EC3333 Tutorial 8 Suggested Answers

1. A newly issued bond has a maturity of 10 years and pays a 7% coupon rate (with coupon payments coming once annually). The bond sells at par value.
   a. What are the convexity and the duration of the bond?
   b. Find the actual price of the bond assuming that its yield to maturity immediately increases from 7% to 8% (with maturity still 10 years).
   c. What price would be predicted by the duration rule? What is the percentage error of that rule?
   d. What price would be predicted by the duration-with-convexity rule? What is the percentage error of that rule?

a.

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

$$\frac{\partial P}{\partial y} = -\sum_{t=1}^{T} t \times \frac{C_t}{(1+y)^{t+1}}$$

$$\text{Convexity} = \frac{1}{P} \times \frac{d^2P}{dy^2}$$

$$\text{Convexity} = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{T} \left[\frac{C_t}{(1+y)^t}(t^2+t)\right]$$

The following spreadsheet shows that the convexity of the bond is 64.933. The present value of each cash flow is obtained by discounting at 7%. (Since the bond has a 7% coupon and sells at par, its YTM is 7%.) Convexity equals: the sum of the last column (7,434.175) divided by $[P \times (1+y)^2] = 100 \times (1.07)^2 = 114.49$

Time (t)	Cash Flow	PV(CF)	$t^2 + t$	$(t^2 + t) \times$ PV(CF)
1	$7	$6.542	2	13.084
2	7	6.114	6	36.684
3	7	5.714	12	68.569
4	7	5.340	20	106.805
5	7	4.991	30	149.727
6	7	4.664	42	195.905
7	7	4.359	56	244.118
8	7	4.074	72	293.333
9	7	3.808	90	342.678
10	107	54.393	110	5,983.271
Sum:		$100.000		7,434.175
			Convexity:	64.933

$$D = \sum_{t=1}^{T} \frac{PV(C_t)}{P} \times t$$

$$PV(C_t) = \frac{C_t}{(1+y)^t} \quad \text{and} \quad P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

The duration of the bond is:

(1) Time until Payment (Years)	(2) Cash Flow	(3) PV of CF (Discount Rate = 7%)	(4) Weight	(5) Column (1) × Column (4)
1	$7	$ 6.542	0.06542	0.06542
2	7	6.114	0.06114	0.12228
3	7	5.714	0.05714	0.17142
4	7	5.340	0.05340	0.21361
5	7	4.991	0.04991	0.24955
6	7	4.664	0.04664	0.27986
7	7	4.359	0.04359	0.30515
8	7	4.074	0.04074	0.32593
9	7	3.808	0.03808	0.34268
10	107	54.393	0.54393	5.43934
Column sums		$100.000	1.00000	7.51523

D = 7.515 years

b.

$$P_B = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{\text{Par Value}}{(1+r)^T} \text{ or } P_B = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right)}_{\text{The Annuity Formula}} + \frac{\text{Par Value}}{(1+r)^T}$$

If the yield to maturity increases to 8%, the bond price will fall to 93.29% of par value, a percentage decrease of 6.71%.

c.

$\frac{\Delta P}{P} = -D^* \times \Delta y$, where $D^* = \mathrm{D}/(1+y) =$ Modified duration

The Modified duration rule predicts a percentage price change of

$$\left(-\frac{D}{1.07}\right) \times 0.01 = \left(-\frac{7.515}{1.07}\right) \times 0.01 = -0.0702, or - 7.02\%$$

The price predicted by the duration rule is 7.02% less than face value, or 92.98% of face value.

This overstates the actual percentage decrease in price by (93.29-92.98)/93.29 *100%= 0.33%.

One can get an estimate the percentage error using 93.29-92.98=0.31% (since the initial price of the bond is 100).

d.

$\frac{\Delta P}{P} = -D^* \times \Delta y + \frac{1}{2} \times \text{Convexity} \times (\Delta y)^2$

The duration-with-convexity rule predicts a percentage price change of

$$\left[\left(-\frac{7.515}{1.07}\right) \times 0.01\right] + [0.5 \times 64.933 \times 0.01^2] = -0.0670, or - 6.70\%$$

The price predicted by the duration with convexity rule is 6.70% less than face value, or 93.30% of face value.

This understates the actual percentage decrease in price by (93.30-93.29)/92.98 *100%= 0.01%.

One can get an estimate the percentage error using 93.30-93.29 = 0.01% (since the initial price of the bond is 100).

The percentage error is 0.01%, which is substantially less than the error using the duration rule.

2. Pension funds pay lifetime annuities to recipients. If a firm will remain in business indefinitely, the pension obligation will resemble a perpetuity. Suppose, therefore, that you are managing a pension fund with obligations to make perpetual payments of $2 million per year to beneficiaries. The yield to maturity on all bonds is 16%.

   a. If the duration of 5-year maturity bonds with coupon rates of 12% (paid annually) is 4 years and the duration of 20-year maturity bonds with coupon rates of 6% (paid annually) is 11 years, how much of each of these coupon bonds (in market value) will you want to hold to both fully fund and immunize your obligation?

   b. What will be the par value of your holdings in the 20-year coupon bond?

a.

$PV(C \text{ in perpetuity}) = \sum_{n=1}^{\infty} \frac{C}{(1+r)^n} = \frac{C}{r}$

The duration of a level perpetuity is equal to: $\frac{1+y}{y}$

PV of obligation = $2 million/0.16 = $12.5 million
Duration of obligation = 1.16/0.16 = 7.25 years
Call w the weight on the five-year maturity bond (which has duration of four years). Then
(w × 4) + [(1 – w) × 11] = 7.25
w = 0.5357
Therefore: 0.5357 × $12.5 = $6.7 million in the 5-year bond and
0.4643 × $12.5 = $5.8 million in the 20-year bond.

b.

$$P_B = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{\text{Par Value}}{(1+r)^T}$$

or

$$P_B = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right)}_{\text{The Annuity Formula}} + \frac{\text{Par Value}}{(1+r)^T}$$

The price of the 20-year bond is

$$P_B = \underbrace{\frac{60}{0.16}\left(1 - \frac{1}{(1+0.16)^{20}}\right)}_{\text{The Annuity Formula}} + \frac{1000}{(1+0.16)^{20}} = \$407.12$$

Therefore, the bond sells for 0.4071 times its par value, and
Market value = Par value × 0.4071
$5.8 million = Par value × 0.4071
Par value = $14.25 million
Another way to see this is to note that each bond with par value $1,000 sells for $407.12. If total market value is $5.8 million, then you need to buy approximately 14,250 bonds, resulting in total par value of $14.25 million

3. Rank the durations of the following pairs of bonds:

a. Bond A is a 6% coupon bond, with a 20-year time to maturity selling at par value. Bond B is a 6% coupon bond, with a 20-year maturity time selling below par value.

b. Bond A is a 20-year noncallable coupon bond with a coupon rate of 6%, selling at par. Bond B is a 20-year callable bond with a coupon rate of 7%, also selling at par.
(A callable bond is a bond that the issuer may repurchase at a given call price in some specified period prior to the bonds' maturity date. When an issuer calls its bonds, it pays investors the call price (usually the face value of the bonds) together with accrued interest to date and, at that point, stops making interest payments.)

a.
Bond B has a higher yield to maturity than bond A since its coupon payments and maturity are equal to those of A, while its price is lower. (Perhaps the yield is higher because of differences in credit risk.) Therefore, the duration of Bond B must be shorter.

b.
Bond A has a lower yield and a lower coupon, both of which cause Bond A to have a longer duration than Bond B. Moreover, A cannot be called, so that its maturity is at least as long as that of B, which generally increases duration.

4. An insurance company must make payments to a customer of $10 million in one year and $4 million in five years. The yield curve is flat at 10%.

   a. If it wants to fully fund and immunize its obligation to this customer with a single issue of a zero-coupon bond, what maturity bond must it purchase?

   b. What must be the face value and market value of that zero-coupon bond?

a.

$$D = \sum_{t=1}^{T} \frac{PV(C_t)}{P} \times t$$

$$PV(C_t) = \frac{C_t}{(1+y)^t} \quad \text{and} \quad P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

(1) Time until Payment (Years)	(2) Cash Flow	(3) PV of CF (Discount Rate = 10%)	(4) Weight	(5) Column (1) × Column (4)
1	$10 million	$ 9.09 million	0.7857	0.7857
5	4 million	2.48 million	0.2143	1.0715
Column sums		$11.57 million	1.0000	1.8572

D = 1.8572 years = required maturity of zero coupon bond.

b.

The market value of the zero must be $11.57 million, the same as the market value of the obligations. Therefore, the face value must be:

$$\$11.57 \text{ million} \times (1.10)^{1.8572} = \$13.81 \text{ million}$$