LECTURE 10

Option Valuation –
Black-Scholes Option Pricing

EC3333 Financial Economics I

Learning Objectives

- Apply the Black-Scholes Option Pricing formula to value a European option.
- Construct Black-Scholes replicating portfolio for a European option.
- Explain the idea behind dynamic hedging, option delta and gamma.

Review: Intrinsic and Time Values

- **Intrinsic value** – the value the option would have if it expired immediately
  - It is the amount by which the option is currently in-the-money, or zero if the option is out-of-the-money.
  - **In-the-money call:**
    - current stock price - exercise price = $S_0 – X$
  - **In-the-money put:**
    - exercise price – current stock price = $X – S_0$

- **Time value** - the difference between the current option price and its intrinsic value
  - Arises because the option is yet to expire and from “volatility value”

Review: Arbitrage Bounds on Option Prices

- A put option cannot be worth more than its strike price.
  - The maximum payoff for a put is when the stock becomes worthless, in which case the payoff is equal to the strike price. This is the highest payoff, so it cannot be worth more than the strike price.

- A call option cannot be worth more than the stock itself.
  - The lower the strike price, the more valuable the call option. If the call option had a strike price of zero, the holder would always exercise the option and receive the stock at zero cost. (Payoff of the call on a stock is maximum when the strike is zero.)

Review: Arbitrage Bounds on Option Prices

- An American option is worth at least as much as its European counterpart.
  - Option price cannot be negative
  - American option carries the same rights and privileges as an equivalent European option, in addition to an early exercise feature absent in the European option
- An American option cannot be worth less than its intrinsic value.
  - If an American option is worth less than its intrinsic value, you could make arbitrage profits by purchasing the option and immediately exercising it.
- Because an American option cannot be worth less than its intrinsic value, it cannot have a negative time value.

Recall: Binominal Option Pricing

- As the number of subperiods increases, the distribution approaches the skewed log-normal distribution (longer right tail).
- Even if the stock price were to decline in each subinterval, it can never drop below 0.
- But there is no corresponding upper bound on its potential upward performance.

Black-Scholes Option Pricing Model

- A technique for pricing European-style options when the stock can be traded continuously.
- It can be derived from the Binomial Option Pricing Model by:
  - allowing the length of each period to shrink to zero and
  - letting the number of periods grow infinitely large.
- All techniques used in financial engineering to price financial securities can be traced to the Black-Scholes formula.
- It uses the Law of One Price, without the need to model preferences.

Black-Scholes Formula for a Call

$$C_0 = S_0 \times N(d_1) - Xe^{-rT} \times N(d_2)$$

- Where: $$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

C_o = Current call option value

S_o = Current stock price

$N(d)$ = The cumulative normal distribution = $Pr(z \leq d)$
i.e., the probability that a random draw from a standard normal distribution will be less than d

X = Exercise price

e = 2.71828, the base of the natural log

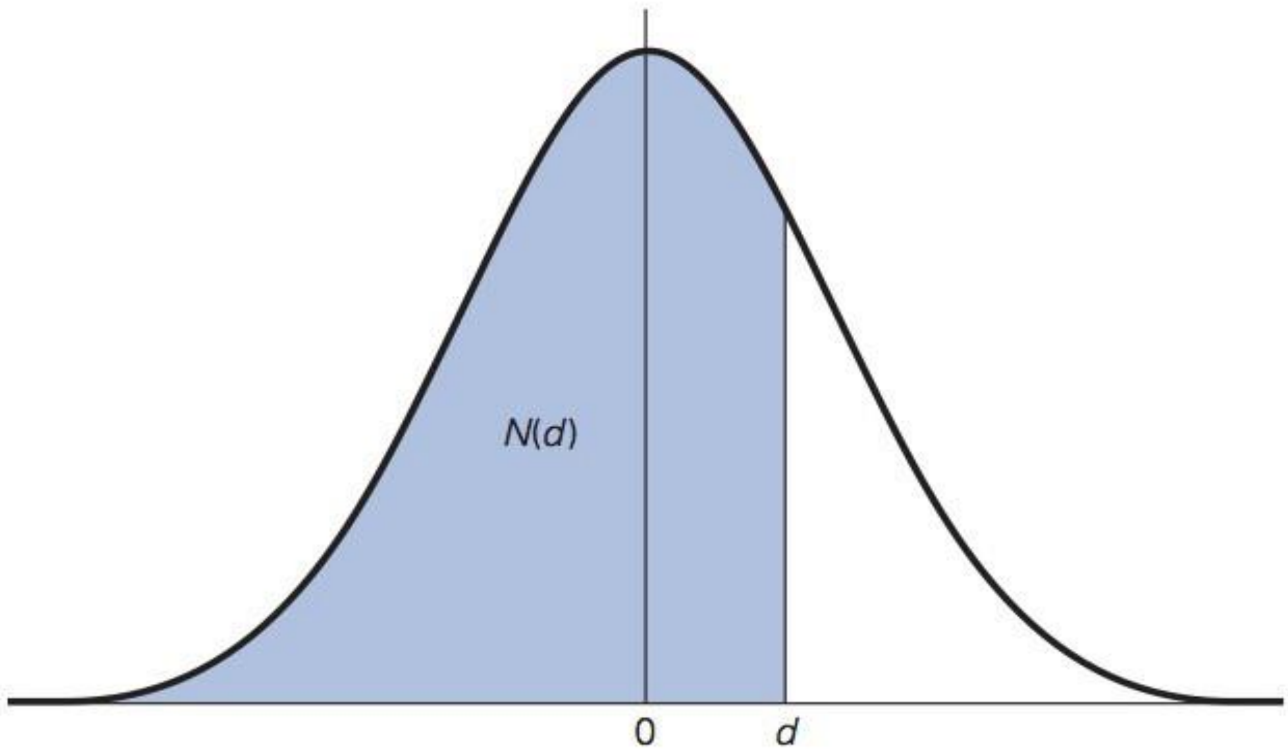
r = Risk-free interest rate

T = time to expiration of the option in years

ln = Natural log function

σ = Annual volatility (standard deviation) of the stock's return

Figure 21.3 A Standard Normal Distribution
from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e



Black-Scholes Formula for a Call

Example 21.4 (from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

- Value a European call option under the following circumstances:
  - S_0 = 100
  - X = 95
  - r = 10% per year or 0.10
  - T = 0.25 (3-months or a quarter of a year)
  - σ = 50% per year or 0.50
- First, calculate:

$$d_1 = \frac{\ln(100/95) + \left(0.10 + (0.5)^2/2\right) \times 0.25}{0.5\sqrt{0.25}} = 0.43$$

$$d_2 = 0.43 - 0.5\sqrt{0.25} = 0.18$$

Black-Scholes Formula for a Call

- Next, using a Standard Normal table (will be provided in exam) or the NORMSDIST function in Excel,

$$N(0.43) = 0.6664$$
$$N(0.18) = 0.5714$$

- Therefore, the value of a European call is:

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$
$$C_0 = 100 \times 0.6664 - 95 \times e^{-0.10 \times 0.25} \times 0.5714$$
$$C_0 = \$13.70$$

Black-Scholes Model with Dividends

- The previous formula applies to stocks that do not pay dividends
- What if dividends ARE paid?
- Replace stock price with a dividend adjusted stock price
- Replace S_0 with S_0 – PV(Dividends)
    - Because a European call option is the right to buy the stock without the dividends paid during the life of the option
    - Because stock price tends to drop by the amount of the dividend when the stock goes ex-dividend

The Black-Scholes Formula for a Put

- Use the put-call parity theorem

$$P = C + Xe^{-rT} - S_0$$

- The value of a European put option is:

$$P = Xe^{-rT}[1 - N(d_2)] - S_0[1 - N(d_1)]$$

- Using Example 21.4 data:
- S_0 = 100, r = 0.10, X = 95, σ = 0.5, and T = 0.25

$$P = \$95\mathrm{e}^{-0.10\times0.25}(1 - 0.5714) - \$100(1 - 0.6664)$$
$$P = \$6.35$$

Put Option: Using Put-Call Parity

$$\begin{aligned} P &= C + PV(X) - S_0 \\ &= C + Xe^{-rT} - S_0 \end{aligned}$$

- Using the data in the example,

$$P = 13.70 + 95e^{-0.10 \times 0.25} - 100$$
$$P = \$6.35$$

The Option Delta Δ

- From the B-S formula, it can be proved that

$$\frac{\partial C}{\partial S} = N(d_1)$$

- Delta is the number of shares in the replicating portfolio for the option.
- When the option value is plotted as a function of the stock value, it is the slope of the value curve evaluated at the current stock price.
- Option delta changes with the stock price $\rightarrow$ Dynamic Hedging.

Figure 21.9 Call option value and hedge ratio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

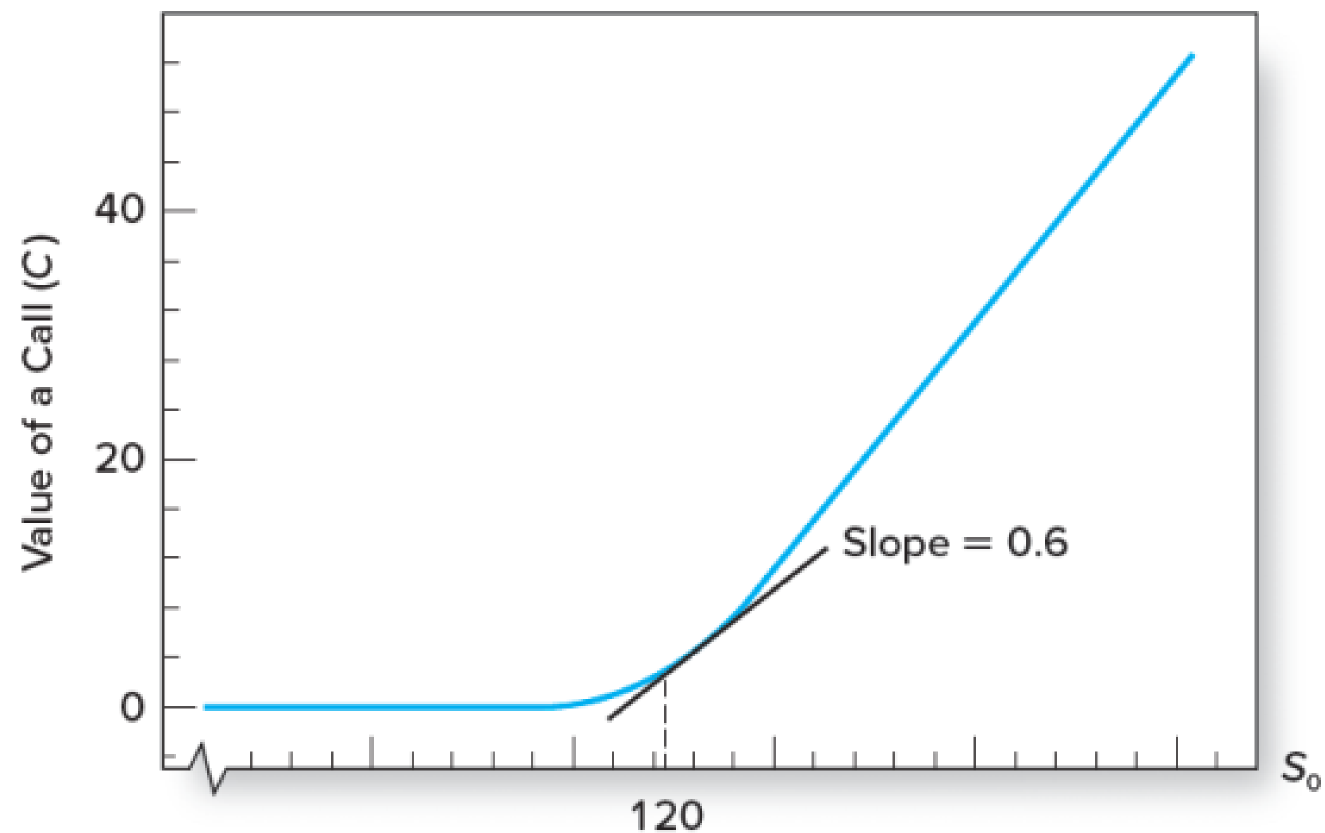


Figure 21.9 Call option value and hedge ratio

The Option Delta Δ and the Replicating Portfolio

- Recall
  - Option Price in the Binomial Model

$$C = S\Delta + B$$

- Then
  - Black-Scholes Replicating Portfolio of a Call Option

$$\Delta = N(d_1)$$
$$B = -PV(X)N(d_2)$$

  - Black-Scholes Replicating Portfolio of a Put Option

$$\Delta = -[1 - N(d_1)]$$
$$B = PV(X)[1 - N(d_2)]$$

The Option Delta Δ and the Replicating Portfolio

- Option Delta (Δ) is the change in the price of an option given a $1 change in the price of the stock. The number of shares in the replicating portfolio for the option.
- For a call option, Δ is always less than 1; the change in the call price is always less than the change in the stock price.
- The replicating portfolio of a call option always consists of a long position in the stock and a short position in the bond.
- The replicating portfolio of a put option always consists of a long position in the bond and a short position in the stock.

Computing the Replicating Portfolio

Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
- PNA Systems pays no dividends and has a current stock price of $10 per share. If its returns have a volatility of 40% and the risk-free rate is 5%, what portfolio would you hold today to replicate a one-year at-the-money call option on the stock?

Computing the Replicating Portfolio

Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- We can apply the Black-Scholes formula with $S_0 = 10$,

$$PV(X) = \frac{10}{1.05} = 9.524, \text{ and}$$

$$d_1 = \frac{\ln\left[\frac{S_0}{PV(X)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = \frac{\ln\left(\frac{10}{9.524}\right)}{40\%} + \frac{40\%}{2} = 0.322$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.322 - 0.40 = -0.078$$

Computing the Replicating Portfolio

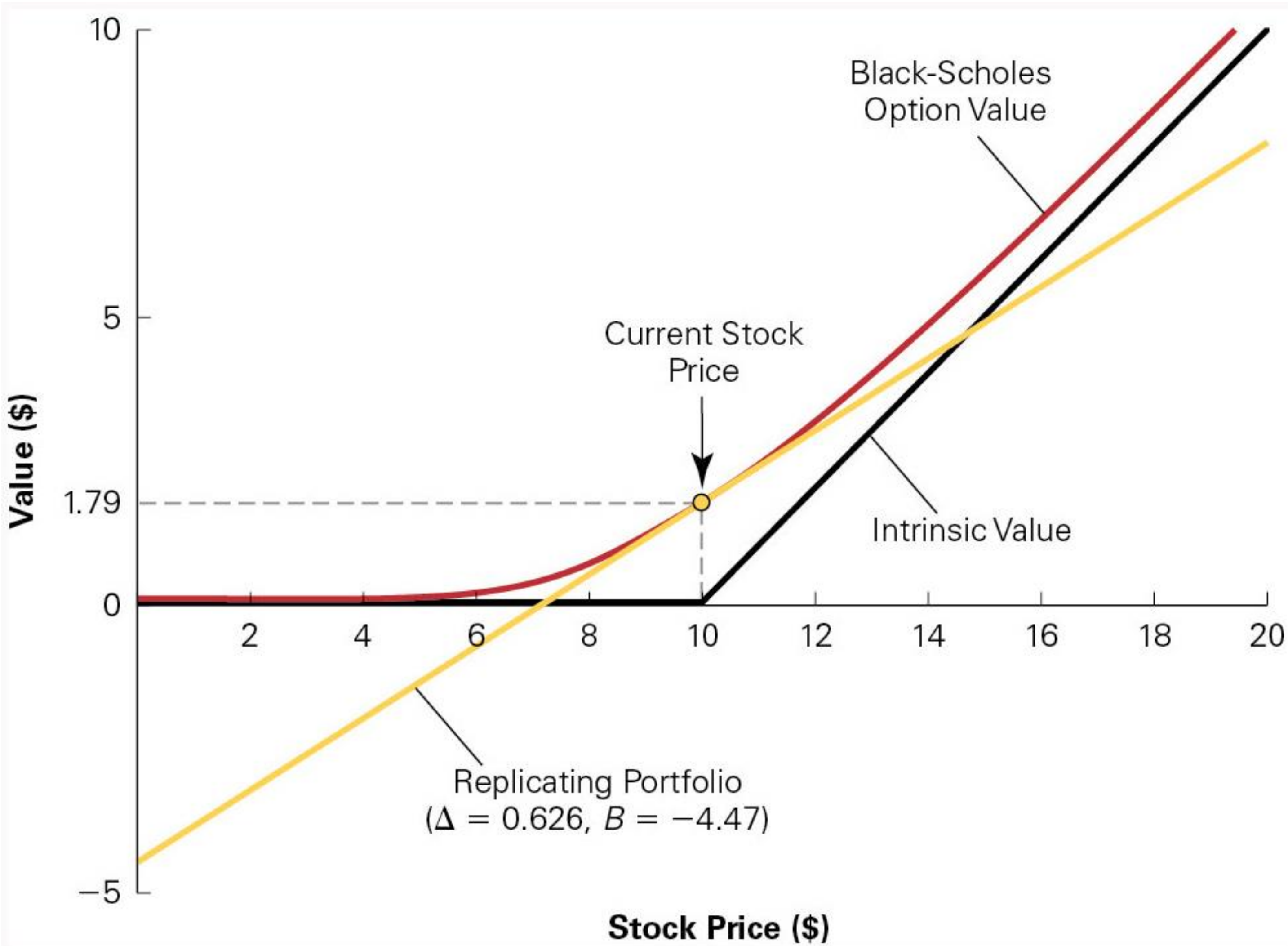
Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - Black-Scholes Replicating Portfolio of a Call Option

$$\Delta = N(d_1) = N(0.322) = 0.626$$
$$B = -PV(X)N(d_2) = -9.524 \times N(-0.078) = -4.47$$

  - That is, we should buy 0.626 shares of the PNA stock, and borrow \$4.47, for a total cost of \$10(0.626) − 4.47 = \$1.79, which is the Black-Scholes value of the call option.

Figure 21.6 Replicating portfolio of a call option in Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e



Hedging and Delta

- As the stock price changes, so do the deltas used to calculate the hedge ratio.
- **Gamma** = sensitivity of the delta to the stock price
- Gamma is similar to bond convexity.
- The hedge ratio will change with market conditions.
- Rebalancing is necessary.

Figure 21.11 Deltas change as stock price fluctuates
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

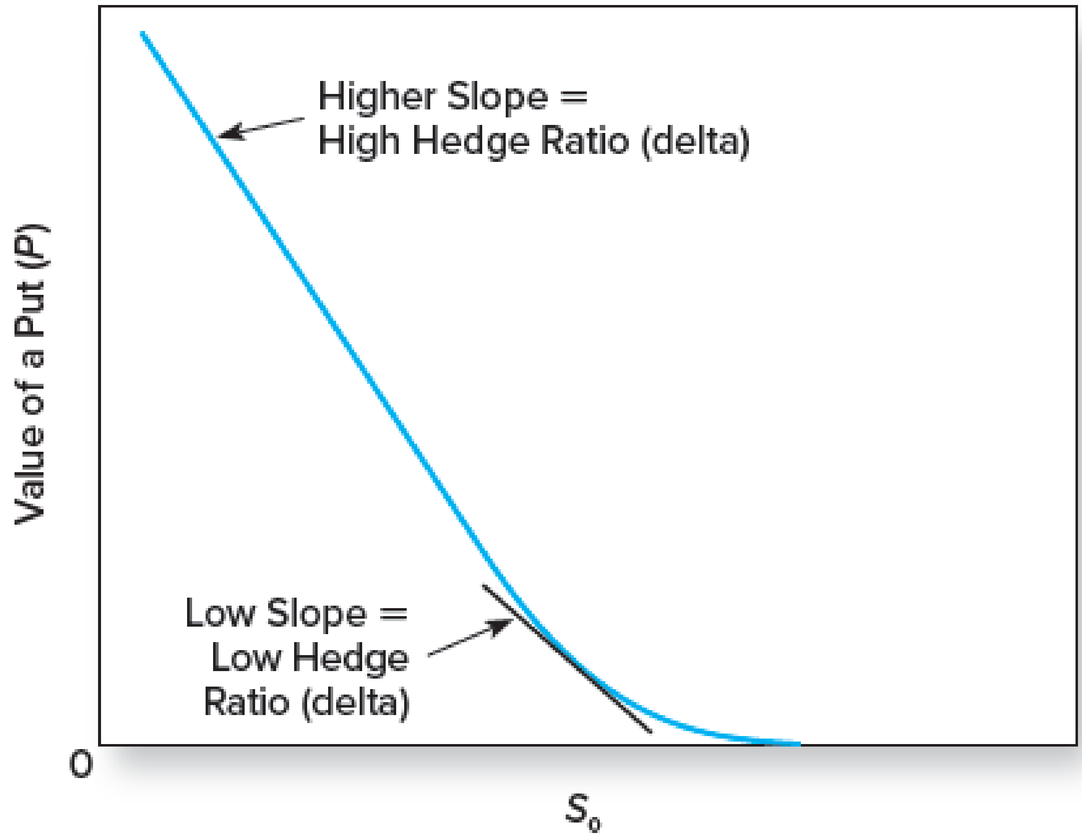


Figure 21.11 Deltas change as the stock price fluctuates

Delta Neutral

- When you establish a position in stocks and options that is hedged with respect to fluctuations in the price of the underlying asset, your portfolio is said to be delta neutral.

- In other words, a delta-neutral portfolio has no tendency to either increase or decrease in value when the stock price fluctuates.

Portfolio Insurance

- Protective Put
  - A long position in a put held on a stock you already own
  - Can also be achieved by purchasing a bond and a call option
- Portfolio Insurance
  - A protective put written on a portfolio rather than a single stock
- Limitations
  - The put corresponding to the investor's portfolio may not be available for purchase
  - Maturity of puts may be too short
- But the good news is that when the put does not itself trade, portfolio insurance can be synthetically created by constructing a replicating portfolio and dynamic hedging.

Black-Scholes Formula for a Call

- Only 5 inputs are needed for the Black-Scholes formula.
    1. Stock price
    2. Strike price
    3. Exercise date
    4. Risk-free rate
    5. Volatility of the stock

- We do not need to know the expected return of the stock.
    - Expected return is already reflected in the current stock price (which is the present value of its future payoffs).

Implied Volatility

- Of the five required inputs in the Black-Scholes formula, only σ is not observable directly.

- The volatility of an asset's returns that is consistent with the quoted price of an option on the asset.

- Using the Black-Scholes formula and the actual price of the option, solve for volatility implied by the price observed in the market.