EC3312: Game Theory & Applications to Economics

Lecture 2: Dominance, mixed equilibrium

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A large game

		2			
		A	B	C	D
	A	5 5	0 5	2 2	4 2
	B	4 0	6 6	5 4	3 0
1	C	2 6	1 5	4 3	1 0
	D	2 4	0 3	0 1	6 1

How should we analyse this?

Dominance

2.1. Definition. $s'_i \in S_i$ strictly dominates $s_i \in S_i$ if $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

2.2. Definition. $s_i \in S_i$ is strictly dominated if it is strictly dominated by some $s'_i \in S_i$.

2.3. Definition. $s'_i \in S_i$ weakly dominates $s_i \in S_i$ if $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

2.4. Definition. $s_i \in S_i$ is weakly dominated if it is weakly dominated by some $s'_i \in S_i$.

The large game again

		2			
		A	*B*	*C*	*D*
	A	5 5	0 5	2 2	4 2
	B	4 0	6 6	5 4	3 0
1	*C*	2 6	1 5	4 3	1 0
	D	2 4	0 3	0 1	6 1

Notice that A strictly dominates D for player 1.

The large game again

		2			
		A	B	C	D
1	A	5 5	0 5	2 2	4 2
	B	4 0	6 6	5 4	3 0
	C	2 6	1 5	4 3	1 0
	D				

Notice that A strictly dominates D for player 1.

The large game again

		2			
		A	B	C	D
1	A	5, 5	5, 0	2, 2	2, 4
	B	0, 4	6, 6	4, 5	0, 3
	C	6, 2	5, 1	3, 4	0, 1
	D				

Now A strictly dominates D for player 2.

The large game again

		2			
		A	B	C	D
1	A	5, 5	5, 0	2, 2	
	B	0, 4	6, 6	4, 5	
	C	6, 2	5, 1	3, 4	
	D				

Now A strictly dominates D for player 2.

The large game again

			2		
		A	B	C	D
	A	5, 5	5, 0	2, 2	
	B	0, 4	6, 6	4, 5	
1	C	6, 2	5, 1	3, 4	
	D				

Now C strictly dominates A for player 1.

The large game again

1 \ 2	A	B	C	D
A				
B	0, 4	6, 6	4, 5	
C	6, 2	5, 1	3, 4	
D				

Now C strictly dominates A for player 1.

The large game again

		2			
		A	B	C	D
	A				
	B	4 0	6 6	5 4	
1	C	2 6	1 5	4 3	
	D				

Now C strictly dominates A for player 2.

The large game again

		2			
		A	B	C	D
1	A				
	B		6, 6	4, 5	
	C		5, 1	3, 4	
	D				

Now C strictly dominates A for player 2.

The large game again

		2			
		A	B	C	D
	A				
	B		6, 6	4, 5	
1	C		5, 1	3, 4	
	D				

Now B strictly dominates C for player 1.

The large game again

			2		
		A	B	C	D
	A				
	B		6, 6	4, 5	
1	C				
	D				

Now B strictly dominates C for player 1.

The large game again

2

		A	B	C	D
	A				
	B		6 6	5 4	
1	C				
	D				

Now B strictly dominates C for player 2.

The large game again

			2		
		A	B	C	D
	A				
1	B		6, 6		
	C				
	D				

Now B strictly dominates C for player 2.

The large game again

		2			
		A	B	C	D
1	A				
	B		6, 6		
	C				
	D				

The unique Nash equilibrium of the original game is (B, B).

Iterated deletion of strictly dominated strategies

Follow the following algorithm:

1. Start by deleting any strictly dominated strategies
2. Now delete any strategies that are strictly dominated in the 'reduced' game
3. Continue until there are no more strictly dominated strategies.

All Nash equilibria of the original game survive this process.

Under what assumptions is it reasonable to assume that players will choose strategies that survive iterated deletion of strictly dominated strategies?

Iterated deletion of strictly dominated strategies

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Under what assumptions is it reasonable to assume that players will choose strategies that survive iterated deletion of strictly dominated strategies?

$\Longrightarrow$ common knowledge of rationality

Iterated deletion doesn't always help

2

1		L	M	R
	T	5 4	0 3	3 1
	B	0 0	5 2	3 5

Iterated deletion doesn't always help

		2		
		L	M	R
1	T	4, 5	3, 0	1, 3
	B	0, 0	2, 5	5, 3

No strategy is dominated. The unique Nash equilibrium is (T, L).

Iterated deletion of weakly dominated strategies?

		2	
		A	B
1	A	1, 1	2, 0
	B	0, 0	2, 2

Iterated deletion of weakly dominated strategies?

		2	
		A	B
1	A	1 1	0 2
	B	0 0	2 2

A weakly dominates B for player 1, but the Nash equilibria are (A, A) and (B, B). Nash equilibria may not survive iterated deletion of weakly dominated strategies!

Iterated deletion of weakly dominated strategies?

		2	
		A	B
1	A	1, 1	2, 0
	B	0, 0	2, 2

A weakly dominates B for player 1, but the Nash equilibria are (A, A) and (B, B). Nash equilibria may not survive iterated deletion of weakly dominated strategies!

Is (B, B) an unreasonable prediction?

Iterated deletion of weakly dominated strategies?

1 \ 2	A	B
A	1, 1	2, 0
B	0, 0	2, 2

A weakly dominates B for player 1, but the Nash equilibria are (A, A) and (B, B). Nash equilibria may not survive iterated deletion of weakly dominated strategies!

Is (B, B) an unreasonable prediction? If player 1 knew 2 was going to choose B, would it make sense to choose A?

Matching pennies

Players A and B choose heads or tails. If they choose the same, A pays \$1 to B; otherwise, B pays \$1 to A.

		B	
		H	T
A	H	1 -1	-1 1
	T	-1 1	1 -1

Mixed strategies

If S_i is finite, a mixed strategy $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{i|S_i|})$ is a distribution over S_i. Let Σ_i be the set of such distributions and let $\Sigma = \times_{i \in N} \Sigma_i$ be the set of mixed strategy profiles.

We extend the utility function to expected utilities in the natural way:

$$u_i(\sigma) = \sum_{s \in S} u_i(s) \prod_{i \in N} \sigma_{is_i}$$

Example: In matching pennies, let p be the probability that A chooses H and q be the probability that B chooses H. Then

$$u_i(\sigma) = p(1-q) + (1-p)q - pq - (1-p)(1-q).$$

Elements of S_i are pure strategies. A mixed strategy that places probability on only one pure strategy is degenerate.

Mixed-strategy equilibrium

2.5. Definition. A mixed strategy profile σ^* is a Nash equilibrium if

$$u_i(\sigma^*) \geq u_i(s_i, \sigma^*_{-i})$$

for all $s_i \in S_i$ and $i \in N$.

Note that we only consider deviations to pure strategies. Why? (See tutorial.)

Equilibrium in matching pennies

The fact that there are no equilibria in pure strategies suggests that we should search for nondegenerate mixed equilibria.

Under what conditions is it a best response for A to play a nondegenerate mixed strategy?

Equilibrium in matching pennies

The fact that there are no equilibria in pure strategies suggests that we should search for nondegenerate mixed equilibria.

Under what conditions is it a best response for A to play a nondegenerate mixed strategy?

A must be indifferent between H and T. That is, we must have

$$\begin{aligned} & u_A(H, \sigma_B) = u_A(T, \sigma_B) \\ \iff & (1-q) - q = q - (1-q) \\ \iff & q = 1/2. \end{aligned}$$

So B must play a nondegenerate mixed strategy. But for this to be a best response for B, we must have

$$\begin{aligned} & u_B(H, \sigma_A) = u_B(T, \sigma_A) \\ \iff & p - (1-p) = (1-p) - p \\ \iff & p = 1/2. \end{aligned}$$

Hence the unique Nash equilibrium of matching pennies is $((1/2, 1/2), (1/2, 1/2))$.

Best replies

2.6. Definition. Given $\sigma_{-i} \in \Sigma_{-i}$, let $B_i(\sigma_{-i})$ be the set of best responses for i; that is,

$$B_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}).$$

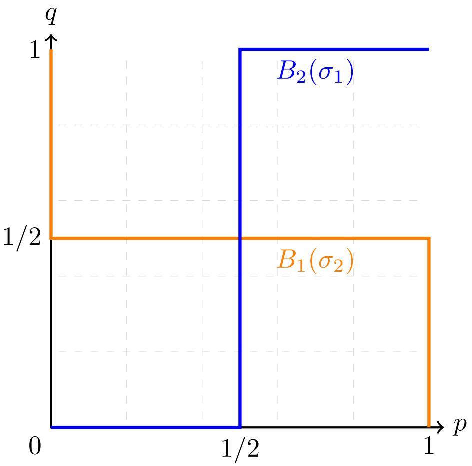
Matching pennies

Suppose $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$. Then

$$B_1(\sigma_2) = \begin{cases} \{T\} & \text{if } q > 1/2 \\ \Sigma_1 & \text{if } q = 1/2 \\ \{H\} & \text{if } q < 1/2 \end{cases}$$

$$B_2(\sigma_1) = \begin{cases} \{H\} & \text{if } p > 1/2 \\ \Sigma_2 & \text{if } p = 1/2 \\ \{T\} & \text{if } p < 1/2 \end{cases}$$

The unique equilibrium is $((1/2, 1/2), (1/2, 1/2))$.



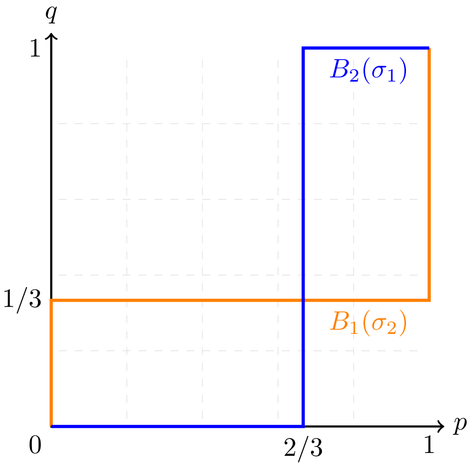
Battle of the sexes

Suppose $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$. Then

$$B_1(\sigma_2) = \begin{cases} \{T\} & \text{if } q > 1/3 \\ \Sigma_1 & \text{if } q = 1/3 \\ \{Y\} & \text{if } q < 1/3 \end{cases}$$

$$B_2(\sigma_1) = \begin{cases} \{T\} & \text{if } p > 2/3 \\ \Sigma_2 & \text{if } p = 2/3 \\ \{Y\} & \text{if } p < 2/3 \end{cases}$$

In addition to (T, T) and (Y, Y), there is a mixed equilibrium: $((2/3, 1/3), (1/3, 2/3))$, which yields only $2/3$ for each player.



Mixed equilibria in real life: the Hawk–Dove game

Gouldian finches are native to northern Australia. The red-headed variety is more aggressive than the black-headed one. The former tends to win contests for preferred nesting spaces, whereas the latter spends more time rearing its offspring.

We can model this situation as follows: finches can inherit the aggressive 'Hawk' trait or the passive 'Dove' trait. A population of finches is randomly matched to play the following game. The payoffs represent reproductive success.

		2	
		H	D
1	H	0, 0	4, 2
	D	2, 4	3, 3

Kokko, Griffith, and Pryke. 2014. 'The hawk–dove game in a sexually reproducing species explains a colourful polymorphism of an endangered bird'. *Proceedings of the Royal Society B.*

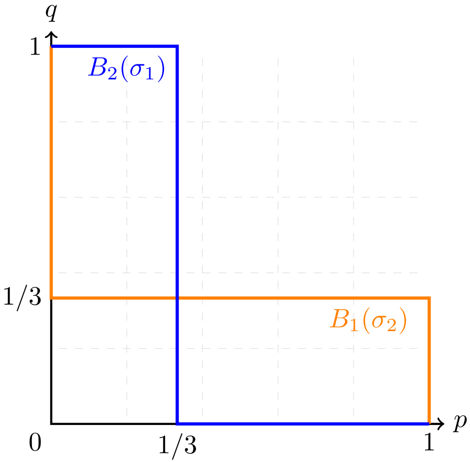
Equilibria of the Hawk–Dove game

$$B_1(\sigma_2) = \begin{cases} \{D\} & \text{if } q > 1/3 \\ \Sigma_1 & \text{if } q = 1/3 \\ \{H\} & \text{if } q < 1/3 \end{cases}$$

$$B_2(\sigma_1) = \begin{cases} \{D\} & \text{if } p > 1/3 \\ \Sigma_2 & \text{if } p = 1/3 \\ \{H\} & \text{if } p < 1/3 \end{cases}$$

There are three equilibria: (H, D), (D, H), and $((1/3, 2/3), (1/3, 2/3))$.

The population will tend towards the mixed equilibrium. (Why?)



Mixed strategies and dominance

		2		
		L	M	R
1	T	4, 10	3, 0	1, 3
	B	0, 0	2, 10	10, 3

No pure strategy dominates any other strategy.

Mixed strategies and dominance

		2		
		L	M	R
1	T	10 4	0 3	3 1
	B	0 0	10 2	3 10

No pure strategy dominates any other strategy.

The mixed strategy $(1/2, 1/2, 0)$ for player 2 yields expected payoff 5 regardless of what 1 chooses. So it strictly dominates R.

Mixed strategies and dominance

2

		L	M	R
1	T	10 4	0 3	3 1
	B	0 0	10 2	3 10

No pure strategy dominates any other strategy.

The mixed strategy $(1/2, 1/2, 0)$ for player 2 yields expected payoff 5 regardless of what 1 chooses. So it strictly dominates R.

Once we eliminate R, we see that the unique Nash equilibrium is (T, L).

Pick a number