NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 4**

1. Let $A = \{\ (1+t, 1+2t, 1+3t) \mid t \in \mathbb{R}\ \}$ be a subset in $\mathbb{R}^3$.

   (a) Describe A geometrically.

   (b) Show that $A = \{\ (x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0\ \}$.

   (c) Write down a matrix equation $\mathbf{Mx} = \mathbf{b}$ where $\mathbf{M}$ is a 3×3 matrix and $\mathbf{b}$ is a 3×1 matrix such that its solution set is A.

2. Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

   (a) If possible, express each of the following vectors as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

   (i) $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix}$

   (b) Is it possible to find 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are not a multiple of each other, and both are not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

3. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ x - y - z = 0 \right\}$ be a subset of $\mathbb{R}^3$.

   (a) Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\}$. Show that $\text{span}(S) = V$.

   (b) Let $T = S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Show that $\text{span}(T) = \mathbb{R}^3$.

4. Which of the following sets S spans $\mathbb{R}^4$?

   (i) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

   (ii) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(iii) $S = \left\{ \begin{pmatrix} 6 \\ 4 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix} \right\}.$

(iv) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$

5. Determine whether $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

(a) $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

(b) $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}.$

6. Determine which of the following sets are subspaces. For those sets that are subspaces, express the set as a linear span. For those sets that are not, explain why.

(a) $S = \left\{ \begin{pmatrix} p \\ q \\ p \\ q \end{pmatrix} \middle| \; p, q \in \mathbb{R} \right\}.$

(b) $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a \geq b \text{ or } b \geq c \right\}.$

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; 4x = 3y \text{ and } 2x = -3w \right\}.$

(d) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}.$

(e) $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \; w + x = y + z \right\}.$

(f) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; ab = cd \right\}$.

(g) S is the solution set of $\mathbf{A}\mathbf{x} = \mathbf{0}$ where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

Extra problems

1. (a) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$. Show that $\det(\mathbf{A}) = \det(\mathbf{D})$.
   (b) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$ and $\mathbf{D}$ is a diagonal matrix. Show that $\mathbf{A}$ is invertible if and only if all the diagonal entries of $\mathbf{D}$ is nonzero.
   (c) Recall that a square matrix $\mathbf{A}$ is nilpotent if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\det(\mathbf{A}) = 0$.
   (d) A square matrix is an *orthogonal* matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that if $\mathbf{A}$ is orthogonal, then $\det(\mathbf{A}) = \pm 1$.

2. (a) Show that the solution set to any homogeneous linear system

$$V = \left\{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{Av} = \mathbf{0} \ \right\}$$

   is a subspace.
   (b) Let $V = \left\{ \ \mathbf{v} \in \mathbb{R}^n \ \middle| \ \mathbf{Av} = \mathbf{0} \ \right\}$. Show that if $\mathbf{Ax} = \mathbf{b}$ is consistent, then the solutions set is

$$\mathbf{u}_p + V = \left\{ \ \mathbf{u}_p + \mathbf{v} \ \middle| \ \mathbf{v} \in V \ \right\},$$

   where $\mathbf{u}_p$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$. (cf. Tutorial 1 Question 1)

   A subset of $\mathbb{R}^n$ is called an *affine space* if it is of the form $\left\{ \ \mathbf{u} + \mathbf{v} \ \middle| \ \mathbf{v} \in V \ \right\}$ for some subspace $V \subseteq \mathbb{R}^n$. Geometrically, an affine space is a subset of $\mathbb{R}^n$ that is parallel to a subspace. This exercise shows that the solution set to the linear system $\mathbf{Ax} = \mathbf{b}$ is an affine space $\left\{ \ \mathbf{u}_p + \mathbf{v} \ \middle| \ \mathbf{v} \in V \ \right\}$, where V is the solutions to homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{u}_p$ is any particular solution.

3. Determine which of the following statements are true. Justify your answer.
   (a) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\operatorname{span}(S_1 \cap S_2) = \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.
   (b) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\operatorname{span}(S_1 \cup S_2) = \operatorname{span}(S_1) \cup \operatorname{span}(S_2)$.

4. In computers, information is stored and processed in the form of strings of binary digits, 0 and 1. For this exercise, we will work in the "world" of binary digits

$$\mathbb{B} = \{0, 1\}.$$

   Addition in $\mathbb{B}$ works just as it does in $\mathbb{R}$, save for one special rule:

$$1 + 1 = 0.$$

   We can similarly perform scalar multiplication in $\mathbb{B}$—however, note that in our "binary world", we only have two possible scalars: 0 and 1 (as opposed to any real number).

   Remark. The special rule for binary addition is equivalent to performing our standard operations **modulo 2**. That is, in our "binary world," we evaluate a sum according to its remainder when divided by 2: if the remainder is 0 (i.e., when a number is even), then it corresponds to the binary digit 0, and if the remainder is 1 (i.e., when a number is odd), then it corresponds to the binary digit 1.

1. Using the rules on the basic operations in $\mathbb{B}$, complete the addition and multiplication tables below.

$+$	0	1
0		
1		

$\times$	0	1
0		
1		

2. Recall that we created the Euclidean space $\mathbb{R}^n$ by taking the set of all n-vectors with real components (i.e., with components in $\mathbb{R}$). We can create the set $\mathbb{B}^n$ in a similar fashion, by taking the set of all n-vectors whose components are binary digits, 0 or 1. Observe, then, that the basic properties of addition and scalar multiplication in $\mathbb{R}^n$ directly apply to $\mathbb{B}^n$, as long as we remember that $1 + 1 = 0$ and the only scalars we are allowed to multiply by are 0 and 1.

   (a) Consider the Euclidean 3-space $\mathbb{R}^3$, which has infinitely many vectors. How many vectors does $\mathbb{B}^3$ have?

   (b) A *byte*—the fundamental unit of data used by many computers—is a string of 8 binary digits. Observe that we can treat each byte as a vector in $\mathbb{B}^8$. How many distinct bytes exist; that is, how many vectors are there in $\mathbb{B}^8$? How does this compare to Euclidean 8-space $\mathbb{R}^8$?

   (c) The Euclidean n-space $\mathbb{R}^n$ has infinitely many vectors. More generally, how many vectors are there in $\mathbb{B}^n$?

   For the purposes of this exercise, you may assume that $\mathbb{B}^n$ has all the properties of a subspace—that is, $\mathbb{B}^n$ is closed under addition and scalar multiplication. (Try to prove this yourself!)

3. To get a sense of how vectors work in $\mathbb{B}^n$, we take a simple example. Let's begin by working in $\mathbb{B}^3$—the set of all 3-vectors whose components are binary digits.

   (a) Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be the set of standard unit vectors in $\mathbb{R}^3$. Show that S forms a basis for $\mathbb{B}^3$.