LECTURE 5

Time Value of Money

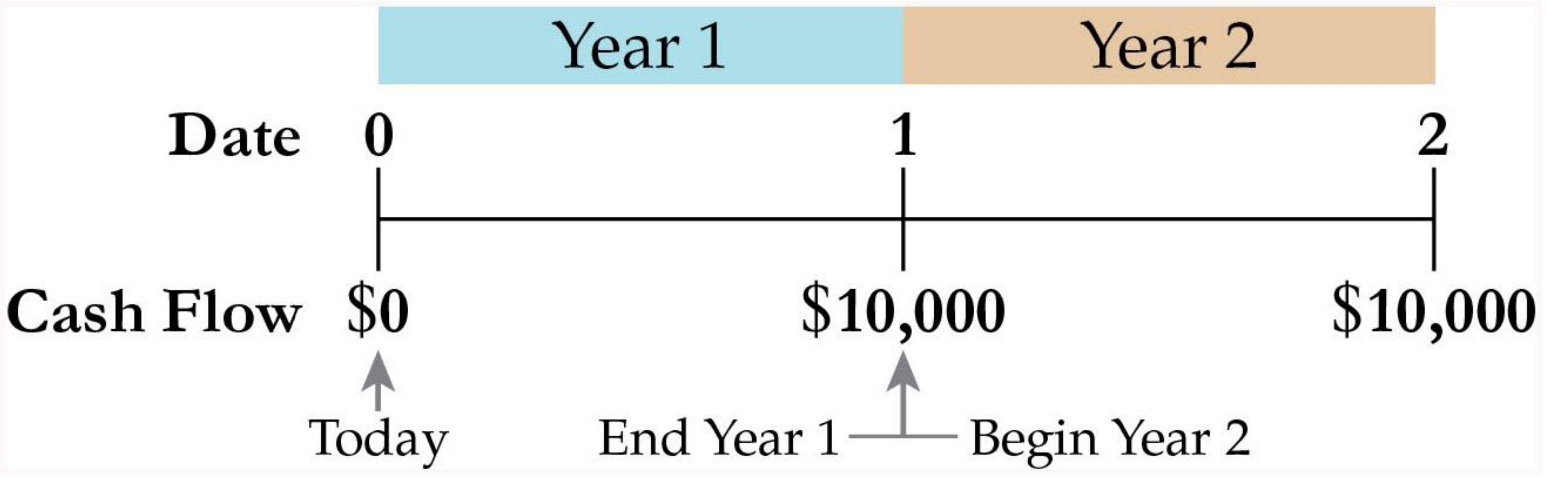
EC3333 Financial Economics I

Learning Objectives

- Define effective annual rate and annual percentage rate and their relationship.
- Describe the relation between nominal and real rates of interest.
- Compute the present value or future value of a stream of cashflows such as a perpetuity and an annuity.

The Timeline

- A linear representation of the timing of cash flows
- Suppose you will receive two payments, one at the end of each year over the next two years
  - Inflows are positive (+) cash flows
  - Outflows are negative (–) cash flows
  - Time 0 is now / today



Three Rules of Time Travel

- Financial decisions often require combining cash flows (CFs) or comparing values
- Never add/subtract CFs from different time periods without first converting them to a common time unit!

Table 4.1 The Three Rules of Time Travel
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

Rule1	Only values at the same point in time can be compared or combined	
Rule2	To move a cash flow forward in time, you must compound it.	Future value of a Cash flow
Rule3	To move a cash flow backward in time, you must discount it.	Present value of a Cash flow

Effective Annual Rate

- Indicates the total amount of interest that will be earned at the end of one year
- Considers the effect of compounding
- a.k.a. the Effective Annual Yield (EAY) or the Annual Percentage Yield (APY)
- For investments that last < 1 year, we compound the per-period return for a full year

$$\text{EAR} = (1 + \text{ rate for period })^{\text{Number of periods per year}} - 1$$

Effective Annual Rate

- Because of compounding, earning a 5% return annually is not the same as earning 2.5% every six months.
- Suppose one is earning 2.5% every 6 months
  - $1+EAR = (1+0.025)^2 = 1.0506$
  - $EAR = 0.0506 = 5.06\%$
- Earning 2.5% every six months = earning 5.0525% a year
- It is greater than 5% because you earn interest on interest
- Suppose one is earning EAR = 5%, it is approximately the same as earning 2.47% every six month.
  - $(1+r)^2 = 1.05$, where r = interest rate for six months
  - $r = (1.05)^{1/2} – 1 = 1.0247 – 1 = 0.0247 = 2.47\%$

Annual Percentage Rate

- The annual percentage rate (APR), indicates the amount of simple interest earned in one year.
- Simple interest is the amount of interest earned without the effect of compounding.
- The APR is typically less than the EAR.
- Rates on investments that last < 1 year are often annualized using simple rather than compound interest.

Annual Percentage Rate

- The APR itself cannot be used as a discount rate
  - This is because the APR does not reflect the true amount you will earn over one year
  - It has to be converted to a discount rate
  - The APR with k compounding periods per year is a way of quoting the actual interest earned each compounding period

$$\text{Interest Rate per Compounding Period} = \frac{\text{APR}}{k \text{ periods / year}}$$

Annual Percentage Rate

- Suppose a bank advertises savings accounts with an interest rate of “6% APR with monthly compounding”
  - In this case, you will earn 6%/12 = 0.5% every month
  - Because the interest rate compounds each month, you will earn $(1.005)^{12} = 1.061678$ at the end of one year, for an EAR of 6.1678%
  - In later months, you earn interest on the interest paid in earlier months

Converting APR to EAR

$$1 + EAR = \left(1 + \frac{APR}{k}\right)^k$$

- The EAR increases with the frequency of compounding.
- As the compounding frequency grows and the compounding interval gets smaller, in the limit we get continuous compounding (CC).
- Continuous compounding is compounding every instant.
- For the continuously compounded case (denoted by r_{cc})

$$1 + EAR = e^{r_{cc}}$$

Fisher Relation

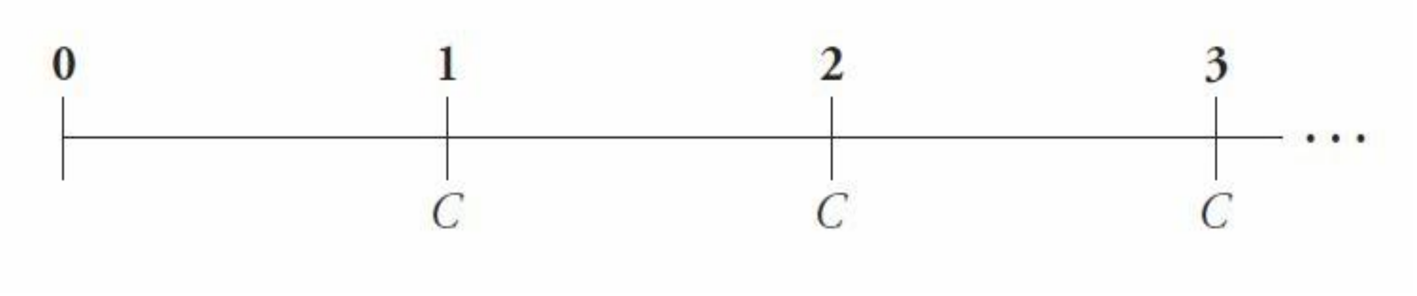
- Real interest rate: r
- Nominal interest rate: i
- Inflation rate: π

$$1 + r = \tfrac{1+i}{1+\pi} \quad \Leftrightarrow 1 + r + \pi + r\pi = 1 + i$$

$$r = \tfrac{i-\pi}{1} \qquad \text{or} \qquad r \approx i - \pi$$

Perpetuities

- When a constant cash flow will occur at regular intervals forever it is called a perpetuity
  - E.g., Consol bonds in the U.K.



Perpetuities

- The value of a perpetuity is simply the cash flow divided by the interest rate

$$PV(C \text{ in perpetuity }) = \sum_{n=1}^{\infty} \frac{C}{(1+r)^n}$$

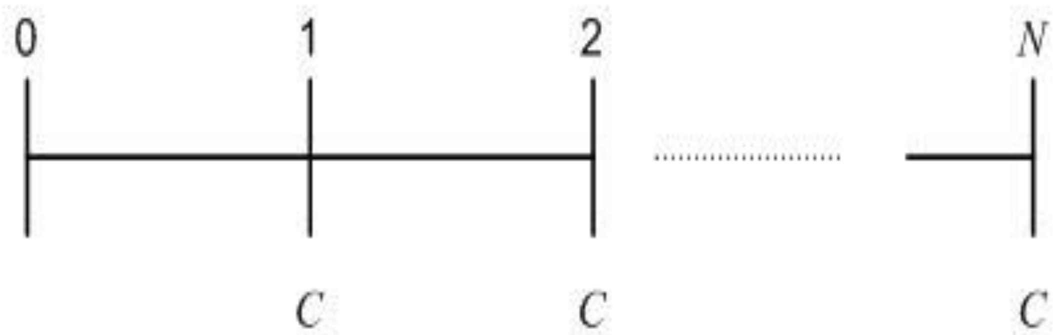
$$PV(C \text{ in perpetuity }) = \frac{C}{(1+r)} \left[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \dots \right]$$

$$PV(C \text{ in perpetuity }) = \frac{C}{(1+r)} \frac{1}{(1 - \frac{1}{1+r})} = \frac{C}{(1+r)} \frac{1}{(\frac{1+r-1}{1+r})}$$

$$PV(C \text{ in perpetuity }) = \frac{C}{r}$$

Annuities

- When a constant cash flow will occur at regular intervals for a finite number of N periods, it is called an annuity



$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^N}$$

$$PV = \sum_{n=1}^{N} \frac{C}{(1+r)^n}$$

Present Value of an Annuity

- Use formula for geometric progression

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^N}$$

$$PV = \frac{C}{(1+r)} \left[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{N-1}} \right]$$

$$PV = \frac{C}{(1+r)} \left[\frac{1 - \frac{1}{(1+r)^N}}{\left(1 - \frac{1}{(1+r)}\right)} \right]$$

$$PV = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right)}_{\textbf{The Annuity Formula}}$$

Present Value and Future Value of an Annuity

- To recap, the present value of annuity of C for N period equals

$$PV = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right)}_{\text{The Annuity Formula}}$$

- Thus, the future value of an annuity equals

$$\begin{aligned} FV(\text{ annuity }) &= P\,\mathrm{V} \times (1+r)^N \\ &= \frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right) \times (1+r)^N \end{aligned}$$

$$FV = \frac{C}{r}\left((1+r)^N - 1\right)$$

Solving for the Present Value

Example 4.14 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - You are about to purchase a new car and have two options to pay for it. You can pay $20,000 in cash immediately, or you can get a loan that requires you to pay $500 each month for the next 48 months (four years). If the monthly interest rate you earn on your cash is 0.5%, which option should you take?

Solving for the Present Value – an annuity with monthly cash flows

Example 4.14 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

Solution

- Let’s start by writing down the timeline of the loan payments:

- The timeline shows that the loan is a 48-period annuity. Using the annuity formula the present value is

$$\text{PV}(48 - \text{ period annuity of } \$500) = \$500 \times \frac{1}{0.005}\left(1 - \frac{1}{1.005^{48}}\right) = \$21{,}290$$

Solving for the Cash Payments

- Sometimes we know the present value or future value, but do not know one of the variables we have previously been given as an input.
- E.g., when you take out a loan you may know the amount you would like to borrow, but may not know the loan payments that will be required to repay it.
- To solve for the loan payment, think of the amount borrowed (the loan principal P) as the present value of the payments when evaluated at the loan rate: P = PV(annuity of C for N periods)
- Loan or Annuity Payment $$C = \frac{P}{\left(\frac{1}{r}\right)\left(1 - \frac{1}{(1+r)^N}\right)}$$

Computing Loan Payments

- Payments are made at a set interval, typically monthly
- Many loans, such as mortgages and car loans, are amortizing loans, which means that each payment made includes the interest on the loan plus some part of the loan balance
- All payments are equal and the loan is fully repaid with the final payment
- When the compounding interval for the APR is not stated explicitly, it is equal to the interval between the payments

Computing a Loan Payment

Example 4.15 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - Your biotech firm plans to buy a new DNA sequencer for $500,000. the seller requires that you pay 20% of the purchase price as a down payments, but is willing to finance the remainder by offering a 48-month loan with equal monthly payments and an interest rate of 0.5% per month. What is the monthly loan payment?

Computing a Loan Payment

Example 4.15 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - Given a down payment of 20% × \$500,000 = \$100,000, your loan amount is \$400,000. We start with the timeline (from the seller's perspective), where each period represents one month:

  - We can solve for the loan payment, *C*, as follows:

$$C = \frac{P}{\frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)} = \frac{4000{,}000}{\frac{1}{0.005}\left(1 - \frac{1}{(1.005)^{48}}\right)}$$
$$= \$9394$$

Valuing Monthly Cash flows

Example 5.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - Suppose your bank account pays interest monthly with the interest rate quoted as an effective annual rate (EAR) of 6%. What amount of interest will you earn each month? If you have no money in the bank today, how much will you need to save at the end of each month to accumulate $100,000 in 10 years?

Valuing Monthly Cash flows

Example 5.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - A 6% EAR is equivalent to earning $(1.06)^{\frac{1}{12}} - 1 = 0.4868\%$ per month. We can write the timeline for our savings plan using **monthly** periods as follows:

Month:	0	1	2	...	120
Cash flow:		C	C		C

  - That is, we can view the savings plan as a monthly annuity with 10 × 12 = 120 monthly payments. We can calculate the total amount saved as the future value of this annuity:

$$\mathrm{FV(annuity)} = \mathrm{C} \times \frac{1}{\mathrm{r}}[(1+\mathrm{r})^{\mathrm{n}} - 1]$$

  - We can solve for the **monthly** payment C using the equivalent **monthly** interest rate r = 0.4868%, and n = 120 months:

$$C = \frac{FV\ (\text{annuity})}{\frac{1}{r}[(1+r)^n - 1]} = \frac{\$100,000}{\frac{1}{0.004868}[(1.004868)^{120} - 1]} = \$615.47 \text{ per month}$$

Computing the Outstanding Loan Balance

- The outstanding balance on a loan, also called the outstanding principal, is equal to the present value of the remaining future loan payments, again evaluated using the loan interest rate.

Computing the Outstanding Loan Balance

Example 5.3 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

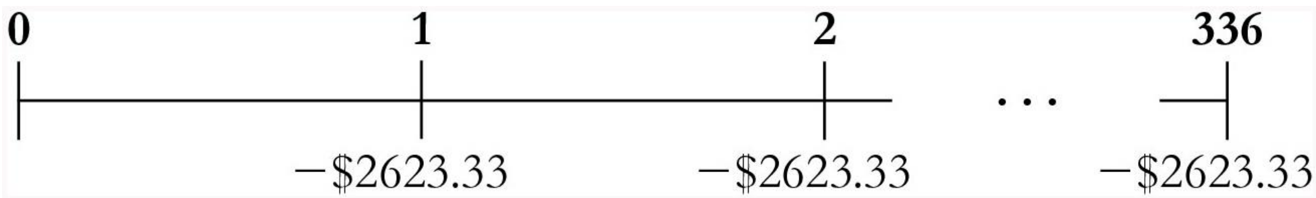
- **Problem**
  - Two years ago your firm took out a 30-year amortizing loan to purchase a small office building. The loan has a 4.80% APR with monthly payments of $2,623.33. How much do you owe on the loan today? How much interest did the firm pay on the loan in the past year?

Computing the Outstanding Loan Balance

Example 5.3 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

Solution

- After 2 years, the loan has 28 years, or 336 months, remaining:



- The remaining balance on the loan is the present value of these remaining payments, using the loan rate of

$$\frac{4.8\%}{12} = 0.4\% \text{ per month:}$$

$$\text{Balance after 2 years} = \$2623.33 \times \frac{1}{0.004}\left(1 - \frac{1}{1.004^{336}}\right)$$

$$= \$484{,}332$$

Computing the Outstanding Loan Balance

Example 5.3 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

Solution

- During the past year, your firm made total payments of \$2,623.33 × 12 = \$31,480 on the loan. To determine the amount that was interest, it is easiest to first determine the amount that was used to repay the principal. Your loan balance one year ago, with 29 years (348 months) remaining, was

$$\text{Balance after one year} = \$2623.33 \times \frac{1}{0.004}\left(1 - \frac{1}{1.004^{348}}\right) = \$492{,}354$$

- Therefore, the balance declined by \$492,354 − \$484,332 = \$8,022 in the past year. Of the total payments made, \$8,022 was used to repay the principal and the remaining \$31,480 − \$8,022 = \$23,458 was used to pay interest.

Solving for the Internal Rate of Return

- In some situations, you know the present value and cash flows of an investment opportunity but you do not know the internal rate of return (IRR)
- IRR is the interest rate that sets the net present value of the cash flows equal to zero

Computing the IRR for a Perpetuity

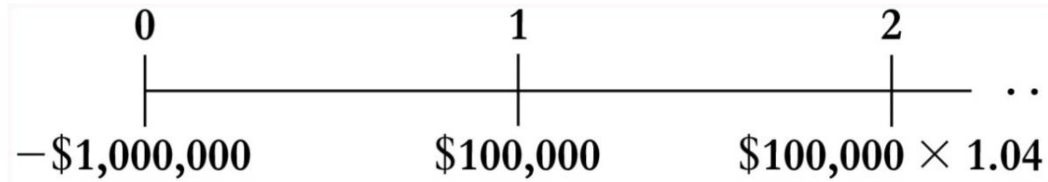
Example 4.16 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - Jessica has just graduated with her MBA. Rather than take the job she was offered at a prestigious investment bank—Baker, Bellingham, and Botts—she has decided to go into business for herself. She believes that her business will require an initial investment of $1 million. after that, it will generate a cash flow of $100,000 at the end of one year, and this amount will grow by 4% per year thereafter. What is the IRR of this investment opportunity?

Computing the IRR for a Perpetuity

Example 4.16 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - The timeline is



  - The timeline shows that the future cash flows are a growing perpetuity with a growth rate of 4%.
  - PV of a growing perpetuity is $\frac{C}{(r-g)}$.
  - Thus, the NPV of this investment would equal zero if

$$1{,}000{,}000 = \frac{100{,}000}{r - 0.04}$$

  - We can solve this equation for *r*

$$r = \frac{100{,}000}{1{,}000{,}000} + 0.04 = 0.14$$

  - So, the IRR on this investment is 14%.

Computing the Internal Rate of return for an Annuity

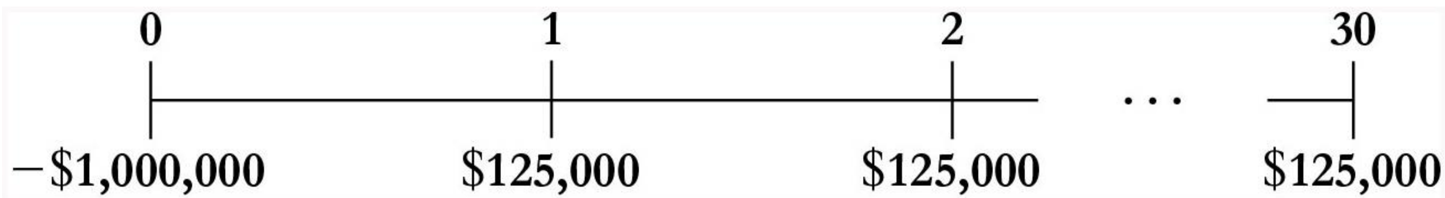
Example 4.17 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - Baker, Bellingham, and Botts, was so impressed with Jessica that it has decided to fund her business. In return for providing the initial capital of $1 million, Jessica has agreed to pay them $125,000 at the end of each year for the next 30 years. What is the internal rate of return on Baker, Bellingham, and Botts's investment in Jessica's company, assuming she fulfills her commitment?

Computing the Internal Rate of return for an Annuity

Example 4.17 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - Here is the timeline (from Baker, Bellingham, and Botts's perspective):



- The timeline shows that the future cash flows are a 30-year annuity. Setting the NPV equal to zero requires

$$1{,}000{,}000 = 125{,}000 \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^{30}}\right)$$

- The IRR on this investment is 12.09%. In this case, we can interpret the IRR of 12.09% as the effective interest rate of the loan.

Solving for the Number of Periods

- In addition to solving for cash flows or the interest rate, we can solve for the amount of time it will take a sum of money to grow to a known value.

Solving for the number of periods in a savings plan

Example 4A.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - You are saving for a down payment on a house. You have $10,050 saved already, and you can afford to save an additional $5,000 per year at the end of each year. If you earn 7.25% per year on your savings, how long will it take you to save $60,000?

Solving for the number of periods in a savings plan

Example 4A.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - The timeline for this problem is

  - We need to find *N* so that the future value of our current savings plus the future value of our planned additional savings (which is an annuity) equals our desired amount:

$$10{,}050 \times 1.0725^{N} + 5000 \times \frac{1}{0.0725}\left(1.0725^{N} - 1\right) = 60{,}000$$

  - To solve mathematically, rearrange the equation to

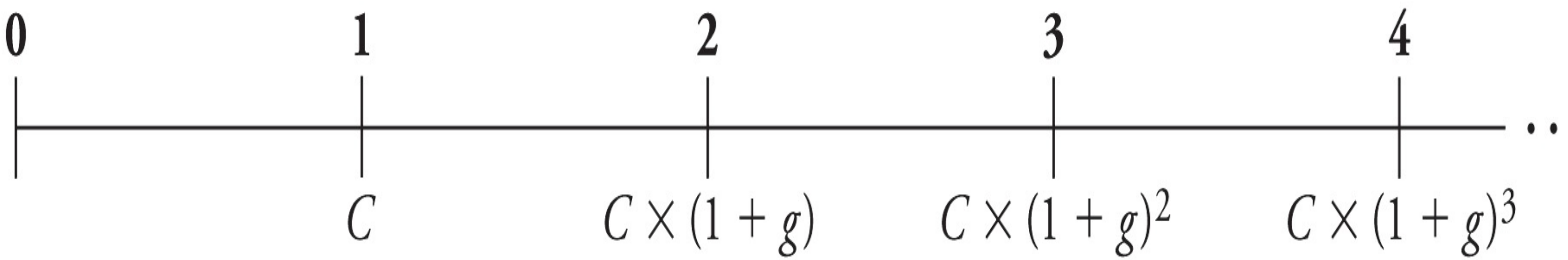
$$1.0725^{N} = \frac{60{,}000 \times 0.0725 + 5000}{10{,}050 \times 0.0725 + 5000} = 1.632$$

  - We can then solve for *N:*

$$N = \frac{\ln(1.632)}{\ln(1.0725)} = 7.0 \text{ years}$$

Growing Perpetuity

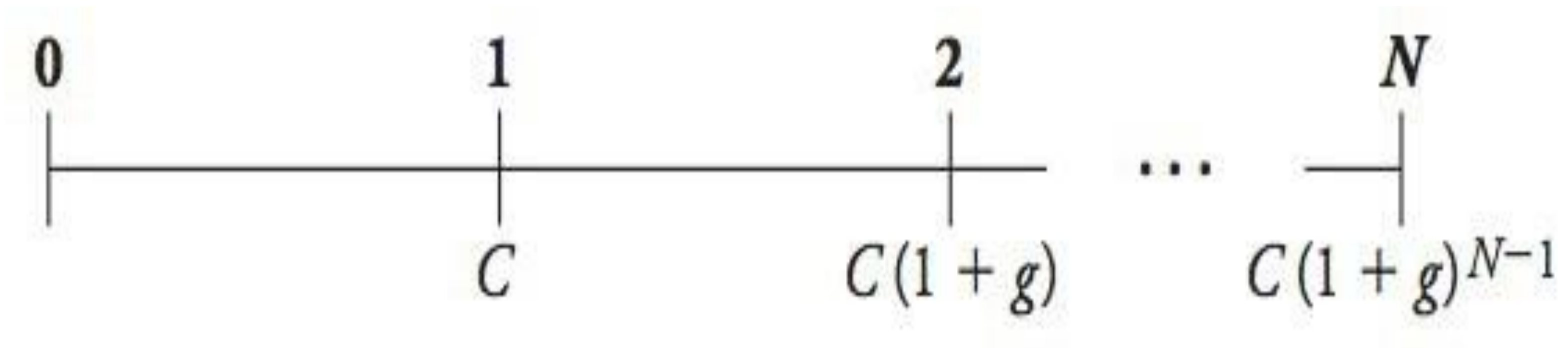
- Assume you expect the amount of your perpetual payment to increase at a constant rate, *g*.



$$PV\ (\text{growing perpetuity}) = \frac{C}{r-g}$$

Growing Annuity

- The present value of a growing annuity with the initial cash flow c, growth rate g, and interest rate r is defined as:



$$PV\ (\text{growing annuity}) = \frac{C}{(r-g)}\left(1 - \left(\frac{1+g}{1+r}\right)^{N}\right)$$