MA1522 Linear Algebra for computing
Chapter 1
Linear Equations in Linear Algebra

August 21, 2023

Our Vital Statistics

Lecturer	:	Professor Loke Hung Yean.
Office	:	S17-06-03.
Office Tel	:	6516-8009.
Email	:	matlhy@nus.edu.sg
Consultation	:	By appointment.

Lecturer	:	Dr Jonathon Teo Yi Han.
Office	:	S17-07-01.
Office Tel	:	6601-2659.
Email	:	jonathonteo@nus.edu.sg
Consultation	:	By appointment.

Let us watch a recorded video of Dr Teo.

Assessments

Lecture Quiz	10 %
Midterm test	25%
Three Homework	15%
Final exam	50%
Total	100%

- Loke Hung Yean will lecture Week 1 to Week 4, first half of Week 5, Week 12, Week 13.
- Jonathon Teo will lecture from second half of Week 5 to Week 11.
- There is a module page at Canvas:

https://canvas.nus.edu.sg

- There are 9 Quizzes posted at Canvas.

  The deadline for each quiz submission is listed there.

  You may start attempting the quizzes today.

  Every student is allowed to try as many times as he/she wishes.

  You will receive a feedback after each submission.

  We will only record the scores of the last attempt before the deadline.

  Thus do attempt and submit the quizzes earlier.

- Tutorials will begin in Week 3.

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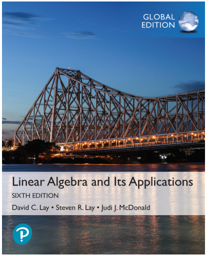
References

1. *Linear Algebra and Its Applications*, Global Edition Paperback by David Lay, Steven Lay and Judi McDonald. Publisher: Pearson, ISBN-10:1292351217.

Purchase your module's chosen Pearson textbook

Module	MA1522 - Linear Algebra for Computing
Title	Linear Algebra and Its Applications, Global Edition 6th Edition
Print	ISBN: 9781292351216 (https://shopee.sg/product/849371650/18882505199)
eBook	ISBN: 9781292351223 (https://shopee.sg/product/849371650/18882505199)



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Warning

There are many linear algebra textbooks in the library and online lectures at Youtube.

I hesitate listing them here because these textbooks/lectures have their own sets of notation and organizations of topics. While is good to learn linear algebra from a variety of sources, you have to be mindful of the differences and not be confused.

Telegram® Chat

There is an MA1522 Telegram Chat where students could post questions and comments on linear algebra. You could find instruction to join the chat at NUS Canvas.

The software MATLAB®

We will be using the mathematics software MATLAB frequently in this course.

There is pdf file *Introduction to MATLAB* placed at MA1522 module page at NUS Canvas.

It contains instructions on how to install MATLAB on your laptop and how to use the commands.

Please learn how to use this software because you will need to use it in your tutorials, homework and maybe your exam. ☺

1.1 System of Linear Equations

A *linear equation* in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where b and the coefficients $a_1, a_2, \ldots, a_n$ are real or complex numbers.

Example.

$$2x_1 - 5x_2 + \frac{1}{2}x_3 = 5.$$

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Example.

$$2x_1 \; - \; 5x_2 \; + \; \frac{1}{2}x_3 \; = \; 5.$$

A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations involving the same variables, say, $x_1, x_2, \ldots, x_n$.

Example.

$$\begin{array}{rcrcrcr} 2x_1 & - & 5x_2 & + & \frac{3}{2}x_3 & = & 5 \\ x_1 & + & x_2 & + & x_3 & = & -2 \\ 3x_1 & & & + & x_3 & = & 0. \end{array}$$

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Solutions of systems

A *solution* of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that satisfies every equation when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Example. The array $(x, y, z) = (0, 0, 1)$ is a solution of the system

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 1 \\ 2x & + & y & + & 3z & = & 3 \\ -x & + & 2y & & & = & 0. \end{array}$$

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The set of all possible solutions is called the *solution set* of the linear system.

Equivalent linear systems

Two linear systems are called *equivalent* if they have the same solution set.

Example. The two linear systems below are equivalent (Why?):

$$\begin{array}{rcrcr} x & - & y & = & 1 \\ -x & + & 2y & = & 0 \end{array}$$

and

$$\begin{array}{rcrcr} 2x & - & 2y & = & 2 \\ x & - & 2y & = & 0. \end{array}$$

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A system of linear equations has

- no solution, or
- exactly one solution, or
- infinitely many solutions.

1. The system is said to be *consistent* if it has at least one solution. In this case it has either one solution or infinitely many solutions.
2. The system is said to be *inconsistent* if it has no solution.

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Example

This linear system has no solution:

$$\begin{aligned} x - y &= 0 \\ x - y &= 1. \end{aligned}$$

(Why?)

The system is *inconsistent*.

Example

This linear system has infinitely many solutions:

$$\begin{aligned} x - y &= 0 \\ z &= 2. \end{aligned}$$

Why?

The system is *consistent*.

Matrix Notation

A linear system can be recorded in a rectangular array called *the coefficient matrix of the system* or simply *the matrix*.

Example.

$$\begin{array}{rcrcrcr} 2x_1 & - & 5x_2 & + & \frac{3}{2}x_3 & = & 5 \\ x_1 & + & x_2 & + & x_3 & = & -2 \\ 3x_1 & & & + & x_3 & = & 0. \end{array}$$

The matrix is

$$\begin{pmatrix} 2 & -5 & \frac{3}{2} \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

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An *augmented matrix* of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

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The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & -5 & \frac{3}{2} & 5 \\ 1 & 1 & 1 & -2 \\ 3 & 0 & 1 & 0 \end{array}\right).$$

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Remark

We compare the two matrices

$$\left(\begin{array}{ccc|c} 2 & -5 & \frac{3}{2} & 5 \\ 1 & 1 & 1 & -2 \\ 3 & 0 & 1 & 0 \end{array}\right) \text{ and } \begin{pmatrix} 2 & -5 & \frac{3}{2} & 5 \\ 1 & 1 & 1 & -2 \\ 3 & 0 & 1 & 0 \end{pmatrix}.$$

We will see later that with or without the vertical line, it does not affect our computations.

We prefer to keep the vertical line to indicate that it is an augmented matrix.

Matrix size

A matrix with m rows and n columns is called an m by n matrix.

Example.

$$A = \begin{pmatrix} 2 & -5 & \frac{3}{2} & 5 \\ 1 & 1 & 1 & -2 \\ 3 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix A is a 3 by 4 matrix.

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Square Matrices

An n by n matrix is called a *square matrix* of *order* n.

Here n is the number of rows which is also the same as the number of columns.

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is a square matrix of order 3.

Strategy for solving a linear system

Replace one square with an equivalent system (i.e., one with the same solution set) that is easier to solve.

Example 1

Solve the given system of equations.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9. \end{array}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right).$$

We add 4 times Equation 1 to Equation 3 to eliminate the variable x_1.

$$
\begin{array}{rrcrcrcr}
 & 4x_1 & - & 8x_2 & + & 4x_3 & = & 0 \\
+) & -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \\
\hline
 & & & -3x_2 & + & 13x_3 & = & -9.
\end{array}
$$

We replace the original third equation with the result.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9. \end{array}$$

$$\underset{\sim}{R_3+4R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right).$$

Multiply Equation 2 by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2.

$$\underset{\sim}{\tfrac{1}{2}R_2}\quad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & -3x_2 & + & 13x_3 & = & -9. \end{array}$$

$$\underset{\sim}{\tfrac{1}{2}R_2}\quad \left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right).$$

Use the x_2 in Equation 2 to eliminate the x_2 in Equation 3.

$$\begin{array}{rrcrcr} & 3x_2 & - & 12x_3 & = & 12 \\ +) & -3x_2 & + & 13x_3 & = & -9 \\ \hline & & & x_3 & = & 3. \end{array}$$

The new system has a triangular form.

$$\underset{\sim}{R_3+3R_2} \quad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3. \end{array}$$

$$\underset{\sim}{R_3+3R_2} \quad \left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Using Equation 3, we eliminate the x_3 term from Equation 1 and Equation 2.

$$\begin{array}{rcl} 4x_3 & = & 12 \\ x_2 - 4x_3 & = & 4 \\ \hline x_2 & = & 16 \end{array}$$

and

$$\begin{array}{rcl} -x_3 & = & -3 \\ x_1 - 2x_2 + x_3 & = & 0 \\ \hline x_1 - 2x_2 & = & -3. \end{array}$$

Combining the results, we get

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned}$$

$$\underset{\sim}{R_2+4R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right) \underset{\sim}{R_1-R_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

We use Equation 2 to eliminate the x_2 in Equation 1 and we get

$$\underset{\sim}{R_1+2R_2} \quad \begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3. \end{aligned}$$

$$\underset{\sim}{R_1+2R_2} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

If there is a solution to the system, then it has to be $(29, 16, 3)$.
However we cannot say for sure that $(29, 16, 3)$ is a solution.

To verify that $(29, 16, 3)$ is a solution, we substitute these values into the original system, and compute.

$$\begin{array}{rcrcrcr} 29 & - & 2(16) & + & 3 & = & 0 \\ & & 2(16) & - & 8(3) & = & 8 \\ -4(29) & + & 5(16) & + & 9(3) & = & -9. \end{array}$$

The results agree with the right side of the original system, so $(29, 16, 3)$ is a solution of the system. $\square$

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The results agree with the right side of the original system, so $(29, 16, 3)$ is a solution of the system. $\square$

Remark

In solving the last example, we wrote the systems of equations as well as the augmented matrices side by side in blue.

We could have worked exclusively with the augmented matrices to get to the answers.

Indeed we will use this approach in Example 2 below.

Elementary row operations

Elementary row operations include the following:

- (Replacement) Replace one row by the sum of itself and a multiple of another row.
- (Interchange) Interchange two rows.
- (Scaling) Multiply all entries in a row by a *nonzero* constant.

Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other.

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Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other.

Notations

We have seen in the last example that it is a good practice to record the elementary row operations used.

Example. We use

- $R_3 - 5R_2$ to denote replacing Row 3 by the sum of itself and -5 times Row 2;
- $R_3 \leftrightarrow R_5$ to denote interchanging Row 3 and Row 5 and
- $2R_3$ to denote multiplying all entries in Row 3 by the constant 2.

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Important facts

1. The row operations are reversible.
2. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

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Fundamental questions about a linear system

Suppose we are given a linear system.

1. Is the system *consistent*; that is, does it have at least one solution?
2. If a solution exists, is it the only one; that is, is the solution *unique*?

Example 2

Determine if the following system is consistent.

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1. \end{array}$$

Solution. The augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right).$$

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Solution. The augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right).$$

To obtain an x_1 in the first equation, we interchange Rows 1 and 2.

$$\stackrel{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right).$$

To eliminate the $5x_1$ term in the third equation, add $-5/2$ times Row 1 to Row 3.

$$\stackrel{R_3 - \frac{5}{2}R_1}{\sim} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right).$$

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$$\overset{R_3 - \frac{5}{2}R_1}{\sim} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right).$$

We use second equation to eliminate the $-\frac{1}{2}x_2$ term from the third equation. Add $\frac{1}{2}$ times Row 2 to Row 3 to get the matrix

$$\underset{\sim}{R_3+\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right).$$

The augmented matrix is now in triangular form.

To interpret it correctly, we convert into equations.

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The augmented matrix means

$$\begin{array}{rcrcrcl} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & \frac{5}{2}. \end{array} \qquad (*)$$

The last equation is $0 = \frac{5}{2}$.
There are no values of x_1, x_2, x_3 could satisfy $0 = \frac{5}{2}$ because $0 \neq \frac{5}{2}$.

Hence the system of equations (*) is inconsistent (i.e., it has no solution).

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$$
\begin{array}{rcrcrcl}
2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\
 & & x_2 & - & 4x_3 & = & 8 \\
0x_1 & + & 0x_2 & + & 0x_3 & = & \frac{5}{2}.
\end{array}
\qquad (*)
$$

The last equation is $0 = \frac{5}{2}$.
There are no values of x_1, x_2, x_3 could satisfy $0 = \frac{5}{2}$ because $0 \neq \frac{5}{2}$.

Hence the system of equations (*) is inconsistent (i.e., it has no solution).

Since the system of equations in Example 2 and the system (*) have the same solution set, the original system is inconsistent (i.e., it has no solution). $\square$

1.2 Row Reduction and Echelon Forms

A rectangular matrix is in *echelon form* (or *row echelon form*) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Example

This 6 by 9 matrix is in echelon form:

$$\begin{pmatrix} 0 & 6 & * & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -5 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 3 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $*$ denotes an arbitrary real number.

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or *reduced row echelon form*):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An echelon matrix (respectively, reduced echelon matrix) is one that is in echelon form (respectively, reduced echelon form.)

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Example

This 6 by 9 matrix is in reduced echelon form:

$$\begin{pmatrix} 0 & 1 & 0 & * & 0 & 0 & * & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $*$ denotes an arbitrary real number.

Pick an arbitrary nonzero matrix A.
Using elementary row operations we could always transform A into a matrix in echelon form.

However different people may use different sequences of row operations on A and get different matrices in echelon form.

Fortunately the next theorem says starting from the matrix A, everyone gets the same the matrix in reduced echelon form.

Theorem (Uniqueness of the Reduced Echelon Form)

Each matrix is row equivalent to one and only one reduced echelon matrix.

- If a matrix A is row equivalent to an echelon matrix U, we call U **an** *echelon form* (or *row echelon form*) of A.
  If U is in reduced echelon form, we call U **the** *reduced echelon form* of A.

- A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
  A *pivot column* is a column of A that contains a pivot position.

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- A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
  A *pivot column* is a column of A that contains a pivot position.

Example 1

Let

$$A = \begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}.$$

Row reduce the matrix A to echelon form, and locate the pivot columns of A.

Solution

The top of the leftmost nonzero column is the first pivot position.

A nonzero entry, or pivot, must be placed in this position.

Interchange Row 1 and Row 4.

$$\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}.$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

$$\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}.$$

Choose 2 in the second row as the next pivot.

Add $-5/2$ times Row 2 to Row 3, and add $3/2$ times Row 2 to Row 4.

$$\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}.$$

There is no way a leading entry can be created in Column 3.

Add $-5/2$ times Row 2 to Row 3, and add $3/2$ times Row 2 to Row 4.

$$\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}.$$

There is no way a leading entry can be created in Column 3.

We interchange Row 3 and Row 4 to produce a leading entry in Column 4.

$$\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix is in echelon form.

Columns 1, 2, and 4 of A are pivot columns.

The pivots in the example are 1, 2 and -5.

We interchange Row 3 and Row 4 to produce a leading entry in Column 4.

$$\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix is in echelon form.

Columns 1, 2, and 4 of A are pivot columns.

The pivots in the example are 1, 2 and -5. $\square$

Row Reduction Algorithm

Example 2.

Let

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}.$$

Apply elementary row operations to transform the matrix A first into echelon form and then into reduced echelon form.

Solution

Step 1: Begin with the leftmost nonzero column.

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}.$$

This is a pivot column.

The pivot position is at the top.

Step 2: Select a nonzero entry in the pivot column as a pivot.

If necessary, interchange rows to move this entry into the pivot position.

Interchange Row 1 and Row 3.

(Rows 1 and 2 could have also been interchanged instead.)

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}.$$

Step 3: Use row replacement operations to create zeros in all positions below the pivot.

We could have divided the top row by the pivot, 3.

However with two 3s in Column 1, it is just as easy to add -1 times Row 1 to Row 2.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}.$$

Step 4: Cover the row containing the pivot position, and cover all rows, if any, above it.

Apply Steps 1–3 to the submatrix that remains.

Repeat the process until there are no more nonzero rows to modify.

With Row 1 covered, Step 1 shows that Column 2 is the next pivot column; for Step 2, select as a pivot the "top" entry in that column.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}.$$

For Step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2.

Instead, we add $-3/2$ times the "top" row to the row below.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

When we cover the row containing the second pivot position, we are left with a new submatrix that has only one row.

Steps 1–3 require no work for this submatrix (i.e. we do nothing).

We have reached an echelon form of the full matrix.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

We perform one more step to obtain the reduced echelon form.

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Step 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot.

If a pivot is not 1, make it 1 by a scaling operation.

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If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in Row 3.

Create zeros above it, adding suitable multiples of Row 3 to Row 2 and Row 1.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

The next pivot is in Row 2.

Scale this row, dividing by the pivot.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Create a zero in Column 2 by adding 9 times Row 2 to Row 1.

$$\begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

The next pivot is in Row 2.

Scale this row, dividing by the pivot.

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Create a zero in Column 2 by adding 9 times Row 2 to Row 1.

$$\begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Finally, scale Row 1, dividing by the pivot, 3.

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

This is the reduced echelon form of the original matrix A. $\square$

Remark

The combination of Steps 1–4 is called the **forward phase** of the row reduction algorithm.

Step 5 is called the **backward phase**. It produces the unique reduced echelon form.

Solutions of Linear Systems

The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

There are 3 variables because the augmented matrix has four columns.

The associated system of equations is

$$\begin{aligned} x_1 \quad - \; 5x_3 &= 1 \\ x_2 \; + \; x_3 &= 4 \\ 0 &= 0. \end{aligned} \tag{*}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**.

The other variable x_3 is called a **free variable**.

Whenever a system is consistent, as in (*), the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

In (*) we ignore the third equation because it offers no restriction on the variables.
We solve the first and second equations for x_1 and x_2.

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &\text{ is free.} \end{aligned} \qquad (**)$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Then the formulas in (**) determine the values for x_1 and x_2.

For example, when $x_3 = 0$, the solution is $(1,4,0)$.
When $x_3 = 1$, the solution is $(6,3,1)$.

Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3. $\square$

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Parametric descriptions of solutions

- The description in (**) is a *parametric description* of solutions sets in which the free variables act as parameters.
- Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

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Warning

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

Example. In System (*), add 5 times Equation 2 to Equation 1 to get the following equivalent system.

$$\begin{aligned} x_1 \;+\; 5x_2 &= 21 \\ x_2 \;+\; x_3 &= 4 \\ 0 &= 0. \end{aligned}$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have another description of the solution set.

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When a system is inconsistent, the solution set is empty, even when the system has free variables.
In this case, the solution set has no parametric representation.

Theorem (Existence and Uniqueness Theorem)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix **does not have** *row of the form*

$$[0 \;\; 0 \;\; \dots \;\; 0 \,|\, b]$$

with b nonzero.

If a linear system is consistent, then the solution set contains either

(i) a unique solution when there are no free variables, or

(ii) infinitely many solutions when there is at least one free variable.

Use Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form.
3. Decide whether the system is consistent.
   If there is no solution, stop.
   Otherwise, go to the next step.

4. Continue row reduction to obtain the reduced echelon form.
5. Write the system of equations corresponding to the matrix obtained in Step 4.
6. Rewrite each nonzero equation from Step 5 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.3 Vector Equations

A matrix with only one column is called a *column vector*, or simply a *vector*.

Example

A vector with two entries is

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where w_1 and w_2 are any real numbers.

The set of all vectors with 2 entries is denoted by $\mathbb{R}^2$.

Remarks

1. The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector.
2. The exponent 2 indicates that each vector contains 2 entries.
3. Two vectors in $\mathbb{R}^2$ are equal if and only if their corresponding entries are equal.

4. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, their sum is the vector

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

obtained by adding corresponding entries of $\mathbf{u}$ and $\mathbf{v}$.

5. Given a vector $\mathbf{u}$ and a real number c, the scalar multiple of $\mathbf{u}$ by c is the vector

$$c\mathbf{u} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$$

obtained by multiplying each entry in $\mathbf{u}$ by c.

4. Given two vectors **u** and **v** in $\mathbb{R}^2$, their sum is the vector

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obtained by multiplying each entry in **u** by c.

Example 1

Given $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, find

$$4\mathbf{u}, \quad (-3)\mathbf{v} \quad \text{and} \quad 4\mathbf{u} + (-3)\mathbf{v}.$$

Solution

We have

$$4\mathbf{u} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}, \quad (-3)\mathbf{v} = \begin{pmatrix} -6 \\ 15 \end{pmatrix}$$

and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{pmatrix} 4 \\ -8 \end{pmatrix} + \begin{pmatrix} -6 \\ 15 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}.$$

Solution

We have

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and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{pmatrix} 4 \\ -8 \end{pmatrix} + \begin{pmatrix} -6 \\ 15 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}.$$

Geometric descriptions of $\mathbb{R}^2$

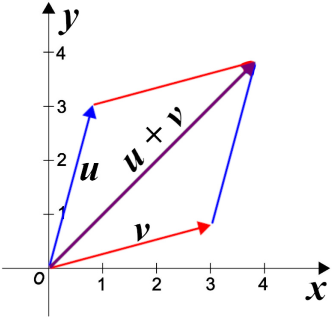
Consider a rectangular coordinate system in the plane.

Each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point (a, b) with the column vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

Hence we may regard $\mathbb{R}^2$ as the set of all points in the plane.

Parallelogram rule for vector addition

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$.



Vectors in $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3 by 1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, $\mathbb{R}^n$ denotes the collection of all lists (or ordered n-tuples) of n real numbers, usually written as n by 1 column matrices, such as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

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$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

The vector

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is called the *zero vector*.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$.

(iv) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ where $-\mathbf{u}=(-1)\mathbf{u}$.

Algebraic Properties of $\mathbb{R}^n$

(v) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$.

(vi) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$.

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$.

(viii) $1\mathbf{u} = \mathbf{u}$.

Linear combinations

Definition.
Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

is called a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$.

The weights in a linear combination can be any real numbers, including zero.

Example 2

Let

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}.$$

Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

That is, do weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}? \tag{1}$$

If Equation (1) has a solution, find it.

Solution

Equation (1) is

$$x_1 \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}.$$

This is same as

$$\begin{pmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}.$$

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Equation (1) is

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The last equation is the same as

$$\begin{pmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}.$$

The three coordinates give three equations.

$$\begin{aligned} x_1 \ + \ 2x_2 &= 7 \\ -2x_1 \ + \ 5x_2 &= 4 \\ -5x_1 \ + \ 6x_2 &= -3. \end{aligned} \tag{2}$$

We have to solve for x_1 and x_2

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We have to solve for x_1 and x_2

To solve this system, row reduce the augmented matrix of the system as follows.

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

The solution of Equation (2) is

$$x_1 = 3 \quad \text{and} \quad x_2 = 2.$$

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The solution of Equation (2) is

$$x_1 = 3 \text{ and } x_2 = 2.$$

We put $x_1 = 3$ and $x_2 = 2$ into Equation (1) to get

$$3\begin{pmatrix}1\\-2\\-5\end{pmatrix}+2\begin{pmatrix}2\\5\\6\end{pmatrix}=\begin{pmatrix}7\\4\\-3\end{pmatrix}.$$

Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Remark

We observe that the original vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right) = \left(\begin{array}{cc|c} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \\ | & | & | \end{array}\right) \tag{3}$$

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The general situation

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \\ | & | & \cdots & | & | \end{array}\right). \tag{4}$$

In particular, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (4).

The general situation

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \\ | & | & \cdots & | & | \end{array}\right). \tag{4}$$

In particular, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (4).

Definition

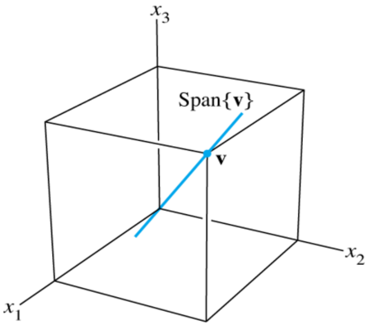
If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the subset of *spanned* (or *generated*) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \left\{\, c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p \ : c_p, \ldots, c_p \text{ are scalars}\right\}.$$

A geometric description of Span$\{V\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$.

Then Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.



Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$.

Then $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$.

The plane $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$.