Macroeconomics analysis II, EC3102

Tutorial 9 Solution

Question 1

Under the Romer model setting, we have:

$$\Delta A_{t+1} = \bar{z} A_t L_{at}$$

Let g_{A_t} and g_{y_t} be the growth rates of knowledge and output per capita respectively. From the law of motion of ideas (idea production function), we have:

$$\Delta A_{t+1} = \bar{z} A_t L_{at}$$
$$\frac{A_{t+1} - A_t}{A_t} = \bar{z} L_{at}$$
$$\frac{A_{t+1} - A_t}{A_t} = \bar{z}\, \bar{l}\, \bar{L}$$
$$\Longrightarrow g_{A_t} = \bar{z}\, \bar{l}\, \bar{L} \tag{1}$$

From the aggregate production function, and taking the $\ln(\quad)$ and differentiating both sides with respect to time,

$$y_t = \frac{Y_t}{\bar{L}} = A_t \cdot \frac{(1-\bar{l})\bar{L}}{\bar{L}} \tag{2}$$

$$\Longrightarrow g_{y_t} = g_{A_t} = \bar{z}\, \bar{l}\, \bar{L} \tag{3}$$

Thus, if the research productivity increases from $\bar{z}$ and $\bar{z}'$, the new growth rates of knowledge and output per capita are:

$$g_{y_t} = g_{A_t} = \bar{z}'\, \bar{l}\, \bar{L} \tag{4}$$

From (2) and (4), we can write:

$$y_t = A_0 (1 + g_A)^t \cdot (1 - \bar{l}) \tag{5}$$

where A_0 is the state of technology in period 0 or the initial state of technology. (This state of technology grows at the rate of g_{A_t}. As such, $A_0(1+g_A)^t$ is A_t because after t periods of time, the state of technology grows to A_t at the rate of g_{A_t}.)

Comment: *Take $ln(\quad)$ on both sides of equation (5), we have:*

$$ln\, y_t = ln\, A_0 + t \cdot \ln(1 + g_{At}) + ln(1 - \bar{l})$$

$$ln\, y_t = \boldsymbol{t} \cdot \ln(1 + \bar{z}\, \bar{l}\, \bar{L}) + ln\, A_0 + ln(1 - \bar{l}) \tag{6}$$

Now, plot $ln(y_t)$ against t. We will have a straight line with the slope $\ln(1 + \bar{z}\, \bar{l}\, \bar{L})$ and the intercept is $ln\, A_0 + ln(1 - \bar{l})$.

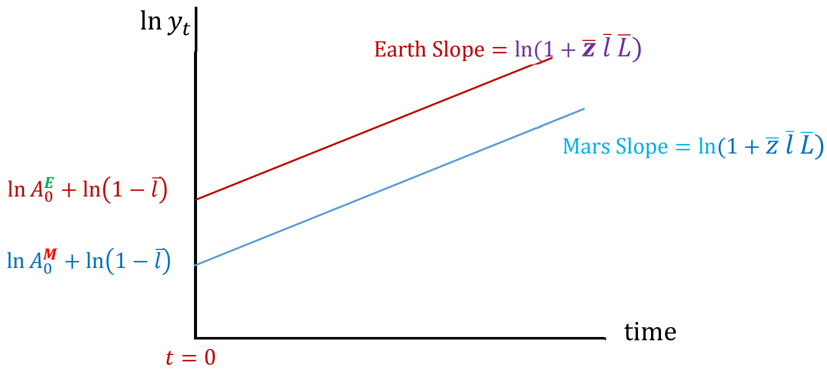
Apply equation (6) to the context of Mars (M) and Earth (E) we have:

$$\begin{cases} \ln y_t = \boldsymbol{t} \cdot \ln(1 + \bar{z}\,\bar{l}\,\bar{L}) + \ln A_0^{\boldsymbol{E}} + \ln(1 - \bar{l}) & \textit{for the Earth economy} \\ \ln y_t = \boldsymbol{t} \cdot \ln(1 + \bar{z}\,\bar{l}\,\bar{L}) + \ln A_0^{\boldsymbol{M}} + \ln(1 - \bar{l}) & \textit{for the Mars economy} \end{cases}$$

Note that the two economy differs only by their initial state of technology (A_0).

Comment: the above recap is to ensure that all students understand what is going on. That is, you do not need to include it in your answer. Just showing the graphs is enough.

Plotting the two curves on the same graph we have:



***Comment**: What we can tell from the above graph? Even though the two economies have the same growth rate (growth effects are the same), but the levels, which are the output levels, are different. This is because they start out with different A_0.*

<u>Question 2</u>

<u>Part a.</u>

Over the last sixty years, the Intellectual Property Products' share of GDP has increased. Based on our textbook model, this trend increase could be due to an increase in the productivity of production of ideas or an increase in the proportion of labour force engaged in the production of ideas in the United States, from the equation governing the production of ideas.

<u>Part b.</u>

Based on the textbook model, an increase in the productivity of production of ideas or an increase in the proportion of labour force engaged in the production of ideas will lead to an increase in the growth rate of real GDP per person. However, there does not seem to be an increased in the long run growth rate of real GDP per person as intellectual property's share of GDP increase. One plausible explanation is that the production of ideas runs into diminishing marginal returns on Ideas (you can refer to tutorial 8 Question 2 part a).

Question 3.

Part a.

From the law of motion of capital (capital accumulation equation),

$$\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t \tag{7}$$

$$K_{t+1} - K_t = \bar{s}Y_t - \bar{d}K_t$$

$$\frac{K_{t+1} - K_t}{K_t} = \frac{\bar{s}Y_t}{K_t} - \bar{d} \tag{8}$$

$$\Rightarrow \underbrace{g_{K_t}}_{\text{constant on balanced growth path}} = \underbrace{\bar{s}}_{\text{constant}} \frac{Y_t}{K_t} - \underbrace{\bar{d}}_{\text{constant}} \tag{9}$$

Since all except for $\frac{Y_t}{K_t}$ are constant along the balance growth path, it is necessary for $\frac{Y_t}{K_t}$ to be a constant along the balance growth path as well. Thus, *along the balance growth path* we have:

$$\Rightarrow \frac{Y_t}{K_t} = \text{a constant} \tag{10}$$

$$\Rightarrow \frac{Y_t}{K_t} \text{ is unchanging } \boldsymbol{or} \text{ the growth rate of } \frac{Y_t}{K_t} = 0 \tag{11}$$

$$\Rightarrow g_{\frac{Y_t}{K_t}} = g_{Y_t} - g_{K_t} = 0 \tag{12}$$

$$\Rightarrow g_{Y_t} = g_{K_t} \; (along\ the\ balance\ growth\ path) \tag{13}$$

$$\Rightarrow g_Y^\star = g_K^\star \tag{14}$$

Comment: First, take note that equations (10) to (14) are along the balance growth path.
Secondly, you can understand balance growth path as the long run path. In the long run (or along the balance growth path), it is necessary for $\frac{Y_t}{K_t}$ to be a constant. Why? If it is not a constant, there are two possible scenarios:

- $\frac{Y_t}{K_t}$ grows in the long run $\Rightarrow g_{\frac{Y_t}{K_t}} > 0 \Rightarrow g_{Y_t} - g_{K_t} > 0 \Rightarrow g_{Y_t} > g_{K_t}$. That means that Y_t will grow to be infinitely bigger than K_t as time continues.

- $\frac{Y_t}{K_t}$ decreases in the long run $\Rightarrow g_{\frac{Y_t}{K_t}} < 0 \Rightarrow g_{Y_t} - g_{K_t} < 0 \Rightarrow g_{Y_t} < g_{K_t}$. That means that K_t will grow to be infinitely bigger than Y_t as time continues.

Both of the scenarios are not realistic and not consistent with stylized facts of an economy.

Part b.

From the law of motion of ideas (idea production function),

$$\Delta A_{t+1} = \bar{z}A_t L_{at} \tag{15}$$

$$\frac{A_{t+1} - A_t}{A_t} = \bar{z}L_{at} \tag{16}$$

$$\frac{A_{t+1} - A_t}{A_t} = \bar{z}\,\bar{l}\,\bar{L} \tag{17}$$

$$g_{A_t} = \underbrace{\bar{z}\,\bar{l}\,\bar{L}}_{\text{this is a constant}} \tag{18}$$

From the aggregate production function, taking $\ln(\quad)$ and differentiate both sides with respect to time,

$$Y_t = A_t K_t^{\frac{1}{3}} L_{yt}^{\frac{2}{3}}$$

$$\ln Y_t = \ln\left(A_t K_t^{\frac{1}{3}} L_{yt}^{\frac{2}{3}}\right) \tag{19}$$

$$\ln Y_t = \ln A_t + \frac{1}{3}\ln K_t + \frac{2}{3}\ln L_{yt} \tag{20}$$

$$\frac{d\ \ln Y_t}{dt} = \frac{d\ \ln A_t}{dt} + \frac{1}{3}\frac{d\ \ln K_t}{dt} + \frac{2}{3}\frac{d\ \ln L_{yt}}{dt} \tag{21}$$

$$\frac{1}{Y_t}\frac{dY_t}{dt} = \frac{1}{A_t}\frac{dA_t}{dt} + \frac{1}{3}\frac{1}{K_t}\frac{dK_t}{dt} + \frac{2}{3}\frac{1}{L_{yt}}\frac{dL_{yt}}{dt} \tag{22}$$

$$\frac{\frac{dY_t}{Y_t}}{dt} = \frac{\frac{dA_t}{A_t}}{dt} + \frac{1}{3}\frac{\frac{dK_t}{K_t}}{dt} + \frac{2}{3}\frac{\frac{dL_{yt}}{L_{yt}}}{dt} \tag{23}$$

$$g_{Y_t} = g_{A_t} + \frac{1}{3}g_{K_t} + \frac{2}{3}g_{Lyt} \tag{24}$$

$$g_{Y_t} = g_{A_t} + \frac{1}{3}g_{K_t} + \frac{2}{3}\cdot 0 \tag{25}$$

$$g_{Y_t} = g_{A_t} + \frac{1}{3}g_{K_t} \tag{26}$$

Comment: You can skip equations (20) -(23). I write in details in order for us to revise what we have discussed in tutorial 6 concerning the growth rates.

Since we are evaluating equation (26) along the balance growth path, we can apply the balance growth path condition stated in equation (14) here. So along the balance growth path,

$$g_Y^\star = g_A^\star + \frac{1}{3}g_K^\star$$

$$g_Y^\star = g_A^\star + \frac{1}{3}g_Y^\star$$

$$\Rightarrow g_Y^\star - \frac{1}{3}g_Y^\star = g_A^\star$$

$$\Rightarrow \frac{2}{3}g_Y^\star = g_A^\star$$

$$g_Y^\star = \frac{3}{2}g_A^\star = \frac{3}{2}\bar{z}\,\bar{l}\,\bar{L} \tag{27}$$

Thus, the growth rate of $\frac{Y_t}{L_t}$ (or the output per capita) along the balance growth path is:

$$g_{\frac{Y_t}{L_t}}^\star = g_y^\star = g_Y^\star - \underbrace{g_{\bar{L}}}_{\substack{=0 \\ \text{since } \bar{L} \\ \text{is a} \\ \text{constant}}} = \frac{3}{2}g_A^\star = \frac{3}{2}\bar{z}\,\bar{l}\,\bar{L} \tag{28}$$

$$\text{or } g_y^\star = \frac{3}{2}\bar{z}\,\bar{l}\,\bar{L} \tag{29}$$

Part c.

Comment: in the target expression, $y_t^\star = A_t^{\star\frac{3}{2}}\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{2}}(1-\bar{l})$, we have $\frac{\bar{s}}{g_y^\star+\bar{d}}$. Where can it come from? We can trace back the equations and find equation (9) is a promising one ($g_{K_t} = \bar{s}\frac{Y_t}{K_t} - \bar{d}$.) We also know that $g_K^\star = g_y^\star$ along the balance growth path.

From equations (9) and (14), the capital ratio is proportional to the investment rate along the balance growth path. That is:

$$\begin{aligned}\frac{K_t^\star}{Y_t^\star} &= \frac{\bar{s}}{g_K^\star + \bar{d}} \\ &= \frac{\bar{s}}{g_Y^\star + \bar{d}} \\ &= \frac{\bar{s}}{\underbrace{g_y^\star}_{\text{see equation 28}} + \bar{d}}\end{aligned} \tag{30}$$

Comment: Again, you might not know how to continue with this question, my suggestion is to look at the end result $\frac{Y_t^\star}{\bar{L}} = y_t^\star = A_t^{\star\frac{3}{2}}\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{2}}(1-\bar{l})$. We have shown $\frac{\bar{s}}{g_y^\star+\bar{d}} = \frac{K_t^\star}{Y_t^\star}$. That means that the final expression involves $A_t^\star$, $K_t^\star$, $Y_t^\star$ and also $\bar{L}$. So the very good candidate to explore is the aggregate production function.

From the aggregate production function,

$$\begin{aligned}Y_t &= A_t K_t^{\frac{1}{3}} L_{yt}^{\frac{2}{3}} \\ &= A_t K_t^{\frac{1}{3}} \left[(1-\bar{l})\,\bar{L}\right]^{\frac{2}{3}}\end{aligned}$$

Along the balance growth path, output per capita is:

$$\begin{aligned}y_t^\star = \frac{Y_t^\star}{\bar{L}^\star} &= A_t^\star K_t^{\star\frac{1}{3}}(1-\bar{l})^{\frac{2}{3}}\,\bar{L}^{-\frac{1}{3}} \\ &= A_t^\star\left(\frac{K_t^\star}{Y_t^\star}Y_t^\star\right)^{\frac{1}{3}}(1-\bar{l})^{\frac{2}{3}}\,\bar{L}^{-\frac{1}{3}} \\ &= A_t^\star\left(\frac{\bar{s}}{g_y^\star+\bar{d}}Y_t^\star\right)^{\frac{1}{3}}(1-\bar{l})^{\frac{2}{3}}\,\bar{L}^{-\frac{1}{3}} \\ &= A_t^\star\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{3}}(Y_t^\star)^{\frac{1}{3}}(1-\bar{l})^{\frac{2}{3}}\,\bar{L}^{-\frac{1}{3}} \\ &= A_t^\star\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{3}}(1-\bar{l})^{\frac{2}{3}}\left(\frac{Y_t^\star}{\bar{L}}\right)^{\frac{1}{3}} \\ &= A_t^\star\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{3}}(1-\bar{l})^{\frac{2}{3}}(y_t^\star)^{\frac{1}{3}}\end{aligned} \tag{31}$$

$$\Rightarrow \frac{y_t^\star}{(y_t^\star)^{\frac{1}{3}}} = A_t^\star \left(\frac{\bar{s}}{g_y^\star + \bar{d}}\right)^{\frac{1}{3}} (1-\bar{l})^{\frac{2}{3}}$$

$$\Rightarrow (y_t^\star)^{\frac{2}{3}} = A_t^\star \left(\frac{\bar{s}}{g_y^\star + \bar{d}}\right)^{\frac{1}{3}} (1-\bar{l})^{\frac{2}{3}}$$

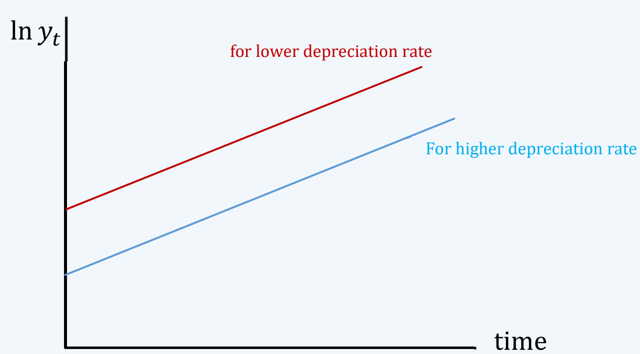
$$\Rightarrow y_t^\star = \left(A_t^\star \left(\frac{\bar{s}}{g_y^\star + \bar{d}}\right)^{\frac{1}{3}} (1-\bar{l})^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$\Rightarrow y_t^\star = A_t^{\star\frac{3}{2}} \left(\left(\frac{\bar{s}}{g_y^\star + \bar{d}}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} \left((1-\bar{l})^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$\Rightarrow y_t^\star = A_t^{\star\frac{3}{2}} \left(\frac{\bar{s}}{g_y^\star + \bar{d}}\right)^{\frac{1}{2}} (1-\bar{l}) \quad \text{shown} \tag{32}$$

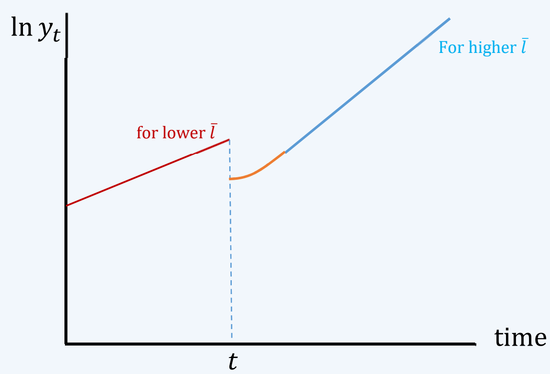
Comment: So what the point of working out the equation (32)? If you look at it hard enough, and I hope you do without straining your eyes, then you can see that on the right-hand side (RHS), all are constant in the long run (along the balance growth path) except for $A_t^\star$. On the left-hand side, we have $\boldsymbol{y_t^\star}$. So this means that $y_t^\star$ is growing at the rate of $A_t^{\star\frac{3}{2}}$ (I am not working out the math, by now you should be able to know why). As such, $g_y^\star = \frac{3}{2} g_A^\star$ and this is consistent with that of equation (29). Also, as for the level effect, $y_t^\star$ is affected by $\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{2}} (1-\bar{l})$:

- If $\bar{d}$ increases, the level of $y_t^\star$ decrease. For example, if there are two economies which are exactly the same, except for the depreciation rate (one higher, one lower). Then the graph of $\ln y_t$ against time t for the two economies are as shown below. As you can see, though they have same growth rate (slope), the one with have higher depreciation rate has lower output **level**.



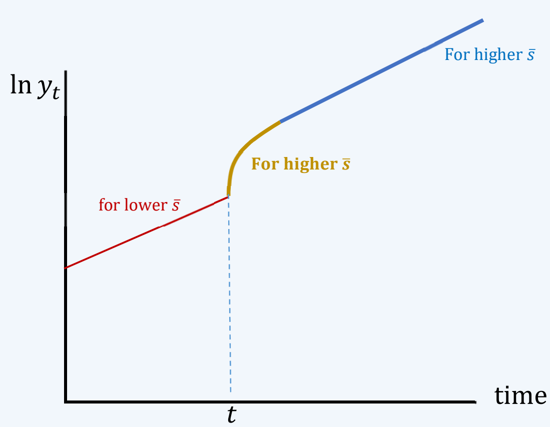
- If $\bar{l}$ increases, it is very interesting. $\bar{l}$ is the proportion of labour in research. So as $\bar{l}$ increases, $g_y^\star$ will be higher because $g_y^\star = g_A^\star = \bar{z}\,\bar{l}\,\bar{L}$. This means that $y_t^\star$ will have higher growth rate (growth effect). However, $\bar{l}$ increases $\Rightarrow (1-\bar{l})$ decreases & $\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)^{\frac{1}{2}}$ decreases $\because g_y^\star \uparrow$. So this means that level wise we have lower output. This makes sense because as more labour joins research, there would be less working in production, causing the output level to fall. However, the economy enjoys higher growth. So let's suppose that there are some policy that causes $\bar{l}$ to increase. Then the graph of $\ln y_t$ against time t for the economy is as shown below.

[the orange curved part of the graph below is where we have both capital accumulation effect (initially negative as the production is lowered due to less production labour) and growth effect combined (higher growth of ideas due to more research labour), as such the growth during this time is lowered for some time and slowly picks up then converges to the new BGP. The degree of the curvature depends on which effect dominates. ***You can treat this part as out of scope*** *for this course (I will not test you on this situation). I write this part for curious students to gain more insight into the model.]*



Suppose that at time t, $\bar{l}$ increases, immediately

- If $\bar{s}$ increases, $\left(\frac{\bar{s}}{g_y^\star+\bar{d}}\right)$ increases and so the output $y_t^\star$ will increase in level. But there is no effect on the growth rate. So let's suppose that at time t, there is an increase in the saving rate. Similar to the pure Solow model, there will be an accumulation of capital as saving rate rises. This capital accumulation will lead to higher output level. So output during these times will increase due to both capital accumulation and growth rate in idea. See the graph below.



Suppose that at time t, $\bar{s}$ increases, there will be capital accumulation. During these period (brown curve), the growth rate is higher (the economy enjoys growth both from capital accumulation and g_A). This growth rate will decrease because capital accumulation will wear out. In the long run, when there is no more capital accumulation, the economy only enjoy the growth rate from g_A. This means we are on the new balance growth path (the blue line). This new balance growth path is on a higher level as compared to the old one (dark red line) but has the same growth rate, g_A, with the old one.