s.s. output per capita) is a constant and similarly $k^\star$ (s.s. capital per capita) is also a constant. In other words, we can pin down the steady state for analysis.

Case Study: Population Growth …—4

- **Recall**: Our production function is $Y_t = F(K_t, L_t)$, and this, by assumption, is constant returns to scale. So if we scale down by $\frac{1}{L_t}$, we get:

$$\frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f\left(\frac{K_t}{L_t}\right) \Rightarrow y_t = f(k_t)$$

- Apply to Cobb-Douglas function that we have:

$$Y_t = F(K_t, L_t) = \bar{A}K_t^{\alpha}L_t^{1-\alpha}$$

$$\frac{Y_t}{L_t} = \frac{\bar{A}K_t^{\alpha}L_t^{1-\alpha}}{L_t} = \bar{A}\frac{K_t^{\alpha}}{L_t^{\alpha}} = \bar{A}k_t^{\alpha}$$

$$y_t = f(k_t) = \bar{A}k_t^{\alpha}$$

Case Study: Population Growth …—5

- Since our output function is now of per-capita form, we should convert the saving curve to per-capita form as well:

$$\frac{I_t}{L_t} = I_t^L = \bar{s}\,\frac{Y_t}{L_t}$$

$$\Longrightarrow I_t^L = \bar{s} f(k_t)$$

$$\Longrightarrow I_t^L = \bar{s}\,\bar{A} k_t^{\alpha}$$

- The per-capita output function $\boldsymbol{y_t} = f(k_t) = \bar{A}\boldsymbol{k_t^{\alpha}}$ is a concave function in k and the per-capita saving function $\boldsymbol{I_t^L} = \bar{s}\,\bar{A}\boldsymbol{k_t^{\alpha}}$ is also a concave function in k.

Case Study: Population Growth …—6

- Since we are drawing graphs with k_t or $\frac{K_t}{N_t}$ as the x-axis, the break-even investment is also "per-capita" investment. That is: *the amount of investment that is needed to maintain the capital **per capita** (k) at the same level.*

- So to maintain the level of capital per capita, we not only need make up for the loss due to depreciation, but also for *the growth of labour*.

- To make it clearer, consider a scenario: Let's assume the capital, K_t is 100 and L_t is 10, and the $\bar{d}$ is 10%.

Case Study: Population Growth …—7

First, consider NO population growth case:

- We have: $\frac{K_t}{L_t} = \frac{100}{10} = 10$;
- and at time t+1, in order to maintain $\frac{K_{t+1}}{L_{t+1}} = k_{t+1} = 10$,
  - **we just need** to invest for the loss due to depreciation (= 10% x 100 = 10)

$$\boxed{\underbrace{\frac{K_{t+1}}{L_{t+1}} = \frac{K_t - 10\%K_t + 10}{L_t}}_{L_{t+1} = L_t \textit{ since } g_L = 0\%,} = \frac{K_t - 10 + 10}{L_t} = \frac{100}{10} = 10}$$

Case Study: Population Growth …—8

Now, consider a labour growth of 10%:

- So: $L_{t+1} = L_t(1 + 10\%) = 10 \times 1.1 = 11$
- if we invest only 10 ($I_t = 10$) to make up for the depreciation **without considering population growth**, then $K_{t+1} = 100$, but $\frac{K_{t+1}}{\mathrm{L_{t+1}}} = k_{\mathrm{t+1}} = 100/11 = 9.091.$

$$\frac{K_{t+1}}{L_{t+1}} = \frac{K_t - 10\%K_t + 10}{L_t \times 1.1} = \frac{K_t - 10 + 10}{10 \times 1.1} = \frac{100}{11} = 9.091$$

- Then, the investment is not enough to maintain the per-capita capital level at 10.

Case Study: Population Growth …—9

- Thus, to maintain $\frac{K_{t+1}}{L_{t+1}} = 10$, the amount of investment needed is $(\delta + g_L) \times K_t = (10\% + 10\%) \times 100 = 20.$

$$\frac{K_{t+1}}{L_{t+1}} = \frac{K_t - 10\%K_t + \mathbf{20}}{L_t \times 1.1} = \frac{K_t - 10 + \mathbf{20}}{10 \times 1.1} = \frac{\mathbf{110}}{11} = 10$$

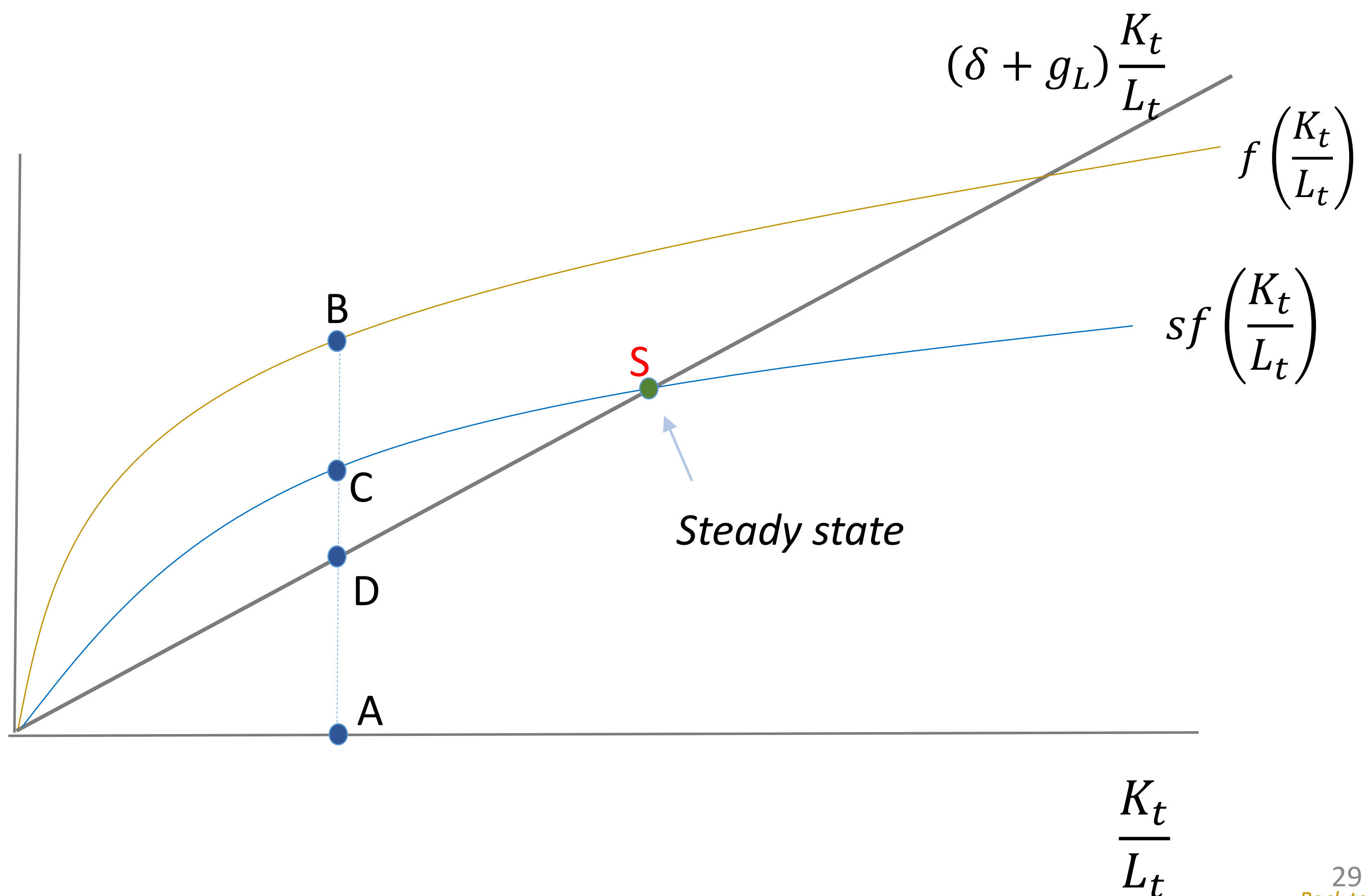
- Therefore, the amount we need to invest is: $\mathrm{I_t} = (\delta + g_L)K_t$

- And so **per capita investment** needed to maintain the level of $\frac{K}{\mathrm{L}}$ is equal to: $\frac{\mathrm{I_t}}{\mathrm{L_t}} = \mathrm{I}_{\mathrm{t}}^{L} = (\delta + g_{\mathrm{L}}) \times \frac{K_{\mathrm{t}}}{\mathrm{L_t}} = (\delta + g_L) \times k_t$

- So the per-capita break even line is, $(\delta + g_{\mathrm{L}})k_t$

Case Study: Population Growth …—10

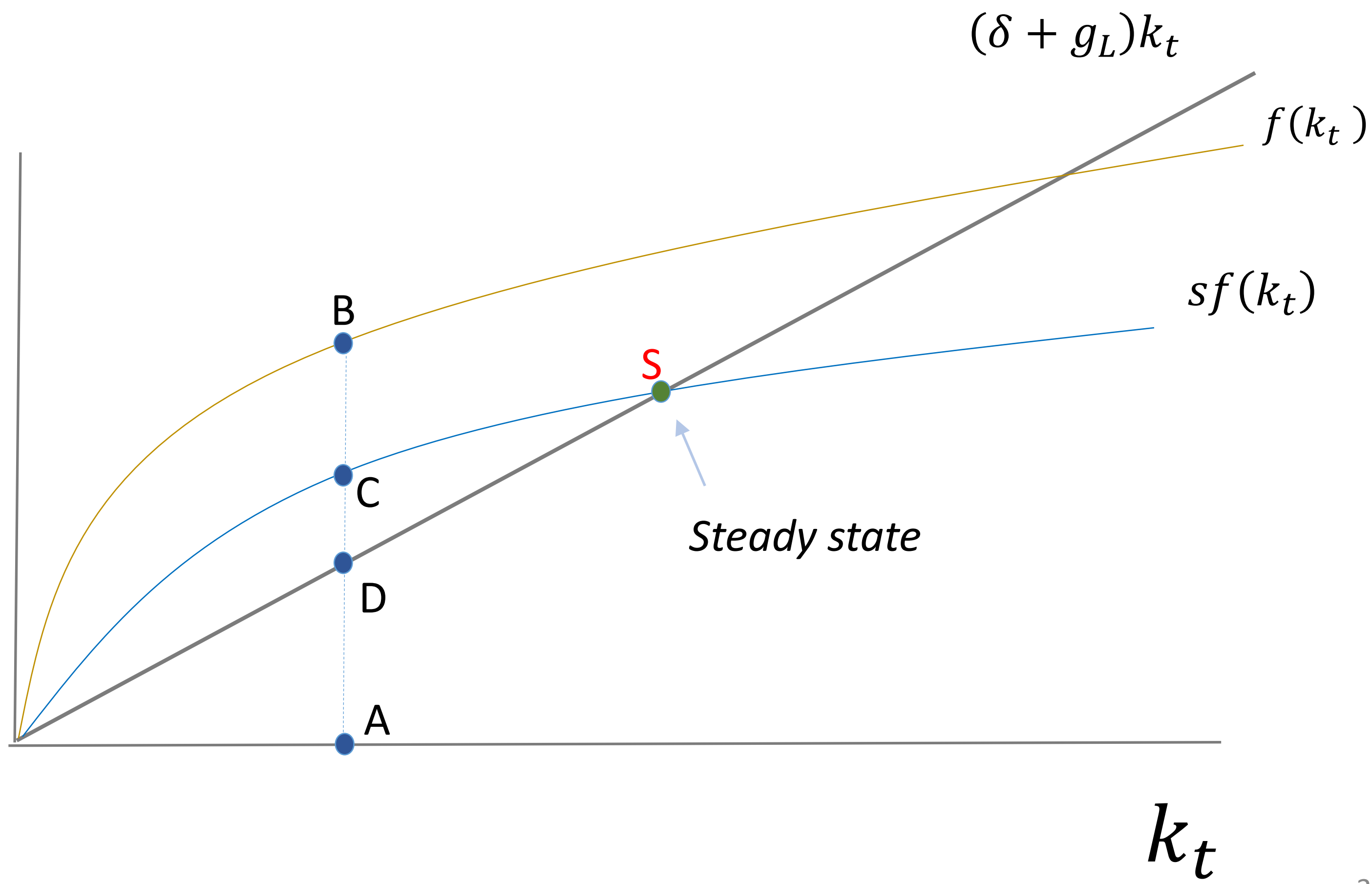
So if we draw the three graphs, $f\left(\frac{K_t}{L_t}\right)$, $sf\left(\frac{K_t}{L_t}\right)$ and $(\delta + g_L)\frac{K_t}{L_t}$, we still get the similar graph. And the steady state would be at S.



Graph 9

Case Study: Population Growth … — 11

Since $y = \frac{Y}{L}$ $and\ k = \frac{K}{L}$



Graph 9

Interpreting graph 9 — 1

- Suppose the economy is at point A.

- At this time, the ***amount of output per person*** produced is BA, the amount of saving is at CA, the required investment to maintain the same ***level of capital per worker*** is DA.

- So the amount of net ***investment per worker*** is CD.

- The analysis of how the economy progresses to the steady state will be similar to what we did for ‘no population growth’. The economy will progress to point S.

Interpreting graph 9 — 2

- What has changed for this diagram?

  - Firstly, $\frac{K}{L}$ is constant the steady state (point S) $\Longrightarrow$ K has to grow at rate of g_L at point S; since:

$$g_{K/L} = g_k = g_K - g_L = 0$$

  - Secondly, $\frac{Y}{L}$ is constant the steady state; but for this to happen, ***Y has to grow at rate of g_n*** at point S.

$$g_{Y/L} = g_y = g_Y - g_L = 0$$

5.8 Some Economic Experiments

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- The Solow model:
  - Does not explain long-run economic growth
  - Does help explain some differences across countries

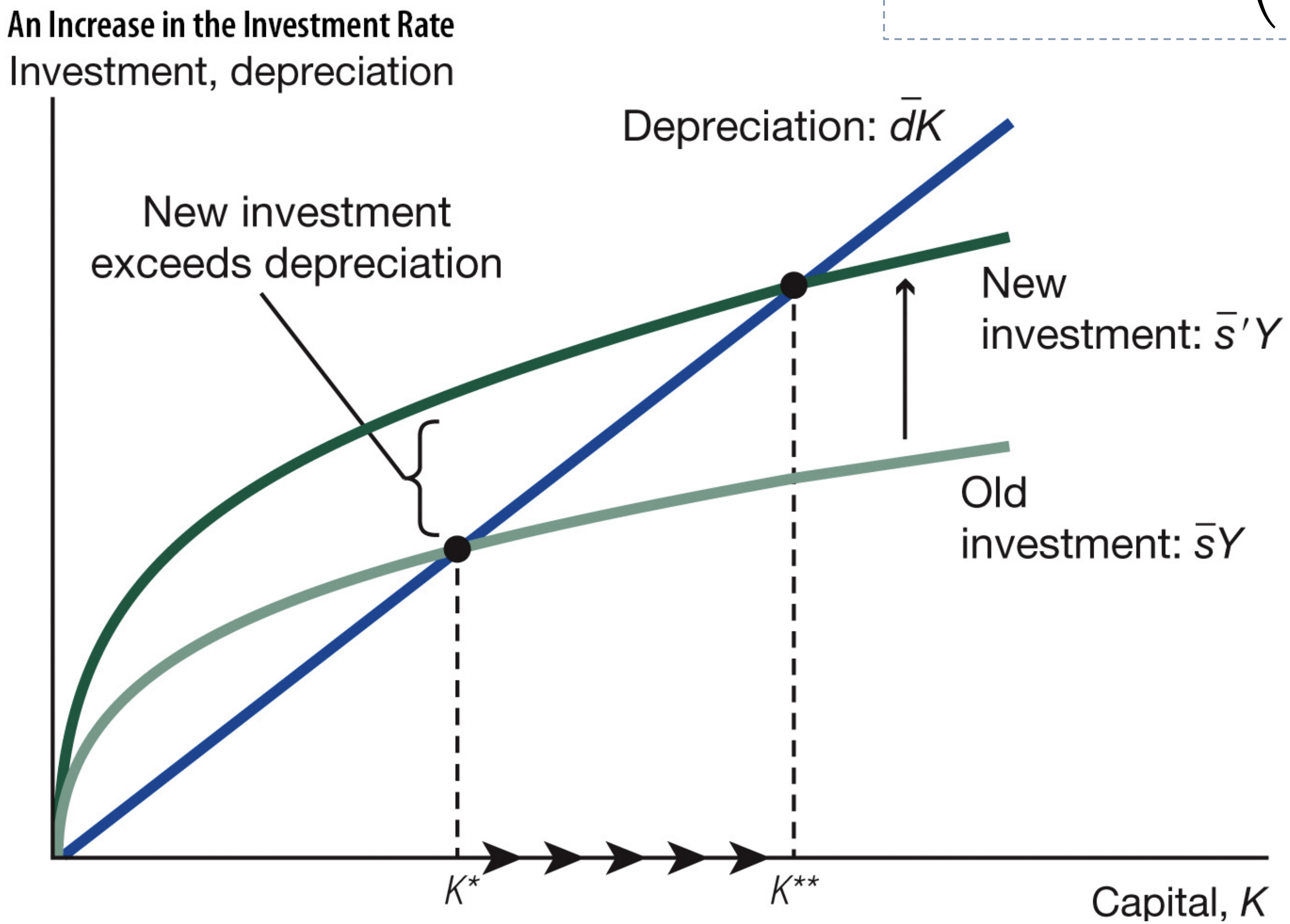
- Economists can experiment with the model by changing parameter values.

An Increase in the Investment Rate—1

- Suppose the investment rate increases permanently for exogenous reasons.
  - The investment curve rotates upward.
  - The depreciation curve remains unchanged.
  - The capital stock
    - *increases* by transition dynamics to reach the new steady state
    - this happens because investment exceeds depreciation
  - The new steady state
    - is located to the right
    - investment exceeds depreciation

An Increase in the Investment Rate—2

Recall: $K_t^* = \bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}$



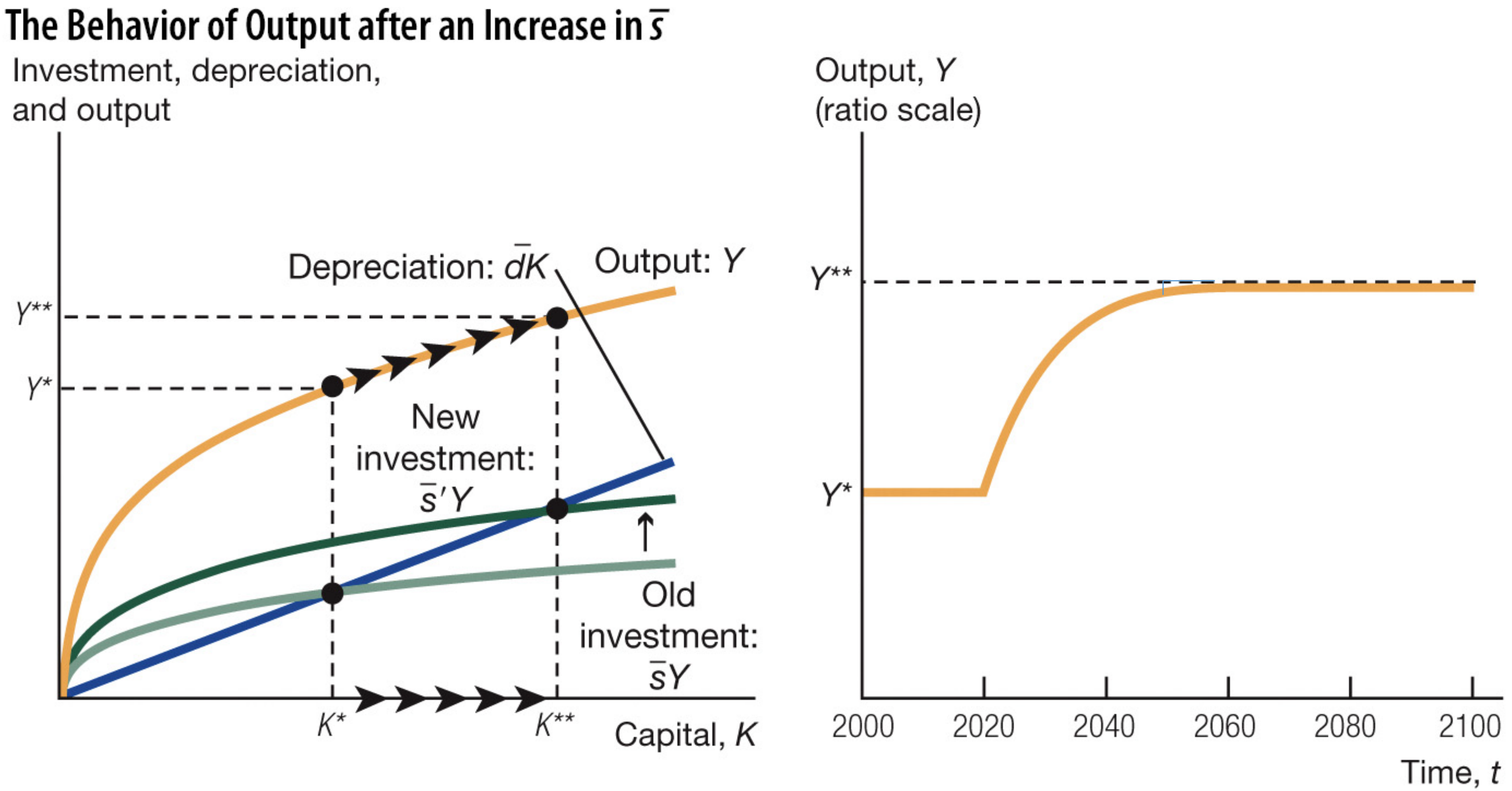
Graph 10

E.g. Compulsory savings plan (CPF)

An Increase in the Investment Rate—3

- What happens to output in response to this increase in the investment rate?
  - The rise in investment leads capital to accumulate over time.
  - This higher capital causes output to rise as well.
  - Output increases from its initial steady state level Y^* to the new steady state Y^{**}.

The Behavior of Output after an Increase in $\bar{s}$



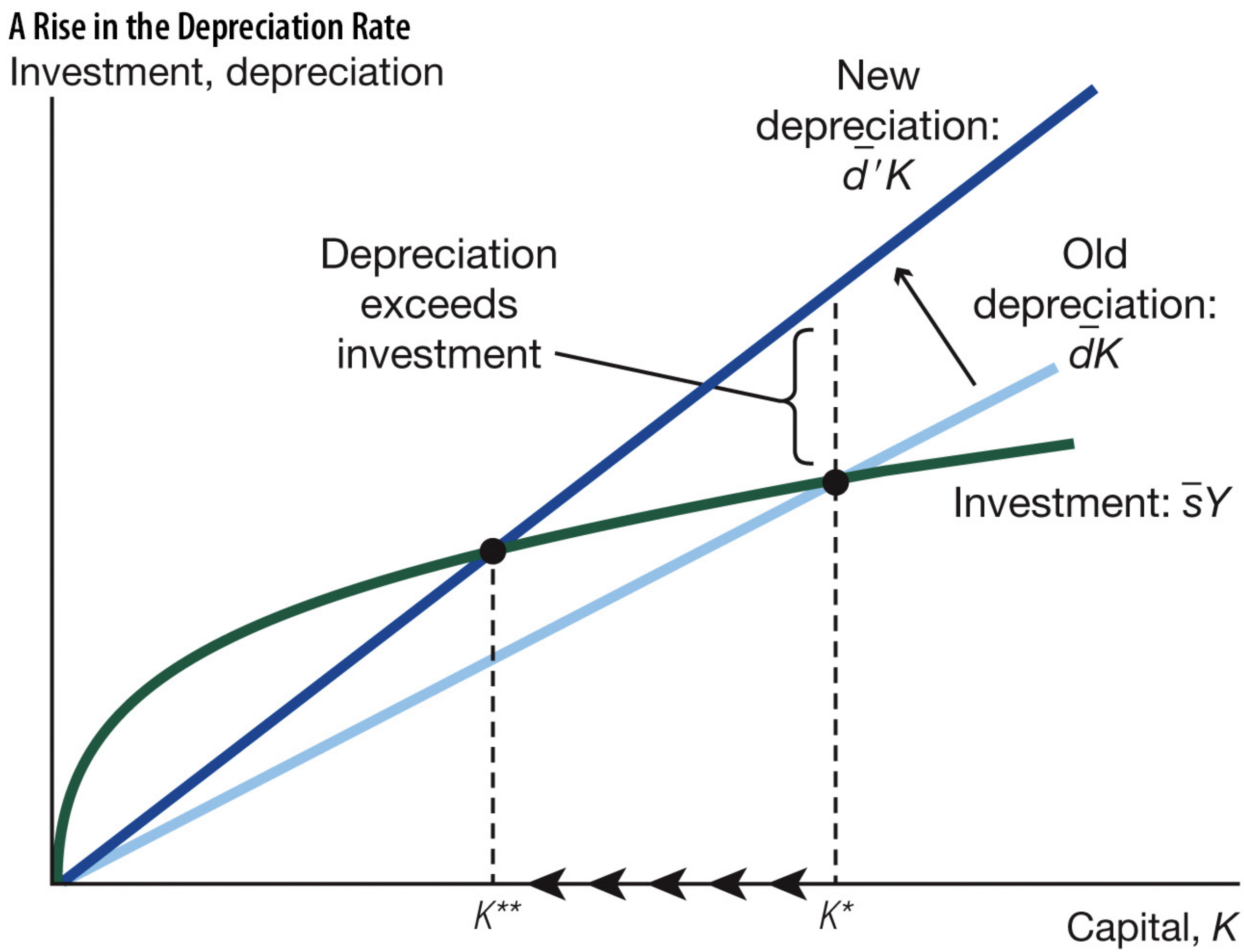
(a) The Solow diagram with output. (b) Output over time.

Graph 11

A Rise in the Depreciation Rate—1

- Suppose the depreciation rate is exogenously shocked to a higher rate.
  - The depreciation curve rotates upward.
  - The investment curve remains unchanged.
  - The capital stock
    - declines by transition dynamics until it reaches the new steady state
    - this happens because depreciation exceeds investment
  - The new steady state
    - is located to the left

A Rise in the Depreciation Rate—2



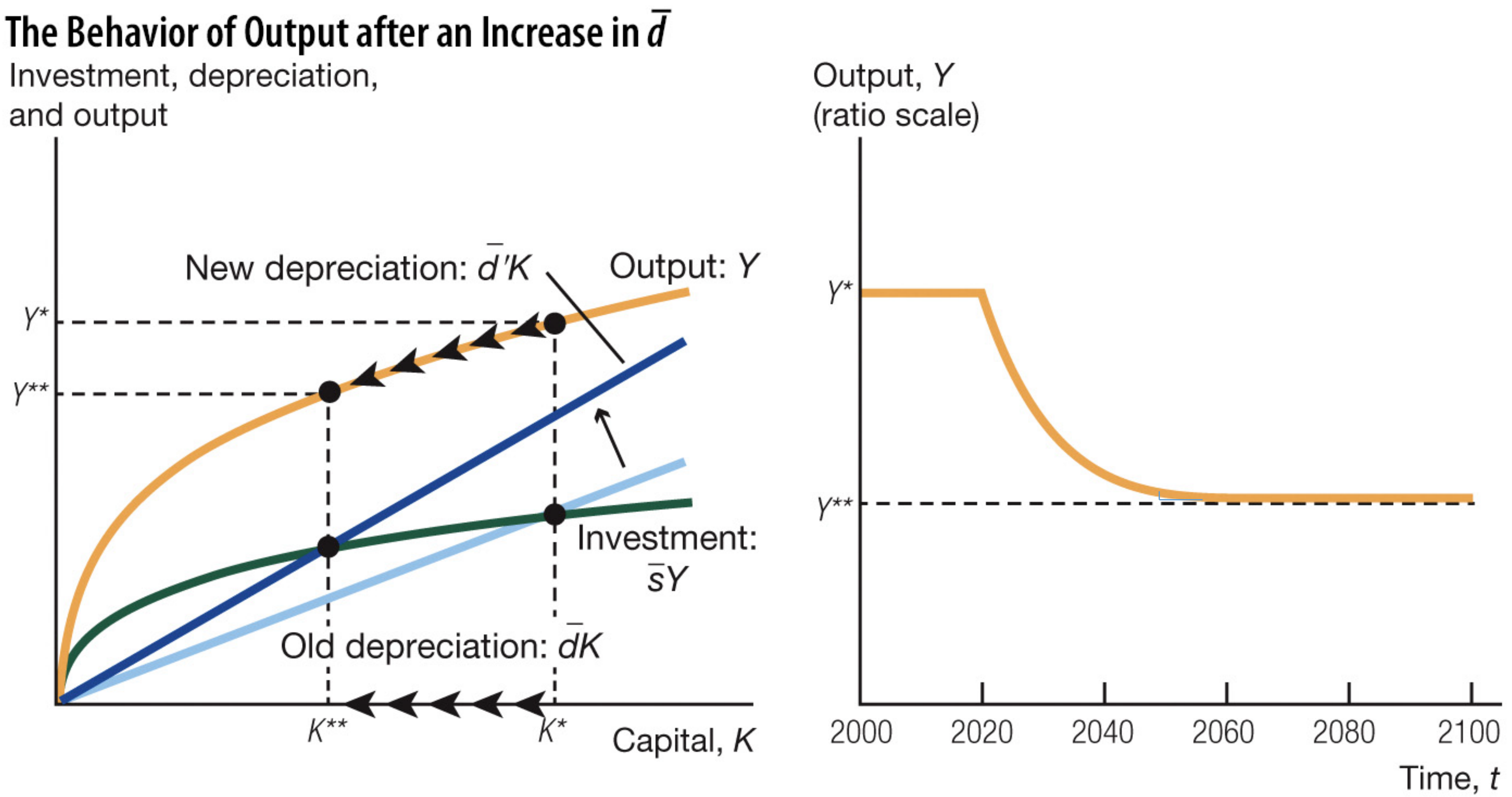
Graph 12

Back to outline

A Rise in the Depreciation Rate—3

- What happens to output in response to this increase in the depreciation rate?
  - The decline in capital reduces output.
  - Output declines rapidly at first, and then gradually settles down at its new, lower steady state level Y^{**}.

Behavior of Output after an Increase in $\bar{d}$



(a) The Solow diagram with output.

(b) Output over time.

Graph 13

Experiments on Your Own

- Try experimenting with all the parameters in the model:
  - Figure out which curve (if either) shifts
  - Follow the transition dynamics of the Solow model
  - Analyze steady state values of:
    - capital (K^*)
    - output (Y^*)
    - output per person (y^*)

5.8 The Principle of Transition Dynamics

5.9 The Principle of Transition Dynamics

- If an economy is below steady state
  - It will grow
- If an economy is above steady state
  - Its growth rate will be negative
- When graphing this, a ratio scale is used.
  - Output changes more rapidly if we are further from the steady state.
  - As the steady state is approached, growth shrinks to zero.

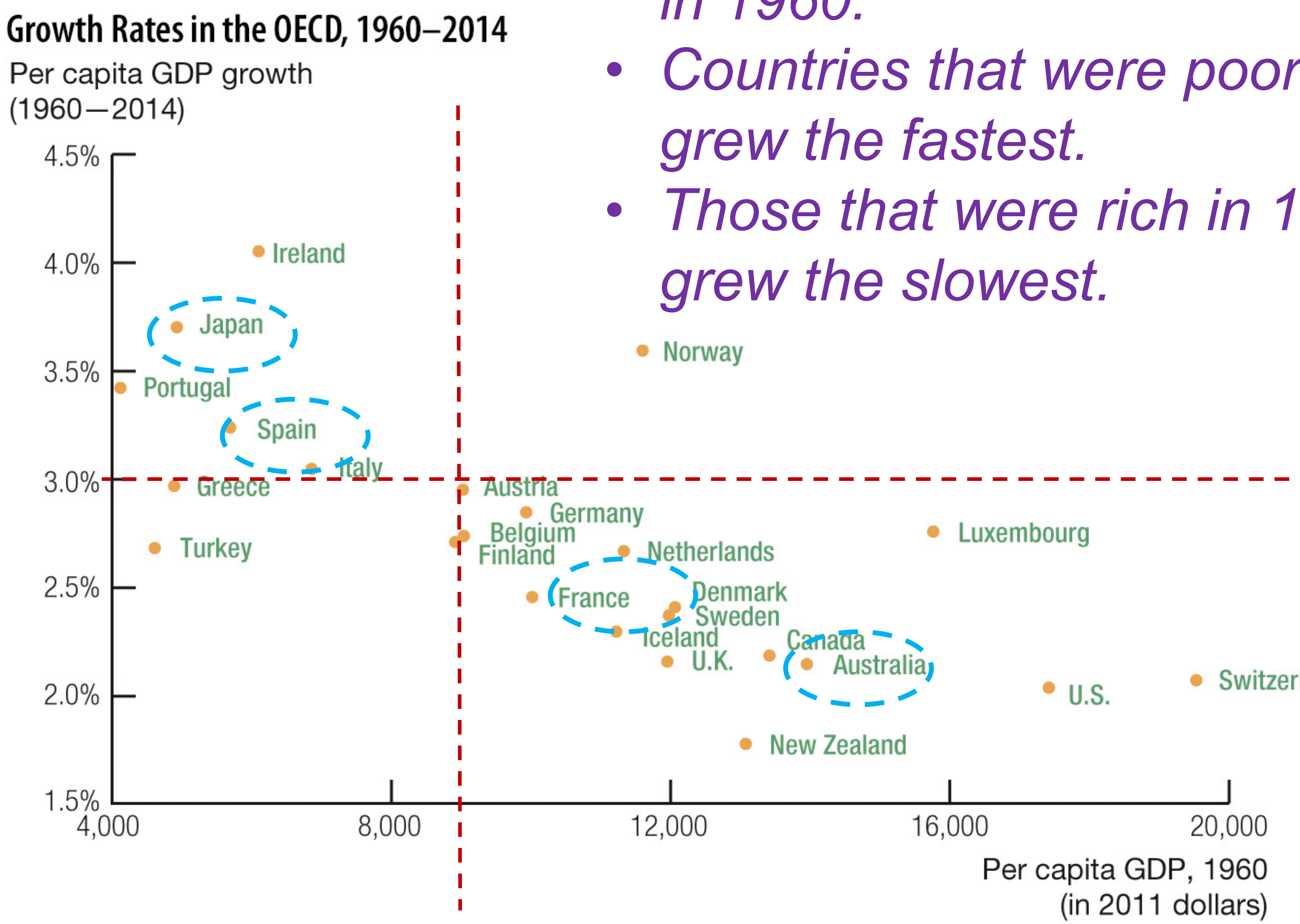
The Principle of Transition Dynamics

- The farther below its steady state an economy is, (in percentage terms)
  - the faster the economy will grow.
- The closer to its steady state,
  - the slower the economy will grow.
- Allows us to understand why economies grow at different rates

Understanding Differences in Growth Rates

- Empirically, for OECD countries, transition dynamics holds:
  - Countries that were poor in 1960 grew quickly
  - Countries that were relatively rich grew slower
- For the world as a whole, on average, rich and poor countries grow at the same rate.
  - Two implications of this:
    - most countries (rich and poor) have already reached their steady states
    - countries are poor not because of a bad shock, but because they have parameters that yield a lower steady state (determinants of the steady state invest rates and *A*)

Growth Rates in the OECD, 1960–2014



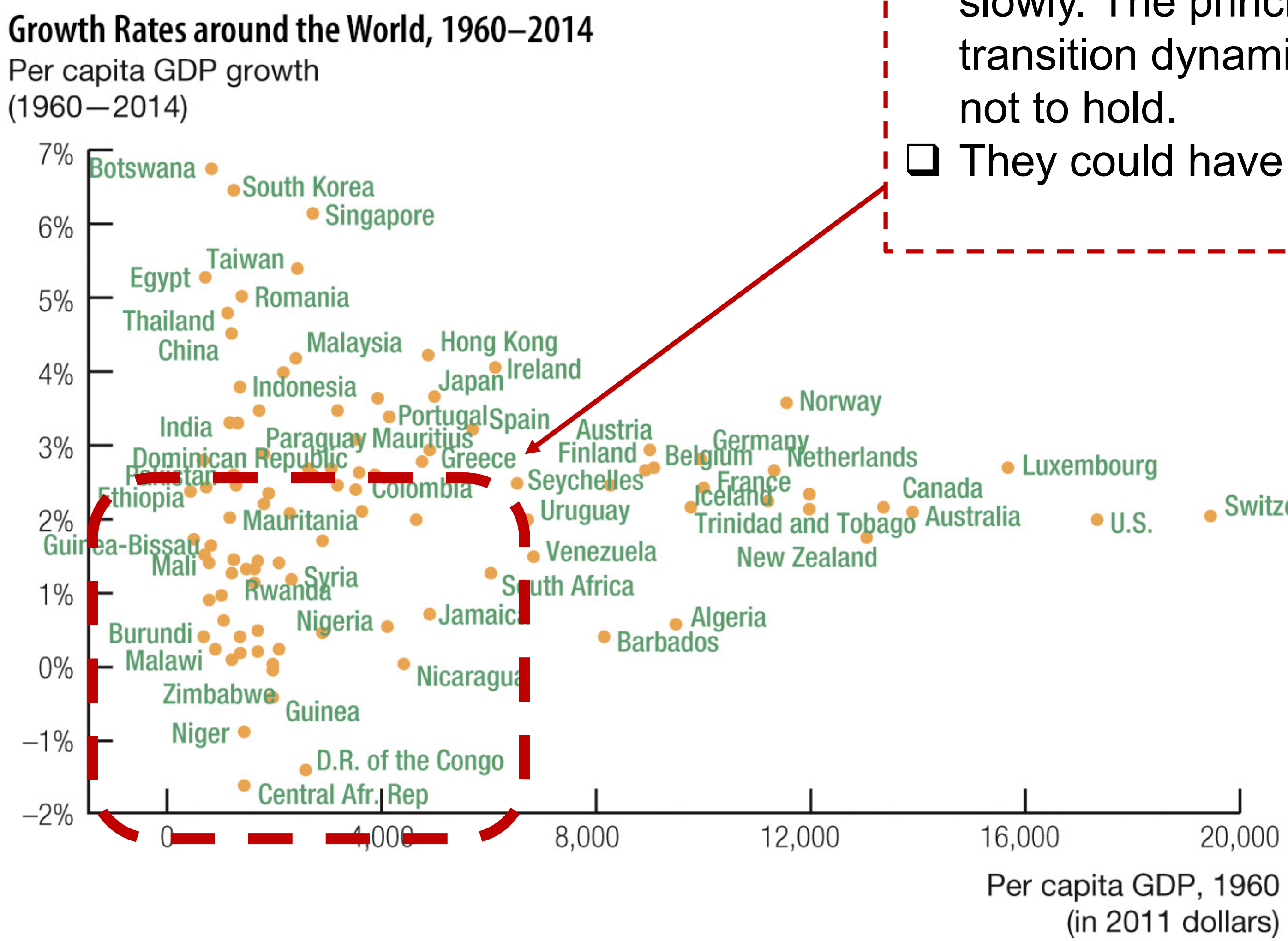
- *The x axis shows GDP per capita in 1960.*
- *Countries that were poor in 1960 grew the fastest.*
- *Those that were rich in 1960 grew the slowest.*

Growth Rates in the OECD, 1960–2014

Suppose all the countries of the OECD will have similar incomes in the long run.

- In this case, countries that are poor in 1960 would be far below their steady state.
- Countries that are rich would be closer to (or even above) their steady state.
- The principle of transition dynamics predicts that the poorest countries should grow quickly while the richest should grow slowly.
- This is evident in the negative slope of the countries.

Growth Rates around the World, 1960–2014



- These are poor and yet grow slowly. The principle of transition dynamics seems not to hold.
- They could have different $\bar{A}$

Case Study: South Korea and the Philippines—1

- South Korea
  - 6 percent per year
  - Increased from 10 percent of U.S. income to 75 percent
- Philippines
  - 2.4 percent per year
  - Stayed around 10 percent of U.S. income
- Transition dynamics predicts
  - South Korea was far below its steady state.
  - Philippines is already at steady state.

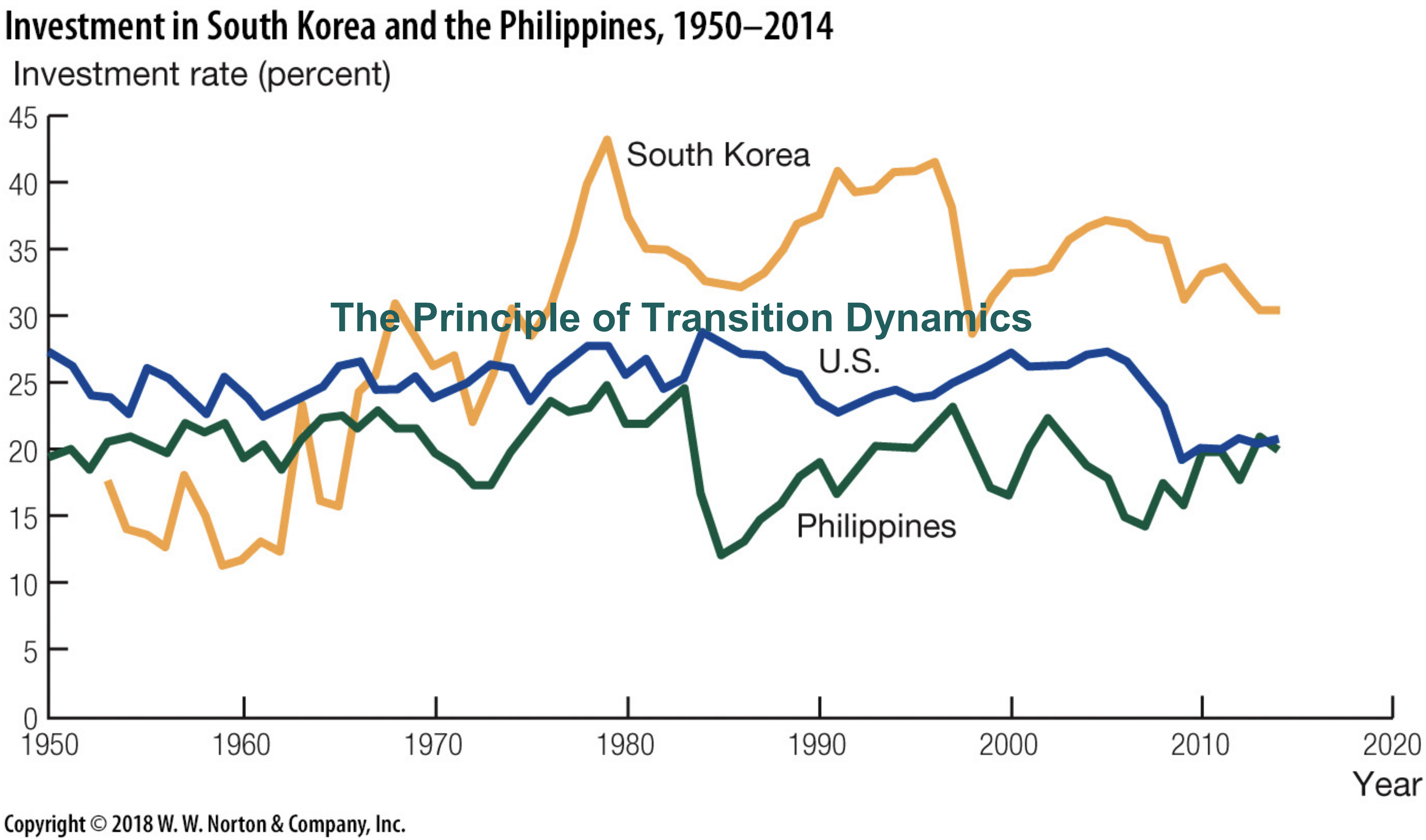
Case Study: South Korea and the Philippines—2

- Assuming equal depreciation rates

$$\frac{y^*_{\text{Korea}}}{y^*_{\text{U.S.}}} = \left(\frac{\bar{A}_{\text{Korea}}}{\bar{A}_{\text{U.S.}}}\right)^{3/2} \times \left(\frac{\bar{s}_{\text{Korea}}}{\bar{s}_{\text{U.S.}}}\right)^{1/2}$$

- The long-run ratio of per capita incomes depends on:
  - The ratio of productivities (TFP levels)
  - The ratio of investment rates

Investment in South Korea and the Philippines, 1950–2014



The Principle of Transition Dynamics

Back to outline

5.10 Strengths and Weaknesses of the Solow Model

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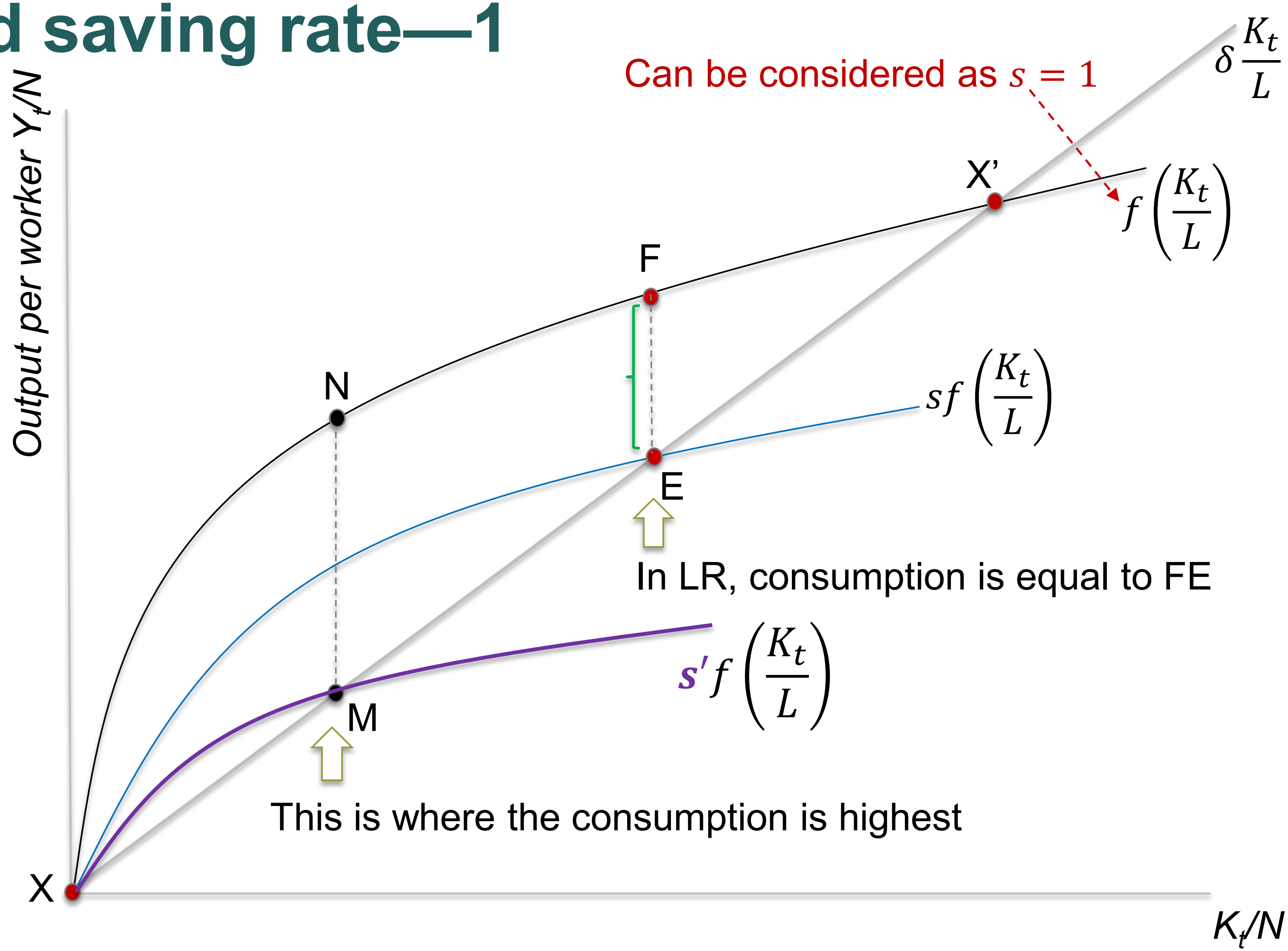
- The strengths of the Solow Model:
  - It provides a theory that determines how rich a country is in the long run.
    - long run = steady state
  - The principle of transition dynamics
    - allows for an understanding of differences in growth rates across countries
    - a country further from the steady state will grow faster

Strengths and Weaknesses of the Solow Model

- The weaknesses of the Solow Model:
  - It focuses on investment and capital
    - the much more important factor of TFP is still unexplained
  - It does not explain why different countries have different investment and productivity rates.
    - a more complicated model could endogenize the investment rate
  - The model does not provide a theory of sustained long-run economic growth.

The Saving Rate and Consumption

Dynamics of steady state consumption and saving rate—1



Graph 14

Dynamics of steady state consumption and saving rate—2

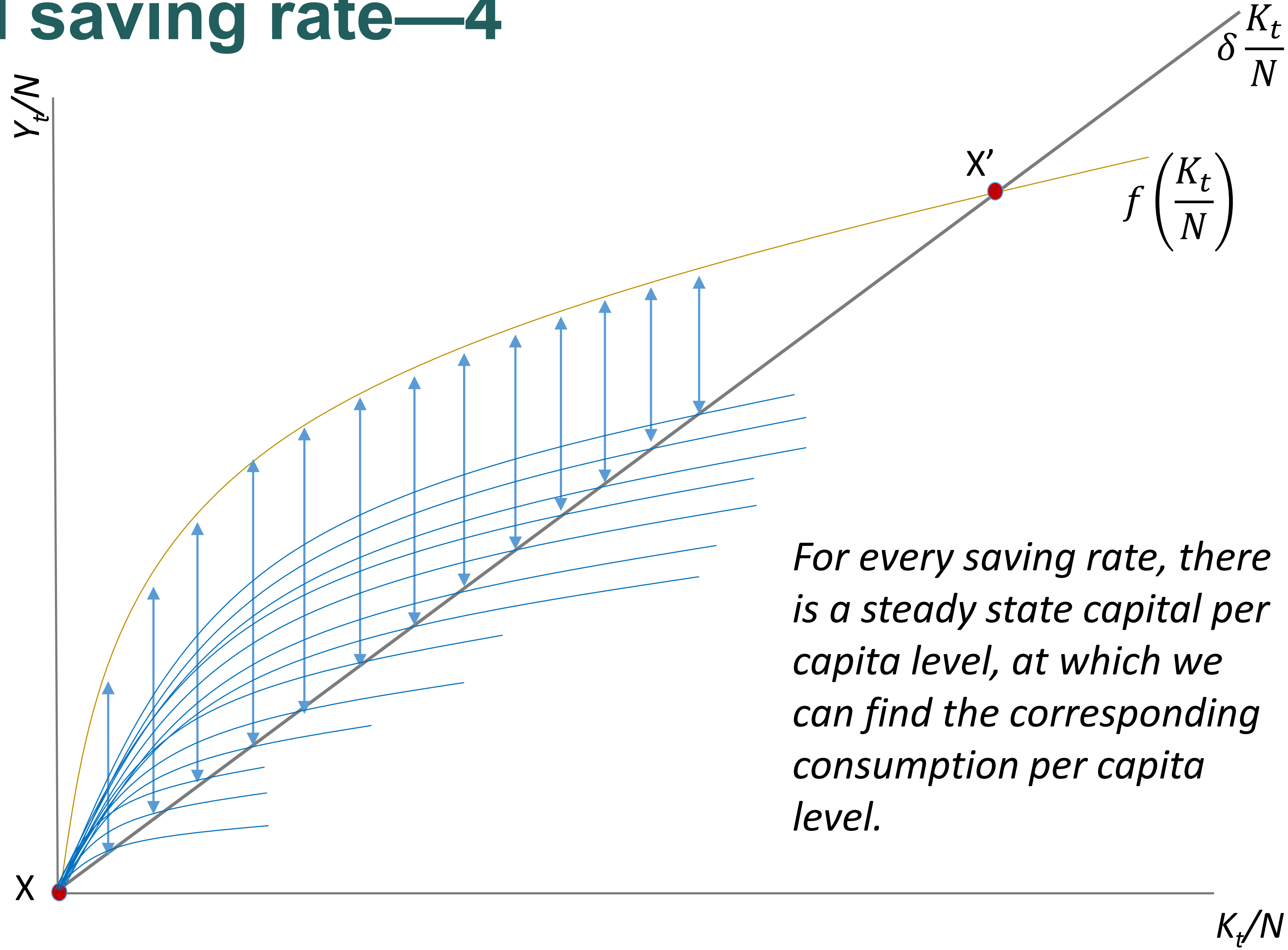
An increase in the saving rate always leads to an increase in the level of *output* per worker. But output is not the same as consumption. To see why, consider what happens for two **extreme values** of the saving rate:

- An economy in which the saving rate is (and has always been) 0 is an economy in which capital is equal to zero. In this case, output is also equal to zero, and so is consumption. A saving rate equal to zero implies zero consumption in the long run. (Point X on graph 7)
- Now consider an economy in which the saving rate is equal to one: People save all their income. The level of capital, and thus output, in this economy will be very high. But because people save all their income, consumption is equal to zero. A saving rate equal to one also implies zero consumption in the long run. (Point X' on graph 7)

Dynamics of steady state consumption and saving rate—3

- The level of capital associated with the value of the saving rate that yields the highest level of consumption in steady state is known as the golden-rule level of capital.

Dynamics of steady state consumption and saving rate—4



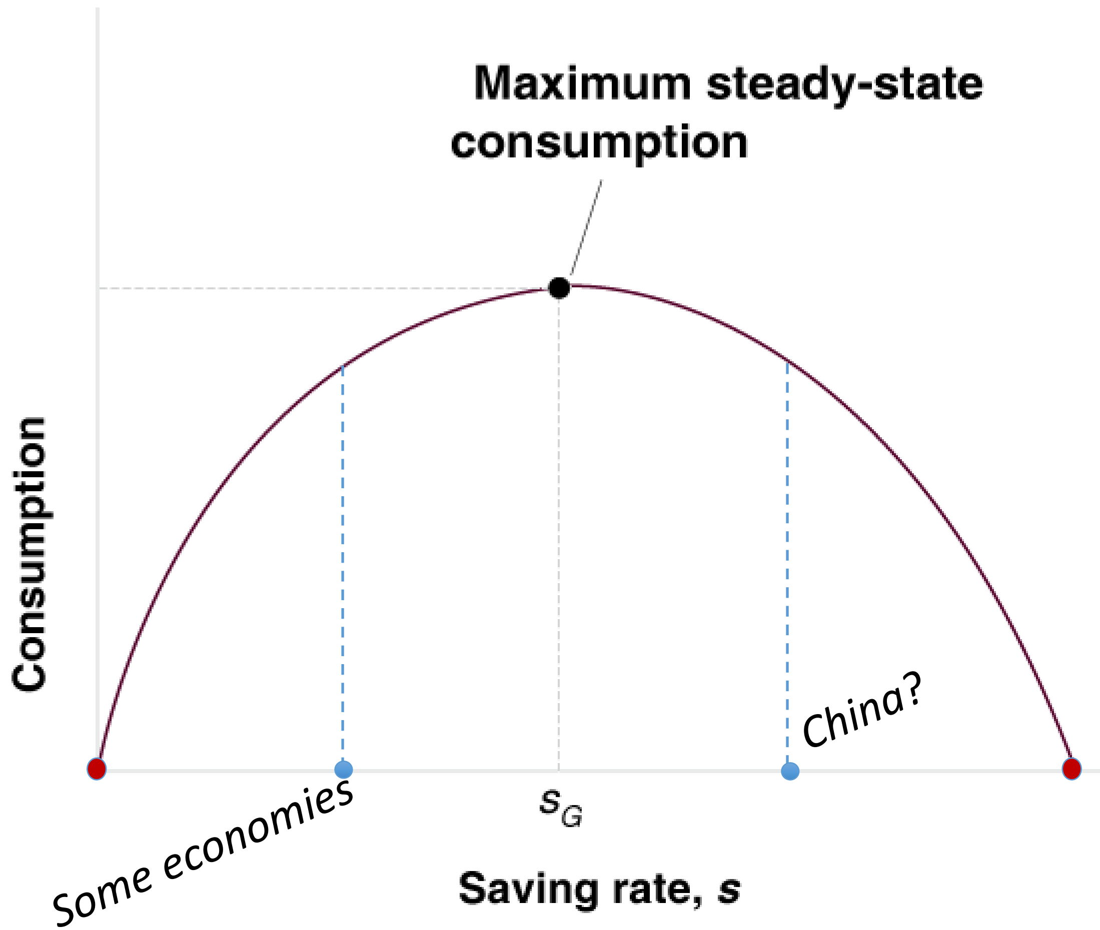
For every saving rate, there is a steady state capital per capita level, at which we can find the corresponding consumption per capita level.

Graph 15

Dynamics of steady state consumption and saving rate—5

The Effects of the Saving Rate on Steady- State Consumption

An increase in the saving rate leads to an increase and then to a decrease in steady-state consumption.



Graph 16

The Saving Rate and Consumption (Policy)—1

Government can affect the saving rates in various way:

- Vary public saving: if public saving is positive, overall saving would be higher and vice versa.
- Vary taxes: e.g. tax breaks to provide more incentive to save

- Government should care about the consumption per worker, not the output per worker because consumption per worker reflects the welfare of the people.
- The level of capital per capita that yields highest level of consumption per capita in *steady state* is known as the **golden-rule level of capital.**

The Saving Rate and Consumption (Policy)—2

- If the capital per capita is below the golden rule level, there should be an increase in the saving rate in the economy.
  - But immediately after the increase in saving rate, there would be lower consumption in the short run.

Policy debates on Saving Rate and Consumption

- However, in the long run, there would be higher consumption per capita. **So there is a problem:**
  - Should the governments care about the future generations more or the current generation more?
  - Because by increasing saving rate (through policies) the OECD governments can help the future generations but hurt the current generation.
  - But the politics is about votes, rights to vote belong to the current generation, not the future generations → it is unlikely that the governments will ask the current generations to make large sacrifices → capital level would be still below the golden rule level.