**NATIONAL UNIVERSITY OF SINGAPORE**
**Department of Mathematics**

**MA1522 Linear Algebra for Computing** **Tutorial 11**

1. (i) Determine whether the following are linear transformations.
   (ii) Write down the standard matrix for each other the linear transformations.
   (iii) Find a basis for the range for each of the linear transformations.
   (iv) Find a basis for the kernel for each of the linear transformations.

   (a) $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (b) $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (c) $T_3\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (d) $T_4\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   (e) $T_5\colon \mathbb{R}^5 \to \mathbb{R}$ such that $T_5\left(\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}\right) = x_3 + 2x_4 - x_5$ for $\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^5$.

   (f) $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$.

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations such that

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 \\ x_1 + x_2 - 3x_3 \\ 5x_2 - x_3 \end{pmatrix} \text{ and } G\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_3 - x_1 \\ x_2 + 5x_1 \\ x_1 + x_2 + x_3 \end{pmatrix},$$

   and let $\mathbf{A}_F$ and $\mathbf{B}_G$ be the standard matrix of $F$ and $G$, respectively.

   (a) Find $\mathbf{A}_F$ and $\mathbf{B}_G$.

   (b) Define
$$(F+G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^3.$$
   Is $(F+G)$ a linear transformation? If it is, find its standard matrix.

   (c) Write down the formula for $F(G(\mathbf{x}))$ and find its standard matrix.

   (d) Find a linear transformation $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$H(G(\mathbf{x})) = \mathbf{x}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

3. For each of the following linear transformations, (i) determine whether there is enough information for us to find the formula of $T$; and (ii) find the formula and the standard matrix for $T$ if possible.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\\0\\1\end{pmatrix},\ T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\\-1\\4\end{pmatrix},\ \text{and } T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\4\\1\\6\end{pmatrix}.$$

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\0\end{pmatrix},\ T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\2\end{pmatrix},\ \text{and } T\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}.$$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right) = -1,\ T\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\right) = 1 \text{ and } T\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}\right) = 0.$$

4. For each of the following linear transformations $T$, determine its rank and nullity, and whether it is one-to-one, and/or onto.

   (a) $T\colon \mathbb{R}^4 \to \mathbb{R}^6$ such that the rank is 4.

   (b) $T\colon \mathbb{R}^6 \to \mathbb{R}^4$ such that the nullity is 2.

   (c) $T : \mathbb{R}^4 \to \mathbb{R}^6$ such that the reduce row-echelon form of its standard matrix has 3 nonzero rows.

   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T$ is one-to-one.

# Extra problems

1. (a) Show that if $\lambda$ is an eigenvalue of a stochastic matrix $\mathbf{P}$, then $|\lambda| \leq 1$.

   **Hint:** Pick an eigenvector $\mathbf{v}$ of $\mathbf{P}^T$ associated with $\lambda$. Let $k \in \{1, 2, \ldots, n\}$ be a coordinate of $\mathbf{v}$ with the maximum absolute value, $|v_k| \geq |v_i|$ for all $i = 1, \ldots, n$. Consider the $k$-th coordinate of the equation $\mathbf{P}^T\mathbf{v} = \lambda\mathbf{v}^T$.

   (b) Let $\mathbf{P}$ be a stochastic matrix. For any vector $\mathbf{v}$, define $\mathbf{v}^{(k)} = \mathbf{P}^k\mathbf{v}$. Show that if $\mathbf{v}$ is an eigenvector of $\mathbf{P}$ that is not associated to eigenvalue 1, then $\mathbf{v}^{(k)} \to 0$ as $k \to \infty$.

2. Let $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$ and $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be some vectors in $\mathbb{R}^m$. Prove that there is a unique transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1, ..., n$.

3. Prove that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective (one-to-one and onto) linear transformation if and only if there is a basis $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ such that the standard matrix of $T$ is a transition matrix from $S$ to $E$, where $E$ is the standard basis for $\mathbb{R}^n$.