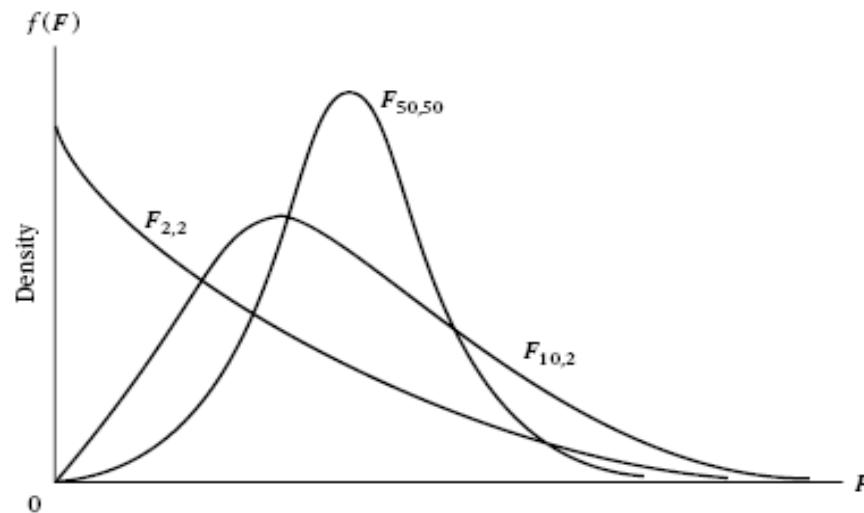


EC 3303: Econometrics I

Linear Regression with Multiple Regressors (Part 3)



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AY 2022/2023, Semester 2

Outline

1. Joint hypothesis tests
2. Other types of hypotheses involving multiple coefficients
3. What variables to include

Test of Joint Hypotheses

$$TestScore_i = \beta_0 + \beta_1 STR_i + \beta_2 Expn_i + \beta_3 PctEL_i + u_i$$

- Parent claims that *neither* class size *nor* expenditures per pupil have an effect on test scores. “school resources don’t matter”.
- null hypothesis that “**school resources don’t matter**”, and the alternative that “they do”, corresponds to:

$$H_0: \beta_1 = 0 \text{ and } \beta_2 = 0$$
$$H_1: \textit{either } \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \text{ or both}$$

$$H_0: \beta_1 = 0 \text{ and } \beta_2 = 0$$
$$H_1: \textit{either } \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \text{ or both}$$

- The hypothesis that ***both*** the coefficient on *STR* (β_1) and the coefficient on expenditures per pupil (β_2) are zero is an example of a joint hypothesis.
- A ***joint hypothesis*** is a hypothesis that specifies a value for two or more coefficients, that is, it imposes a ***restriction*** on two or more coefficients.
- A “common sense” idea is to reject H_0 if either of the individual t -statistics exceeds 1.96 in absolute value.
 - But this “one at a time” test is ***not valid***.

Why can't we just test the individual coefficients one at a time?

- Because the probability that you will reject the null hypothesis when it is in fact true is greater than the desired level (e.g. 5%)
 - Lecture 3: **Significance level, α** : pre-specified probability of rejecting the null when it is actually true; usually 5% or 1%.
- Let t_1 & t_2 be the *t-statistics*:

$$t_1 = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)}, t_2 = \frac{\hat{\beta}_2 - 0}{SE(\hat{\beta}_2)}$$

- The “one at a time” test is

Reject $H_0: \beta_1 = \beta_2 = 0$ if $|t_1| > 1.96$ and/or $|t_2| > 1.96$

- What is the probability that this “one at a time” test rejects H_0 , when H_0 is actually true? (It *should* be 5%).

- To simplify calculation, suppose that t_1 & t_2 are independent.
- probability of rejecting the null hypothesis when it is actually true, using the “one at a time” test is:
 - null is not rejected only if **both** $|t_1| < 1.96$ **and** $|t_2| < 1.96$.
 - So the probability of **not rejecting** the null when it is true is:

$$\begin{aligned}\Pr(|t_1| < 1.96 \text{ and } |t_2| < 1.96) &= \Pr(|t_1| < 1.96) \times \Pr(|t_2| < 1.96) \\ &= 0.95 \times 0.95 = 0.9025\end{aligned}$$

[first equality holds because the t-statistics are independent]

- So the probability of rejecting the null when it is true is:

$$1 - 0.9025 = 0.0975 \text{ (or 9.75\%), and } \textbf{not} \text{ (the desired) 5\%}.$$

Solution?

- Use a different test statistic – the ***F*-statistic** – that tests both β_1 and β_2 *at the same time*...

F-Statistic

- The *F-Statistic* is used to *test joint hypotheses* about regression coefficients.
- In general, the formula for the *F-statistic* is complicated (and difficult to write down), but for the *special case* of the joint hypothesis: $\beta_1 = 0$ and $\beta_2 = 0$ (i.e. 2 restrictions imposed) formula is:

$$F = \frac{1}{2} \left(\frac{t_1^2 + t_2^2 - 2\hat{\rho}_{t_1, t_2} t_1 t_2}{1 - \hat{\rho}_{t_1, t_2}^2} \right)$$

where $\hat{\rho}_{t_1, t_2}$ is an estimator of the correlation between t_1 and t_2

$$F = \frac{1}{2} \left(\frac{t_1^2 + t_2^2 - 2\hat{\rho}_{t_1, t_2} t_1 t_2}{1 - \hat{\rho}_{t_1, t_2}^2} \right)$$

- To simplify understanding the *F-Statistic*, consider the case where the t-statistics are uncorrelated ($\hat{\rho}_{t_1, t_2} = 0$). The *F-Statistic* simplifies:

$$F = \frac{1}{2} (t_1^2 + t_2^2)$$

- Under $H_0: \beta_1 = 0$ **and** $\beta_2 = 0$ while under H_1 : **either** $\beta_1 \neq 0$ **or** $\beta_2 \neq 0$ **or both**
- If **either** t_1^2 **or** t_2^2 **or both** are large, this is evidence that either $\beta_1 \neq 0$ or $\beta_2 \neq 0$ (or both), so we should reject H_0 .
- **Inference:** Accordingly, we **reject** H_0 if the *F-Statistic* is **large**.

But how large is large?

- First, need to know how the *F-Statistic* is distributed...
- In large samples, the *F-Statistic* has an $F_{2,\infty}$ distribution for the case where the joint null has 2 *restrictions*.
- In general, in large samples,

the *F-Statistic* is distributed as $F_{q,\infty}$

(where $F_{q,\infty}$ reads “*F distribution* with q numerator degrees of freedom and ∞ denominator degrees of freedom”)

where the joint null hypothesis has q *restrictions*.

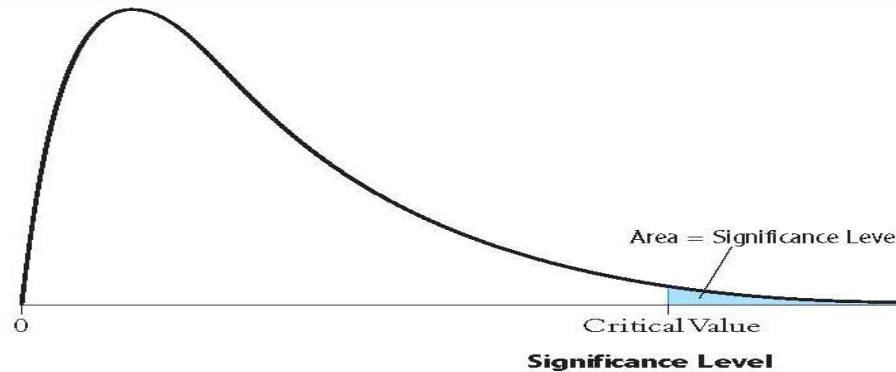
Inference: *reject* H_0 when the *F-Statistic* is *large* enough – where “large enough” is determined by critical values.

- Appendix table 4 (pg 807) provides the critical values for the *F distribution* for the various values of q and significance levels α .

Implementation in STATA:

- Use the “test” command after the regression.

TABLE 4 Critical Values for the $F_{m,\infty}$ Distribution



Degrees of Freedom	10%	5%	1%
1	2.71	3.84	6.63
2	2.30	3.00	4.61
3	2.08	2.60	3.78
4	1.94	2.37	3.32
5	1.85	2.21	3.02
6	1.77	2.10	2.80
7	1.72	2.01	2.64
8	1.67	1.94	2.51
9	1.63	1.88	2.41
10	1.60	1.83	2.32
11	1.57	1.79	2.25
12	1.55	1.75	2.18
13	1.52	1.72	2.13
14	1.50	1.69	2.08
15	1.49	1.67	2.04
16	1.47	1.64	2.00
17	1.46	1.62	1.97
18	1.44	1.60	1.93
19	1.43	1.59	1.90
20	1.42	1.57	1.88
21	1.41	1.56	1.85
22	1.40	1.54	1.83
23	1.39	1.53	1.81
24	1.38	1.52	1.79
25	1.38	1.51	1.77
26	1.37	1.50	1.76
27	1.36	1.49	1.74
28	1.35	1.48	1.72
29	1.35	1.47	1.71
30	1.34	1.46	1.70

This table contains the 90th, 95th, and 99th percentiles of the $F_{m,\infty}$ distribution. These serve as critical values for tests with significance levels of 10%, 5%, and 1%.

$$TestScore_i = \beta_0 + \beta_1 STR_i + \beta_2 Expn_i + \beta_3 PctEL_i + u_i$$

- Claim: *neither* class size *nor* expenditures per pupil have an effect on test scores.

$$H_0: \beta_1 = 0 \text{ and } \beta_2 = 0$$

$$H_1: \textit{either } \beta_1 \neq 0 \textit{ or } \beta_2 \neq 0 \textit{ or both}$$

F-test example, California data:

```
reg testscr str expn_stu pctel, r
```

Regression with robust standard errors

Number of obs = 420
F(3, 416) = 147.20
Prob > F = 0.0000
R-squared = 0.4366
Root MSE = 14.353

		Robust				
testscr		Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
-----+-----						
str		-.2863992	.4820728	-0.59	0.553	-1.234001 .661203
expn_stu		.0038679	.0015807	2.45	0.015	.0007607 .0069751
pctel		-.6560227	.0317844	-20.64	0.000	-.7185008 -.5935446
_cons		649.5779	15.45834	42.02	0.000	619.1917 679.9641

NOTE

```
test str expn_stu
```

The test command follows the regression

```
( 1) str = 0.0  
( 2) expn_stu = 0.0
```

There are q=2 restrictions being tested

```
F( 2, 416) = 5.43  
Prob > F = 0.0047
```

*The 5% critical value for q=2 is 3.00
Stata computes the p-value for you*

P-value of the F-Statistic

$$F(2, 416) = 5.43$$

The 5% critical value for $q=2$ is 3.00

$$\text{Prob} > F = 0.0047$$

Stata computes the p-value for you

- *P-value* of the *F-Statistic* is the probability of drawing a sample which produces an *F-Statistic* as large or larger than the one actually computed using your sample, assuming H_0 is indeed true.
- The smaller the p-value, the stronger the evidence *against* H_0 .
- Reject H_0 if the p-value of the *F-Statistic* is small ($< 5\%$ or 1%).

$$F(2, 416) = 5.43$$

The 5% critical value for $q=2$ is 3.00

$$\text{Prob} > F = 0.0047$$

Stata computes the p -value for you

- In this application, the (heteroskedasticity-robust) *F-Statistic* of the test that $\beta_1 = 0$ *and* $\beta_2 = 0$ is 5.43.
- In large samples, the *F-Statistic* has an $F_{2,\infty}$ distribution.
- The 5% (1%) critical value of the $F_{2,\infty}$ distribution is 3.00 (4.61).
- Since the value of the *F-Statistic* computed = 5.43 > 4.61, *reject H_0* at the 1% level.
- Alternatively, since the *p-value* of the *F-Statistic* = 0.0047 < 0.01, *reject H_0* at the 1% level.
- Conclude: There is evidence that **school resources do have an effect on test scores**. In other words, there is evidence that *either $\beta_1 \neq 0$ or $\beta_2 \neq 0$ or both*.

Heteroskedasticity-Robust vs Homoskedasticity-Only *F-Statistic*

- You *should use* the *heteroskedasticity-robust F-statistic*, with $F_{q,\infty}$ critical values. These are valid whether the population errors are homoskedastic or heteroskedastic.
- Heteroskedasticity-robust *F*-statistics are automatically computed by STATA if you use the “**,robust**” option when running the regression.
- If you do not use the “,robust” option, STATA computes homoskedasticity-only standard errors and *F-Statistics*. These are *not valid* if the population errors are heteroskedastic.

- More generally, can have a joint null hypothesis that imposes q restrictions on the coefficients:

$H_0: \beta_j = \beta_{j,0}, \beta_m = \beta_{m,0}, \dots, \text{for a total of } q \text{ restrictions}$

vs.

$H_1: \text{one or more of the } q \text{ restrictions under } H_0 \text{ does not hold}$

where β_j, β_m, \dots , refer to different regression coefficients and $\beta_{j,0}, \beta_{m,0}, \dots$, refer to the values of these coefficients under H_0 .

Example

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \beta_5 X_{5i} + u_i$$

where X_1, X_4 , and X_5 represent a collection of variables deemed to be “school resources” & Y represents test score.

- claim that school resources do not have an effect on test score can be tested:

$H_0: \beta_1 = 0, \beta_4 = 0, \text{ and } \beta_5 = 0$ (*how many restrictions here?*)

$H_1: \text{one or more of the restrictions under } H_0 \text{ do not hold}$

Testing Single Restrictions Involving Multiple Coefficients

- Sometimes, economic theory might suggest a *single restriction* that involves *two coefficients*.

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i$$

- E.g.: theory might suggest $\beta_1 = \beta_2$.

$$H_0: \beta_1 = \beta_2 \quad vs \quad H_1: \beta_1 \neq \beta_2$$

- This null hypothesis imposes a *single restriction* ($q = 1$) on *multiple* coefficients – it is not a joint hypothesis with multiple restrictions.

Implementation in STATA

$$TestScore_i = \beta_0 + \beta_1 STR_i + \beta_2 Expn_i + \beta_3 PctEL_i + \beta_4 LchPct_i + u_i$$

where *LchPct* is the percentage of students who are eligible for free lunch in the i^{th} district (more on this variable later)

- Suppose (unrealistically), that theory leads us to believe that the effect on test score of a unit change in *PctEL* is the same as that of a unit change in *LchPct*....

- In STATA, to test $\beta_3 = \beta_4$ vs $\beta_3 \neq \beta_4$ (two sided):

regress testscr str expn_stu el_pct meal_pct, robust

Linear regression

Number of obs = 420

F(4, 415) = 364.67

Prob > F = 0.0000

R-squared = 0.7834

Root MSE = 8.9096

		Robust				
testscr		Coef.	Std. Err.	t	P> t	[95% Conf. Interval]

str		-.2353884	.3248148	-0.72	0.469	-.8738757 .4030989
expn_stu		.003622	.0009447	3.83	0.000	.001765 .0054791
el_pct		-.1283415	.0324397	-3.96	0.000	-.1921082 -.0645748
meal_pct		-.5463929	.0231685	-23.58	0.000	-.5919351 -.5008506
_cons		665.9882	10.37683	64.18	0.000	645.5905 686.3859

test el_pct = meal_pct

test el_pct = meal_pct

(1) el_pct - meal_pct = 0

F(1, 415) = 65.56

Prob > F = 0.0000

- In large samples, the *F-Statistic* here will have an $F_{1,\infty}$ distribution since $q = 1$ (1 restriction imposed under H_0).
- The mechanics for statistical inference is unchanged:
 - H_0 is *rejected* if the value of the *F-Statistic* actually computed *exceeds* the critical value under the $F_{1,\infty}$ distribution.
 - Alternatively, H_0 is rejected if the p-value of the *F-Statistic* is small enough ($< 5\%$ or 1%).

```
test    el_pct = meal_pct
```

```
( 1)    el_pct - meal_pct    = 0
```

```
F( 1, 415) = 65.56
```

```
Prob > F = 0.0000
```

- In this application, the *F-Statistic* of the test that $\beta_{PctEL} = \beta_{LchPct}$ is 65.56 and its p-value is 0.00.
- In large samples, the *F-statistic* has an $F_{1,\infty}$ distribution under the null.
- The 5% (1%) critical value of the $F_{1,\infty}$ distribution is 3.84 (6.63).
- Since *F-Statistic* = 65.56 > 6.63, reject H_0 at the 1% level.

Conclude: there is evidence that $\beta_{PctEL} \neq \beta_{LchPct}$ at the 1% significance level.

Deciding which Variables to include in a Regression

- In practice, determining which variables to include in a regression (i.e. choosing a model specification) is challenging.
- In testscore – class size e.g.
 - Apart from STR, PctEL, LchPct, what other variables would you include as regressors?

Why?

- Fortunately, some useful guidelines are available:
 - starting point is to think carefully about *what we are really interested to know* and what the *possible sources of omitted variable bias* are.
- Potential honours thesis topics:

- We are often interested to know (only) how *one variable* affects *another*.
- in test score example,
 - we are interested to know whether changing class size will affect test scores; we want to obtain an *unbiased estimator of the effect of class size on test scores*.
 - we included *PctEL* as a regressor. But are we really interested to know how *PctEL* affects test scores? No.

- Only reason we included it is because, if neglected, it could lead the OLS estimator of the class size effect to suffer from omitted variable bias.
- *PctEL* is a “*control variable*”.
 - A control variable is not the object of interest in a study; it is a regressor included to hold constant factors that, if neglected, could lead the OLS estimator of the effect of interest to suffer from omitted variable bias.
- “*which variables to include so that any omitted variable bias in the OLS estimator of the effect of interest is mitigated?*”

- Want to examine the causal effect of one variable (i.e. variable of interest).
 - obtain an unbiased OLS estimator of this effect.
- The rest of the variables are *control variables* and we are not really interested in them. In practice, there will be omitted variables potentially correlated with the dependent variable Y and the control variables.
- We don't have to worry about this as long as we care only about the effect of interest – just make sure that *the OLS estimator of the coefficient of interest is unbiased!*
 - The OLS estimators of the coefficients on the control variables can be biased (and need not have a causal interpretation).

Relaxing LSA #1 to Conditional Mean Independence

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i$$

LSA#1:

$$E(u_i | X_{1i} = x_1, X_{2i} = x_2, \dots, X_{ki} = x_k) = 0$$

- LSA#1 treats the regressors *symmetrically* – does not distinguish between the variable of interest and the control variables.

- Can *relax LSA #1* by replacing it with the *conditional mean independence* assumption.
- Consider a regression with just two regressors, where X_{1i} is the variable of interest and X_{2i} is the control variable.
- *Conditional mean independence requires that u_i not be correlated with X_{1i} though it can potentially be correlated with X_{2i}*
- If conditional mean independence holds, the OLS estimator, $\hat{\beta}_1$, is unbiased (and has a causal interpretation), but $\hat{\beta}_2$ is, in general, biased (and does not have a causal interpretation).

- More generally, for a regression with k regressors, where X_{1i} is the variable of interest and X_{2i}, \dots, X_{ki} are the control variables:
 - Conditional mean independence requires that u_i not be correlated with X_{1i} , though it can potentially be correlated with X_{2i}, \dots, X_{ki} .
 - As long as u_i is not correlated with X_{1i} (after we have controlled for X_{2i}, \dots, X_{ki}), $\hat{\beta}_1$ will be unbiased.
 - What if we have two variables of interest (say X_{1i} & X_{2i})?

- Correct way to approach problem of model specification is to choose a set of control variables so that we achieve the goal:
 - obtain an unbiased estimator of the coefficient of interest.
- ***Do not*** choose control variables solely to maximize R^2 (or \bar{R}^2)!

Do not choose control variables solely to maximize R^2 (or \bar{R}^2)

- It is easy to fall into the **trap of maximizing R^2** – but this loses sight of the real objective.

R^2 : What it tells you

- A high R^2 means that the regressors are good at predicting Y_i .

R^2 : What it does not tell you

- A high R^2 does not mean that you have eliminated omitted variable bias from the OLS estimator of the effect of interest.

R^2 : What it does not tell you

- A high R^2 does not mean that the regressors are a true *cause* of the dependent variable.
- An increase in R^2 does not mean that an added variable is statistically significant.
 - whether an added regressor is statistically significant must be determined using *hypothesis tests*.

Note on R^2 & \bar{R}^2

- In STATA, you can get the \bar{R}^2 if you run the regression *without* the “,robust” option
- Running the regression with the “,robust” option will only give you the unadjusted R^2 .
 - Note, that the standard errors are incorrect though if u_i is heteroskedastic.

Example: R^2 & \bar{R}^2

With “,robust”

```
. regress testscr str expn_stu el_pct, robust
```

Linear regression

Number of obs = 420
F(3, 416) = 147.20
Prob > F = 0.0000
R-squared = 0.4366
Root MSE = 14.353

testscr	Coef.	Robust Std. Err.	t	P> t	[95% Conf. Interval]	
str	-.2863992	.4820728	-0.59	0.553	-1.234002	.661203
expn_stu	.0038679	.0015807	2.45	0.015	.0007607	.0069751
el_pct	-.6560227	.0317844	-20.64	0.000	-.7185008	-.5935446
_cons	649.5779	15.45834	42.02	0.000	619.1917	679.9641

Without “,robust”

```
. regress testscr str expn_stu el_pct
```

Source	SS	df	MS
Model	66409.8837	3	22136.6279
Residual	85699.7099	416	206.008918
Total	152109.594	419	363.030056

Number of obs = 420
F(3, 416) = 107.45
Prob > F = 0.0000
R-squared = 0.4366
Adj R-squared = 0.4325
Root MSE = 14.353

testscr	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
str	-.2863992	.4805232	-0.60	0.551	-1.230955	.658157
expn_stu	.0038679	.0014121	2.74	0.006	.0010921	.0066437
el_pct	-.6560227	.0391059	-16.78	0.000	-.7328924	-.5791529
_cons	649.5779	15.20572	42.72	0.000	619.6883	679.4676

Back to the test score application:

Variables actually in the California data set:

- student-teacher ratio (*STR*)
- percentage of English learners in the district (*PctEL*)
- name of the district (so we could look up average rainfall, for example)
- percentage of students in the district eligible for free lunch
- percentage of students in the district whose families qualify for income assistance
- average district income

Which of these variables would you include?

Which other variables would you include if you could?

There is no simple recipe for deciding which variables to include in a regression – you must exercise judgement!

A General Approach to *Model Specification*

- 1) Specify a “base” or “benchmark” specification.
 - should contain the variable of primary interest & a core set of control variables, suggested by judgement or theory.
- 2) Specify a range of plausible alternative specifications.
 - which include additional candidate control variables.
- 3) Run the regressions.
 - Pay attention to whether the estimates of the coefficient of interest are numerically similar across the different specifications.
 - Yes: estimates from your base specification are reliable (caveat).
 - No: base specification suffers from omitted variable bias.

Tabular Presentation of Regression Results

- We have a number of regressions and we want to report them.
- It is awkward and difficult to read many regressions written out in equation form, so instead it is conventional to report them in a *table*.
- Table of regression results should include:
 - estimated regression coefficients
 - (robust) standard errors
 - measures of fit
 - number of observations
 - relevant F -statistics, if any
 - any other pertinent information

TABLE 7.1 Results of Regressions of Test Scores on the Student–Teacher Ratio and Student Characteristic Control Variables Using California Elementary School Districts

Dependent variable: average test score in the district.

Regressor	(1)	(2)	(3)	(4)	(5)
Student–teacher ratio (X_1)	−2.28** (0.52)	−1.10* (0.43)	−1.00** (0.27)	−1.31** (0.34)	−1.01** (0.27)
Percent English learners (X_2)		−0.650** (0.031)	−0.122** (0.033)	−0.488** (0.030)	−0.130** (0.036)
Percent eligible for subsidized lunch (X_3)			−0.547** (0.024)		−0.529** (0.038)
Percent on public income assistance (X_4)				−0.790** (0.068)	0.048 (0.059)
Intercept	698.9** (10.4)	686.0** (8.7)	700.2** (5.6)	698.0** (6.9)	700.4** (5.5)
Summary Statistics					
<i>SER</i>	18.58	14.46	9.08	11.65	9.08
\overline{R}^2	0.049	0.424	0.773	0.626	0.773
<i>n</i>	420	420	420	420	420

These regressions were estimated using the data on K-8 school districts in California, described in Appendix 4.1. Standard errors are given in parentheses under coefficients. The individual coefficient is statistically significant at the *5% level or **1% significance level using a two-sided test.

Linear Regression Functions

- So far, we have modelled Y as a *linear* function of the regressors
 - a unit change in the regressor is assumed to have the same effect on Y , regardless of the value of the regressor.
- But sometimes, this assumption might not be reasonable:
 - E.g: the effect on test score of reducing class size by 1 might be different for a district currently with 30 students per teacher than for a district currently with 15 students per teacher.
- Next class, learn how to incorporate such *nonlinearities* in the regression model.

Reminder: Homework 1

- Homework 1 will be posted on 28 February (Tuesday), 12pm and will be due on 3 March (Friday), 7pm.
- It will be done through **Canvas Quiz**.
- Go to Canvas, on the left panel click on “Quizzes” (see next slide) and you will be able to access the Homework.
- Covers Lectures 1-4.
- Students need not complete the homework in 1 sitting. They can save the homework and submit it only later (but by the due date).

Reminder: Midterm Test

- **March 7 (Tuesday), 4.15pm-5.15pm (week 8)**
- 1-hour
- Venue: **MPSH 2A**
- Seating plan in MPSH 2A can now be found in Canvas under “Files”->”Supporting Materials for EC3303 Midterm Test”->”EC3303 Midterm Test Seating Arrangement”
- Please arrive at least 5 mins early
- Covers lecture 1 to lecture 4 (inclusive)
- 20 MCQ questions
- Closed book
- Bring a 2B pencil with you
- Bring a calculator with you
- Normal table will be provided. A sample copy of the normal table which students will be getting can now be found in Canvas under “Files”->”Supporting Materials for EC3303 Midterm Test”->”Normal Tables Sample”

Additional Consultation

- Every Friday, 4-6pm (regular consultation)
- plus 2 March, Thursday, 4-6pm (additional consultation)
- <https://nus-sg.zoom.us/j/83502944248?pwd=K2c1WENobVBXV0YrcEliUlh4Z2s5dz09>
- Meeting ID: 835 0294 4248
- Passcode: 714459