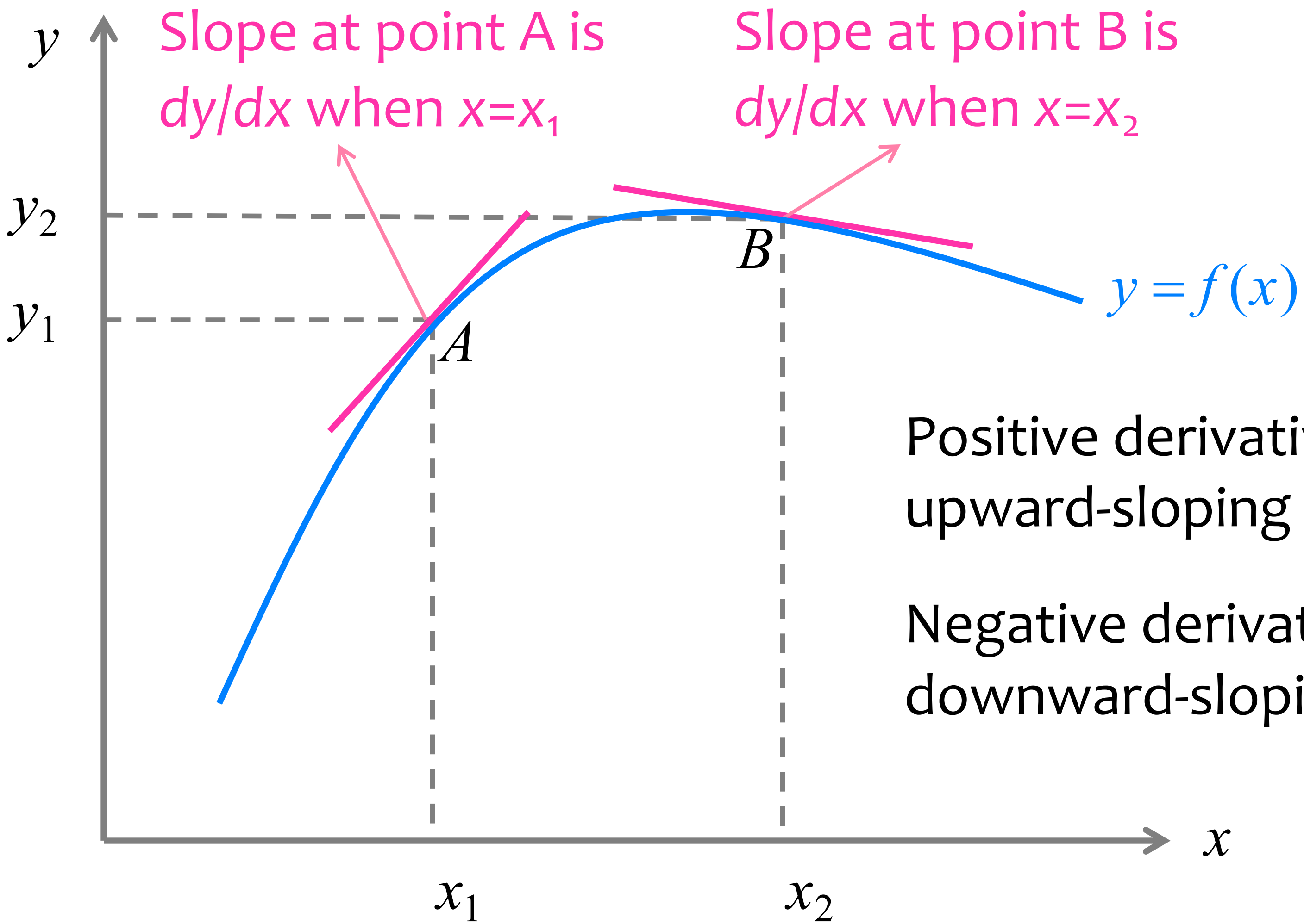
EC2101: Microeconomic Analysis I

Lecture 0

Optimization

- Unconstrained Optimization
- Constrained Optimization

Review: Derivative and Slope



Positive derivative:
upward-sloping curve

Negative derivative:
downward-sloping curve

Unconstrained Optimization with One Variable

Unconstrained Optimization with One Variable

- What is the maximum of the following function?

$$y = -x^2 + 2x + 10$$

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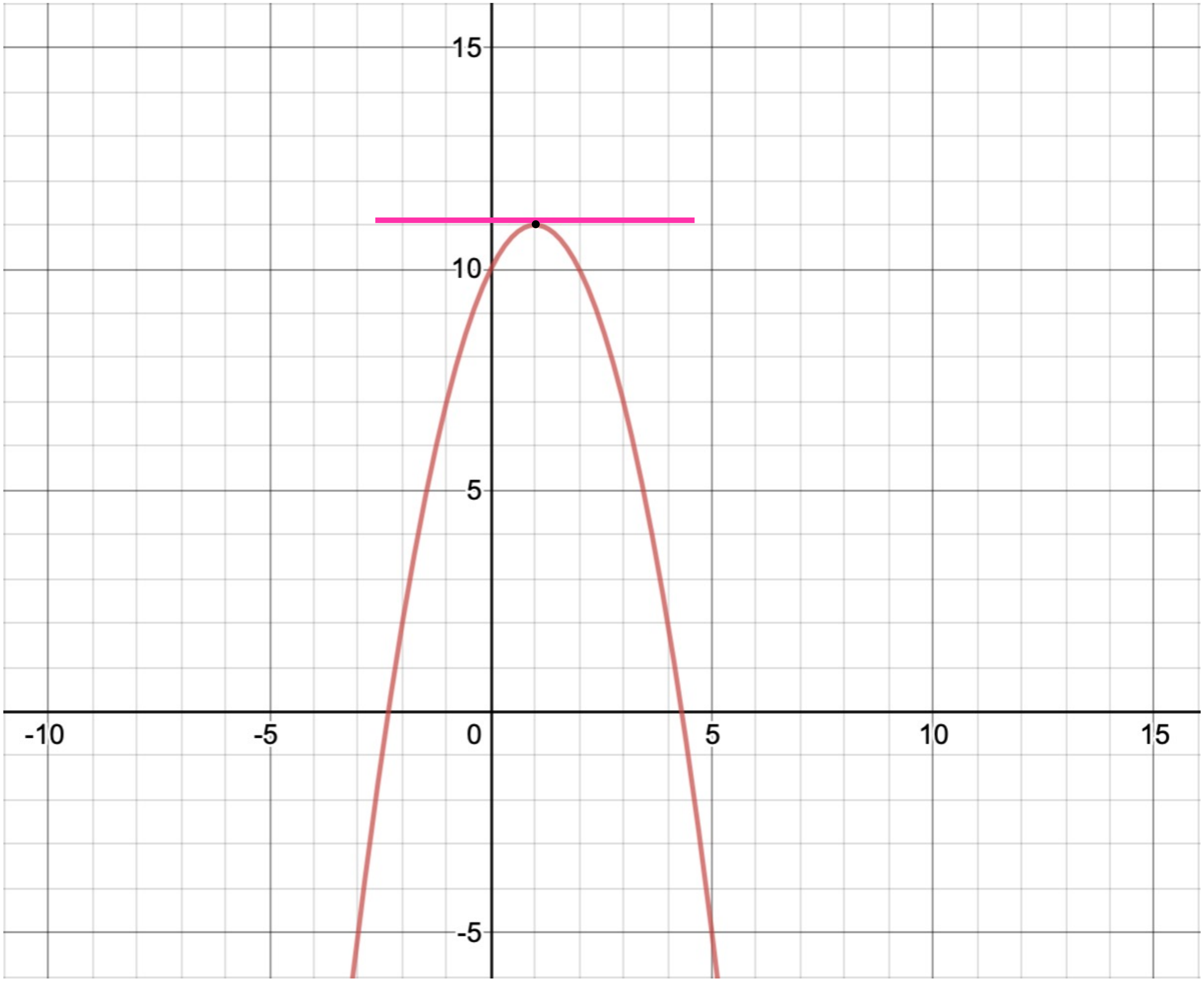


Unconstrained Optimization with One Variable

- What is the maximum of the following function?

$$y = -x^2 + 2x + 10$$

- At the maximum, the slope of the function must be zero.

- First-order condition:

$$\frac{dy}{dx} = 0$$

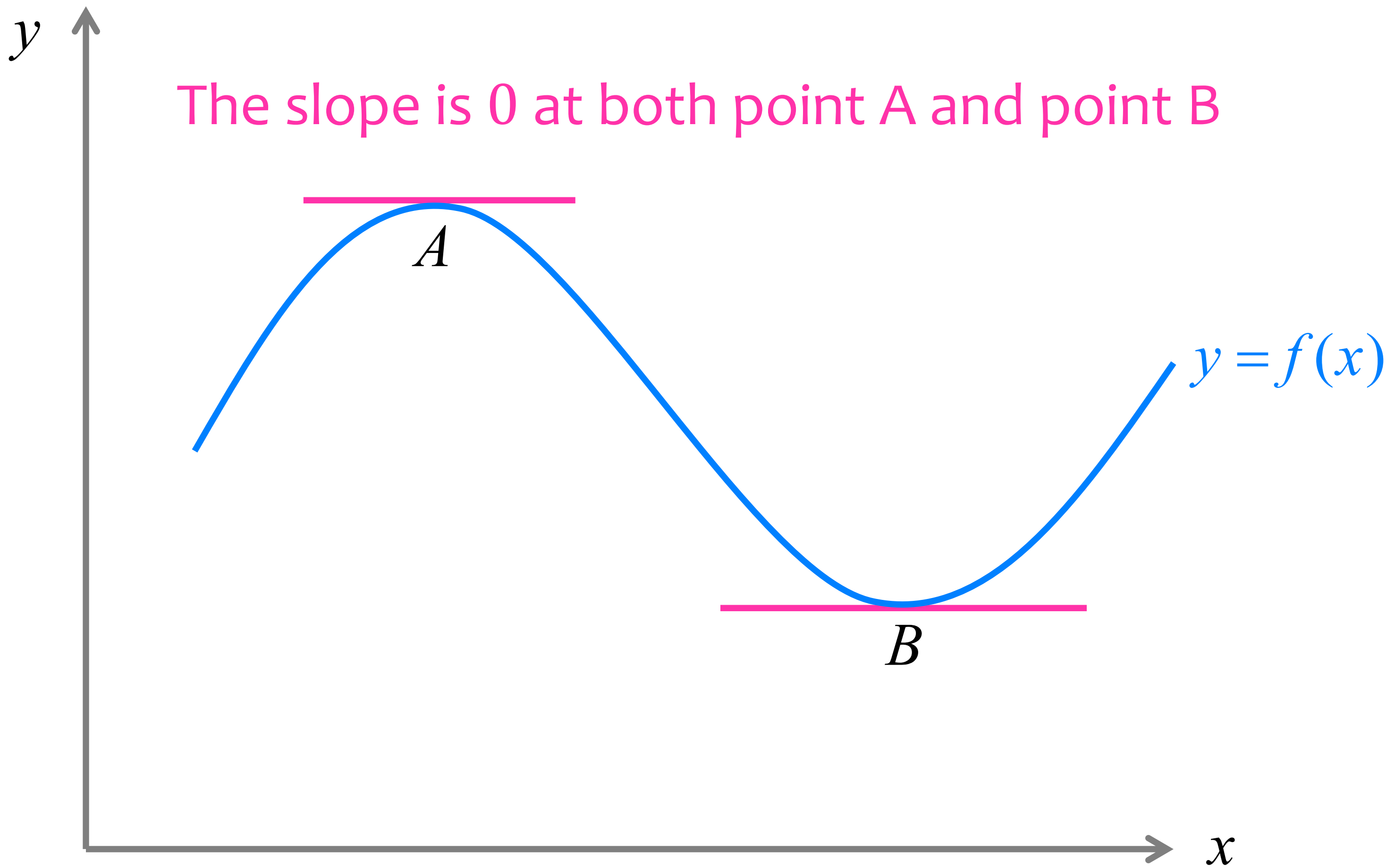
$$-2x + 2 = 0$$

- At the maximum:

$$x = 1$$

$$y = 11$$

Minimum vs. Maximum



Second-Order Condition

- At the maximum:

$$\frac{d^2y}{dx^2} \leq 0$$

- At the minimum:

$$\frac{d^2y}{dx^2} \geq 0$$

- Using our earlier example:

$$\frac{d^2y}{dx^2} = \frac{d(-2x+2)}{dx} = -2$$

Unconstrained Optimization with Two Variables

Unconstrained Optimization with Two Variables

- Suppose you want to find the maximum of:

$$f(x, y) = -x^2 + 2x - y^2 + 4y + 5$$

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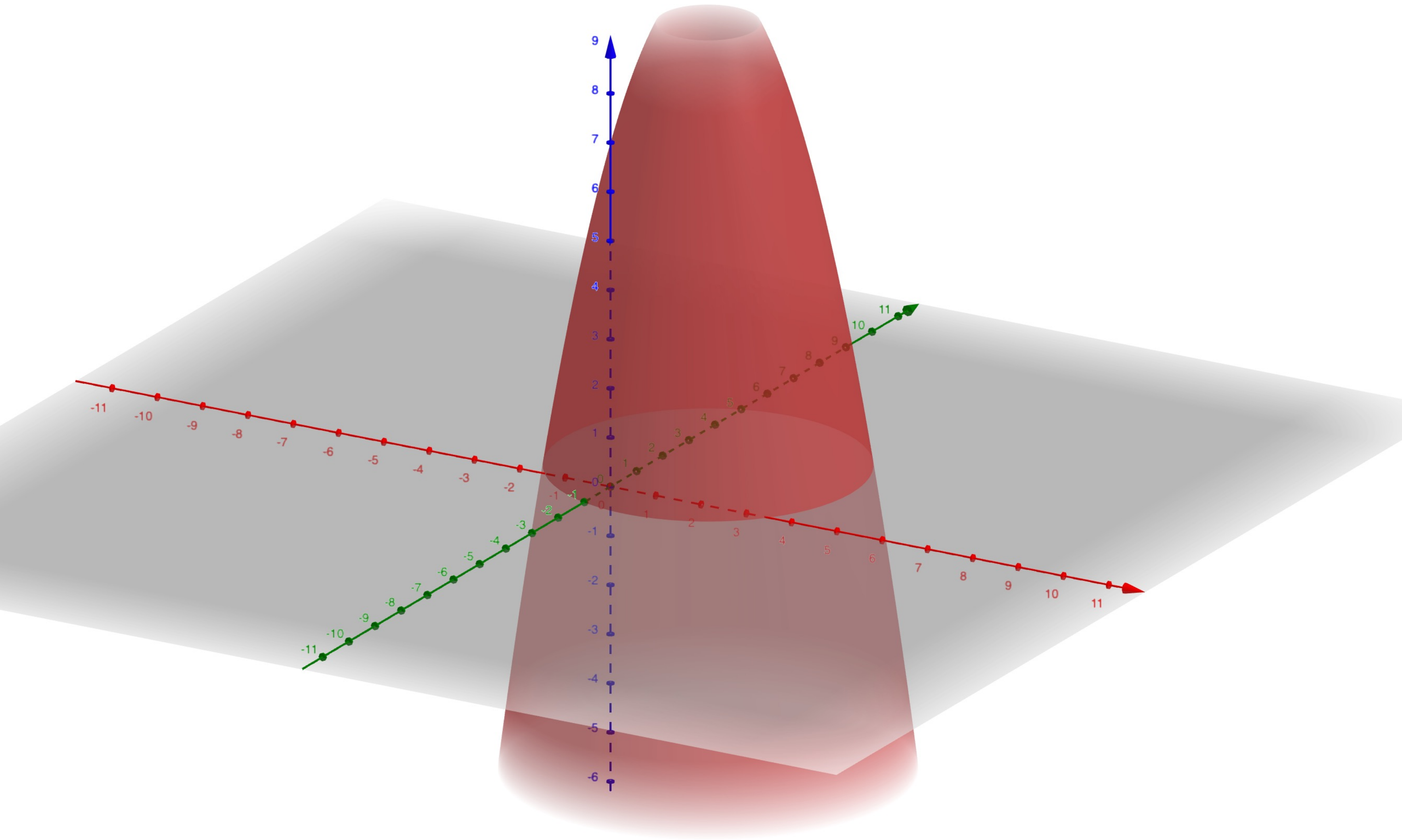


Unconstrained Optimization with Two Variables

- Suppose you want to find the maximum of:

$$f(x,y) = -x^2 + 2x - y^2 + 4y + 5$$

- First-order conditions:

$$\frac{\partial f}{\partial x} = -2x + 2 = 0$$

$$\frac{\partial f}{\partial y} = -2y + 4 = 0$$

- Solving for the two equations, $x = 1$ and $y = 2$.

- The maximum value of the function is $f = 10$.

Constrained Optimization with Two Variables

Constrained Optimization with Two Variables

- Suppose you want to find the maximum of:

$$f(x,y) = -x^2 + 2x - y^2 + 4y + 5$$

- But now you have to satisfy another equation:

$$x + y = 1$$

Constrained Optimization with Two Variables

- This is a constrained maximization problem.
  - The objective function is:

$$f(x, y) = -x^2 + 2x - y^2 + 4y + 5$$

  - The constraint is:

$$x + y = 1$$

Unconstrained Optimization with Two Variables

- Suppose you want to find the maximum of:

$$f(x, y) = -x^2 + 2x - y^2 + 4y + 5$$

- Solving for the two equations, $x = 1$ and $y = 2.$

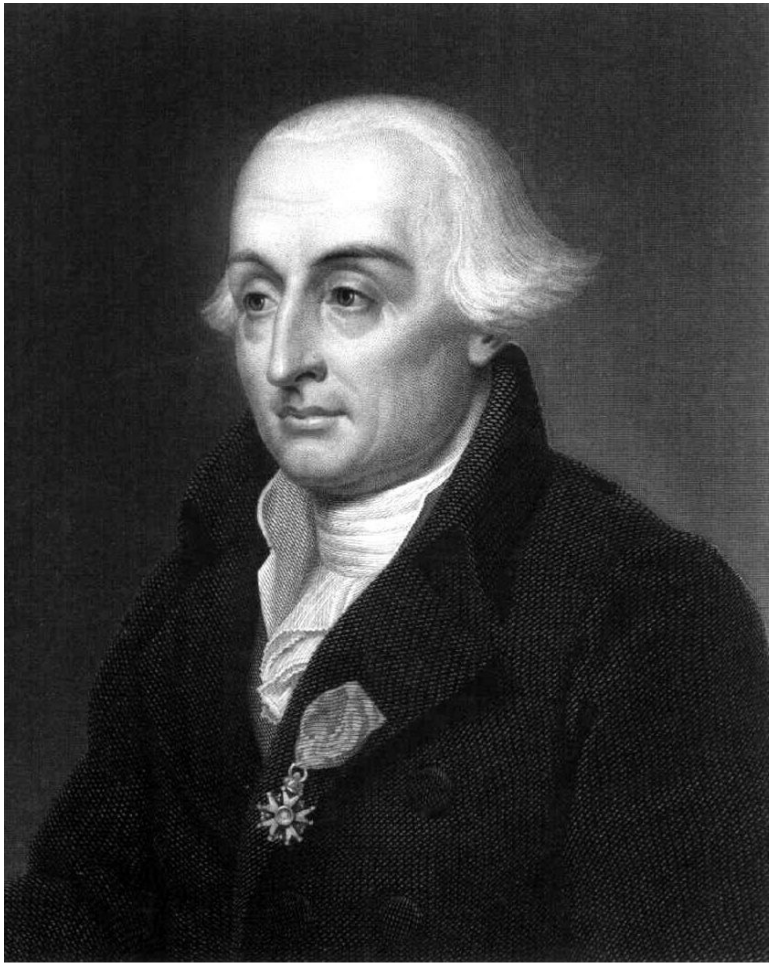
- The maximum value of the function is $f = 10.$

- When we introduce a constraint of:

$$x + y = 1$$

  - The solution is no longer $x = 1$ and $y = 2.$

Joseph-Louis Lagrange



Joseph-Louis Lagrange
1736–1813

- Baptized Giuseppe Lodovico Lagrangia.
- Italian-French mathematician and astronomer.
- Made great contributions to analysis, number theory, and classical and celestial mechanics.

Lagrange Multiplier Method

- We first rewrite the constraint as:

$$1 - x - y = 0$$

- We then construct the Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \lambda(1 - x - y)$$

- The new unknown λ is the Lagrange multiplier.
- To solve the constrained maximization problem, we just need to maximize the Lagrangian function.

Lagrange Multiplier Method

- Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \lambda(1 - x - y)$$

- First-order conditions:

$$\frac{\partial \Lambda}{\partial x} = -2x + 2 - \lambda = 0$$

$$\frac{\partial \Lambda}{\partial y} = -2y + 4 - \lambda = 0$$

$$\frac{\partial \Lambda}{\partial \lambda} = 1 - x - y = 0$$

Lagrange Multiplier Method

- Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \lambda(1 - x - y)$$

- Solving the three equations, $x = 0, y = 1,$ and $\lambda = 2.$
- The maximum value of the function is $\Lambda = 8.$

Lagrange Multiplier Method: General Form

- The constrained optimization problem is:

$$\max_{x,y} f(x, y)$$

$$subject\ to\ g(x, y) = 0$$

  - $f(x, y)$ is the objective function.
  - $g(x, y)$ is the constraint.

Lagrange Multiplier Method: General Form

- The Lagrangian function is:

$$\Lambda(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

- First-order conditions:

$$\frac{\partial \Lambda}{\partial x} = \frac{\partial f(x, y)}{\partial x} + \lambda \frac{\partial g(x, y)}{\partial x} = 0$$

$$\frac{\partial \Lambda}{\partial y} = \frac{\partial f(x, y)}{\partial y} + \lambda \frac{\partial g(x, y)}{\partial y} = 0$$

$$\frac{\partial \Lambda}{\partial \lambda} = g(x, y) = 0$$

- Solve for x, y, and λ.

Exercise 0.1

Lagrange Multiplier Method

- Instead of writing the constraint as

  $$1 - x - y = 0,$$

  can we rewrite the constraint as

  $$x + y - 1 = 0\,?$$

- Maximize the Lagrangian function:

  $$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \lambda(x + y - 1)$$

- Solve for x, y, and λ.

Exercise 0.1

Lagrange Multiplier Method

- Earlier, we maximized the Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \boxed{\lambda(1 - x - y)}$$

- First-order conditions:

$$\frac{\partial \Lambda}{\partial x} = -2x + 2 \boxed{-\lambda} = 0$$

$$\frac{\partial \Lambda}{\partial y} = -2y + 4 \boxed{-\lambda} = 0$$

$$\frac{\partial \Lambda}{\partial \lambda} = \boxed{1 - x - y = 0}$$

Exercise 0.1

Lagrange Multiplier Method

- Earlier, we maximized the Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \boxed{\lambda(1 - x - y)}$$

- Solving the three equations, $x = 0, y = 1,$ and $\boxed{\lambda = 2.}$
- The maximum value of the function is $\Lambda = 8.$

Exercise 0.1

Lagrange Multiplier Method

- In this exercise, we maximize the Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \boxed{\lambda(x + y - 1)}$$

- First-order conditions:

$$\frac{\partial \Lambda}{\partial x} = -2x + 2 \boxed{+ \lambda} = 0$$

$$\frac{\partial \Lambda}{\partial y} = -2y + 4 \boxed{+ \lambda} = 0$$

$$\frac{\partial \Lambda}{\partial \lambda} = \boxed{x + y - 1 = 0}$$

Exercise 0.1

Lagrange Multiplier Method

- In this exercise, we maximize the Lagrangian function:

$$\Lambda(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \boxed{\lambda(x + y - 1)}$$

- Solving the three equations, $x = 0, y = 1,$ and $\boxed{\lambda = -2.}$

- The maximum value of the function is $\Lambda = 8.$