LECTURE 11

Option Valuation –
Risk Neutral Option Pricing

EC3333 Financial Economics I

Learning Objectives

- Discuss what is meant by risk-neutral probabilities.
- Show how these probabilities can be used to price an option.

Risk-Neutral Probabilities

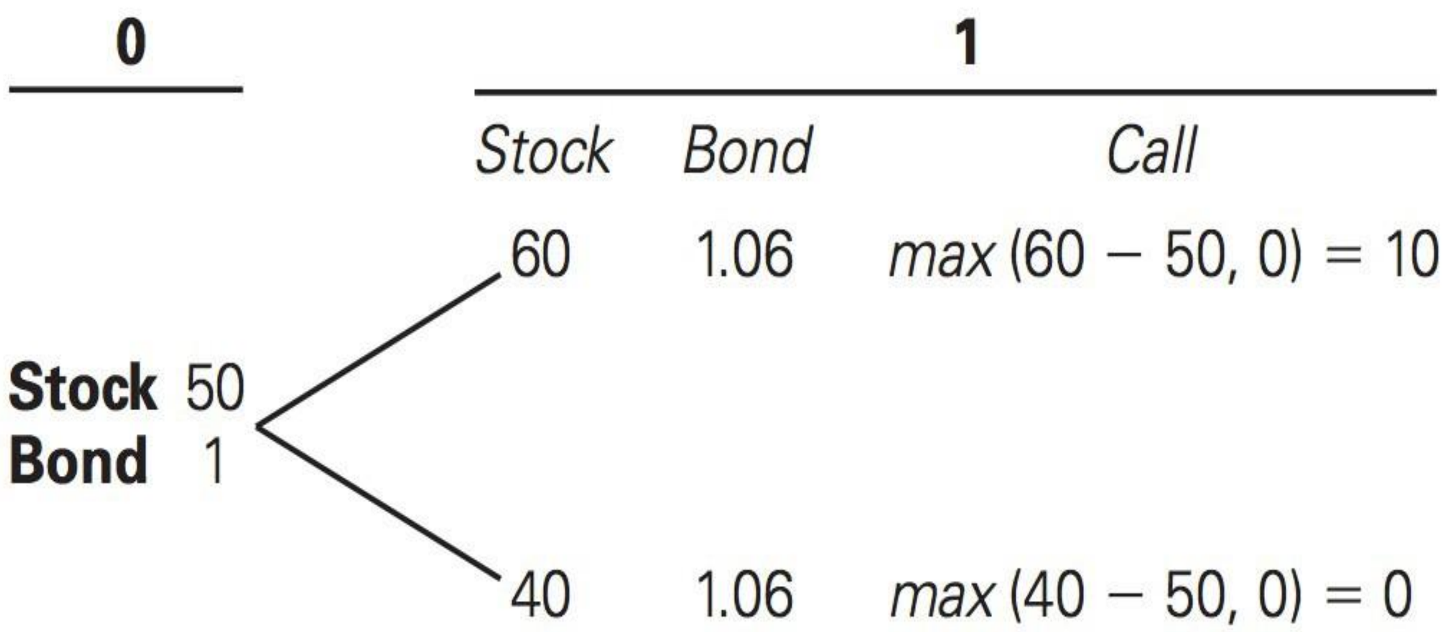
- Imagine a risk-neutral economy, that is, an economy in which all investors are risk-neutral.
- This hypothetical economy must value options the same as the real world because risk aversion cannot affect the valuation formula.
- In a risk-neutral world, investors would not demand risk premiums to compensate for risk.
- Hence, all financial assets (including options) must have the same cost of capital, i.e., the risk-free rate of interest.
- We can therefore value all assets by discounting expected payoffs at the risk-free rate of interest.

A Risk-Neutral Two-State Model

- Assume a world consisting of only risk-neutral investors, and consider the original two-state example.
- The stock price today is equal to $50.
- In one period it will either go up by $10 or go down by $10.
- The one-period risk-free rate of interest is 6%.

A Two-State Single-Period Model

- The payoffs can be summarized in a binomial tree.



Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 796

A Risk-Neutral Two-State Model

- In the risk-neutral world:
  - *ρ* = probability that stock price will increase
  - *$(1 - \rho)$* = the probability that stock price will decrease

- The stock's expected return in the risk-neutral world must equal the risk-free rate

$$\frac{60\rho + 40(1-\rho)}{50} - 1 = 0.06$$

- Alternatively, the value of the stock today must equal the present value of the expected price next period discounted at the risk-free rate

$$50 = \frac{60\rho + 40(1-\rho)}{1.06}$$

- Solving for ρ yields

$$\rho = 0.65$$

A Risk-Neutral Two-State Model

- The call option (X= $50) will be worth either $10 or $0 at expiration. The present value of the expected payoff is:

$$\frac{10(0.65) + 0(1 - 0.65)}{1.06} = 6.13$$

- Same value from the replicating portfolio where it was not assumed that investors were risk neutral!

- The replication mechanism behind option pricing does not depend on assumptions about the investors' risk preference.

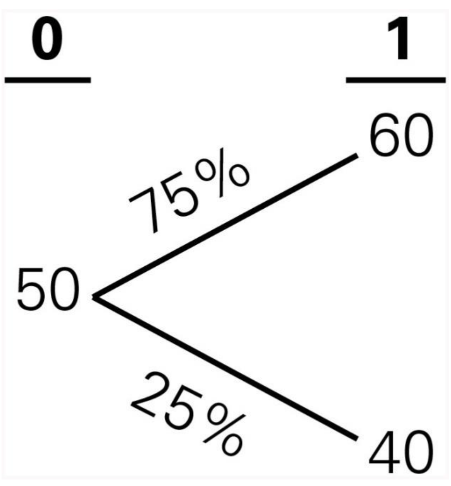
A Risk-Neutral Two-State Model

- Probabilities: ρ and (1 − ρ) are risk-neutral probabilities.
  - The probability of future states that are consistent with current prices of securities assuming all investors are risk neutral.

- In other words, ρ is not the actual probability of the stock price increasing.
  - It represents how the actual probability would have to be adjusted to keep the stock price the same in a risk-neutral world.
  - The risk-neutral probability that makes the stock's expected return equal to the risk-free interest rate.

Implications of the Risk-Neutral World

- Assume from the previous example that the current price of $50 has a true probability of 75% of increasing to $60 and a true probability of 25% of decreasing to $40.



Implications of the Risk-Neutral World

- This stock's true expected return is therefore

$$\frac{60 \times 0.75 + 40 \times 0.25}{50} - 1 = 10\%$$

- Given the risk-free interest rate of 6%, this stock has a 4% risk premium.

- But as calculated earlier, the risk-neutral probability that the stock will increase is 65%, which is less than the true probability.
  - Thus, the expected return of the stock in the risk-neutral world is 6%.

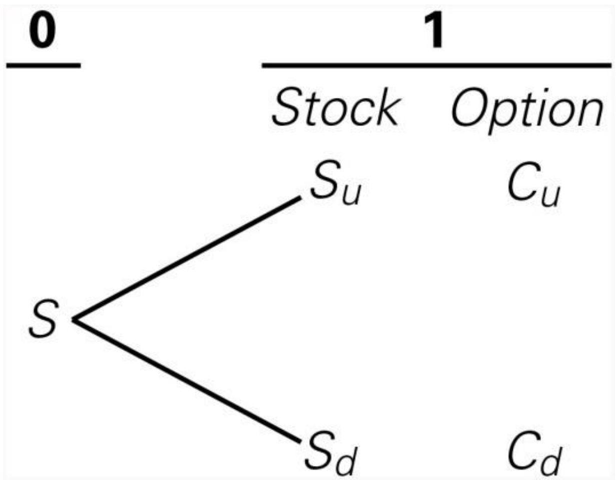
$$\frac{60 \times 0.65 + 40 \times 0.35}{50} - 1 = 6\%$$

Implications of the Risk-Neutral World

- To ensure that all assets in the risk-neutral world have an expected return equal to the risk-free rate, relative to the true probabilities, the risk-neutral probabilities overweight the bad states and underweight the good states.

Option Pricing with Risk-Neutral Probabilities

- Consider again the general binomial stock price tree.



Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 815

- Compute the risk-neutral probability that makes the stock's expected return equal to the risk-free interest rate:

$$\frac{\rho S_u + (1-\rho)S_d}{S} - 1 = r_f$$

Option Pricing with Risk-Neutral Probabilities

- Solving, the risk-neutral probability *ρ*:

$$\rho = \frac{(1+r_f)S - S_d}{S_u - S_d} = \frac{(1+r_f) - d}{u - d}$$

- Define *u* and *d* as follows:

$$S_d = dS \text{ and } S_u = uS$$

- E.g.,
  - $u = 1.2$
  - $d = 0.9$
- The value of the option can be calculated by computing its expected payoff using the risk-neutral probabilities, and discount the expected payoff at the risk-free interest rate.

Option Pricing with Risk-Neutral Probabilities

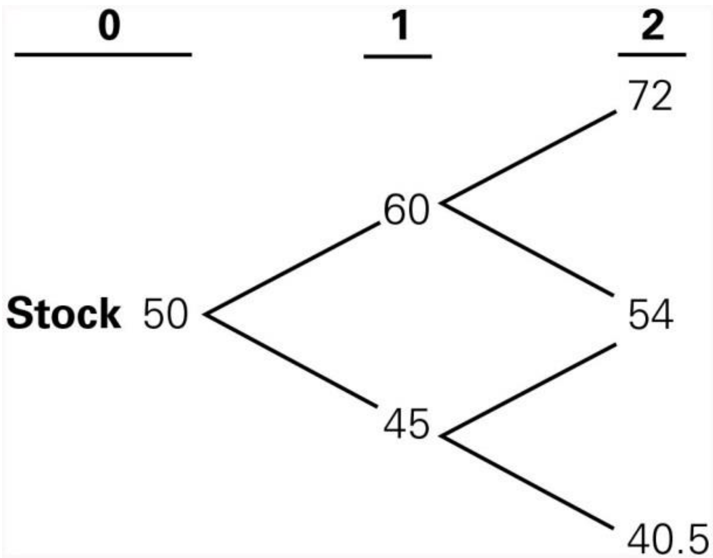
Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
- Suppose the current price of the stock is $50 per share. In each of the next two years, the stock price will either increase by 20% or decrease by 10%. The 3% one-year risk-free rate of interest will remain constant.
- Imagine all investors are risk neutral and calculate the probability of every state in the next two years. Use these probabilities to calculate the price of a two-year call option on this stock with a strike price $60.

Option Pricing with Risk-Neutral Probabilities

Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- The binomial tree in the three-state example is:



- First, compute the risk-neutral probability that the stock price will increase.
- At time 0, we have:

$$\rho = \frac{(1+r_f)S - S_d}{S_u - S_d} = \frac{(1.03)50 - 45}{60 - 45} = 0.433$$

Option Pricing with Risk-Neutral Probabilities

Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- Because the stock has the same returns(up 20% or down 10%) at each date, we can check that the risk neutral probability is the same at each date as well.
- Consider the call option with a strike price of \$60. This call pays \$12 if the stock goes up twice, and zero otherwise.
- The risk-neutral probability that the stock will go up twice is 0.433 × 0.433, so the call option has an expected payoff of:

$$0.433 \times 0.433 \times \$12 = \$2.25$$

- We compute the current price of the call option by discounting this expected payoff at the risk-free rate:

$$C = \frac{\$2.25}{1.03^2} = \$2.12$$

Option Pricing with Risk-Neutral Probabilities

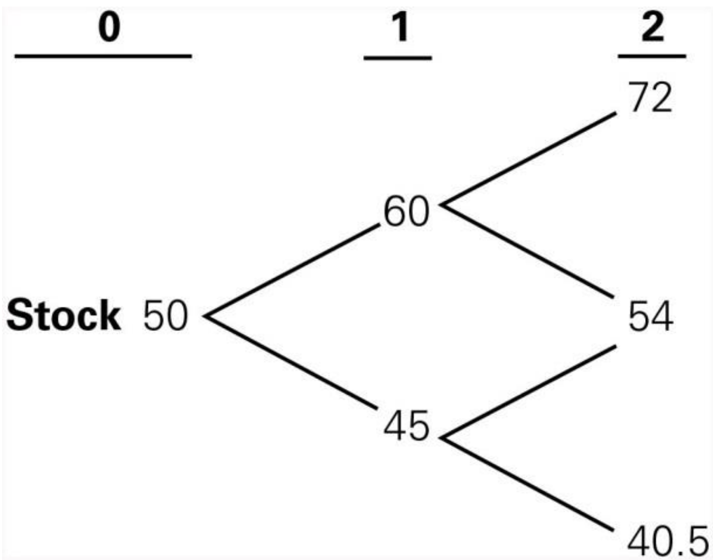
Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
- As in the previous problem, suppose the current price of the stock is $50 per share. In each of the next two years, the stock price will either increase by 20% or decrease by 10%. The 3% one-year risk-free rate of interest will remain constant.
- Imagine all investors are risk neutral and calculate the probability of every state in the next two years. Use these probabilities to calculate the price of a two-year put option on this stock with a strike price $60.

Option Pricing with Risk-Neutral Probabilities

Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- The binomial tree in the three-state example is:



- First, compute the risk-neutral probability that the stock price will increase.
- At time 0, we have:

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Option Pricing with Risk-Neutral Probabilities

Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- Because the stock has the same returns (up 20% or down 10%) at each date, we can check that the risk neutral probability is the same at each date as well.
- The put ends up in the money if the stock goes down twice, if it goes up and then down, or if it goes down and then up.
- Because the risk-neutral probability of a drop in the stock price is: $1 - 0.433 = 0.567$, the expected payoff of the put option is:

$$0.567 \times 0.567 \times \$19.5 + 0.433 \times 0.567 \times \$6 + 0.567 \times 0.433 \times \$6 = \$9.21$$

- The value of the put today is therefore:

$$P = \frac{\$9.21}{1.03^2} = \$8.68$$

Risk-Neutral Probabilities and Derivative Pricing

- Derivative Security
  - A security whose cash flows depend solely on the prices of other marketed assets.

- The probabilities in the risk-neutral world can be used to price any derivative security.

- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate

Risk-Neutral Probabilities and Derivative Pricing

- When we are valuing a derivative in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant for derivative pricing

Following are additional slides for your understanding and interest only.

They will NOT be included in the finals, so that you do not have to worry about access to Excel or other programs for numerical simulations.

Girsanov's Theorem

- Volatility is the same in the real world and the risk-neutral world
- We can therefore measure volatility in the real world and use it to build a tree for the an asset in the risk-neutral world

Choosing tree parameters: p, u and d to match volatility in the Risk Neutral World

- p, u, and d are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world
- For a non-dividend paying stock:

$$\text{Mean:} \quad e^{r\Delta t} = pu + (1-p)d$$

$$\text{Variance:} \quad \sigma^2 \Delta t = pu^2 + (1-p)d^2 - \mathrm{e}^{2r\Delta t}$$

- For small time steps, one way of matching the volatility is to set:

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step.

- This is the approach used by Cox, Ross, and Rubinstein (CRR)

The Probability of an Up Move for various assets in the Risk Neutral World

$$p = \frac{a-d}{u-d}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

For a non-dividend paying stock

$$a = e^{r\Delta t}$$

For a stock index where q is the dividend yield on the index

$$a = e^{(r-q)\Delta t}$$

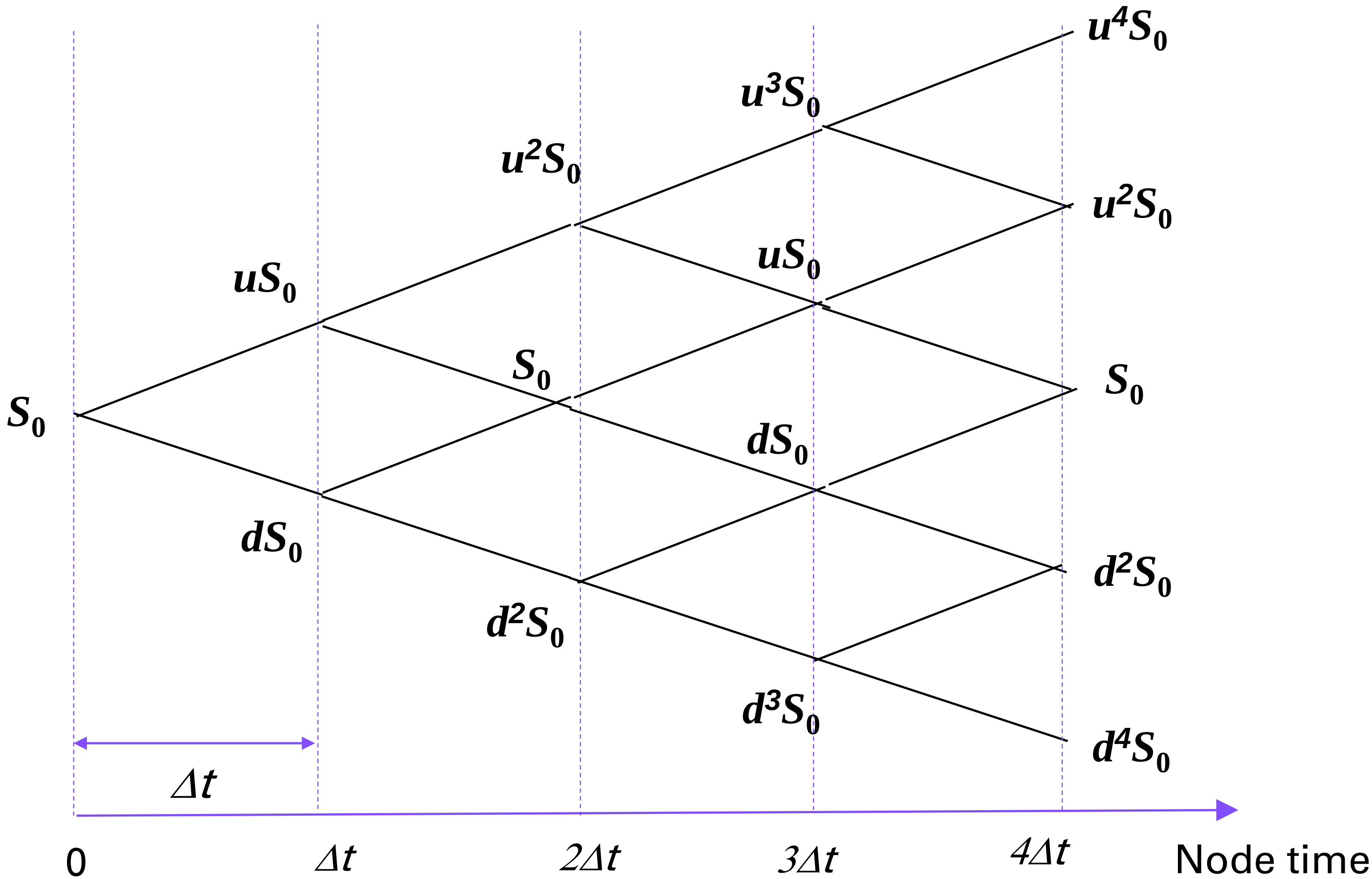
For a currency where r_f is the foreign risk-free rate

$$a = e^{(r-r_f)\Delta t}$$

For a futures contract

$$a = 1$$

Generate the Binomial Tree for the Underlying Asset using CRR set-up



Backward Induction to obtain the option price using Binomial Tree with Risk-Neutral Pricing

- Compute payoff of the option at the final nodes
- Work back through the tree using risk-neutral valuation (risk neutral probabilities) to calculate the value of the option at each node (and testing for early exercise for American options)

Applying Risk-Neutral Valuation with simulation for derivatives

- Monte Carlo Simulation
  - A common technique for pricing derivative assets in which the expected payoff of the derivative security is estimated by calculating its average payoff after simulating many random paths for the underlying stock.
- In the randomization, the risk-neutral probabilities are used and so the average payoff can be discounted at the risk-free rate to estimate the derivative security's value. That is:
  1. Assume that the expected return from the stock price is the risk-free rate
  2. Calculate the expected payoff from the option
  3. Discount at the risk-free rate

Pricing European stock options using Monte Carlo simulation

When used to value European stock options,

Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoffs
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Pricing European stock options using Monte Carlo simulation

- In a risk neutral world the process for a stock price is

$$d \ln S = \left(r - \sigma^2/2\right)dt + \sigma dz$$

where r is the risk-neutral return

- We can simulate a path by choosing time steps of length Δt and using the discrete version of this

$$\ln S(t + \Delta t) - \ln S(t) = \left(r - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

or

$$S(t + \Delta t) = S(t)e^{\left(r - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

where ε is a random sample from $\phi(0,1)$

In Excel =NORMSINV(RAND()) gives a random sample from $\phi(0,1)$

Computing Option Greeks from Monte Carlo Simulation

- To compute the option Greeks:
- Make a small changes to asset price (for delta and gamma), volatility (for vega) and time to maturity (for theta) respectively, holding all else constant
- Use the same random innovations generated earlier to carry out the Monte Carlo pricing.
- From the change in the option price, compute the relevant Greeks - e.g.. Δ as the change in the option price divided by the change in the asset price

Application of Monte Carlo Simulation

- Monte Carlo simulation can be useful for path dependent options, options dependent on several underlying state variables, and options with complex payoffs
- It cannot easily deal with American-style options (due to compare intrinsic and holding value at each node) unlike Binomial Tree method