# Macroeconomics analysis II, EC3102

# Tutorial 2 Solution

**Question 1**

***Note:***

*Given the notation rule in this question, t stands for the present (or now); and so $s$ stands for the number of periods away from "now". In other words, we are using the letter $s$ as a counter for the time index.*

At time $t+s$, the budget for the period is:

$$\underbrace{P_{t+s}c_{t+s} + S_{t+s}\,a_{t+s}}_{\text{outflows}} = \underbrace{S_{t+s}a_{t+s-2} + D_{t+s}\,a_{t+s-2} + Y_{t+s}}_{\text{inflows}} \tag{1}$$

***Comment:*** *At the time $t+s$:*

- *The inflows includes the income for that period, the dividend earned from the asset bought two periods ago (time $t+s-2$) and the nominal value of the assets bought two periods ago (this nominal value is equal to the price of one asset at time $t+s$, which is $S_{t+s}$, multiplied with the amount of assets bought two periods ago – that is $a_{t+s-2}$.)*

- *The outflow includes the consumptions (in nominal values) and the amount of money to buy assets ($a_{t+s}$) the individual wish to hold in this period.*

The problem of the consumer is to maximize lifetime utility from time $t$ onwards subject to an infinite sequence of flow budget constraints, that is:

$$\max_{\{c_{t+s}\,,\,a_{t+s}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

$$\text{subject to:} \left\{ \underbrace{P_{t+s}c_{t+s} + S_{t+s}\,a_{t+s}}_{\textit{outflows}} = \underbrace{S_{t+s}a_{t+s-2} + D_{t+s}\,a_{t+s-2} + Y_{t+s}}_{\textit{inflows}} \right\}_{s=0}^{\infty}$$

***Comment:*** *Our choice variables in this maximization problem are $c_{t+s}$ and $a_{t+s}$ $(\mathrm{s} = 1,2, \dots \infty)$. That is, we need to choose the right amount for all of these variables in order to maximize the lifetime utility of this representative individual. Well, at this point, you might wonder how to solve for all of them because there are infinity numbers of $c_{t+s}$ and $a_{t+s}$ . Do not worry, we are just solving for one set of $c_{t+s}$ and $a_{t+s}$ and then generalize the pattern for the subsequent period's $c$ and $a$.*

The Lagrangian is:

$$\mathcal{L}(c_t, c_{t+1}, \dots; a_t, a_{t+1}, \dots; \lambda_t, \lambda_{t+1} \dots) = \sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

$$-\sum_{s=0}^{\infty} \lambda_{t+s}\beta^{s}\{P_{t+s}\,c_{t+s} + S_{t+s}a_{t+s} - S_{t+s}a_{t+s-2} - D_{t+s}a_{t+s-2} - Y_{t+s}\} \quad (2)$$

Or expanding it, we have:

$$\begin{aligned}
\mathcal{L}(\dots) = {} & u(c_{t+0}) + \beta^{1}u(c_{t+1}) + \beta^{2}u(c_{t+2}) + \cdots \\
& -\lambda_{t+0}\beta^{0}\{P_{t+0}\,c_{t+0} + S_{t+0}a_{t+0} - S_{t+0}a_{t+0-2} - D_{t+0}a_{t+0-2} - Y_{t+0}\} \\
& -\lambda_{t+1}\beta^{1}\{P_{t+1}\,c_{t+1} + S_{t+1}a_{t+1} - S_{t+1}a_{t+1-2} - D_{t+1}a_{t+1-2} - Y_{t+1}\} \\
& -\lambda_{t+2}\beta^{2}\{P_{t+2}\,c_{t+2} + S_{t+2}a_{t+2} - S_{t+2}a_{t+2-2} - D_{t+2}a_{t+2-2} - Y_{t+2}\} \\
& -\lambda_{t+3}\beta^{3}\{P_{t+3}\,c_{t+3} + S_{t+3}a_{t+3} - S_{t+3}a_{t+3-2} - D_{t+3}a_{t+3-2} - Y_{t+3}\} \\
& - \cdots
\end{aligned} \quad (3)$$

***Note:***

*1. Since I have already specified the $\mathcal{L}$ function with all the choice variables in equation (2), so instead of listing out them out again, I just use the ellipsis - that is,$\mathcal{L}(\dots)$ .*

*2. Note carefully these terms. The 2nd ellipsis at the end indicate that the summation continues forever (since the consumer is assumed to maximize lifetime utility), but the terms written down are only ones that are important for the problem at hand: the consumer in period t chooses $c_t$ and $a_t$and there are no other terms in the Lagrangian (i.e., there are no other budget constraints) that contain these quantities. Also note the as we move successive periods into the future, the discount factor $\beta$ is exponentiated further.*

To get the first order conditions (FOCs), differentiate $\mathcal{L}$ with respect to any set of $c_{t+s}$ and $a_{t+s}$ ($s$ can be any number) and equate the partial derivatives to 0.

$$\mathcal{L}_{c_{t+s}}(\dots) = \beta^{s}u_1(c_{t+s}) - \lambda_{t+s}\beta^{s}P_{t+s} = 0 \quad (4)$$

$$\text{and } \mathcal{L}_{a_{t+s}}(\dots) = -\lambda_{t+s}\beta^{s}S_{t+s} - \{-\lambda_{t+s+2}\beta^{s+2}[S_{t+s+2} + D_{t+s+2}]\} = 0 \quad (5)$$

***Mathematical comment:*** *From (4) and (5), we need to get rid of $\lambda_{t+s}$ and $\lambda_{t+s+2}$ because they are not what we want. But clearly, you can see that there is no way to get rid of $\lambda_{t+s+2}$. So we need another equation that involves $\lambda_{t+s+2}$. Observe equation (4), we can try the FOC with respect to $c_{t+s+2}$. Fortunately, there is a pattern. As the matter of fact, you can try to partial differentiate $\mathcal{L}$ with respect to $c_{t+s+2}$ and see that the result is just a shifting the time index by 2 units. So we get:*

$$\mathcal{L}_{c_{t+s+2}}(\dots) = \beta^{s+2}u_1(c_{t+s+2}) - \lambda_{t+s+2}\beta^{s+2}P_{t+s+2} = 0 \quad (6)$$

From (5), we get:

$$\lambda_{t+s}S_{t+s} = \lambda_{t+s+2}\beta^2\{S_{t+s+2} + D_{t+s+2}\} \tag{7}$$
$$\frac{\lambda_{t+s}}{\lambda_{t+s+2}} = \frac{\beta^2\{S_{t+s+2} + D_{t+s+2}\}}{S_{t+s}}$$

From (4) by (6), we get:

$$\frac{u_1(c_{t+s})}{u_1(c_{t+s+2})} = \beta^2 \frac{\lambda_{t+s}}{\lambda_{t+s+2}} \frac{\beta^s P_{t+s}}{\beta^{s+2} P_{t+s+2}} \tag{8}$$

Substituting (7) into (8), we get:

$$\begin{aligned} \frac{u_1(c_{t+s})}{u_1(c_{t+s+2})} &= \beta^2 \frac{\beta^2\{S_{t+s+2} + D_{t+s+2}\}}{S_{t+s}} \frac{\beta^s P_{t+s}}{\beta^{s+2} P_{t+s+2}} \\ &= \frac{\beta^2\{S_{t+s+2} + D_{t+s+2}\}}{S_{t+s}} \frac{P_{t+s}}{P_{t+s+2}} \end{aligned} \tag{9}$$

Expressing $S_{t+s}$ in term of the rest, we get:

$$S_{t+s} = \beta^2 \left[\frac{u_1(c_{t+s+2})}{u_1(c_{t+s})} (S_{t+s+2} + D_{t+s+2}) \frac{P_{t+s}}{P_{t+s+2}} \right] \tag{10}$$

The above formula is a general one for a stock price at any time $t + s$. So the present stock price, or the stock price at time $t$ is:

$$S_t = \beta^2 \left[\frac{u_1(c_{t+2})}{u_1(c_t)} (S_{t+2} + D_{t+2}) \frac{P_t}{P_{t+2}}\right] \tag{11}$$

Economic intuition: if we rewrite equation (9) for time $t$ (that is, now) and also rearranging the term, we get:

c.

From (11), we observe that the stock price in period $t$ is affected by period $t + 2$ marginal utility $u_1(c_{t+2})$, price level $P_{t+2}$, stock price $S_{t+2}$, and the dividend $D_{t+2}$. This is because, by assumption stated in the question, stock purchased in period $t$ does not result in any returns till period $t + 2$. So the decision made in the stock market will be based on the information in period $t + 2$ only. Information from other periods does not affect the decision.

**<u>Question 2</u>**

a.

Note that, in this question, the dividend comes in the following period, $t + 1$, similar to the assumption in the lecture note. With dividend tax, the flow budget constraint for period $t$ (that is, any period) is:

$$\underbrace{P_t c_t + S_t a_t}_{\text{outflows}} = \underbrace{S_t a_{t-1} + (1 - \tau_t^D) D_t a_{t-1} + Y_t}_{\text{inflows}} \tag{12}$$

***Comment:*** *In this question, instead of using the time indexing system of $(t+s)$ where $s$ is the counter (for this system of indexing, the present corresponds to the time index $t$), we are using the time indexing system of $t$ where $t$ is the time index counter. So, in this question, the present corresponds to the time index $0$. You might lament why we have to learn both systems. The reason is these are two ways that economists use in their journal papers and books for time indexing.*

*Also, you might wonder since the life of this representative individual starts at time $0$ $(t=0)$, then why do we have $a_{-1}$. Valid question but we can treat $a_{-1}$ as $0$. But leaving it in the above budget constraint would aid our understanding of the BC better.*

b.

The problem of the consumer is:

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to:} \quad \left\{ \underbrace{P_t c_t + S_t\, a_t}_{outflows} = \underbrace{S_t a_{t-1} + (1-\tau_t^D) D_t\, a_{t-1} + Y_t}_{inflows} \right\}_{t=0}^{\infty} \tag{13}$$

Lagrangian, using the summation, is:

$$\mathcal{L}(c_0, c_1, \dots; a_0, a_1, \dots; \lambda_0, \lambda_1 \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$- \sum_{t=0}^{\infty} \lambda_t \beta^t \{P_t\, c_t + S_t a_t - S_t a_{t-1} - (1-\tau_t^D) D_t a_{t-1} - Y_t\} \tag{14}$$

The expanded Lagrangian is:

$$\begin{aligned}
\mathcal{L}(\dots) = {} & u(c_0) + \beta^1 u(c_1) + \beta^2 u(c_2) + \cdots \\
& -\lambda_0 \beta^0 \{P_0\, c_0 + S_0 \boldsymbol{a_0} - S_0 a_{-1} - (1-\tau_0^D) D_0 a_{-1} - Y_0\} \\
& -\lambda_1 \beta^1 \{P_1\, c_1 + S_1 a_1 - S_1 \boldsymbol{a_0} - (1-\tau_1^D) D_1 \boldsymbol{a_0} - Y_1\} \\
& - \cdots \\
& - \lambda_t \beta^t \{P_t\, c_t + S_t \boldsymbol{a_t} - S_t a_{\boldsymbol{t-1}} - (1-\tau_t^D) D_t a_{\boldsymbol{t-1}} - Y_t\} \\
& -\lambda_{t+1} \beta^{t+1} \{P_{t+1} c_{t+1} + S_{t+1} a_{t+1} - S_{t+1} \boldsymbol{a_t} - (1-\tau_{t+1}^D) D_{t+1} \boldsymbol{a_t} - Y_{t+1}\} \\
& - \cdots
\end{aligned} \tag{15}$$

***Comment:*** *I coloured the terms so that you can see that the asset for a particular period will appear again in the budget constraint 1 period ahead. Also note that, BCs for times $t$ and $t+1$ and are listed out so that you can see a pattern in the BCs is repeated for any period in time.*

Now let's get the first order conditions (FOCs) with respect to the variables at any time period - say period $t$. We have:

$$\mathcal{L}_{c_t}(\dots) = \beta^t u_1(c_t) - \lambda_t \beta^t P_t = 0 \tag{16}$$

$$\text{and } \mathcal{L}_{a_t}(\dots) = -\lambda_t \beta^t S_t - \left\{-\lambda_{t+1}\beta^{t+1}\left[S_{t+1} + (1-\tau_{t+1}^D)D_{t+1}\right]\right\} = 0 \tag{17}$$

$$\mathcal{L}_{c_{t+1}}(\dots) = \beta^{t+1} u_1(c_{t+1}) - \lambda_{t+1}\beta^{t+1} P_{t+1} = 0 \tag{18}$$

From (17), we get:

$$\lambda_t S_t = \lambda_{t+1}\beta[S_{t+1} + (1-\tau_{t+1}^D)D_{t+1}]$$

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{\beta[S_{t+1} + (1-\tau_{t+1}^D)D_{t+1}]}{S_t} \tag{19}$$

From (16) and (18), we get:

$$\frac{\beta^t u_1(c_t)}{\beta^{t+1} u_1(c_{t+1})} = \frac{\lambda_t \beta^t P_t}{\lambda_{t+1}\beta^{t+1} P_{t+1}}$$

$$\frac{u_1(c_t)}{u_1(c_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}}\frac{P_t}{P_{t+1}} \tag{20}$$

Substituting (19) into (20), we get:

$$\frac{u_1(c_t)}{u_1(c_{t+1})} = \frac{\beta[S_{t+1} + (1-\tau_{t+1}^D)D_{t+1}]}{S_t}\frac{P_t}{P_{t+1}}$$

$$S_t = \beta\left[\frac{u_1(c_{t+1})}{u_1(c_t)}[S_{t+1} + (1-\tau_{t+1}^D)D_{t+1}]\frac{P_t}{P_{t+1}}\right] \tag{21}$$

If we are interested in how today's stock is being priced, just replace $t$ with 0 to get:

$$S_0 = \beta\left[\frac{u_1(c_1)}{u_1(c_0)}[S_1 + (1-\tau_1^D)D_1]\frac{P_0}{P_1}\right] \tag{22}$$

From (21), we can observe that the stock price in any period, $t$, is affected by period $t+1$ dividend tax rate $\tau_{t+1}^D$. The dividend tax rate determines the future after-tax income flows. All else equal, a higher dividend tax rate in the future implies a lower stock price today. This is because if people know that here is higher tax on dividend in one period from now, the stock is less attractive to them and thus its demand is lower, resulting in a lower stock price today.

***Comment on the result:***

*If we use the time index system with that of Question 1, then equation (21) will become:*

$$S_{t+s} = \beta\left[\frac{u_1(c_{t+s+1})}{u_1(c_{t+s})}\left(S_{t+s+1} + (1-\boldsymbol{\tau_{t+s+1}^D})D_{t+s+1}\right)\frac{P_{t+s}}{P_{t+s+1}}\right] \tag{23}$$

*Suppose that Question 1's assumption on the dividend is that the dividend is realized after **1 period instead of 2,** then equation (11) will be the same with that in the lecture note. That is:*

$$S_{t+s} = \beta^2 \left[\frac{u_1(c_{t+s+1})}{u_1(c_{t+s})}(S_{t+s+1} + D_{t+s+1}) \frac{P_{t+s}}{P_{t+s+1}}\right] \tag{24}$$

*If you compare the result in (23) with that of part a (24), what difference do you find beside different time index system? Do you realize that the difference is in the dividend term? That is, it changes from* $D_{t+s+1}$ *to* $(\mathbf{1}-\boldsymbol{\tau}_{t+s+1}^{D})D_{t+s+1}$*. This should not be a surprise to you since now instead of considering only the dividend of the stock, the individual now considers the after tax dividend (or the effective dividend) in their asset purchasing decision. So the only place that we need to update in the result is the dividend.*

*So actually without going through the Lagrange analysis again, using the intuition discussed above, we could have written down the new asset pricing equation like that of equation (21). But all is not wasted, at least we went through the chores of mathematics again as a practice, didn't we?*

c.

If we use the intuition discussed at the end of part b, we can straight away write down the solution to the asset pricing for this new context. It should be:

$$S_t = \beta\left[\frac{u_1(c_{t+1})}{u_1(c_t)}[S_{t+1} + (1-\tau_{t+1}^{D})D_{t+1}]\frac{(1+\tau_t^c)P_t}{(1+\tau_{t+1}^c)P_{t+1}}\right] \tag{25}$$

***Comment:*** *Do you see why? The argument is simple. Since there is a tax on the consumption on each period. You can think of it as that the consumer is facing a higher price level (or* $1+\tau_t^c)P$ *for any time t). But for the sake of those who might not be so convinced and are interested in the mathematics behind. We can go through the toils again.*

The problem of the consumer is:

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to:} \quad \left\{\underbrace{(1+\tau_t^c)P_t c_t + S_t\, a_t}_{outflows} = \underbrace{S_t a_{t-1} + (1-\tau_t^D)D_t\, a_{t-1} + Y_t}_{inflows}\right\}_{t=0}^{\infty} \tag{26}$$

Lagrangian, using the summation, is:

$$\mathcal{L}(c_0, c_1, \dots; a_0, a_1, \dots; \lambda_0, \lambda_1 \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$- \sum_{t=0}^{\infty} \lambda_t \beta^t \{(1+\tau_t^c)P_t\, c_t + S_t a_t - S_t a_{t-1} - (1-\tau_t^D)D_t a_{t-1} - Y_t\} \tag{27}$$

The expanded Lagrangian is:

$$
\begin{aligned}
&\mathcal{L}(\dots) = u(c_0) + \beta^1 u(c_1) + \beta^2 u(c_2) + \cdots \\
&-\lambda_0 \beta^0 [(1+\tau_0^c) P_0\, c_0 + S_0 \boldsymbol{a_0} - S_0 a_{-1} - (1-\tau_0^D) D_0 a_{-1} - Y_0] \\
&-\lambda_1 \beta^1 [(1+\tau_1^c) P_1\, c_1 + S_1 a_1 - S_1 \boldsymbol{a_0} - (1-\tau_1^D) D_1 \boldsymbol{a_0} - Y_1] \\
&- \cdots \\
&- \lambda_t \beta^t \quad [(1+\tau_t^c) P_t\, c_t + S_t \boldsymbol{a_t} - S_t a_{\boldsymbol{t-1}} - (1-\tau_t^D) D_t a_{\boldsymbol{t-1}} - Y_t] \\
&-\lambda_{t+1} \beta^{t+1} [(1+\tau_{t+1}^c) P_{t+1} c_{t+1} + S_{t+1} a_{t+1} - S_{t+1} \boldsymbol{a_t} - (1-\tau_{t+1}^D) D_{t+1} \boldsymbol{a_t} - Y_{t+1}] \\
&\qquad\qquad\qquad\qquad - \cdots
\end{aligned} \tag{28}
$$

Now let's get the first order conditions (FOCs) with respect to the variables at any time period - say period $t$. We have:

$$\mathcal{L}_{c_t}(\dots) = \beta^t u_1(c_t) - \lambda_t \beta^t (1+\tau_t^c) P_t \ = 0 \tag{29}$$

$$\text{and } \mathcal{L}_{a_t}(\dots) = -\lambda_t \beta^t S_t - \{-\lambda_{t+1} \beta^{t+1} [S_{t+1} + (1-\tau_{t+1}^D) D_{t+1}]\} = 0 \tag{30}$$

$$\mathcal{L}_{c_{t+1}}(\dots) = \beta^{t+1} u_1(c_{t+1}) - \lambda_{t+1} \beta^{t+1} (1+\tau_{t+1}^c) P_{t+1} = 0 \tag{31}$$

From (30), we get:

$$\lambda_t S_t = \lambda_{t+1} \beta [S_{t+1} + (1-\tau_{t+1}^D) D_{t+1}]$$

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{\beta [S_{t+1} + (1-\tau_{t+1}^D) D_{t+1}]}{S_t} \tag{32}$$

From (29) and (31), we get:

$$
\begin{aligned}
\frac{\beta^t u_1(c_t)}{\beta^{t+1} u_1(c_{t+1})} &= \frac{\lambda_t \beta^t (1+\tau_t^c) P_t}{\lambda_{t+1} \beta^{t+1} (1+\tau_{t+1}^c) P_{t+1}} \\
\frac{u_1(c_t)}{u_1(c_{t+1})} &= \frac{\lambda_t}{\lambda_{t+1}} \frac{(1+\tau_t^c) P_t}{(1+\tau_{t+1}^c) P_{t+1}}
\end{aligned} \tag{33}
$$

Substituting (32) into (33), we get:

$$
\begin{aligned}
\frac{u_1(c_t)}{u_1(c_{t+1})} &= \frac{\beta [S_{t+1} + (1-\tau_{t+1}^D) D_{t+1}]}{S_t} \frac{(1+\tau_t^c) P_t}{(1+\tau_{t+1}^c) P_{t+1}} \\
S_t &= \beta \left[ \frac{u_1(c_{t+1})}{u_1(c_t)} [S_{t+1} + (1-\tau_{t+1}^D) D_{t+1}] \frac{(1+\tau_t^c) P_t}{(1+\tau_{t+1}^c) P_{t+1}} \right]
\end{aligned} \tag{34}
$$

If we are interested in how today's stock is being priced, just replace $t$ with 0 to get:

$$S_0 = \beta \left[ \frac{u_1(c_1)}{u_1(c_0)} [S_1 + (1 - \tau_1^D) D_1] \frac{(1 + \tau_0^c) P_0}{(1 + \tau_1^c) P_1} \right] \tag{35}$$

***Comment:*** For the second part of part c of the question, I treat it as a discussion (for theoretical exercise) because the information of the model is not enough to give a clean solution to this part*:*

*<u>Main answer:</u> From (35), <u>all else equal</u>, an one-off increase in consumption tax rate in period t, $\tau_t^c$, the higher the stock price in period t, $S_t$ . The intuition is that increase in period t consumption tax relative to that of period t+1 would make the consumption in period t costlier than the consumption in period t+1. Substitution effect implies that consumers will consume less in period t, and more in period t+1. Thus, consumers would consume less and save more. The saving in this model setting is through the investments in stocks - the only asset in this economy). Higher demand for stock would drive the stock price, $S_t$, up.*

*<u>Here comes the tricky part:</u> What about the income effect? That is, since the consumers are taxed, they will become poorer and thus their savings in real term which is $y_t - c_t \cdot (1 + \tau_t^c + \Delta)$ can in fact be lower than before the change in tax. However, we are not told what the government does with the tax revenues.*

*If we assume that the government does not change its spending in period 1, then the national savings in real term will be*

$$y_t - c_t \cdot (1 + \tau_t^c + \Delta) + \left( \underbrace{c_t \cdot (\tau_t^c + \Delta)}_{tax\ revenue} - g_t \right) = y_t - \underbrace{c_t}_{\substack{falls\ because\ of \\ consumption \\ tax\ increase}} - \underbrace{g_t}_{\substack{assumed\ to \\ be\ unchanged}} .$$

*Because of consumption tax increase, $c_t$ drops and thus the national savings, $y_t - c_t - g_t$ , increases.*

*All savings have to be invested and since we only have stocks as investment assets, more savings are put in stocks and thus $S_t$ increases. NOTE: the phrase "all else equal" in the Main Answer also means $g_t$ does not change.*

*However, if we assume that the government spending ALSO changes <u>by the same amount</u> as tax collection change, the national savings might decrease, causing $S_t$ to decrease.*

***Thus, since the model does not involve government, the analysis of this part cannot be complete. Thus, I leave it as a theoretical exercise****.*