LECTURE 12

Review

EC3333 Financial Economics I

Topics covered in this course

- Portfolio theory
  - Capital Asset Pricing Model (CAPM)
- Valuation Models of Financial Assets/Securities
  - Fixed-Income Securities: Bonds/Annuities/Perpetuities
  - Derivative Securities: Options

L1: Learning Objectives

- Recognise the roles of financial markets in an economy and the differences between financial and real assets.
- Identify the major components of the investment process.
- Compute the realized or total return for an investment.
- Estimate expected return, variance, and standard deviation (or volatility) of returns using the empirical distribution of realized returns.

Table 10.2 Realized Return for the S&P 500, Microsoft, and Treasury Bills, 2005–2017
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

Year End	S&P 500 Index	Dividends Paid*	S&P 500 Realized Returned	Microsoft Realized Return	1-Month T-Bill Return
2004	1211.92				
2005	1248.29	23.15	4.9 %	−0.9 %	3 %
2006	1418.3	27.16	15.8 %	15.8 %	4.8 %
2007	1468.36	27.86	5.5 %	20.8 %	4.7 %
2008	903.25	21.85	−37 %	−44.4 %	1.5 %
2009	1115.1	27.19	26.5 %	60.5 %	0.1 %
2010	1257.64	25.44	15.1 %	−6.5 %	0.1 %
2011	1257.61	26.59	2.1 %	−4.5 %	0 %
2012	1426.19	32.67	16 %	5.8 %	0.1 %
2013	1848.36	39.75	32.4 %	44.3 %	0 %
2014	2058.9	42.47	13.7 %	27.6 %	0 %
2015	2043.94	43.45	1.4 %	22.7 %	0 %
2016	2238.83	49.56	12 %	15.1 %	0.2 %
2017	2673.61	53.99	21.8 %	40.7 %	0.8 %

Total dividends paid by the 500 stocks in the portfolio, based on the number of shares of each stock in the index, adjusted until the end the year, assuming they were reinvested when paid.
Source: Standard & Poor's, Microsoft and U.S. Treasury Data

Average Annual Return

$$\bar{R} = \frac{1}{T}(R_1 + R_2 + \cdots + R_T) = \frac{1}{T}\sum_{t=1}^{T} R_t$$

- Where R_t is the realized return of a security in year t, for the years 1 through T
- Using the data from Table 10.2, the average annual return for the S&P 500 from 2005 to 2017 is as follows:

$$\bar{R} = \frac{1}{13}(0.049 + 0.158 + 0.055 - 0.37 + 0.265 + 0.151 + 0.021 + 0.160 + 0.324 + 0.137 + 0.014 + 0.120 + 0.218) = 10.0\%$$

- $\bar{R}$ is the **Arithmetic Average Return**

Arithmetic vs. Geometric Average Rates of Return

- We should use the arithmetic average return when we are tying to estimate an investment's expected return over a future horizon based on it past performance
- If past returns are independent draws from the same distribution, then the arithmetic average return provides an unbiased estimate of the true expected return
- But the arithmetic average rate of return fails to capture the effect of compounding (earning interest on interest)
- To capture the effect of compounding, we need to use the geometric average rate of return
- **Geometric Average Return**

$$= [(\mathbf{1} + \boldsymbol{R_1})(\mathbf{1} + \boldsymbol{R_2}) \dots (\mathbf{1} + \boldsymbol{R_T})]^{\mathbf{1}/\boldsymbol{T}} - \mathbf{1}$$

The Variance and Volatility of Returns

- Variance Estimate Using Realized Returns

$$\boldsymbol{Var(R) = \frac{1}{T-1}\sum_{t=1}^{T}(R_t - \bar{R})^2}$$

- The estimate of the standard deviation (volatility) is the square root of the variance
- Earlier, we calculated the average annual returns of the S&P 500 from 2005 to 2017 to be 10.0%. Therefore,

$$\boldsymbol{Var(R) = \frac{1}{T-1}\sum_{t}(R_t - \overline{R})^2}$$

$$\boldsymbol{= \frac{1}{13-1}[(0.049-0.100)^2 + (0.158-0.100)^2 + \ldots + (0.218 - 0.100)^2]}$$

$$\boldsymbol{= 0.029}$$

$\boldsymbol{The\ volatility\ or\ standard\ deviation\ is\ therefore}$

$$\boldsymbol{SD(R) = \sqrt{Var(R)} = \sqrt{0.029} = 17.0\%}$$

Table 10.5 Volatility Versus Excess Return of U.S. Small Stocks, Large Stocks (S&P 500), Corporate Bonds, and Treasury Bills, 1926–2017 (from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)

Investment	Return Volatility (Standard Deviation)	Excess Return (Average Return in Excess of Treasury Bills)
Small stocks	39.2%	15.3%
S&P 500	19.8%	8.6%
Corporate bonds	6.4%	2.9%
Treasury bills (30-day)	3.1%	0.0%

Using Past Returns to Predict the Future

- We can use a security's historical average return to estimate its actual expected return. However, the average return is just an estimate of the expected return.
- **Standard Error**
  - A statistical measure of the degree of estimation error
- Recall that for S&P 500 from 1926-2017 (a sample of 92 years of observations for annual realized returns)
  - N = 92
  - This sample has a mean $\bar{R}$ of 12.0% and a standard deviation $SD(R)$ of 19.8% (See Table 10.3 & 10.5)
  - Standard Error of Average Return:

$$= \frac{\boldsymbol{SD}}{\sqrt{\boldsymbol{N}}} = \frac{\mathbf{19.8\%}}{\sqrt{\mathbf{92}}}$$

Using Past Returns to Predict the Future

- Standard Error of the Estimate of the Expected Return

$$\begin{aligned}&\boldsymbol{SD(Average\ of\ Independent, Identical\ Risks)}\\&= \frac{\boldsymbol{SD(Individual\ Risk)}}{\sqrt{\boldsymbol{Number\ of\ Observations}}}\end{aligned}$$

- 95% Confidence Interval

$$\boldsymbol{Historical\ Average\ Return \pm (2 \times Standard\ Error)}$$

- For the S&P 500 (1926–2017)

$$\boldsymbol{12.0\% \pm 2\left(\frac{19.8\%}{\sqrt{92}}\right) = 12.0\% \pm 4.1\%}$$

- Or a range from 7.9% to 16.1%

Active Recall: The Accuracy of Expected Return Estimates

Problem

- Using the returns for the S&P 500 from 2005 to 2017 only (see Table 10.2), what is the 95% confidence interval you would estimate for the S&P 500's expected return?

The Accuracy of Expected Return Estimates

Solution

- Earlier, we calculated the average return for the S&P 500 during this period to be 10.0%, with a volatility of 17.0%
- The standard error of our estimate of the expected return is

$$\mathbf{17.0\%} \div \sqrt{\mathbf{13}} = \mathbf{4.7\%}$$

- The 95% confidence interval is 10.0% ± (2 × 4.7%), or from 0.6% to 19.4%.

- How does this range compare to the range for the period from 1926 to 2017?
  - With only a few years of data, we cannot reliably estimate expected returns for stocks!

Note: Using Excel

Sourced from:

https://support.microsoft.com/en-us/office

- **STDEV.S** assumes that its arguments are a **sample** of the population. The standard deviation is calculated using the "n-1" method.
- If your data represents the entire **population**, then compute the standard deviation using **STDEV.P**.
- **COVARIANCE.S** function in Microsoft Excel returns the **sample** covariance, the average of the products of deviations for each data point pair in two data sets.
- **COVARIANCE.P** function in Microsoft Excel returns **population** covariance, the average of the products of deviations for each data point pair in two data sets.

L2: Learning Objectives

- Given a portfolio of stocks, including the holdings in each stock, the expected return in each stock, and its variance, compute the following:
  - Expected return on the portfolio;
  - Variance and volatility of the portfolio; and
  - Covariance and correlation of each pair of stocks in the portfolio.
- Define an efficient portfolio and the efficient frontier.
- Describe short sales and how it extends the efficient frontier.
- Define and contrast idiosyncratic (firm specific) and systematic (market) risk.
- Discuss the limits to diversification.

Portfolios of Risky Assets

- Portfolio Weights
  - The fraction of the total investment in the portfolio held in each individual investment in the portfolio.
  - The portfolio weights must add up to 1.

$$x_i = \frac{\text{Value of investment } i}{\text{Total value of portfolio}} = \frac{\text{Value of investment } i}{\sum_i \text{ Value of investment } i}$$

- The return on the portfolio R_p is the weighted average of the returns on the investments in the portfolio.

$$R_P = x_1 R_1 + x_2 R_2 + \ldots + x_n R_n = \sum_i x_i R_i$$

$$E[R_P] = E\left[\sum_i x_i R_i\right] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i]$$

Covariance

- More generally, covariance is the expected product of the deviations of two returns from their means.
- Covariance between Returns *Ri* and *Rj*

$$\mathrm{Cov}(R_i, R_j) = E[(R - E[R_i])(R_j - E[R_j])]$$

- Estimate of the Covariance from Historical Data

$$\mathrm{Cov}(R_i, R_j) = \frac{1}{T-1}\sum_t \left(R_{it} - \bar{R}_i\right)\left(R_{j,t} - \bar{R}_j\right)$$

  - If the covariance is positive, the two returns tend to move together.
  - If the covariance is negative, the two returns tend to move in opposite directions.
  - But high covariance could be due to the returns being more volatile or the returns moving more closely together

Covariance and Correlation

- Covariance of returns on bond and equity

$$\text{Cov}(r_D r_E) = \rho_{DE}\sigma_D\sigma_E$$

  - ρ_{DE} = Correlation coefficient of returns
    - A measure of the common risk shared by assets that does not depend on their volatility
  - σ_D = Standard deviation of bond returns
  - σ_E = Standard deviation of equity returns

- Range of values for the correlation coefficient ρ

$$-1 \leq \rho \leq 1$$

  - If $\rho = 1$, the securities are perfectly positively correlated
  - If $\rho = 0$, the securities are uncorrelated
  - If $\rho = -1$, the securities are perfectly negatively correlated

The Volatility of a Portfolio of Stocks

- The variance of a portfolio is equal to the weighted average covariance of each stock with the portfolio

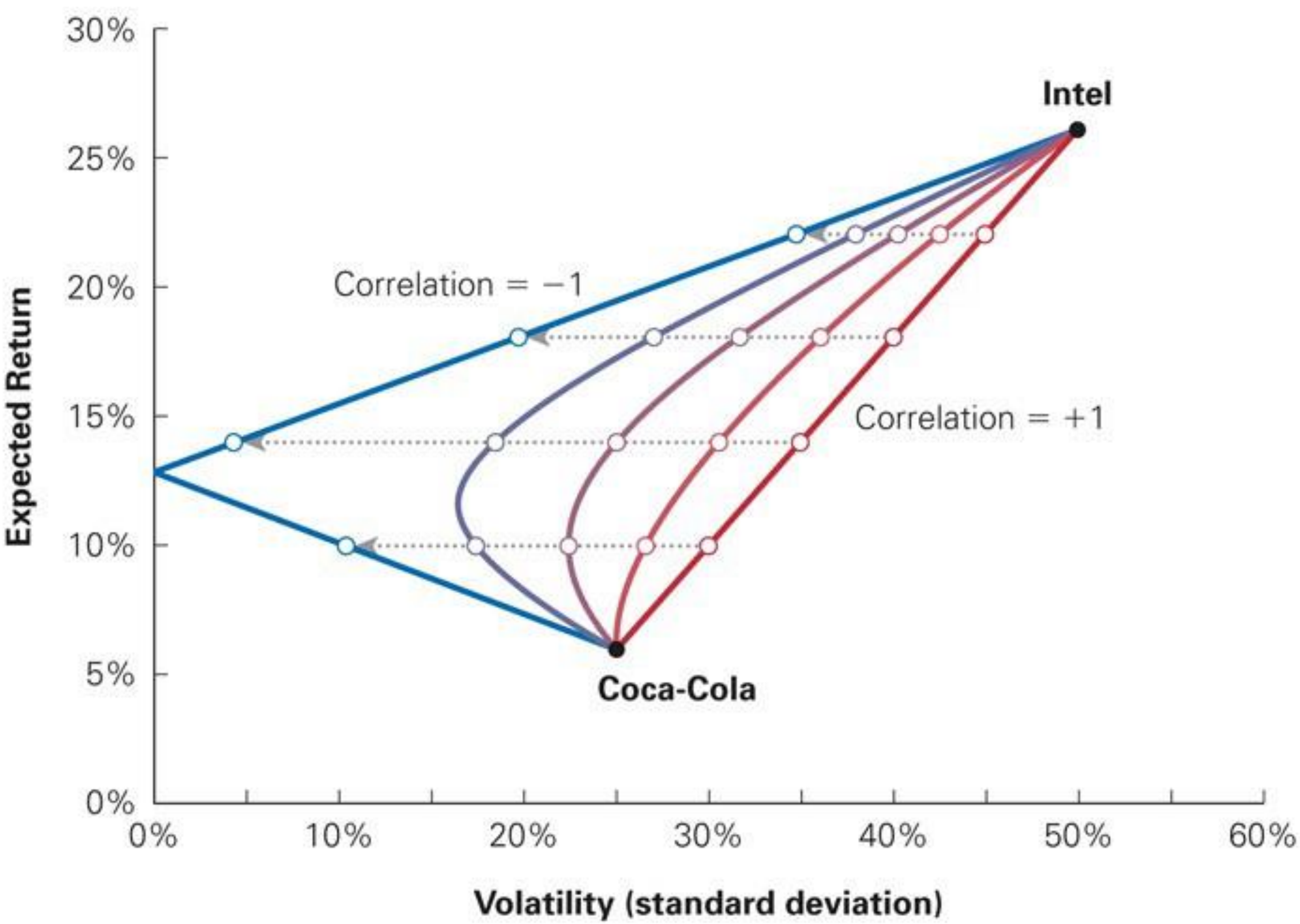
$$\mathrm{Var}(R_P) = \mathrm{Cov}(R_P, R_P) = \mathrm{Cov}\left(\sum_i x_i R_i, R_P\right) = \sum_i x_i \,\mathrm{Cov}(R_i, R_P)$$

- The risk of a portfolio depends on how each stock's return moves in relation to it

$$\begin{aligned}\mathrm{Var}(R_P) &= \sum_i x_i \,\mathrm{Cov}(R_i, R_p) = \sum_i x_i \,\mathrm{Cov}(R_i, \Sigma_j x_j R_j) \\ &= \sum_i \sum_j x_i x_j \,\mathrm{Cov}(R_i, R_j)\end{aligned}$$

- The overall variability of the portfolio depends on the total co-movement of the stocks within it

Figure 11.4 Effect on Volatility and Expected Return of Changing the Correlation between Intel and Coca-Cola Stock
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)



Efficient vs. Inefficient Portfolios

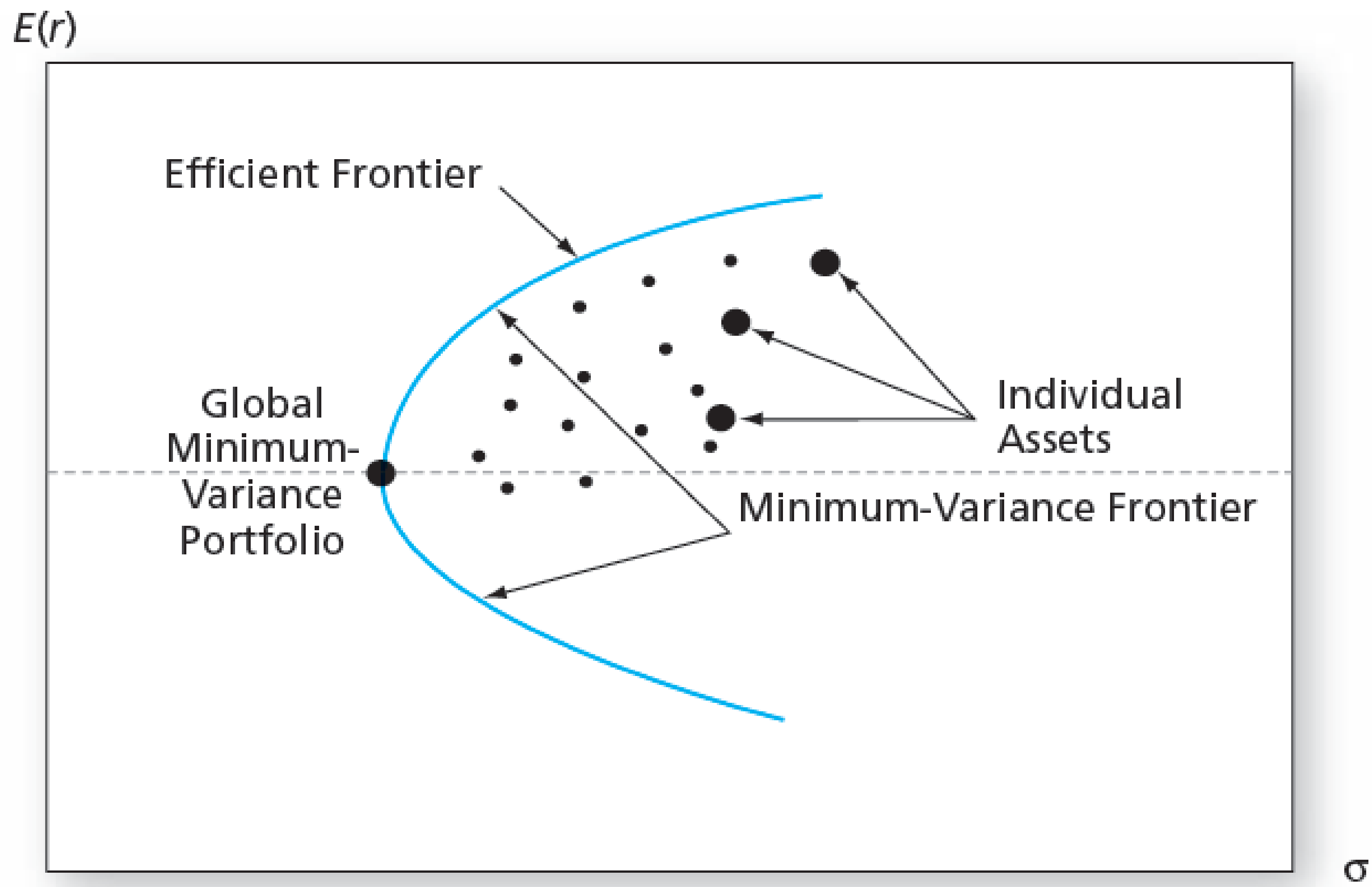
- In an inefficient portfolio, it is possible to find another portfolio that is better in terms of both expected return and volatility.
- The efficient portfolios are the portfolios that offer the highest possible expected return for a given level of volatility
- In other words, in an efficient portfolio there is no way to reduce the volatility of the portfolio without lowering its expected return.

Risk Versus Return: Many Stocks

- The minimum-variance frontier plots the lowest variance that is attainable for every portfolio expected return.

- The efficient portfolios are those on the northwest edge of the minimum-variance frontier, which is called the efficient frontier (of risky assets).
  - It consists of the portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward.

Recall: Figure 7.10 The Efficient Frontier with Multiple Risky Assets
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



The Minimum Variance Portfolio with Two Risky Assets: Derivation

- To minimize portfolio variance, differentiate wrt x_D

$$\sigma_p^2 = x_D^2\sigma_D^2 + (1 - x_D)^2\sigma_E^2 + 2x_D(1 - x_D)\rho_{DE}\sigma_D\sigma_E$$

- Set the derivative equal to zero and solve for x_D

$$x_D^* = \frac{\sigma_E^2 - \rho_{DE}\sigma_D\sigma_E}{\sigma_D^2 + \sigma_E^2 - 2\rho_{DE}\sigma_D\sigma_E}$$

$$x_E^* = 1 - x_D$$

Recall Tutorial 3

Finding the portfolio with minimum portfolio variance is equivalent to solving the following minimization problem, where $\sigma_{SB} = \rho_{BS}\sigma_B\sigma_S$.

$$\min_{x_S, x_B} \sigma_p^2 = x_S^2\sigma_S^2 + x_B^2\sigma_B^2 + 2x_S x_B \sigma_{SB}$$

$$\text{s.t. } x_S + x_B = 1$$

Using substitution method, with $x_B = 1 - x_S$

We get a minimization problem with only one variable.

$$\min_{x_S} \sigma_p^2 = x_S^2\sigma_S^2 + (1 - x_S)^2\sigma_B^2 + 2x_S(1 - x_S)\sigma_{SB}$$

First order condition of the minimization problem is:

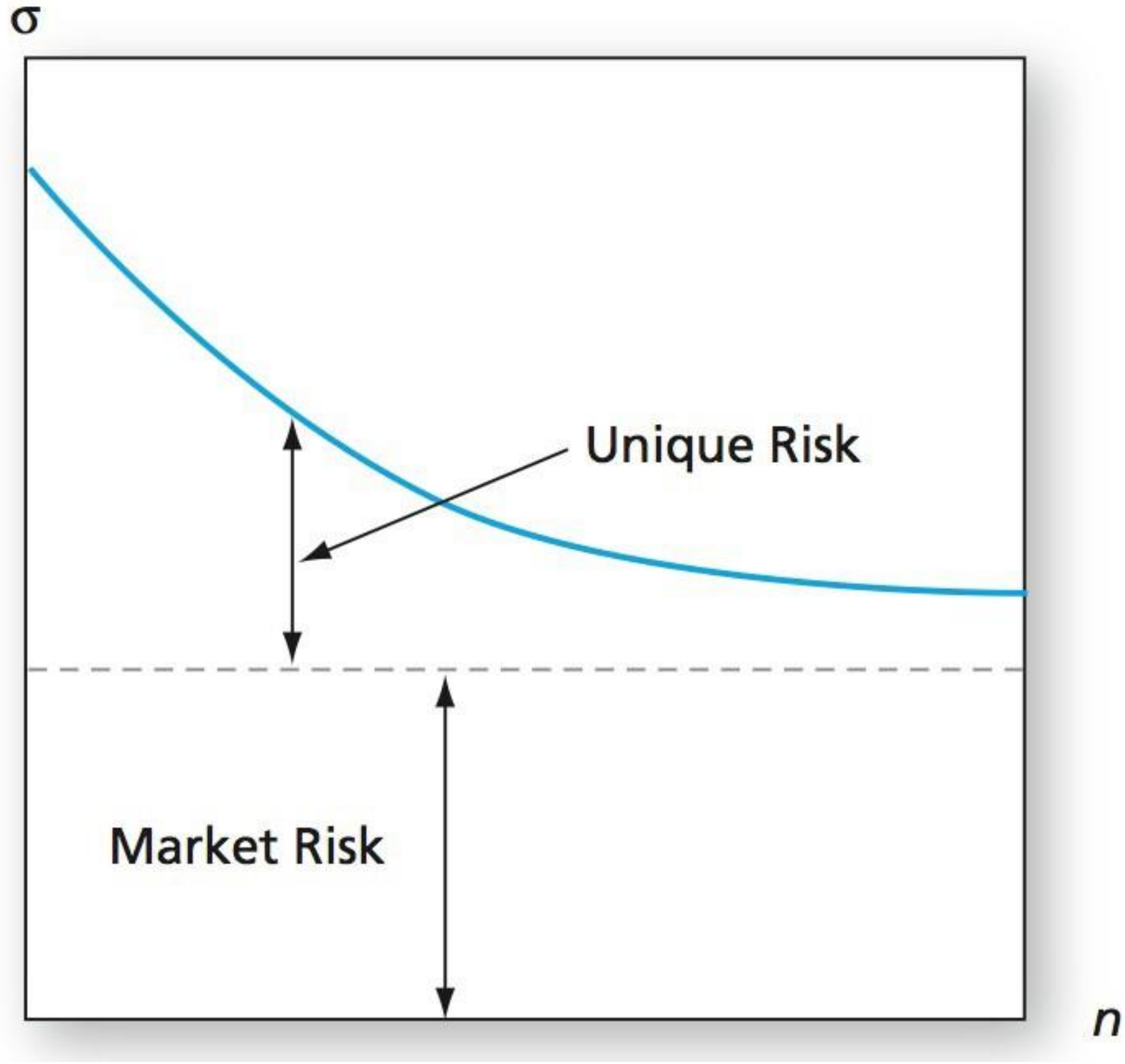
$$\begin{aligned} 0 &= \frac{d}{dx_S}\sigma_p^2 = \frac{d}{dx_S}\left(x_S^2\sigma_S^2 + (1 - x_S)^2\sigma_B^2 + 2x_S(1 - x_S)\sigma_{SB}\right) \\ &= 2x_S\sigma_S^2 - 2(1 - x_S)\sigma_B^2 + 2\sigma_{SB}(1 - 2x_S) \\ &\Rightarrow x_S^{min} = \frac{\sigma_B^2 - \sigma_{SB}}{\sigma_S^2 + \sigma_B^2 - 2\sigma_{SB}}, \qquad x_B^{min} = 1 - x_S^{min} \end{aligned}$$

Diversification & Portfolio Risk

- Market risk
  - Risk attributable to marketwide risk sources and remains even after extensive diversification.
  - Also call systematic or non-diversifiable.

- Firm-specific risk
  - Risk that *can* be eliminated by diversification.
  - Also called diversifiable or nonsystematic.

Figure 7.1 Firm-Specific Risk vs. Market Risk
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



L3: Learning Objectives

- Explain the effect of combining a risk-free asset with a portfolio of risky assets, and compute the expected return and volatility for that combination.
- Define the Sharpe ratio, and explain how it helps identify the portfolio with the highest possible expected return for any level of volatility, and how this information can be used to identify the tangency (efficient) portfolio.
- Show how risk aversion can be characterized by a utility function.
- Demonstrate the 2-step process of portfolio construction:
    1. determine of the tangency (efficient) portion of risky assets in complete portfolio, and
    2. allocate capital in the complete portfolio to risk-free versus tangency (efficient) portfolio of risky assets.

Risk and Return with a Risk-Free Asset

- Consider the following complete portfolio where:
  - x = portfolio weight on the risky portfolio, P
  - $(1 - x)$ = portfolio weight on the risk-free asset, F

- The expected return on the complete portfolio:

$$E(r_c) = (1 - x)r_f + xE(r_P)$$
$$E(r_c) = r_f + x[E(r_P) - r_f]$$

- The standard deviation of the complete portfolio:

$$\sigma_C = x\sigma_P$$

c.f. L2: Portfolios of Risky Assets

- Portfolio Weights
  - The fraction of the total investment in the portfolio held in each individual investment in the portfolio.
  - The portfolio weights must add up to 1.

$$x_i = \frac{\text{Value of investment } i}{\text{Total value of portfolio}} = \frac{\text{Value of investment } i}{\sum_i \text{ Value of investment } i}$$

- The return on the portfolio R_p is the weighted average of the returns on the investments in the portfolio.

$$R_P = x_1 R_1 + x_2 R_2 + \ldots + x_n R_n = \sum_i x_i R_i$$

$$E[R_P] = E\left[\sum_i x_i R_i\right] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i]$$

c.f. L2: The Volatility of a Portfolio of Stocks

- The variance of a portfolio is equal to the weighted average covariance of each stock with the portfolio

$$\mathrm{Var}(R_P) = \mathrm{Cov}(R_P, R_P) = \mathrm{Cov}\left(\sum_i x_i R_i, R_P\right) = \sum_i x_i \,\mathrm{Cov}(R_i, R_P)$$

- The risk of a portfolio depends on how each stock's return moves in relation to it

$$\begin{aligned}\mathrm{Var}(R_P) &= \sum_i x_i \,\mathrm{Cov}(R_i, R_p) = \sum_i x_i \,\mathrm{Cov}(R_i, \Sigma_j x_j R_j)\\ &= \sum_i \sum_j x_i x_j \,\mathrm{Cov}(R_i, R_j)\end{aligned}$$

- The overall variability of the portfolio depends on the total co-movement of the stocks within it

Risk and Return with a Risk-Free Asset

- Rearrange and substitute $x = \sigma_C/\sigma_P$

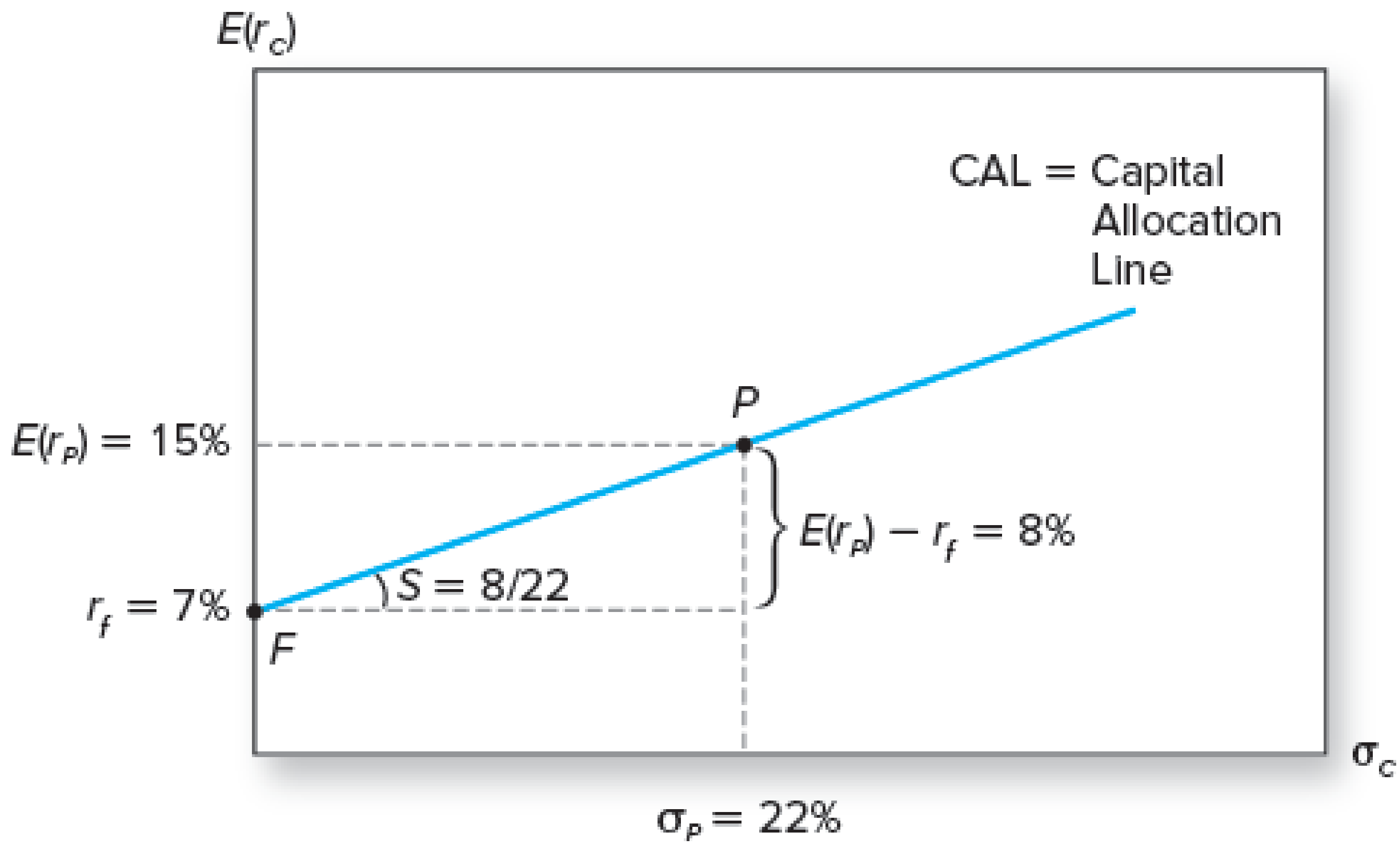
$$E(r_C) = r_f + \frac{[E(r_P) - r_f]}{\sigma_P}\sigma_C$$

- This equation is called the capital allocation line (CAL)

$$\text{Slope} = \frac{E(r_P) - r_f}{\sigma_P}$$

- The slope is the reward-to-volatility ratio: it is also called the **Sharpe ratio**

Figure 6.3 The Capital Allocation Line (CAL)
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



3
2

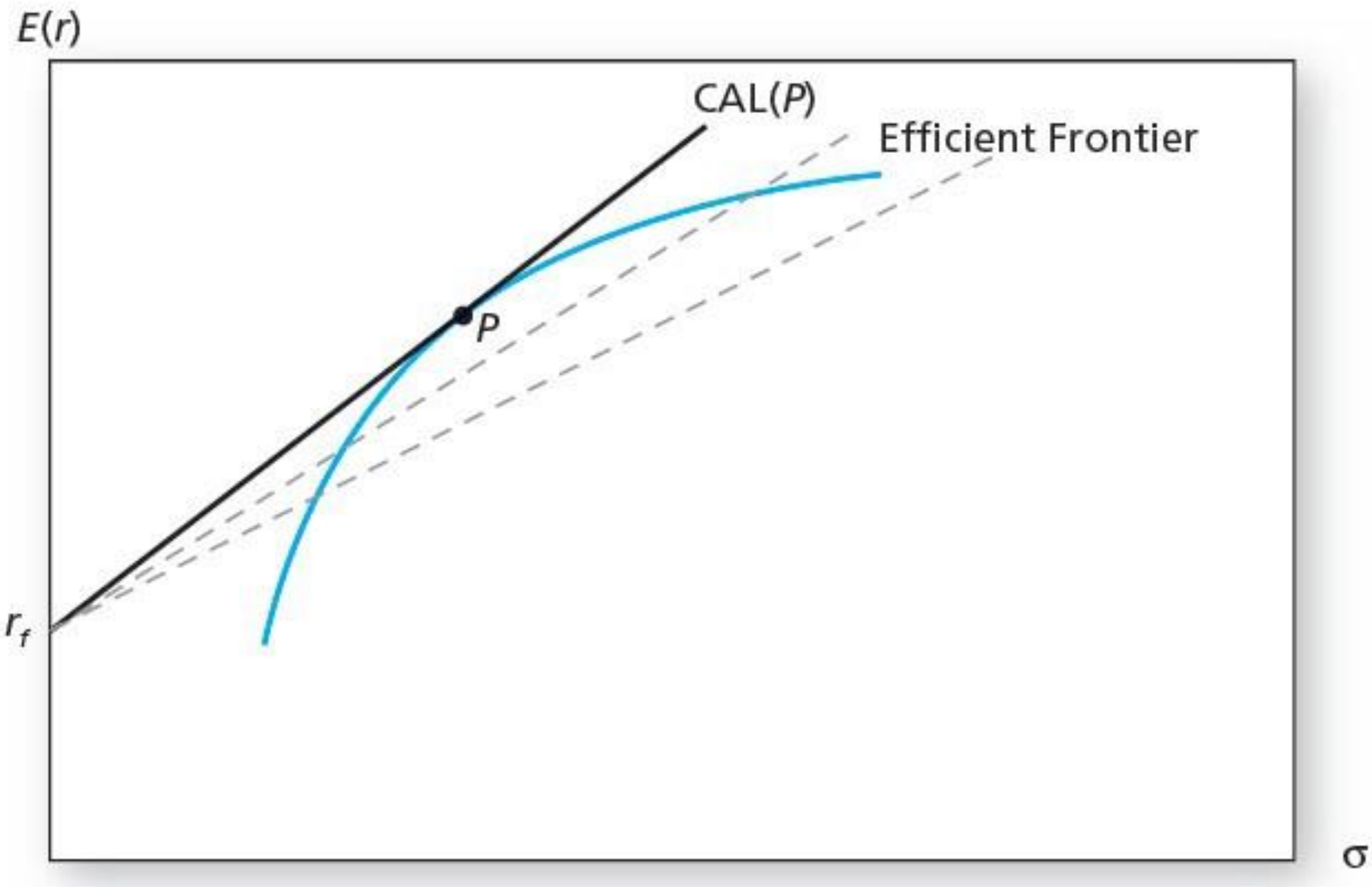
Optimal Risky Portfolio or Tangent Portfolio

- Investors will find the CAL with the highest reward-to-volatility ratio to find the optimal risky portfolio on the efficient frontier
- Maximize the slope of the CAL for any possible portfolio, *P*
- The objective function is the slope:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

- Recall that the slope is also called the Sharpe ratio

Figure 7.11 The Efficient Frontier of Risky Assets with the Optimal CAL (from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Recall Tutorial 3

Finding the tangency portfolio is equivalent to solving the following optimization problem:

$$\max_{x_S, x_B} \mathrm{SR}_p = \frac{E(r_p) - r_f}{\sigma_p} \text{ subject to}$$

$$E(r_p) = x_S E(r_S) + x_B E(r_B)$$

$$\sigma_p^2 = x_S^2 \sigma_S^2 + x_B^2 \sigma_B^2 + 2 x_S x_B \sigma_{SB}$$

$$1 = x_S + x_B$$

Numerical solution

Using the Excel solver function by maximizing the Sharpe ratio, subject to the sum of the weights to1 (see Excel file posted).

Analytical solution

Using substitution method, with $x_B = 1 - x_S$, and from the first order condition of the maximization problem, the weights of the tangent portfolio are given below. (The proof of the following asset weights in the tangent portfolio is not required for the exam.)

$$x_S^{\tan} = \frac{\left(E(r_S) - r_f\right)\sigma_B^2 - \left(E(r_B) - r_f\right)\sigma_{SB}}{\left(E(r_S) - r_f\right)\sigma_B^2 + \left(E(r_B) - r_f\right)\sigma_S^2 - \left(E(r_S) - r_f + E(r_B) - r_f\right)\sigma_{SB}}$$

$$x_B^{\tan} = 1 - x_S^{\tan}$$

Efficient Portfolio & Required Returns

- Consider an arbitrary portfolio of risky securities, P
- Consider whether we could raise its Sharpe ratio by adding investment i to the portfolio P
- Specifically, we will short the risk-free assets (i.e., borrow at the risk-free rate) and invest the proceeds in investment i

Efficient Portfolio & Required Returns

- Adding *i* to the portfolio *P* will improve our Sharpe ratio if

$$E[R_i] - r_f > SD(R_i) \times \text{Corr}(R_i, R_P) \times \frac{E[R_P] - r_f}{SD(R_P)}$$

Efficient Portfolio & Required Returns
Recall Tutorial 4

The portfolio with the highest Sharpe ratio is the portfolio where the line with the risk-free investment is tangent to the efficient frontier of risky investments. The portfolio is known as the **tangent portfolio**.

If you were to purchase more of investment *i* by borrowing, you would earn the expected return of *i* minus the risk-free return. Adding *i* to the portfolio *P* will improve our Sharpe ratio if

$$\underbrace{E[R_i] - r_f}_{\substack{\text{Additional return} \\ \text{from investment } i}} > \underbrace{SD(R_i) \times \mathrm{Corr}(R_i, R_p)}_{\substack{\text{Incemental volatility} \\ \text{from investment i}}} \times \underbrace{\frac{E[R_P] - r_f}{SD(R_P)}}_{\substack{\text{Return per unit of volatilty} \\ \text{available from portfolio P}}}$$

Efficient Portfolio & Required Returns

- Define Beta of Portfolio *i* with Portfolio P

$$\beta_i^P = \frac{SD(R_i) \times \text{Corr}(R_i, R_P)}{SD(R_P)}$$

$$\beta_i^P = \frac{SD(R_i)}{SD(R_P)} \times \frac{\text{Cov}(R_i, R_P)}{SD(R_i)SD(R_p)}$$

$$\beta_i^P = \frac{\text{Cov}(R_i, R_P)}{\text{Var}(R_P)}$$

- For each 1% change in the portfolio's return, investment i's return is expected to change by beta percent due to the risk i has in common with P.

Expected Returns & the Efficient Portfolio

- A portfolio is efficient if and only if the expected return of every available security equals its required return (defined with reference to the portfolio).
- Expected Return of a Security

$$E[R_i] = r_i \equiv r_f + \beta_i^{eff} \times (E[R_{eff}] - r_f)$$

- R_{eff} is the return of the efficient / tangent portfolio

- The required return is the expected return that is necessary to compensate for the risk investment i will contribute to the portfolio.

Risk Aversion and Utility Values

- We assume each investor can assign a welfare, or utility, score to competing portfolios

$$U = E(r) - \frac{1}{2}A\sigma^2$$

- Utility function
- U = Utility value
- E(r) = Expected return
- A = Index of the investor's risk aversion
- σ^2 = Variance of returns
- ½ = Scaling factor

Investor Types

- Risk-averse investors consider risky portfolios only if they provide compensation for risk via a risk premium
  - $A > 0$
- Risk-neutral investors find the level of risk irrelevant and consider only the expected return of risk prospects
  - $A = 0$
- Risk lovers are willing to accept lower expected returns on prospects with higher amounts of risk
  - $A < 0$

Risk Aversion & Mean-Variance (M-V) Criterion

- Investors who are risk averse reject investment portfolios that are fair games or worse
- A fair game is a risky investment with a risk premium or expected excess return of zero
- Mean-Variance (M-V) Criterion: Portfolio A dominates portfolio B if:

$$E(r_A) \geq E(r_B)$$

$$\& \quad \sigma_A \leq \sigma_B$$

Figure 6.1 Trade-off between risk and return
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

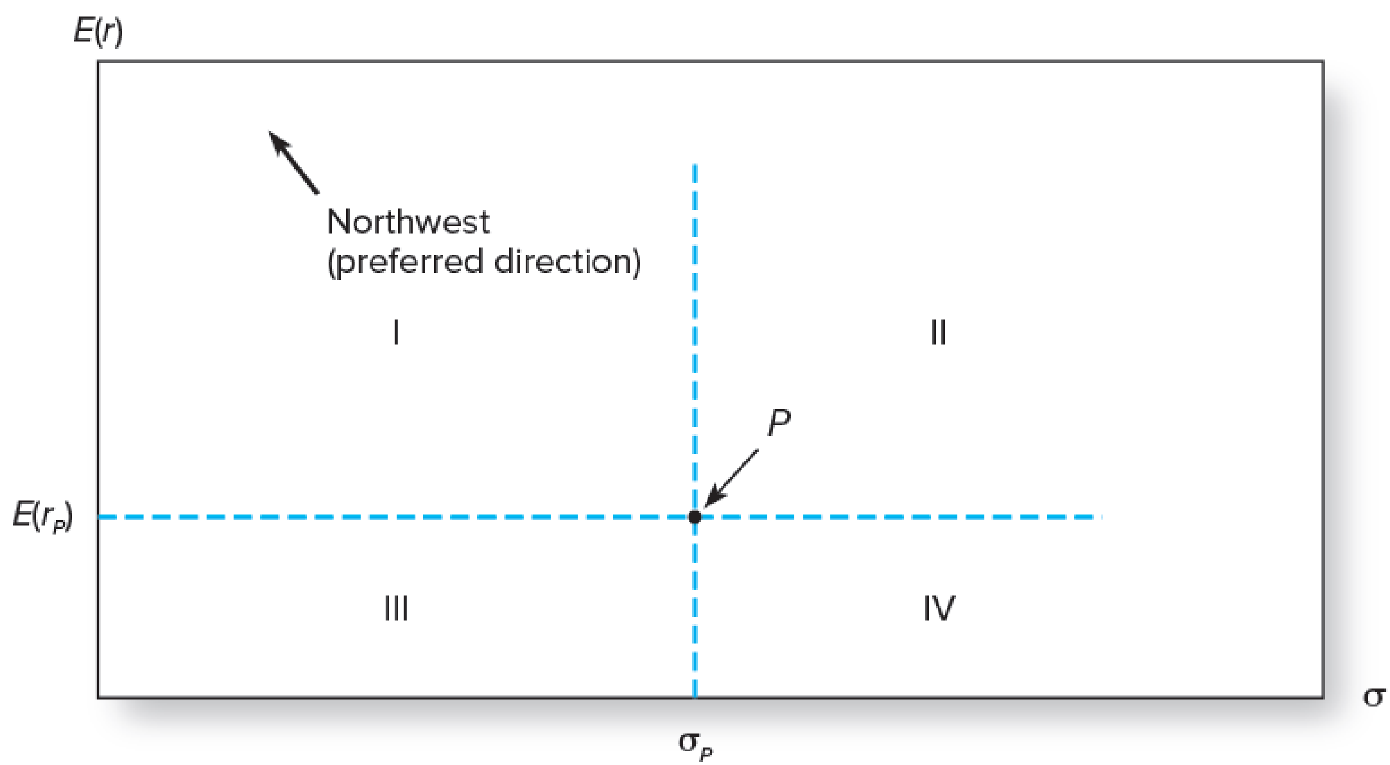


Figure 6.1 The trade-off between risk and return of a potential investment portfolio, *P*

Figure 6.2 The Indifference Curve
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

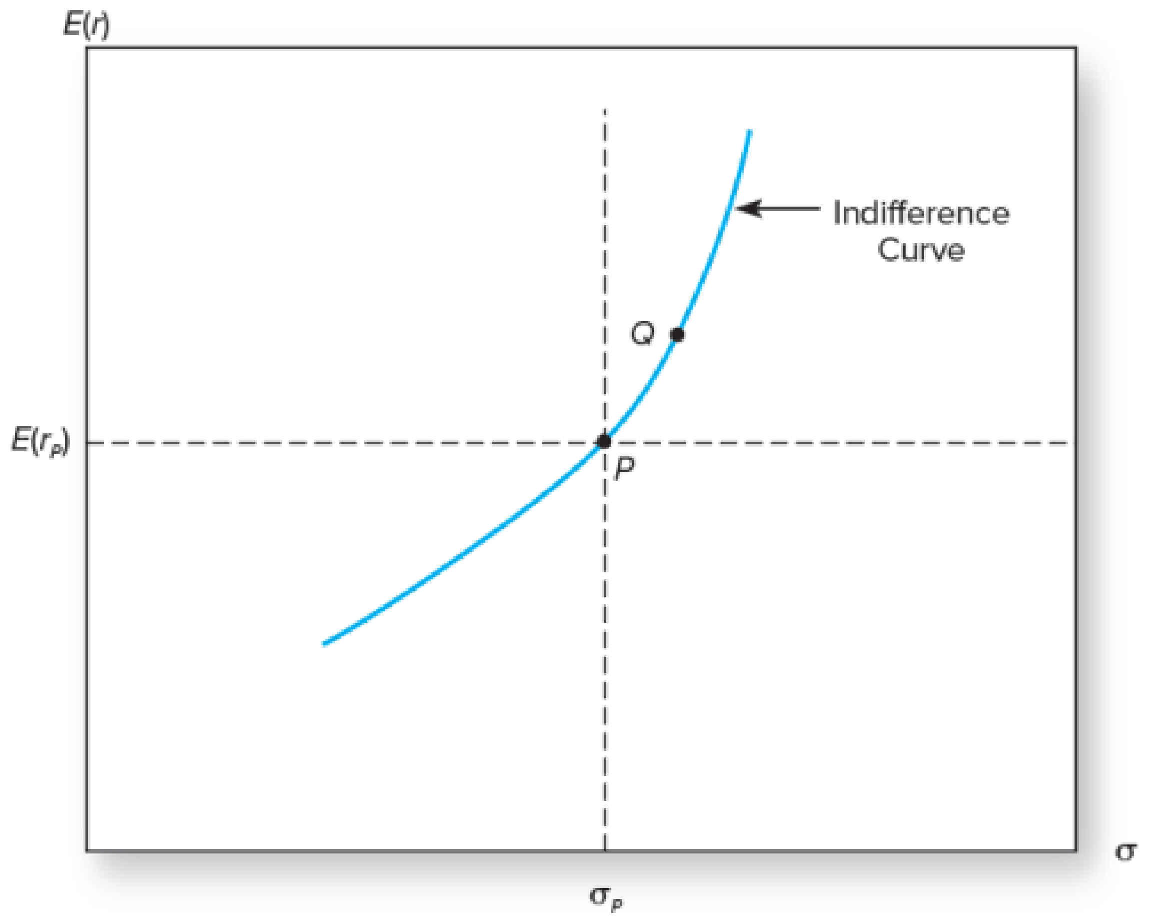


Figure 6.2 The indifference curve

Figure 6.7 Portfolio C is the Optimal Complete Portfolio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

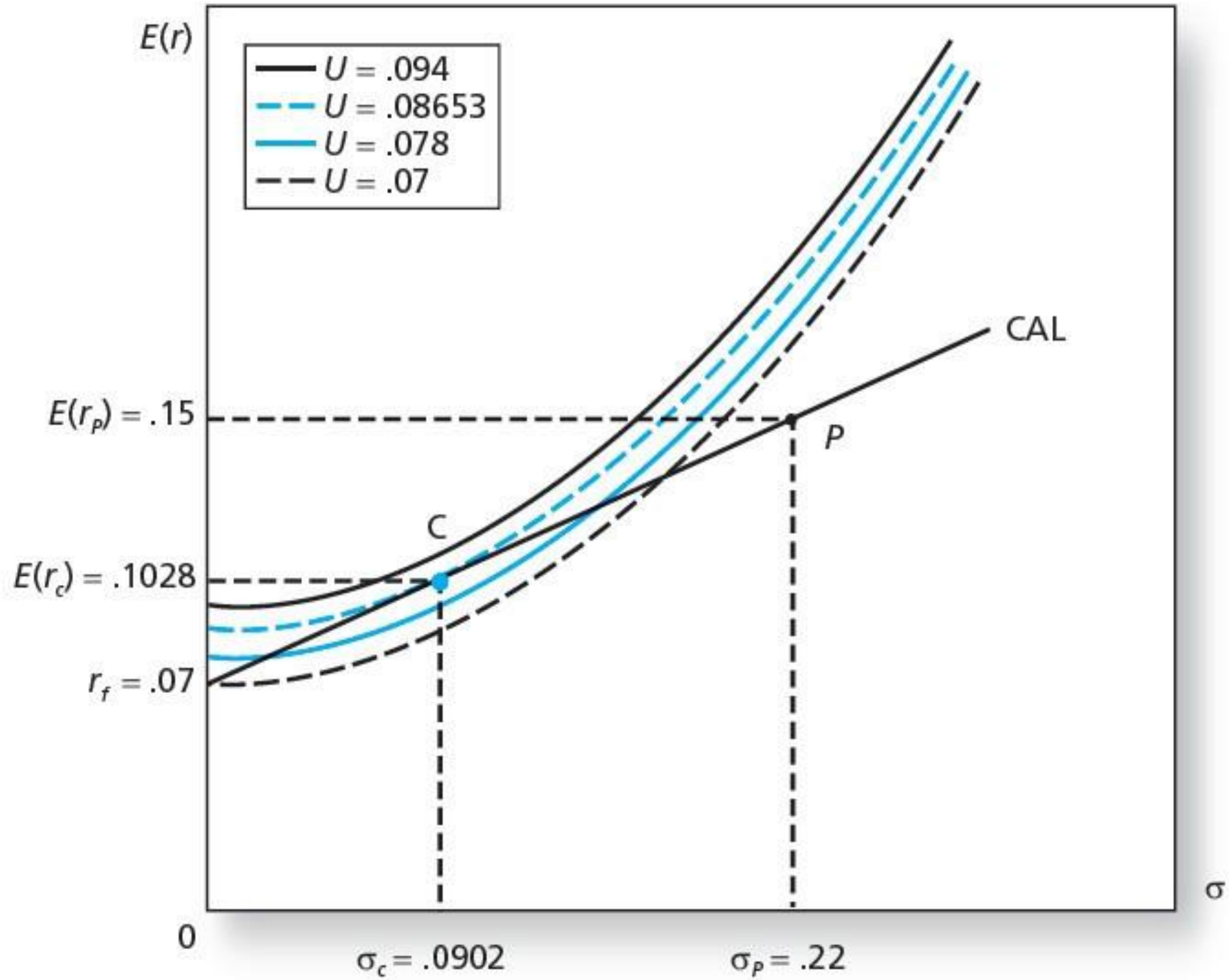
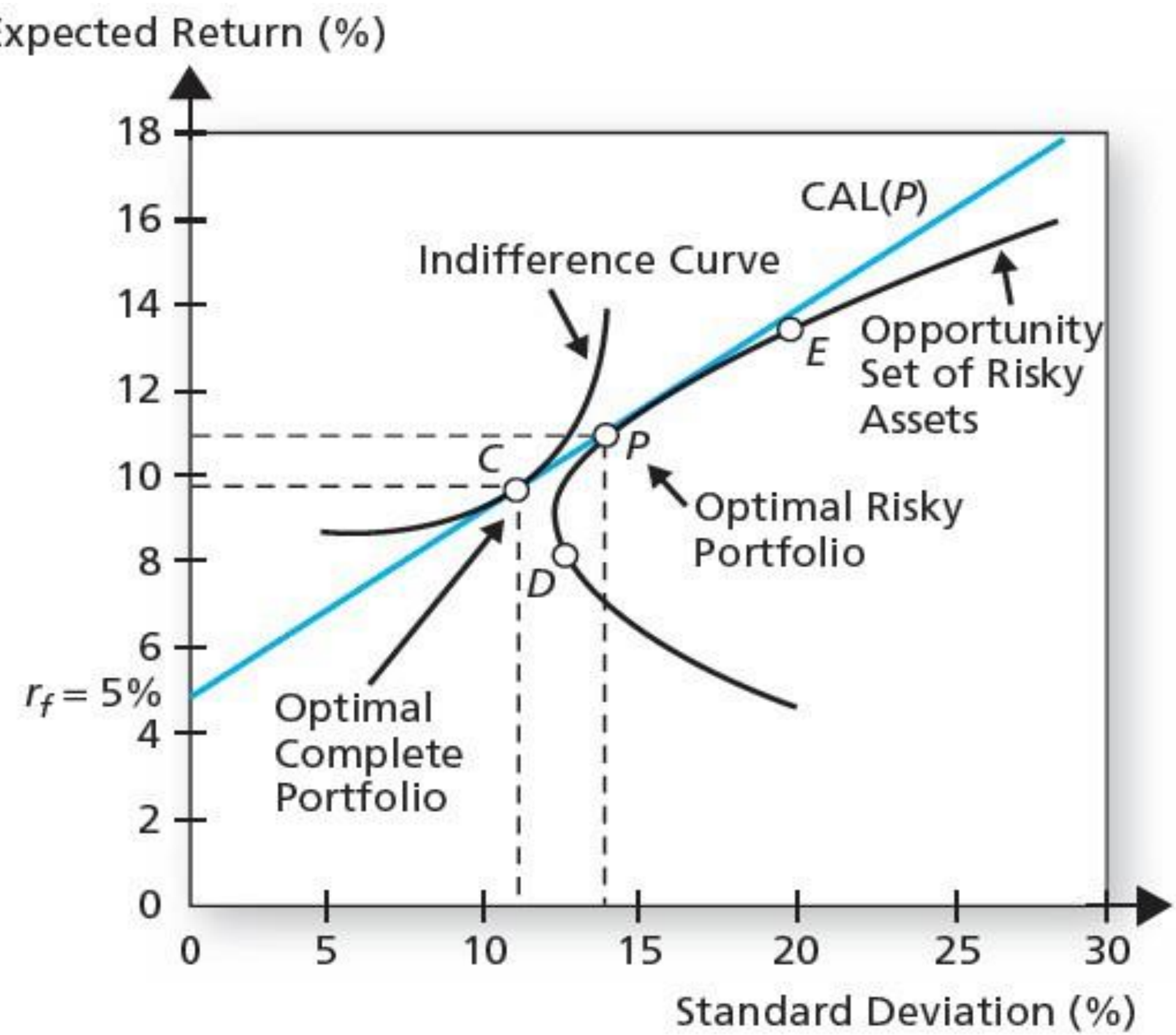


Figure 7.8 Determination of the Optimal Overall Portfolio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



The Separation Property / Theorem

- Portfolio choice problem may be separated into two independent tasks
  - Determination of the optimal risky portfolio is purely technical
    - Everyone invests in *P*, regardless of their degree of risk aversion
  - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference
    - More risk averse investors put more in the risk-free asset

L4: Learning Objectives

- Explain the theory of the capital asset pricing model (CAPM), and be able to construct and use the security market line.
  - State the main assumptions underlying the Capital Asset Pricing Model.
  - Explain why the CAPM implies that the market portfolio of all risky securities is the efficient portfolio.
  - Compare and contrast the capital market line with the security market line.
  - Define beta for an individual stock and for a portfolio.
  - Define alpha for an individual stock and the implications of CAPM on alpha.

The Efficient Portfolio

- To identify the efficient portfolio, we need for every risky asset
    - Expected returns
    - Volatilities
    - Correlations

- A lot of parameters to estimate and a difficult estimation task

- But CAPM allows us to identify the efficient portfolio without knowledge of the expected return of each security

The CAPM Assumptions

- Investors can buy and sell all securities at competitive market prices (without incurring taxes or transactions costs) and can borrow and lend at the risk-free interest rate.

- All investors are rational mean-variance optimizers
  - Investors hold only efficient portfolios of traded securities (i.e. portfolios that yield the maximum expected return for a given level of volatility).

- Investors have homogeneous expectations regarding the volatilities, correlations, and expected returns of securities.

Equilibrium and the Efficiency of the Market Portfolio

- Homogeneous expectations + Mean-variance optimizers + borrowing and lending rate at the risk-free interest rate

  → All investors face identical efficient frontier and CAL

  → All investors hold the same efficient portfolio of risky assets (P)

- In equilibrium, demand = supply

  → The efficient portfolio of risky assets that all investors hold must equal the market portfolio

  → All investors hold the market portfolio in equilibrium

The Market Portfolio

- Market Capitalization
  - The total market value of a firm's outstanding shares

$$MV_i = (\text{ Number of Shares of } i \text{ Outstanding }) \times \ (\text{Price of } i \text{ per Share})$$

- Value-Weighted Portfolio
  - A portfolio in which each security is held in proportion to its market capitalization

$$x_i = \frac{\text{Market Value of } i}{\text{Total Market Value of All Securities}} = \frac{MV_i}{\sum_j MV_j}$$

CAPM: Resulting Equilibrium Conditions

- When the CAPM assumptions hold, all investors will hold combinations of only two portfolios: the risk- free asset and the market portfolio

  - The two mutual fund theorem
    - Mutual funds are financial intermediaries that sell shares to savers and use their funds to buy diversified pools of assets and manage them

  - When the tangent line goes through the market portfolio, it is called the capital market line (CML)

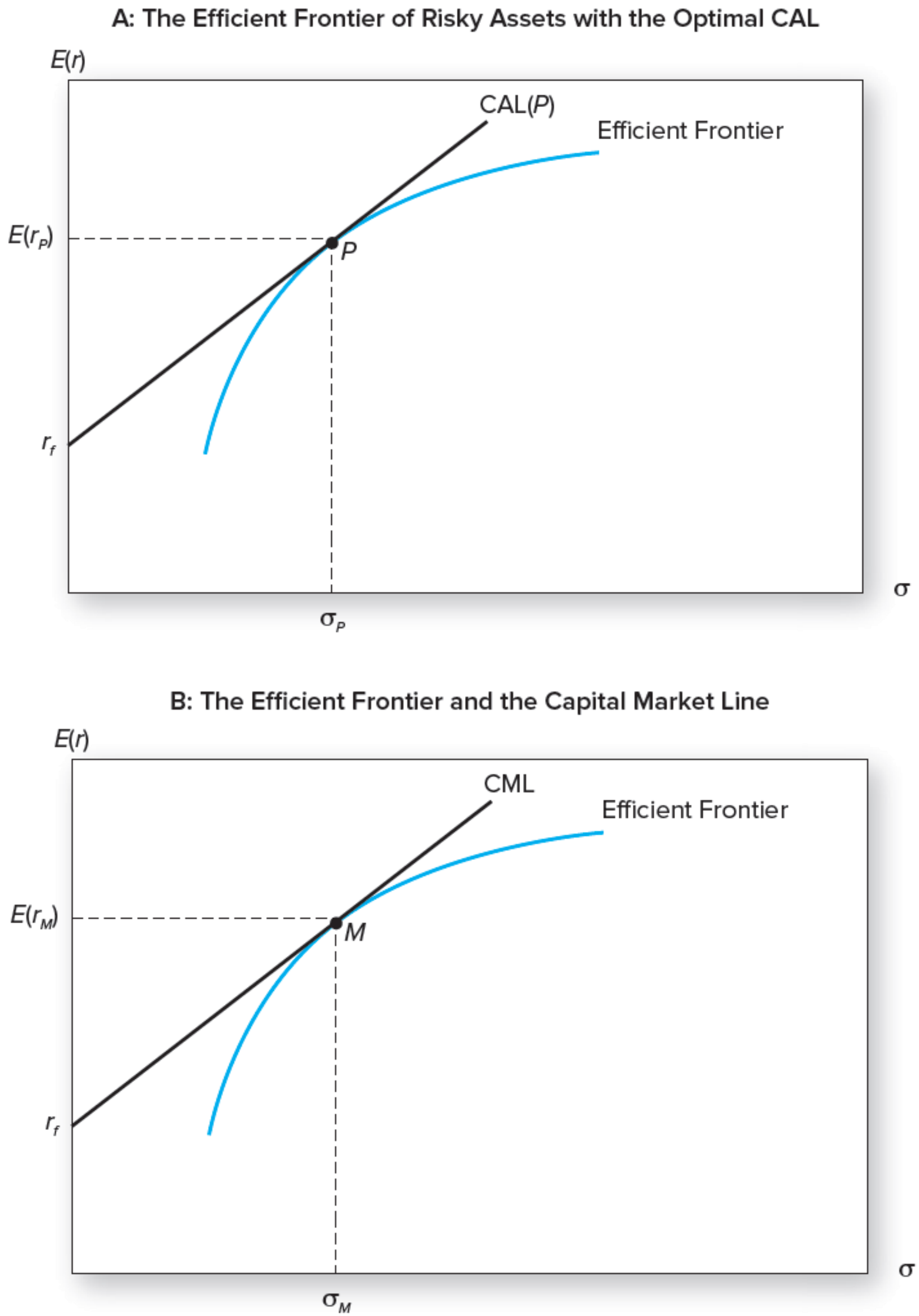


Figure 9.1 Capital allocation line and the capital market line (from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

The Capital Market Line (CML)

- The CML is a capital allocation line formed by investment in two passive portfolios:
  1. Risk-free short-term T-bills (or a money market fund)
  2. Fund of common stocks that mimics a broad market index
- Note that the CML is CAL with the optimal risky portfolio replaced by the market portfolio

The Capital Market Line (CML)

- Suppose
  - *x* = portfolio weight on the market portfolio, M
  - *(1 - x)* = portfolio weight on the risk-free asset, F

- The expected return of a portfolio on the CML:

$$E(r_e) = (1 - x)r_f + xE(r_M)$$
$$E(r_e) = r_f + x[E(r_M) - r_f]$$

- With standard deviation:

$$\sigma_e = x\sigma_M$$

The Capital Market Line (CML)

- Rearrange and substitute $x = \sigma_e/\sigma_M$

$$E(r_e) = r_f + \frac{[E(r_M) - r_f]}{\sigma_M}\sigma_e$$

- This equation is called the capital market line (CML)

$$\text{Slope} = \frac{E(r_M) - r_f}{\sigma_M}$$

Recall L3: Expected Returns & the Efficient Portfolio

- A portfolio is efficient if and only if the expected return of every available security equals its required return (defined with reference to the portfolio).
- Expected Return of a Security

$$E[R_i] = r_i \equiv r_f + \beta_i^{eff} \times (E[R_{eff}] - r_f)$$

- R_{eff} is the return of the efficient / tangent portfolio

- The required return is the expected return that is necessary to compensate for the risk investment i will contribute to the portfolio.

The Return and Risk For Individual Securities: Market Risk and Beta

- Given an efficient market portfolio, the expected return of an investment is:

$$E[R_i] = r_i = r_f + \underbrace{\beta_i^{\text{Mkt}} \left(E[R_{Mkt}] - r_f\right)}_{\text{Risk premium for security } i}$$

Beta

$$\beta_i^{Mkt} = \frac{\mathbf{Cov}(R_i, R_{Mkt})}{\mathbf{Var}(R_{Mkt})}$$

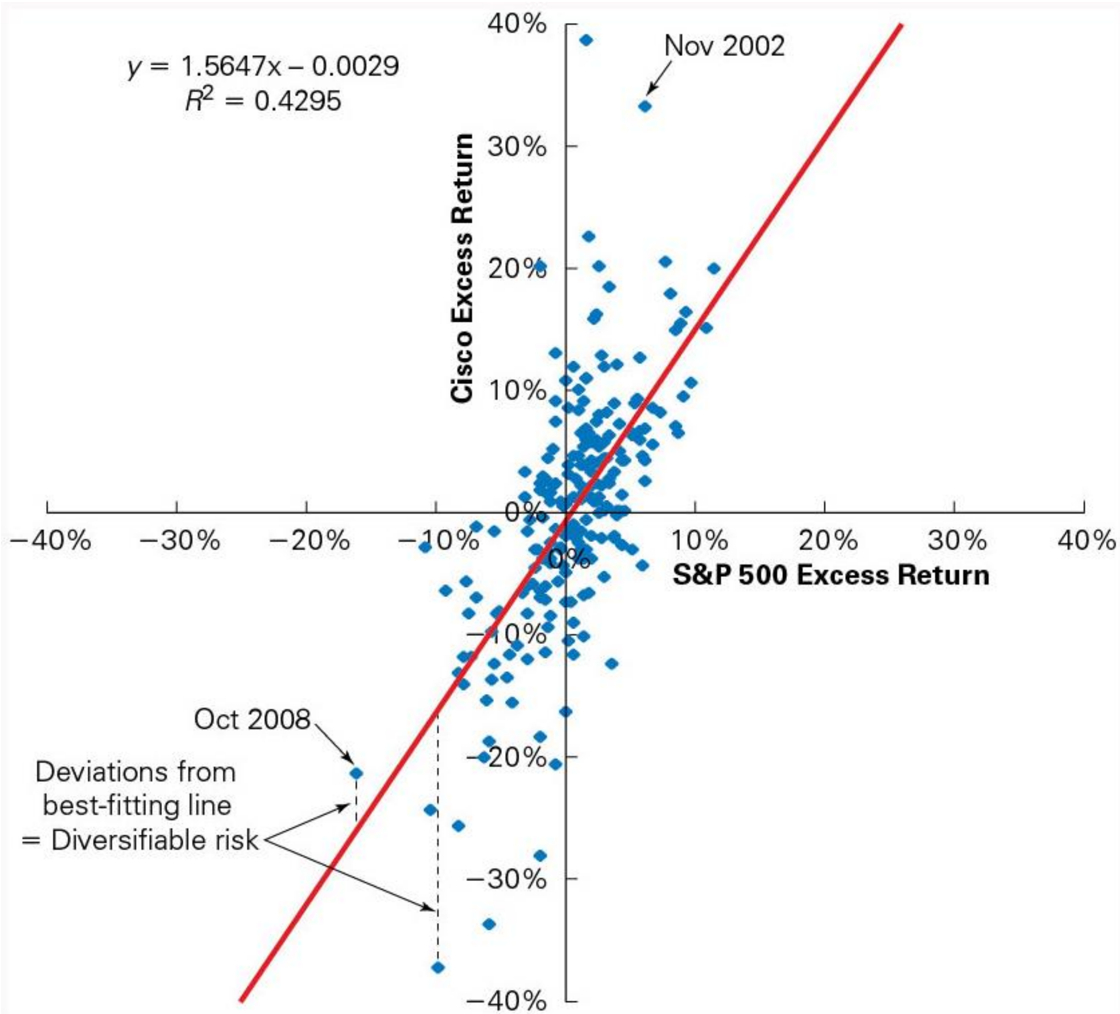
Covariance with the market

Variance of the market

Estimating Beta from Historical Returns

- Beta = the expected percent change in the excess return of the security for a 1% change in the excess return of the market portfolio
  - Consider Cisco Systems stock and how it changes with the market portfolio
  - Cisco tends to be up when the market is up, vice versa
  - A 10% change in the market's return corresponds to about a 15% change in Cisco's return
  - Thus, Cisco's return moves about one and a half times that of the overall market, so Cisco's beta is about 1.5
- Beta corresponds to the slope of the best-fitting line in the plot of the security's excess returns versus the market excess return

Figure 12.2 Scatterplot of Monthly Excess Returns for Cisco Versus the S&P 500, 2000-2017
(from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)



Estimating Beta from Historical Returns

$$(R_i - r_f) = \alpha_i + \beta_i(R_M - r_f) + \varepsilon_i$$

- Using the monthly returns for Cisco during 2000–2017, estimate
  - β =1.56 with a 95% confidence interval of [1.3, 1.8]
  - α = –0.29% with a standard error of 0.5% (statistically insignificant)

The Security Market Line

- The security market line (SML) is graphed as the line through the risk-free investment and the market
  - According to the CAPM, if the expected return and beta for individual securities are plotted, they should all fall along the SML
- There is no clear relationship between an individual stock's volatility (= total risk) and its expected return
- Expected return is determined by only that part of an individual stock's volatility that cannot be diversified away

$$E[R_i] = r_f + \beta_i^{Mkt}(E[R_{Mkt}] - r_f)$$

Figure 9.2 The Security Market Line
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

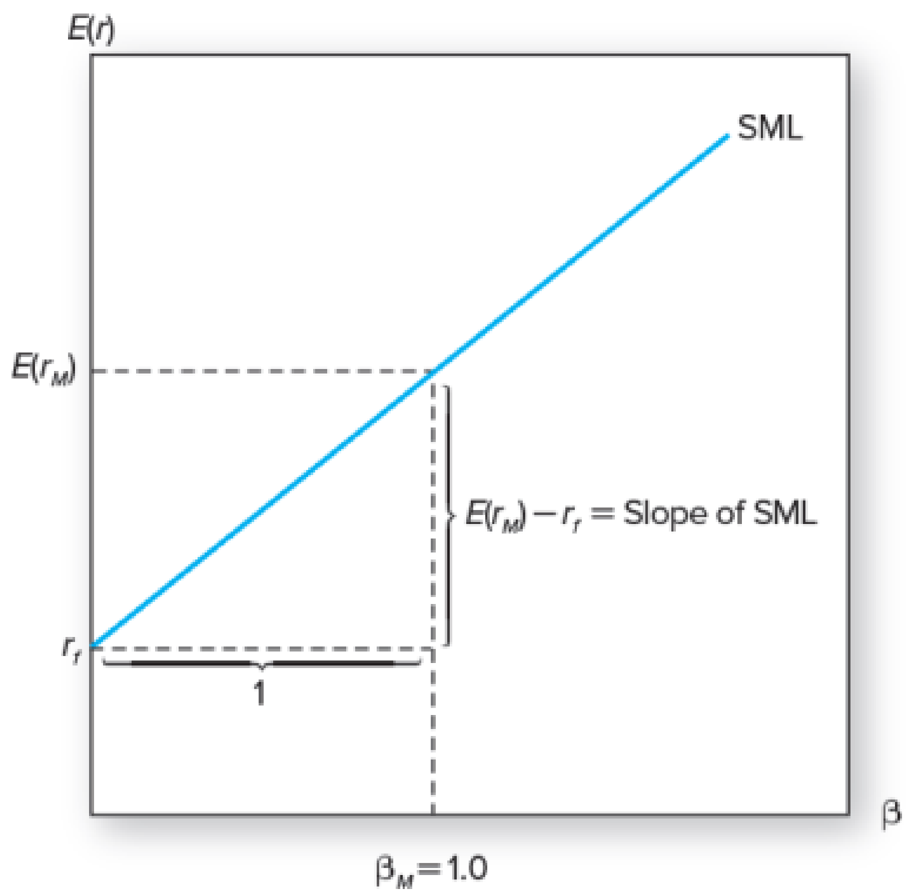


Figure 9.2 The security market line

Beta of a Portfolio

$$\beta_P = \frac{\mathrm{Cov}(R_p, R_M)}{\mathrm{Var}(R_M)}$$

$$= \frac{\mathrm{Cov}(\sum_i x_i R_i, R_M)}{\mathrm{Var}(R_M)}$$

$$= \sum_i x_i \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}$$

$$\beta_P = \sum_i x_i \beta_i$$

- x_i is the portfolio weight on security *i*

SML for a Portfolio of Securities

- CAPM also holds for a portfolio of securities:

$$E(R_P) = \sum_i x_i E(R_i)$$

$$E(R_P) = \sum_i x_i (r_f + \beta_i [E(R_{Mkt}) - r_f])$$

$$E(R_P) = r_f + \beta_P [E(R_{Mkt}) - r_f]$$

$$\text{where } \beta_P = \sum_i x_i \beta_i$$

The SML and Alpha

- Alpha is the difference between a stock's expected return and its required return according to the SML

$$\alpha_i = \underbrace{E(R_i)}_{\text{Expected return}} - \underbrace{(r_f + \beta_i[E(R_M) - r_f])}_{\text{Required return according to SML}}$$

$$E[R_i] = \underbrace{r_f + \beta_i(E[R_M] - r_f)}_{\text{Expected return for } i \text{ from the SML}} + \overbrace{\alpha_i}^{\text{Distance above / below the SML}}$$

- Stocks with non-zero alpha's do not lie on the SML
- alpha ≠ 0 → market portfolio is inefficient

Figure 9.3 The SML and a Positive-Alpha Stock
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

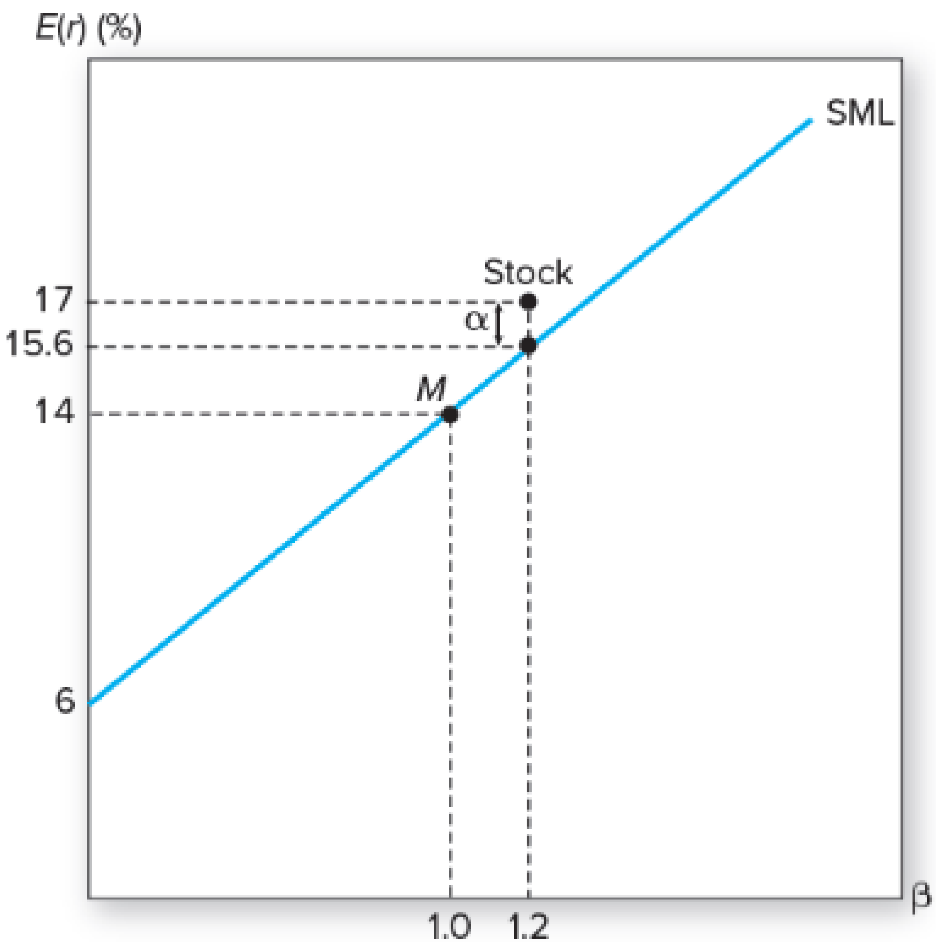


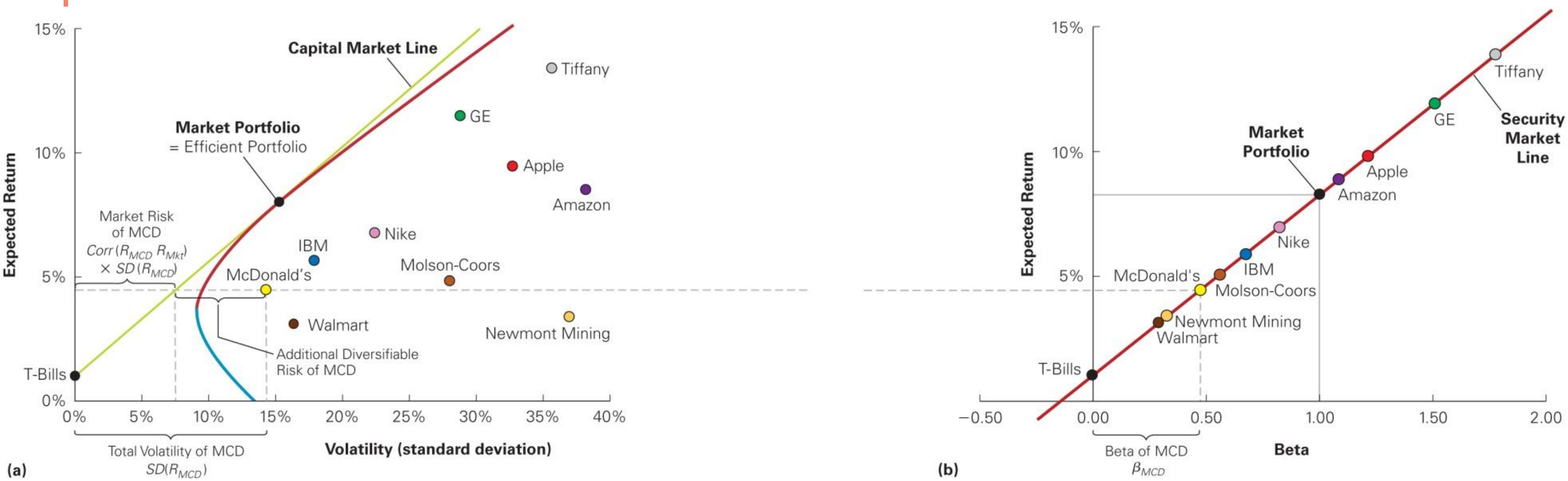
Figure 9.3 The SML and a positive-alpha stock

The SML and Alpha

$$\alpha_i = \underbrace{E(R_i)}_{\text{Expected return}} - \underbrace{(r_f + \beta_i[E(R_M) - r_f])}_{\text{Required return according to SML}}$$

- Thus, α_i represents a risk-adjusted performance measure for the historical returns
  - CAPM → α_i should not be significantly different from zero
- Caveats:
  - Difficult to estimate with accuracy without a very long data series
  - The alphas for individual stocks have very little persistence
  - During 1996-2000, Cisco's return had an alpha of 3% per month
  - This positive alpha did not forecast superior future performance

Figure 11.12 The Capital Market Line and the Security Market Line (from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e)



L5: Learning Objectives

- Define effective annual rate and annual percentage rate and their relationship.
- Describe the relation between nominal and real rates of interest.
- Compute the present value or future value of a stream of cashflows such as a perpetuity and an annuity.

Effective Annual Rate

- Indicates the total amount of interest that will be earned at the end of one year
- Considers the effect of compounding
- a.k.a. the Effective Annual Yield (EAY) or the Annual Percentage Yield (APY)
- For investments that last < 1 year, we compound the per-period return for a full year

$$\mathrm{EAR} = (1 + \text{ rate for period })^{\text{Number of periods per year}} - 1$$

Annual Percentage Rate

- The annual percentage rate (APR), indicates the amount of simple interest earned in one year.
- Simple interest is the amount of interest earned without the effect of compounding.
- The APR is typically less than the EAR.
- Rates on investments that last < 1 year are often annualized using simple rather than compound interest.

Annual Percentage Rate

- The APR itself cannot be used as a discount rate
  - This is because the APR does not reflect the true amount you will earn over one year
  - It has to be converted to a discount rate
  - The APR with k compounding periods per year is a way of quoting the actual interest earned each compounding period

$$\text{Interest Rate per Compounding Period} = \frac{\text{APR}}{k \text{ periods / year}}$$

Converting APR to EAR

$$1 + EAR = \left(1 + \frac{APR}{k}\right)^k$$

- The EAR increases with the frequency of compounding.
- As the compounding frequency grows and the compounding interval gets smaller, in the limit we get continuous compounding (CC).
- Continuous compounding is compounding every instant.
- For the continuously compounded case (denoted by r_{cc})

$$1 + EAR = e^{r_{cc}}$$

Fisher Relation

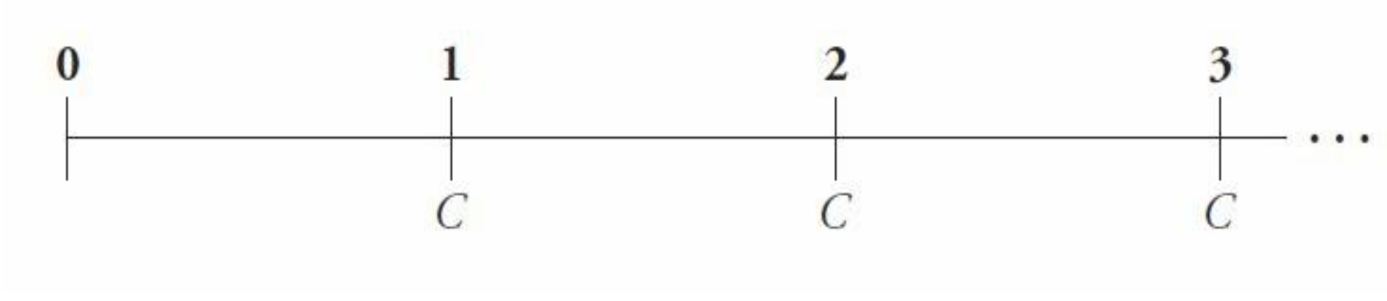
- Real interest rate: r
- Nominal interest rate: i
- Inflation rate: π

$$1+r=\frac{1+i}{1+\pi} \quad \Leftrightarrow 1+r+\pi+r\pi=1+i$$

$$r=\frac{i-\pi}{1} \qquad \text{or} \qquad r\approx i-\pi$$

Perpetuities

- When a constant cash flow will occur at regular intervals forever it is called a perpetuity
  - E.g., Consol bonds in the U.K.



Perpetuities

- The value of a perpetuity is simply the cash flow divided by the interest rate

$$PV(C \text{ in perpetuity }) = \sum_{n=1}^{\infty} \frac{C}{(1+r)^n}$$

$$PV(C \text{ in perpetuity }) = \frac{C}{(1+r)} \left[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \dots\right]$$

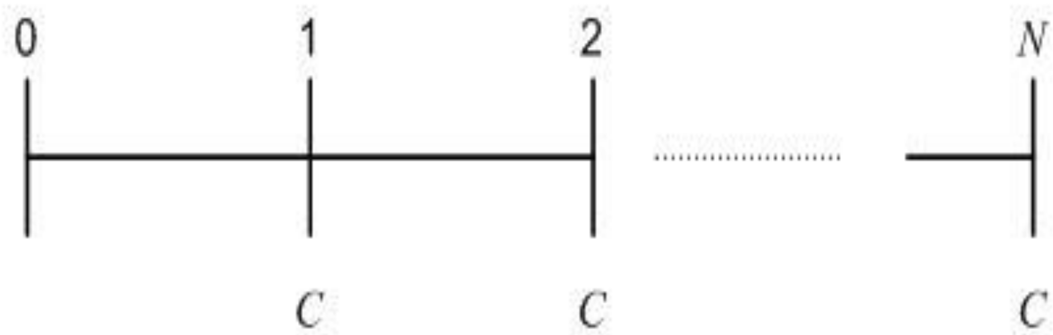
$$PV(C \text{ in perpetuity }) = \frac{C}{(1+r)} \frac{1}{(1 - \frac{1}{1+r})} = \frac{C}{(1+r)} \frac{1}{(\frac{1+r-1}{1+r})}$$

$$PV(C \text{ in perpetuity }) = \frac{C}{r}$$

$$PV(\text{growing perpetuity}) = \frac{C}{(r-g)}$$

Annuities

- When a constant cash flow will occur at regular intervals for a finite number of N periods, it is called an annuity



$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^N}$$

$$PV = \sum_{n=1}^{N} \frac{C}{(1+r)^n}$$

Present Value of an Annuity

- Use formula for geometric progression

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^N}$$

$$PV = \frac{C}{(1+r)}\left[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{N-1}}\right]$$

$$PV = \frac{C}{(1+r)}\left[\frac{1 - \frac{1}{(1+r)^N}}{\left(1 - \frac{1}{(1+r)}\right)}\right]$$

$$PV = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right)}_{\text{The Annuity Formula}}$$

Present Value and Future Value of an Annuity

- To recap, the present value of annuity of C for N period equals

$$PV = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right)}_{\text{The Annuity Formula}}$$

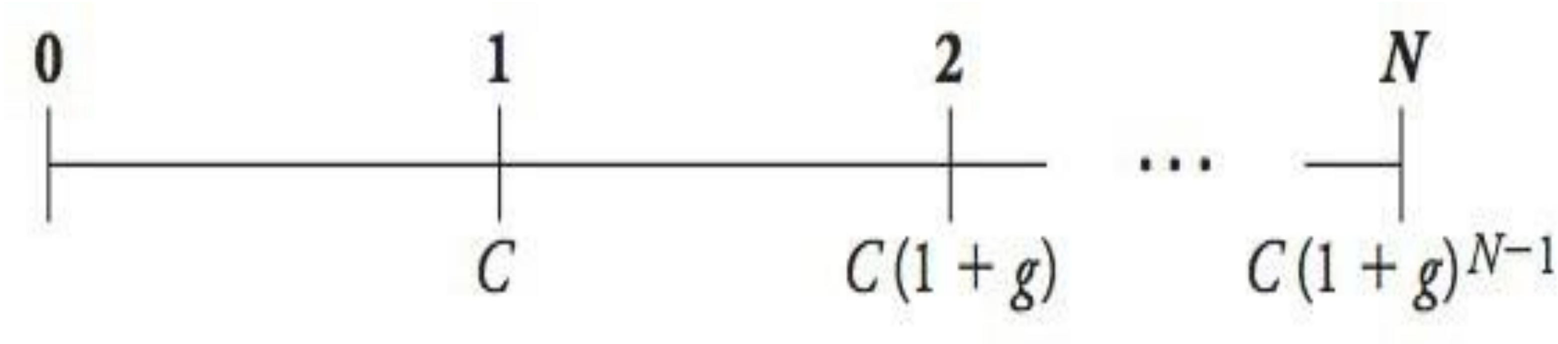
- Thus, the future value of an annuity equals

$$\begin{aligned} FV(\text{ annuity }) &= P\,\mathrm{V} \times (1+r)^N \\ &= \frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right) \times (1+r)^N \end{aligned}$$

$$FV = \frac{C}{r}\left((1+r)^N - 1\right)$$

Growing Annuity

- The present value of a growing annuity with the initial cash flow c, growth rate g, and interest rate r is defined as:



$$PV \text{ (growing annuity)} = \frac{C}{(r-g)}\left(1-\left(\frac{1+g}{1+r}\right)^{N}\right)$$

Solving for the Internal Rate of Return

- In some situations, you know the present value and cash flows of an investment opportunity but you do not know the internal rate of return (IRR)
- IRR is the interest rate that sets the net present value of the cash flows equal to zero

Solving for the Number of Periods

- In addition to solving for cash flows or the interest rate, we can solve for the amount of time it will take a sum of money to grow to a known value.

L6: Learning Objectives

- Calculate yields and prices of bonds
- Understand how term structure concepts apply to valuation of securities
- Calculate forward rates from the term structure
- Describe the major theories of term structure, their assumptions and implications

Bond Pricing

$$P_B = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} + \frac{\text{Par Value}}{(1+r)^T}$$

- P_B = Cash price of the bond
- C_t = Interest or coupon payments
- T = Number of periods to maturity
- r = discount rate per period

$$\text{Coupon Payment} = \frac{\text{Coupon Rate} \times \text{Face Value}}{\text{Number of Coupon Payments per Year}}$$

Bond Pricing

- If you apply the formula for annuity, instead of calculating the sum of (T+1) terms, you only have to calculate the sum of 2 terms

$$P_B = \underbrace{\sum_{t=1}^{T} \frac{C}{(1+r)^t}}_{\substack{\text{A T-period Annuity} \\ \text{that pays C Per period}}} + \frac{\text{Par Value}}{(1+r)^T}$$

$$P_B = \underbrace{\frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right)}_{\text{The Annuity Formula}} + \frac{\text{Par Value}}{(1+r)^T}$$

Yield to Maturity (YTM)

- Interest rate that makes the present value of the bond's payments equal to its price is the yield to maturity (YTM)
- In other words, YTM is the one discount rate that, when applied to the promised cash flows of the bond, recovers the current market price of the bond
- Solve the bond formula for r given the values of P_B and C → The resulting r = YTM

$$P_B = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{\text{Par Value}}{(1+r)^T}$$

Bond: Prices and Yields

- Prices and yields (required rates of return) have an inverse relationship
- The bond price curve (Figure 14.3) is convex

Figure 14.3 The Inverse Relationship Between Bond Prices and Yields
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

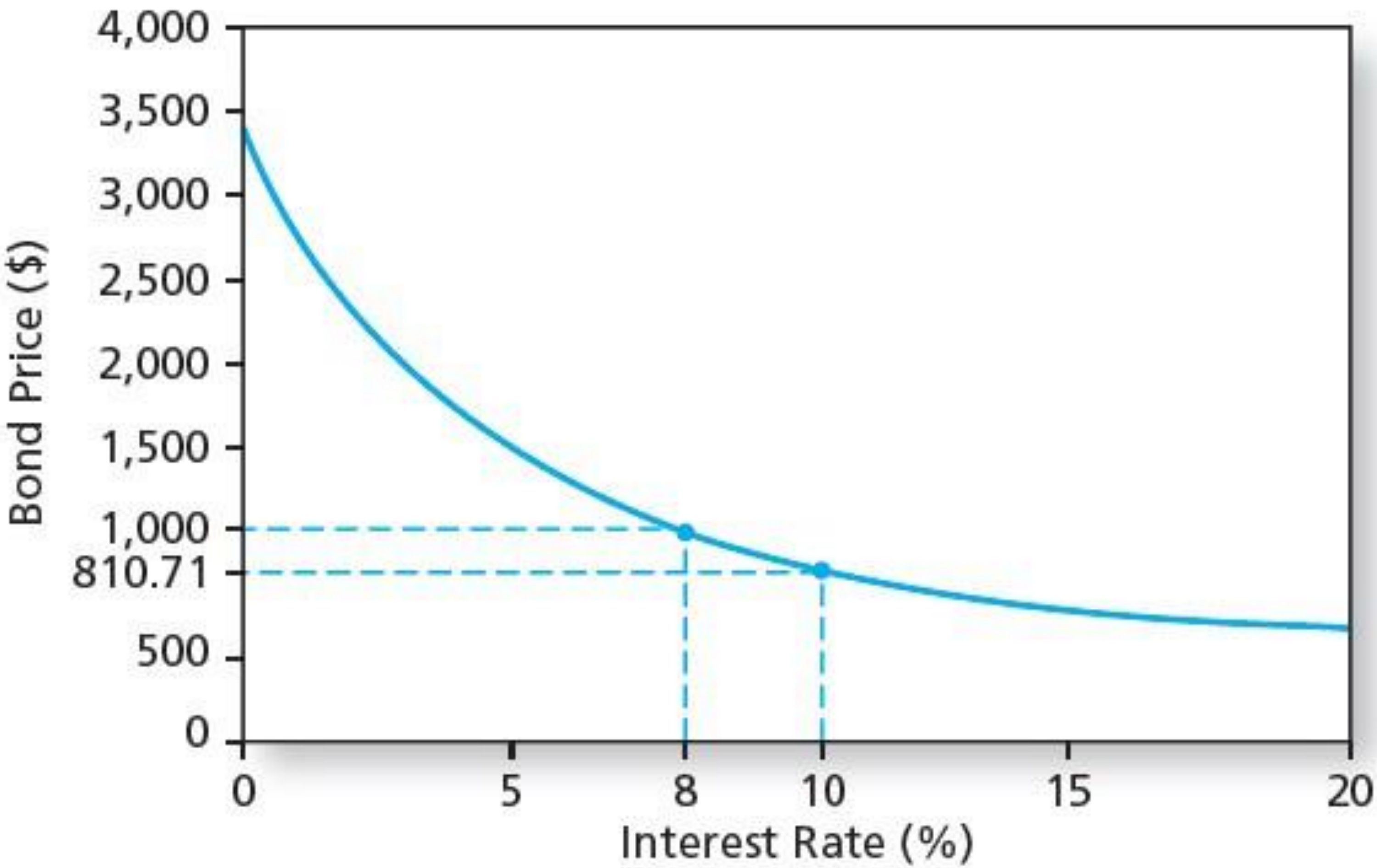
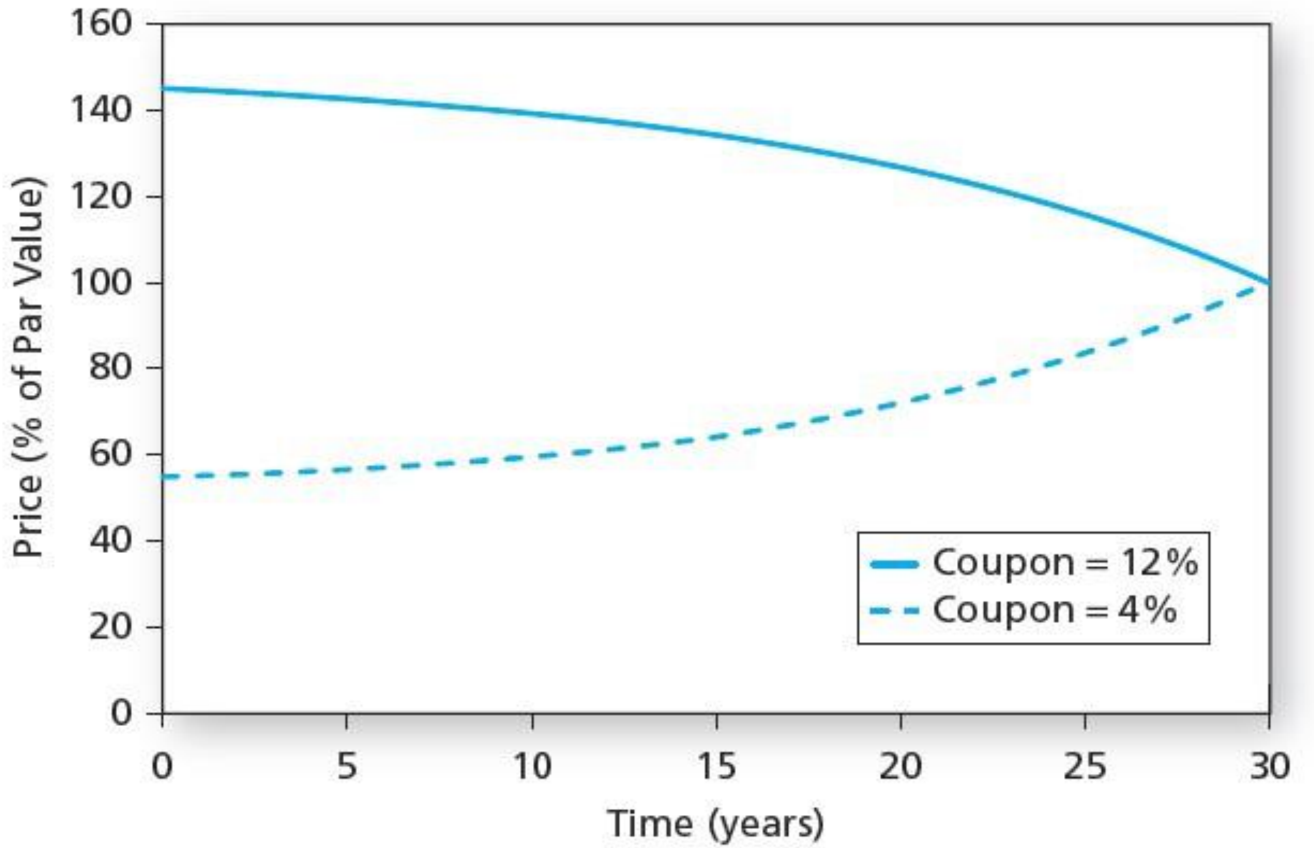


Figure 14.6 Bond Prices over Time
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



- Two 30-year maturity bonds, each selling at a yield to maturity of 8%. Bond price approaches par value as maturity date approaches

Recall: Yield to Maturity (YTM)

- Interest rate that makes the present value of the bond's payments equal to its price is the yield to maturity (YTM)
- In other words, YTM is the one discount rate that, when applied to the promised cash flows of the bond, recovers the current market price of the bond
- Solve the bond formula for r given the values of P_B and C $\rightarrow$ The resulting r = YTM

$$P_B = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{\text{Par Value}}{(1+r)^T}$$

YTM vs. Current Yield

- Yield to Maturity
  - Bond's internal rate of return; accounts for capital gain/loss
  - The interest rate that makes the PV of a bond's payments equal to its price; assumes that all bond coupons can be reinvested at the YTM
- Current Yield
  - Bond's annual coupon payment divided by the bond price
  - For premium bonds (selling above par value)

    Coupon rate > Current yield > YTM
  - For discount bonds (selling bellow par value)

    YTM > Current yield > Coupon rate

Realized Return from Bond

- Assume an annual coupon bond
- The realized return from holding the bond for one year or the holding period return is

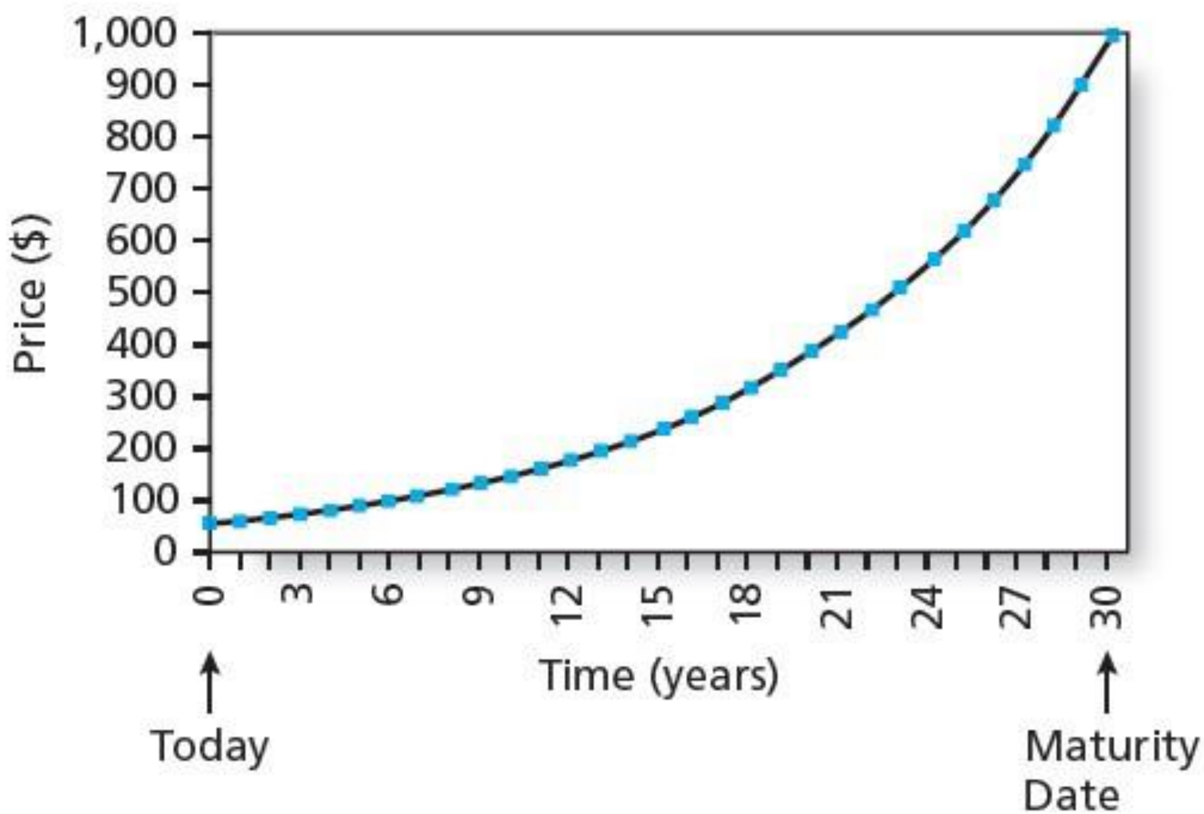
$$
\begin{aligned}
R_{t+1} &= \frac{C_{t+1} + P_{t+1}}{P_t} - 1 = \frac{C_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t} \\
&= \text{ Current Yield } + \text{ Capital Gain/Loss Rate}
\end{aligned}
$$

- P_{t+1} depends on the market interest rate that prevails at t + 1, which is uncertain from today's perspective
- But if the bond is held until maturity, then its value must equal the par value on the maturity date itself.

Zero-Coupon Bonds

- Zero-coupon bond does not make coupon payments
- Always sells at a discount (a price lower than face value)
- So they are also called pure discount bonds
- Treasury Bills are U.S. government zero-coupon bonds with a maturity of up to one year.

Figure 14.7 The Price of a 30-Year Zero-Coupon Bond over Time
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



- The bond prices rise exponentially, not linearly, until its maturity Price = 1,000/$(1+y)^T$, where T = time to maturity, y = 10%

Recall: Yield to Maturity (YTM)

- Interest rate that makes the present value of the bond's payments equal to its price is the yield to maturity (YTM)
- In other words, YTM is the one discount rate that, when applied to the promised cash flows of the bond, recovers the current market price of the bond
- Solve the bond formula for r given the values of P_B and C → The resulting r = YTM

$$P_B = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{\text{Par Value}}{(1+r)^T}$$

Recall from Supplementary notes

Spot Zero-Coupon (or Discount) Rate

- Spot Zero-Coupon (or Discount) Rate, $R_{0,t}$ is the annualized rate on a pure discount bond.

$$\frac{1}{\left(1+R_{0,t}\right)^{t}} = B(0,t)$$

  - where $B(0,t)$ is the market price at date 0 of a bond paying off \$1 at date t, Thus, $B(0,t)$ can be also used as a discount factor to get the present value (time=0) of an equivalent of \$1 at time t.

- The spot rate is calculated by finding the discount rate that makes the present value (PV) of a zero-coupon bond equal to its price. That is, the YTM or IRR of a zero-coupon bond with maturity t.

- Spot rates term structure can show different interest rates for different year-to-maturity.

- Since each zero-coupon bond has only one cashflow, which is on maturity, the spot zero-coupon (discount) rate is thus very useful for us to infer what is the discount rate implied by the market for various maturities (that corresponds to the zero-coupon bond). Therefore, the spot zero-coupon rate is also known as the discount rate. This is used in the general pricing formula of a coupon bond, by discounting each corresponding cashflow as if it is a zero-coupon bond of the same maturity, as in Slide 31 of Lecture 6.

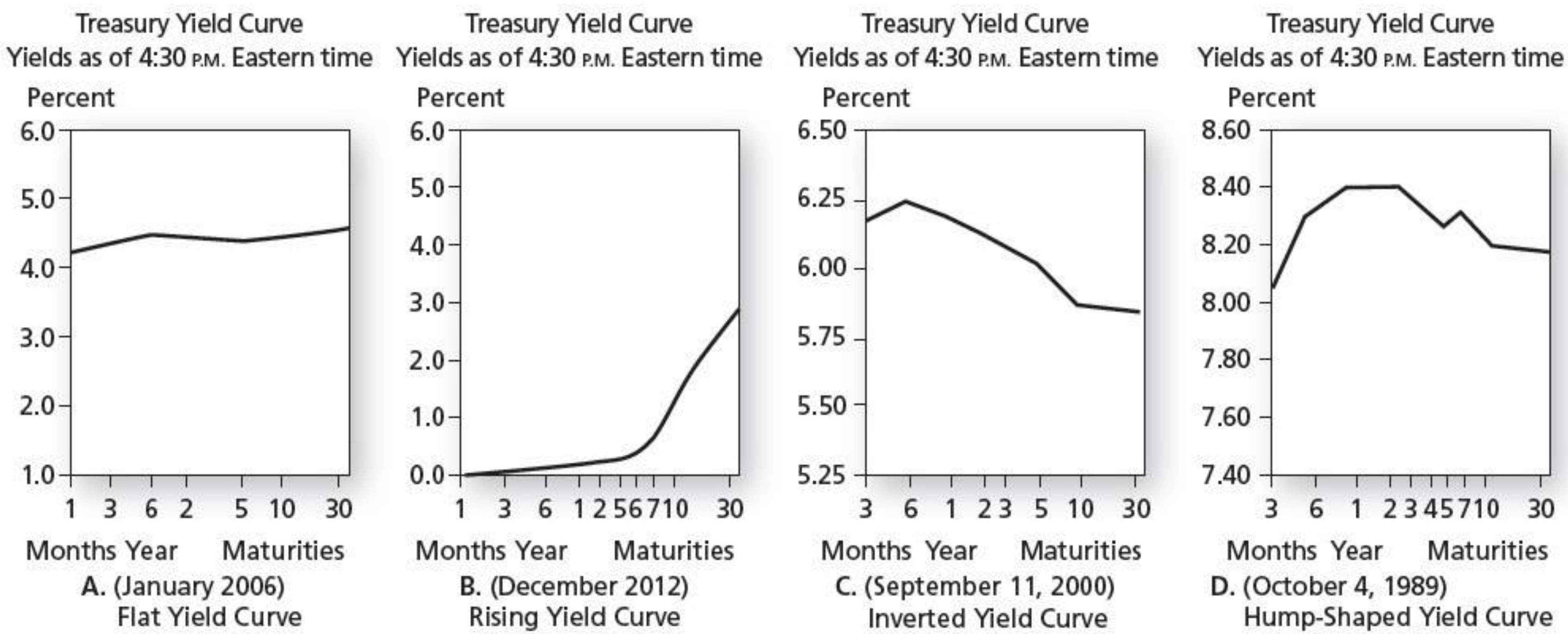
Default Risk

- Rating companies
  - Moody's, Standard & Poor's, Fitch
- Rating Categories
  - Highest rating is AAA or Aaa
  - Investment grade bonds are rated BBB or above (S&P, Fitch) or Baa and above (Moody's)
  - Speculative grade / junk bonds / high-yield bonds have ratings below BBB or Baa
- Higher probability of default for bonds with lower ratings.
  The risk of default rises in a recession

The Yield Curve

- Term structure of interest rates is the structure of interest rates for discounting cash flows of different maturities
- The yield curve is a graph that depicts the relationship between YTM and time to maturity
    - A graph of the term structure
- It is generally upward sloping, but can also be downward sloping or humped-shaped
- Information on expected future short-term rates can be implied from the yield curve
    - E.g., the expectations hypothesis

Figure 15.1 Treasury Yield Curves
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



The Yield Curve & Discount Rates

- The term structure can be used to compute the present and future values of a risk-free cash flow over different investment horizons

$$PV = \frac{C_n}{(1 + r_n)^n}$$

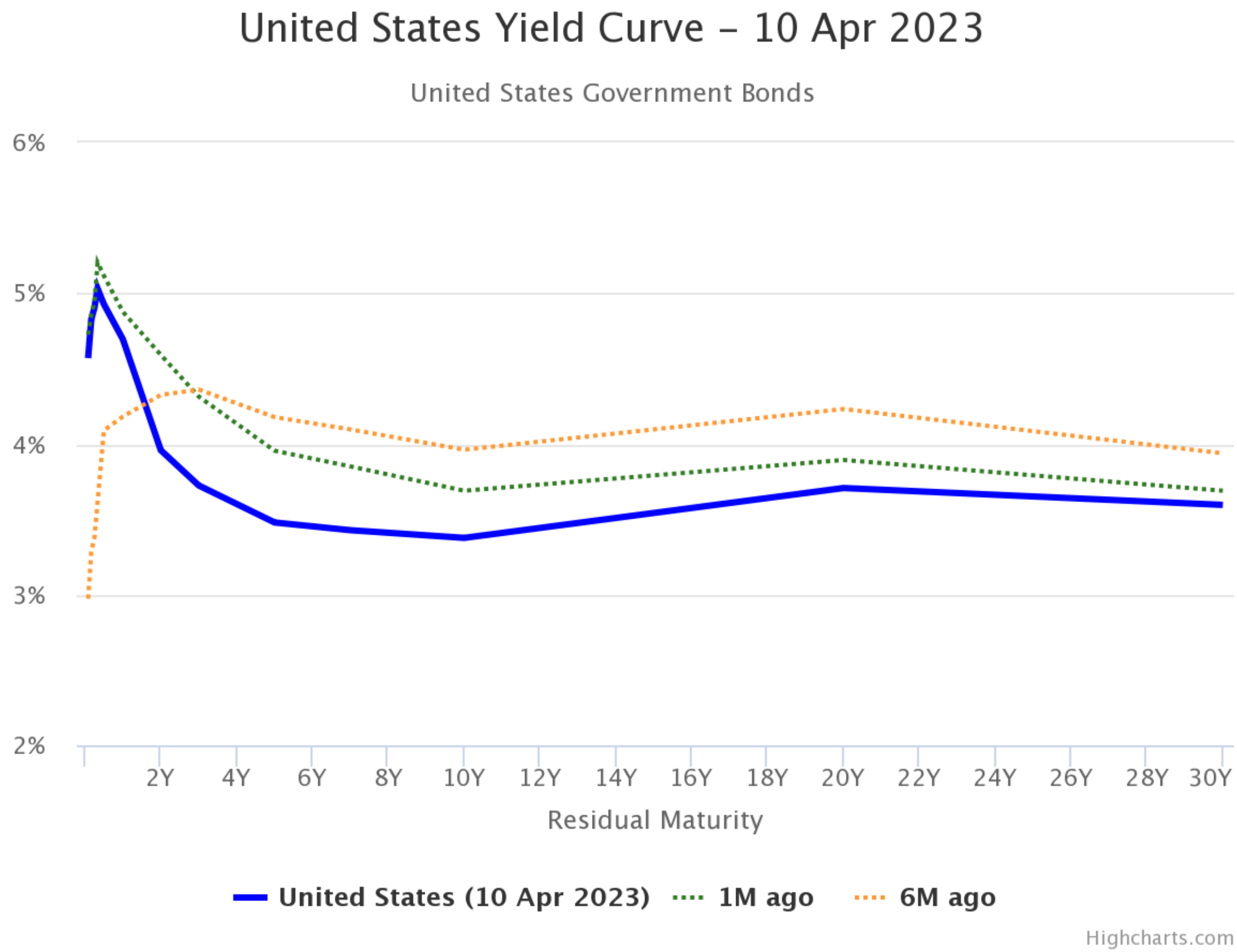
- Present Value of a Cash Flow Stream Using a Term Structure of Discount Rates

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \cdots + \frac{C_N}{(1 + r_N)^N} = \sum_{n=1}^{N} \frac{C_N}{(1 + r_n)^n}$$

The Yield Curve

- The yield curve is a graph that depicts the relationship between YTM and time to maturity
  - A graph of the term structure
- Information on expected future short-term rates can be implied from the yield curve
- Spot rate: YTM on zero-coupon bonds
  - The rate that prevails today for a given maturity
- Short rate: The rate for a given time interval or maturity (e.g. one year) at different points in time

Treasury Yield Curve



Focusing on 2 years Government Bond:

2Y vs 1Y	-73.6 bp	Yield Curve is inverted in Short-Term Maturities
5Y vs 2Y	-47.9 bp	Yield Curve is inverted in Mid-Term vs Short-Term Maturities
10Y vs 2Y	-58.2 bp	Yield Curve is inverted in Long-Term vs Short-Term Maturities

Source: http://www.worldgovernmentbonds.com/country/united-states/

Treasury Yield Curve

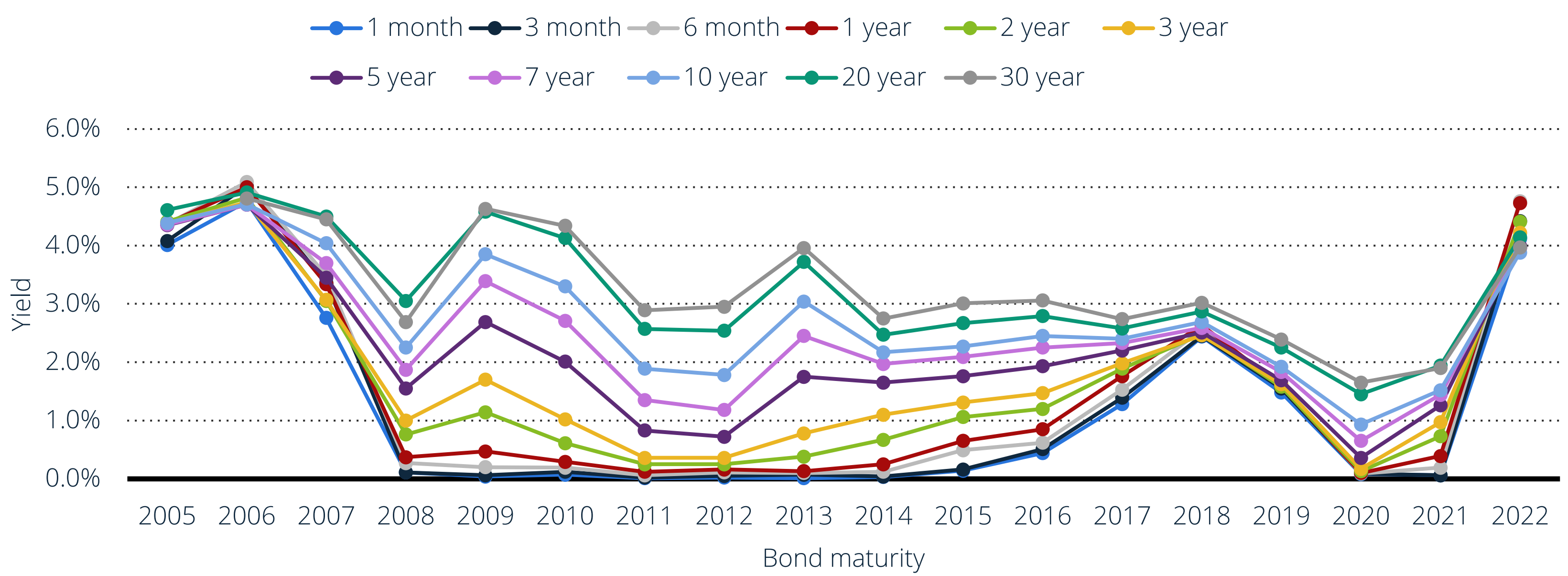
Source: http://www.worldgovernmentbonds.com/country/united-states/

Residual Maturity	Yield			ZC Price			Last Change
	Last	Chg 1M	Chg 6M	Last	Chg 1M	Chg 6M	
1 month	4.569%	-15.6 bp	+159.7 bp				09 Apr
2 months	4.827%	-2.8 bp	+154.1 bp				07 Apr
3 months	4.903%	-3.9 bp	+152.5 bp				10 Apr
4 months	5.030%	-17.3 bp	*n.a.*				07 Apr
6 months	4.927%	-18.9 bp	+83.7 bp				10 Apr
1 year	4.694%	-18.2 bp	+51.3 bp	95.52	+0.18 %	-0.49 %	10 Apr
2 years	3.958%	-63.2 bp	-36.7 bp	92.53	+1.21 %	+0.71 %	10 Apr
3 years	3.724%	-59.0 bp	-63.6 bp	89.61	+1.71 %	+1.85 %	10 Apr
5 years	3.478%	-47.6 bp	-69.9 bp	84.29	+2.33 %	+3.42 %	10 Apr
7 years	3.427%	-42.3 bp	-66.9 bp	78.99	+2.91 %	+4.62 %	10 Apr
10 years	3.376%	-31.3 bp	-58.5 bp	71.75	+3.07 %	+5.81 %	10 Apr
20 years	3.707%	-18.6 bp	-52.4 bp	48.29	+3.65 %	+10.60 %	10 Apr
30 years	3.595%	-9.4 bp	-34.2 bp	34.66	+2.76 %	+10.38 %	10 Apr

Last Update: 10 Apr 2023 2:15 GMT+0

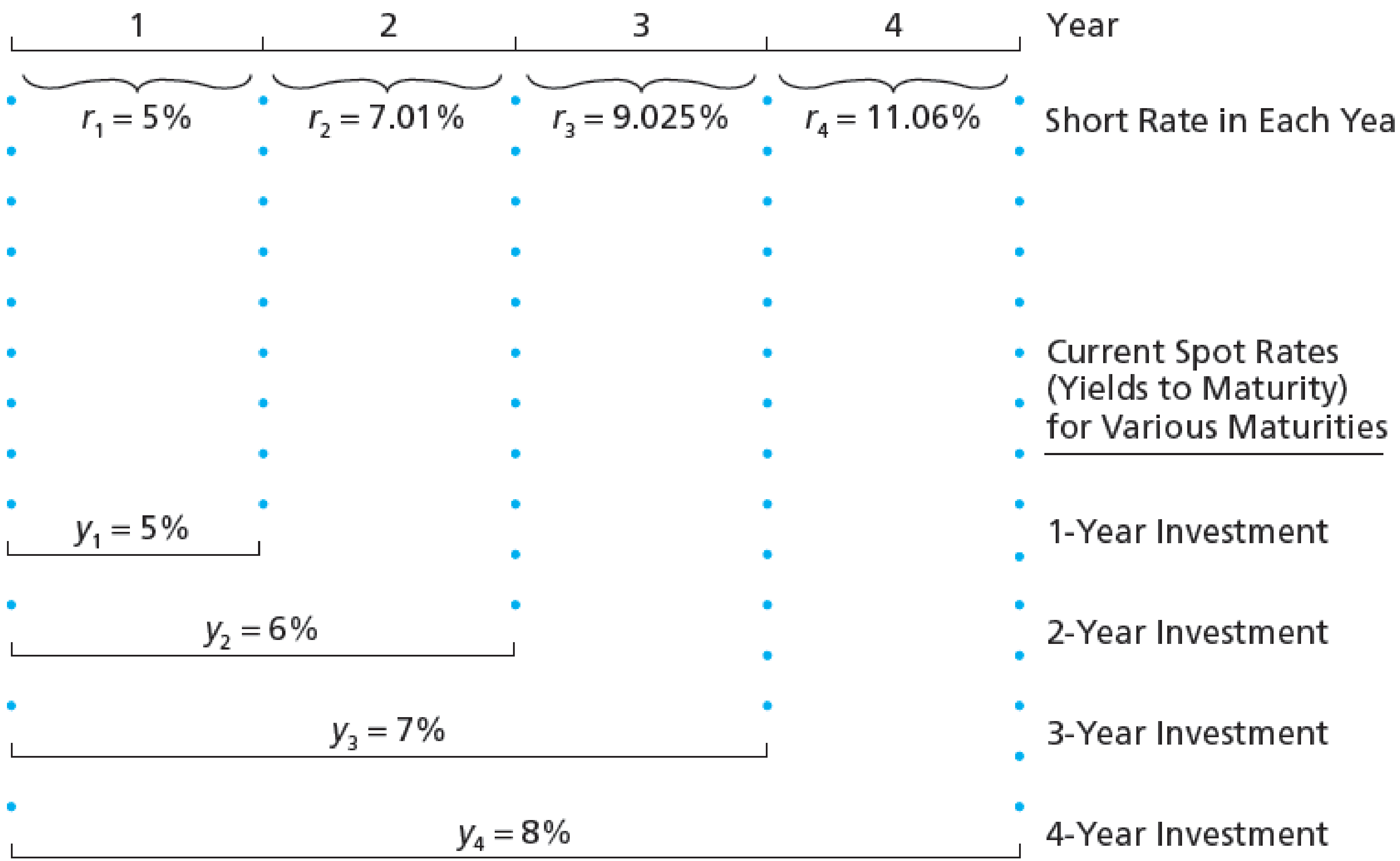
Treasury yield curve in the United States from 2005 to 2022, by maturity

Treasury yield rates in the U.S. 2005-2022, by maturity



Note(s): United States; 2005 to 2022
Further information regarding this statistic can be found on page 8.
Source(s): US Department of the Treasury; ID 1059669

Figure 15.2 Short Rates versus Spot Rates
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Pricing a bond

- Compute the present value of a risk-free three-year annuity of $500 per year, given the following yield curve:

Zero Coupon Treasury Rates	
Term (Years)	**Rate**
1	0.261%
2	0.723%
3	1.244%

- Each cash flow must be discounted by the corresponding interest rate:

$$PV = \frac{\$500}{1.00261} + \frac{\$500}{1.00723^2} + \frac{\$500}{1.01244^3} = \$1,473.34$$

Recall from Tutorial 7:

4. Assume there are four default-free bonds with the following prices and future cash flows:

		Cash Flows		
Bond	**Price Today**	**Year 1**	**Year 2**	**Year 3**
A	$934.58	1000	0	0
B	881.66	0	1000	0
C	1,118.21	100	100	1100
D	839.62	0	0	1000

Do these bonds present an arbitrage opportunity? If so, how would you take advantage of this opportunity? If not, why not?

Recall from Tutorial 7:

To determine whether these bonds present an arbitrage opportunity, check whether the pricing is internally consistent. Calculate the spot rates implied by Bonds A, B, and D (the zero-coupon bonds), and use this to check Bond C. (You may alternatively compute the spot rates from Bonds A, B, and C, and check Bond D, or some other combination.)

$$934.58 = \frac{1000}{(1+YTM_1)} \Rightarrow YTM_1 = 7.0\%$$

$$881.66 = \frac{1000}{(1+YTM_2)^2} \Rightarrow YTM_2 = 6.5\%$$

$$839.62 = \frac{1000}{(1+YTM_3)^3} \Rightarrow YTM_3 = 6.0\%$$

Given the spot rates implied by Bonds A, B, and D, the price of Bond C should be

$$\frac{100}{(1+0.07)} + \frac{100}{(1+0.065)^2} + \frac{1100}{(1+0.06)^3} = \$1105.21$$

Its price really is \$1118.21, so it is overpriced by \$13 per bond. Yes, there is an arbitrage opportunity.

Forward Interest Rates

- An interest rate forward contract is a contract today that fixes the interest rate for a loan or investment in the future.
- A forward rate is an interest rate that we can guarantee today for a loan or investment that will occur in the future.
- Here we consider interest rate forward contracts for one-year investments, so the forward rate for year 5 means the rate available today on a one-year investment that begins four years from today.
- By the law of one price, the forward rate for year 1 (f_1) is equal to the yield to maturity of a one-year zero-coupon bond (y_1).

$$f_1 = y_1$$

Forward Interest Rate

- In general

$$(1+f_n)=\frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}}$$

  - f_n = One-year forward rate for period n
  - y_n = Yield for a security with a maturity of n

$$(1+y_n)^n=(1+y_{n-1})^{n-1}(1+f_n)$$

Comparison

- Spot rate: YTM on zero-coupon bonds
  - The rate that prevails today for a given maturity
- Short rate: The rate for a given time interval or maturity (e.g. one year) at different points in time
- Forward rate: An interest rate forward contract is a contract today that fixes the interest rate for a loan or investment in the future. Here we consider interest rate forward contracts for one-year investments, so the forward rate for year 5 means the rate available today on a one-year investment that begins four years from today.

$$(1+y_n)^n = (1+y_{n-1})^{n-1}(1+f_n)$$

f_n = One-year forward rate for period n

y_n = Yield for a security with a maturity of n

Theories of the Term Structure of Interest Rates – The Expectations Hypothesis

- Risk Neutrality → the Expectations Hypothesis
  - Risk neutral investors → only expected returns matter → Bonds of different maturities are perfect substitutes
  - No arbitrage → $\boldsymbol{f_n = E(r_n)}$

Theories of the Term Structure of Interest Rates – The Expectations Hypothesis

- The Expectations Hypothesis
  - Observed long-term rate is a function of today's short-term rate and expected future short-term rates

$$f_n = E(r_n)$$

  - The interest rate on a long-term bond will be equal to the geometric average (or, *approximately*, arithmetic average) of the short-term interest rates that people expect to occur over the life of the long-term bond

Theories of the Term Structure of Interest Rates – Liquidity Premium Theory

- Risk averse investors require a risk premium to hold a longer-term bond → the Liquidity Preference Theory
- This **liquidity premium** compensates short-term investors for the uncertainty about future prices

Theories of the Term Structure of Interest Rates –
Liquidity Premium Theory

- The Liquidity Preference Theory
- Long-term bonds are more risky

$$f_n > E(r_n)$$

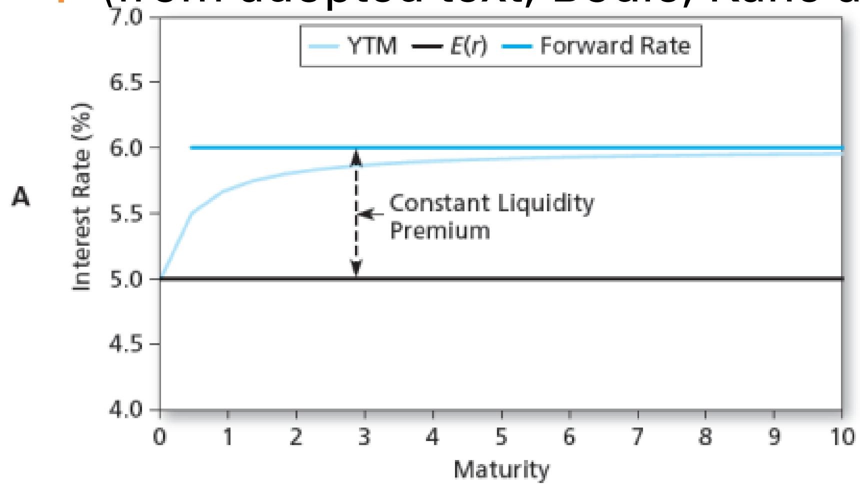
- The excess of f_n over $E(r_n)$ is the *liquidity premium*
  - Predicted to be positive
- Yield curve has an upward bias built into the long-term rates because of the liquidity premium

Interpreting the Term Structure

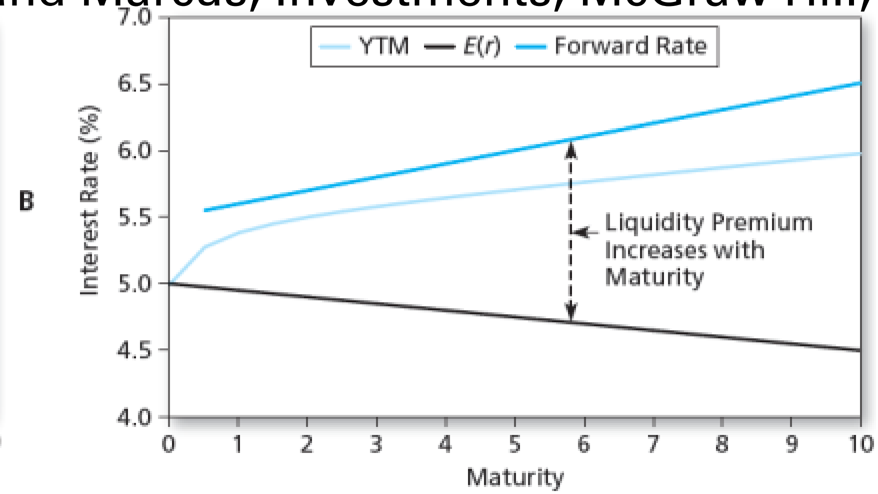
- Yield curve reflects expectations of future short rates, but also reflects other factors such as liquidity premiums
- An upward sloping curve could indicate:
  - Rates are expected to rise

    and/or
  - Investors require large liquidity premiums to hold long term bonds
- The yield curve has been used by the markets as a predictor of the business cycle
  - Long-term rates tend to rise in anticipation of economic expansion
  - Inverted yield curve may indicate that interest rates are expected to fall and signal a recession

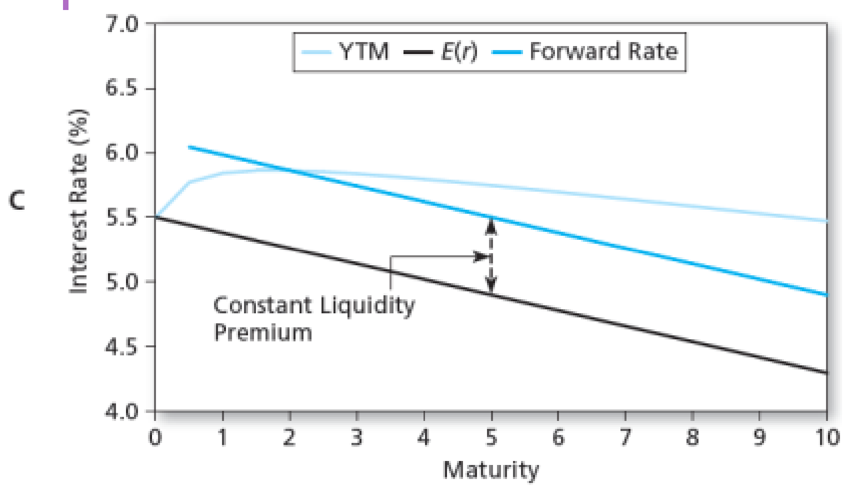
Figure 15.4 Yield curves
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



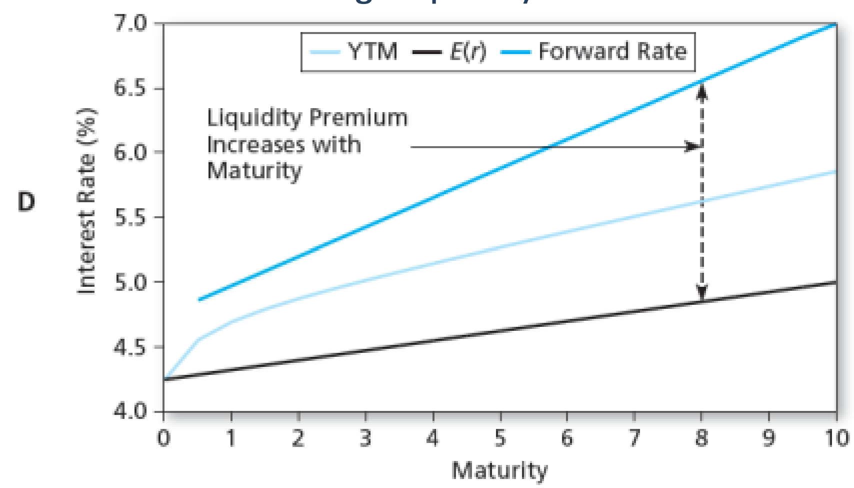
Panel A:
Constant Expected Short Rate.
Liquidity Premium of 1%.



Panel B:
Declining Expected Short Rates.
Increasing Liquidity Premiums.



Panel C:
Declining Expected Short Rates.
Constant Liquidity Premiums.

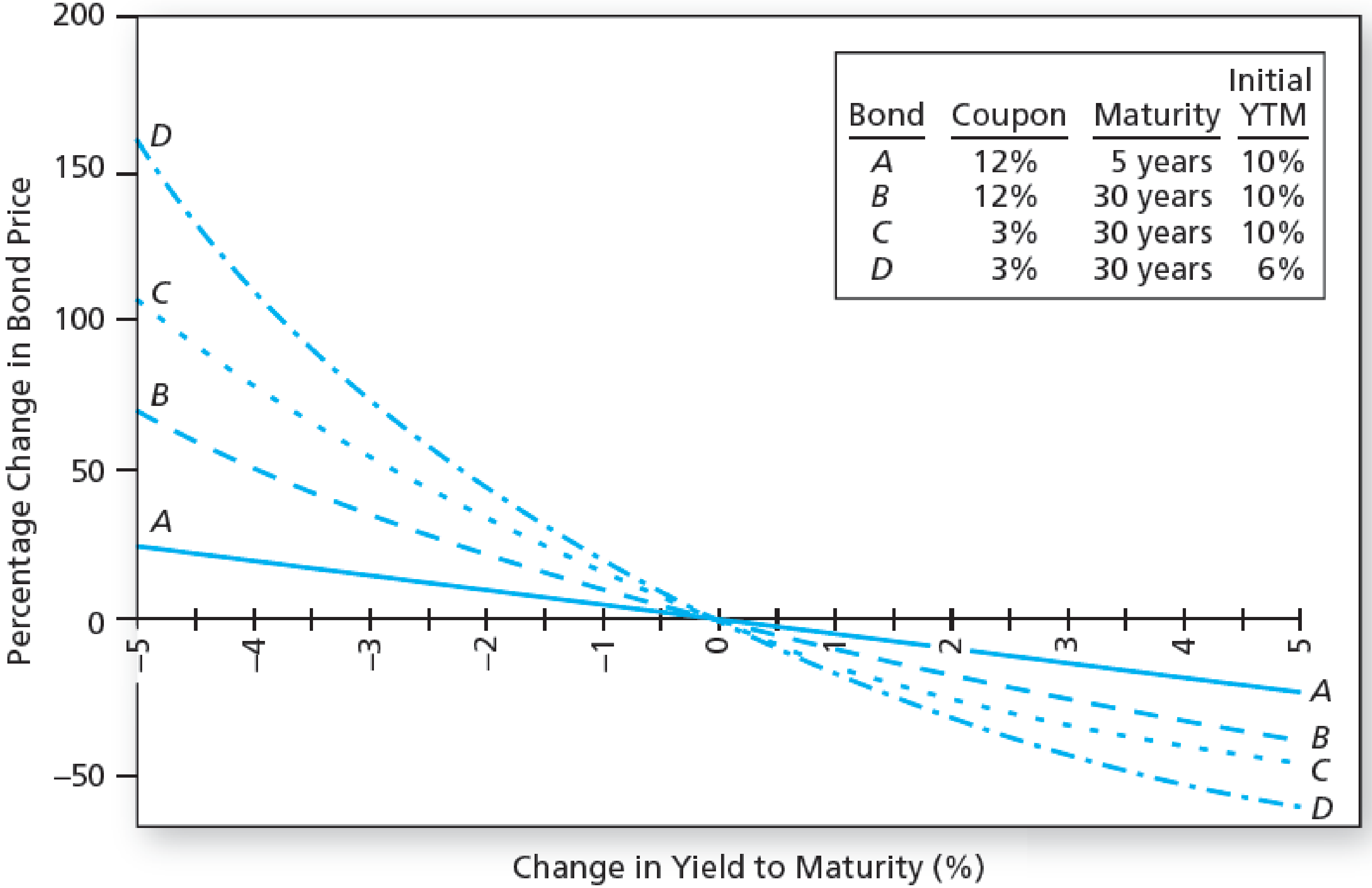


Panel D:
Increasing Expected Short Rates.
Increasing Liquidity Premiums.

L7: Learning Objectives

- Explain the sensitivity of bond prices to interest rates fluctuations.
- Compute various measures of sensitivity in terms of Macauley's duration and modified duration.
- Compute refinements of interest rate sensitivity measures, namely convexity.

Figure 16.1 Change in Bond Price as a Function of Change in Yield to Maturity
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Bond	Coupon	Maturity	Initial YTM
A	12%	5 years	10%
B	12%	30 years	10%
C	3%	30 years	10%
D	3%	30 years	6%

General properties of Interest Rate Sensitivity

Properties 1 to 5 are known as Malkiel's bond-pricing relationships (1962)
Property 6 was demonstrated by Homer and Liebowitz (1972)

1. Bond prices and yields are inversely related
2. An increase in a bond's yield to maturity results in a smaller price change than a decrease in yield of equal magnitude
3. Prices of long-term bonds tend to be more sensitive to interest rate changes than prices of short-term bonds
4. Interest rate risk is less than proportional to bond maturity
5. Interest rate risk is inversely related to the bond's coupon rate
6. The sensitivity of a bond's price to a change in its yield is inversely related to the YTM at which the bond is currently selling

Interest Rate Risk

- Higher-coupon-rate bonds have higher fraction of value tied to coupons rather than the final payment of par value.
- Zero-coupon bond = only one cash flow (CF) at maturity
- Coupon bond = a "portfolio" of CFs; compared to Zero-coupon bond, the portfolio of cashflows is more heavily weighted towards the earlier, short-maturity payments.
- Thus, coupon bond has lower "effective maturity" than zero-coupon bond with the same time to maturity.
- Hence, coupon bond is less price sensitive than zero-coupon bond

Duration

- Macaulay’s duration equals the weighted average of the times to each coupon or principal payment, using the relative present values of the cash flows as weights.
- The weight = the present value of the cash flow divided by the bond price
- It is a measure of the average maturity of a bond’s promised cash flows
- Macaulay’s Duration = Maturity for zero coupon bonds
- Macaulay’s Duration < Maturity for coupon bonds

Macaulay's Duration

$$D = \sum_{t=1}^{T} \frac{PV(C_t)}{P} \times t$$

$$PV(C_t) = \frac{C_t}{(1+y)^t} \quad \text{and} \quad P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

- y = YTM
- C_t = Cash flow at time t

Duration as Price Sensitivity

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

$$\frac{\partial P}{\partial y} = -\sum_{t=1}^{T} t \times \frac{C_t}{(1+y)^{t+1}}$$

$$-\frac{1}{P}\frac{\partial P}{\partial y} = \frac{1}{P}\frac{1}{(1+y)}\sum_{t=1}^{T} t \times \frac{C_t}{(1+y)^t}$$

$$-\frac{1}{P}\frac{\partial P}{\partial y} = \frac{1}{(1+y)}\sum_{t=1}^{T} t \times \frac{PV(C_t)}{P}$$

$$-\frac{1}{P}\frac{\partial P}{\partial y} = D^*$$

Duration as Price Sensitivity – Modified Duration

- Price change is proportional to duration and not to maturity

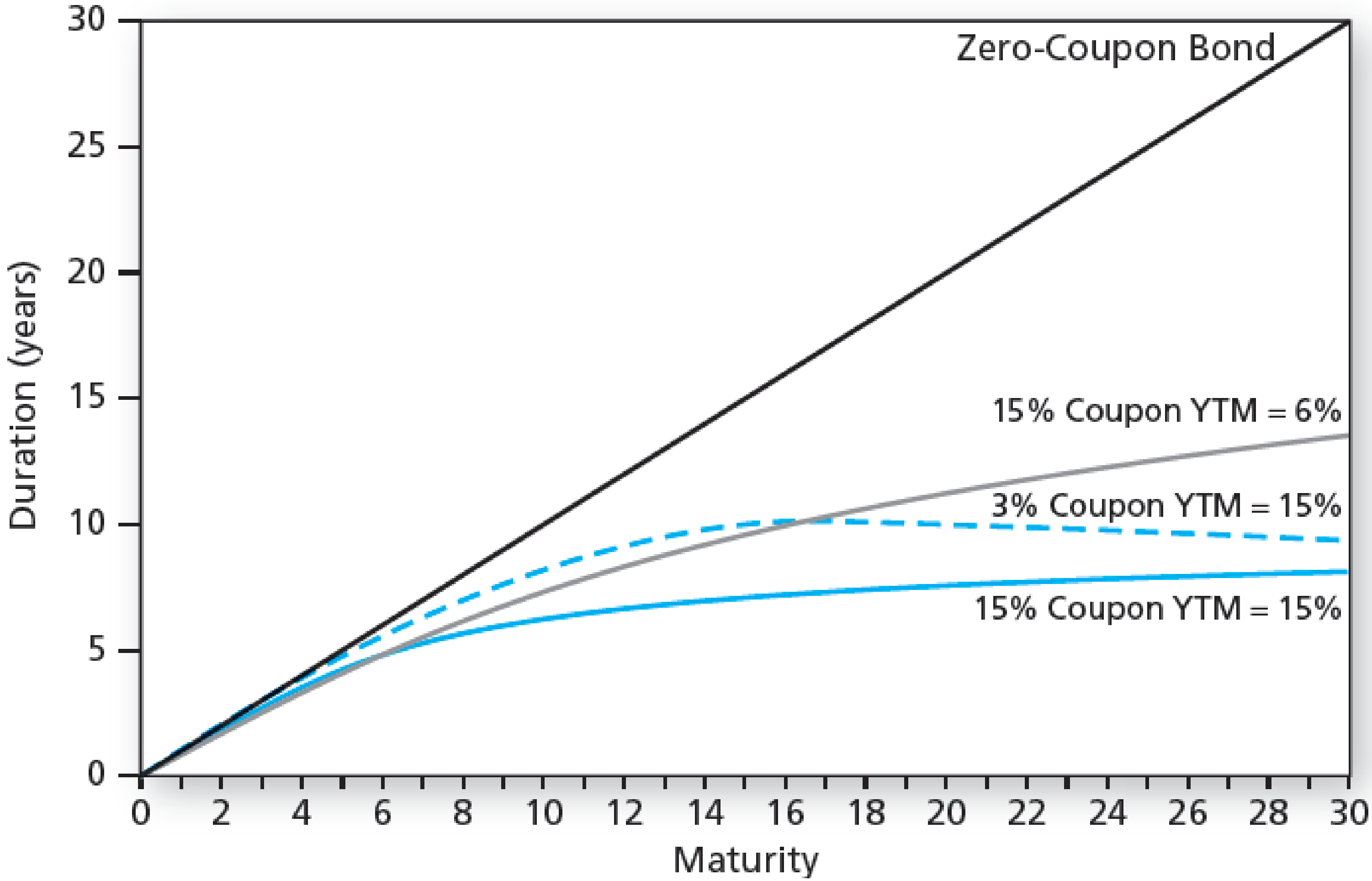
$$\frac{\Delta P}{P} = -D^* \times \Delta y$$

- D^* = D / (1+y) = Modified duration

Duration: Useful Rules of Thumb

1. The duration of a zero-coupon bond equals its time to maturity
2. Holding maturity constant, a bond's duration is lower when the coupon rate is higher
3. Holding the coupon rate constant, a bond's duration generally increases with its time to maturity
4. Holding other factors constant, the duration of a coupon bond is higher when the bond's yield to maturity is lower
5. The duration of a level perpetuity is equal to: $\frac{1+y}{y}$

Figure 16.2 Bond Duration versus Bond Maturity
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



The Duration of a Portfolio

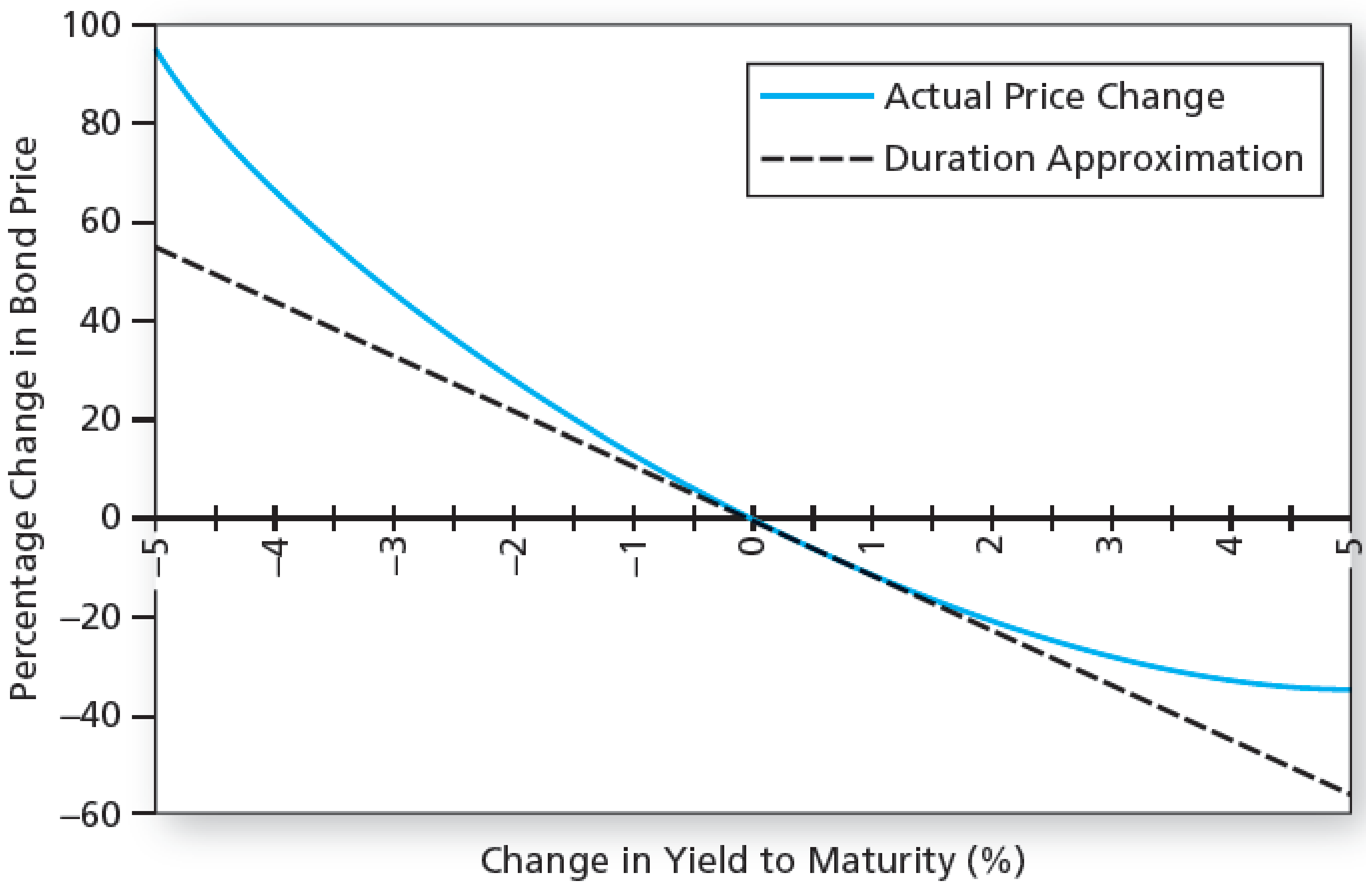
- The duration of a portfolio of investments is the value-weighted average of the durations of each investment in the portfolio.
  - A portfolio of securities with market values A and B and durations D_A and D_B, respectively, has the following duration:

$$D_{A+B} = \frac{A}{A+B} D_A + \frac{B}{A+B} D_B$$

Convexity

- The relationship between bond prices and yields is not linear, but convex
- Duration rule is a good approximation for only small changes in bond yields
- The curvature of the price-yield curve is called the convexity of the bond
- Bonds with greater convexity have more curvature in the price-yield relationship
- Formally, convexity is defined as the rate of change of the slope of the price-yield curve (i.e., second derivative of the price-yield curve), expressed as a fraction of the bond price.

Figure 16.3 Bond Price Convexity
(30-Year Maturity; 8% Coupon; Initial YTM = 8%)
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Convexity

$$\text{Modified Duration } D^* = -\frac{1}{P} \times \frac{dP}{dy}$$

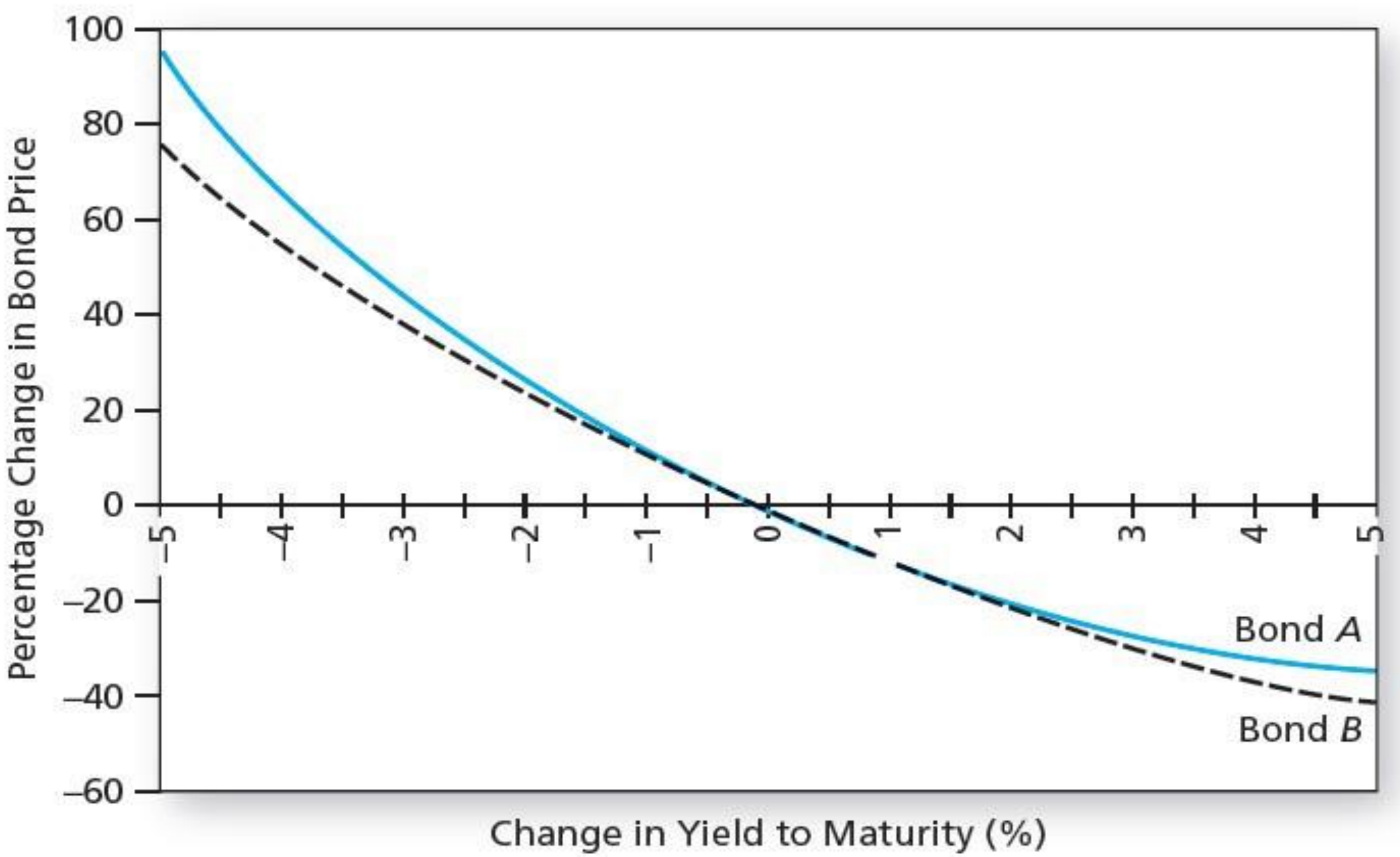
$$\text{Convexity} = \frac{1}{P} \times \frac{d^2P}{dy^2}$$

$$\text{Convexity} = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{n} \left[\frac{C_t}{(1+y)^t} \left(t^2 + t\right) \right]$$

- Correction for Convexity:

$$\frac{\Delta P}{P} = -D^* \times \Delta y + \frac{1}{2} \times \text{ Convexity } \times (\Delta y)^2$$

Figure 16.4 Convexity of Two Bonds
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



- Bond A benefits more from rate decreases and suffer less from rate increases

Immunization

- A way to control interest rate risk that is widely used by pension funds, insurance companies, and banks
- If there is a mismatch between asset and liability maturity structures, the pension fund's or insurance company's ability to meet future obligations will fluctuate with interest rates
- In a portfolio, the interest rate exposure of assets and liabilities are matched by matching the duration of the assets and liabilities
- As a result, value of assets will track the value of liabilities whether rates rise or fall
- A portfolio with a zero duration is called a duration-neutral portfolio or an immunized portfolio

Q&A

- I am not still not very sure what it means to "immunise" your portfolio from lecture 7 - why do we have to match both the duration and present value of the obligation (payment promised to customers) to immunise a portfolio?
- Specifically pertaining to problem set 8 Q2 when you explained an alternative way to solve the problem (from the answer key), I am not quite sure where the value 14,250 bonds comes from.

Another way to see this is to note that each bond with par value \$1,000 sells for \$407.12. If total market value is \$5.8 million, then you need to buy approximately 14,250 bonds, resulting in total par value of \$14.25 million

Duration as Price Sensitivity – Modified Duration

- Price change is proportional to duration and not to maturity

$$\frac{\Delta P}{P} = -D^{*} \times \Delta y$$

- D^{*} = D / (1+y) = Modified duration

The Duration of a Portfolio

- The duration of a portfolio of investments is the value-weighted average of the durations of each investment in the portfolio.
    - A portfolio of securities with market values *A* and *B* and durations D_A and D_B, respectively, has the following duration:

$$D_{A+B} = \frac{A}{A+B} D_A + \frac{B}{A+B} D_B$$

Q&A

- I am not still not very sure what it means to "immunise" your portfolio from lecture 7 - why do we have to match both the duration and present value of the obligation (payment promised to customers) to immunise a portfolio?
- Specifically pertaining to problem set 8 Q2 when you explained an alternative way to solve the problem (from the answer key), I am not quite sure where the value 14,250 bonds comes from.

Another way to see this is to note that each bond with par value $1,000 sells for $407.12. If total market value is $5.8 million, then you need to buy approximately 14,250 bonds, resulting in total par value of $14.25 million

If the face value of each 20-year bond is $1000, and each of this 20-year bond is currently trading at $407.12,
From part (a), you need to buy $5.8 million of 20-year bonds,
Thus, you need to buy $5,800,000/$407.12 = 14,250 units of 20-year bonds, each with a face value of $1000.

Q&A

- As for Q4 b, I am not sure where the formula (highlighted with the red box in the screenshot) comes from either.

b.

The market value of the zero must be \$11.57 million, the same as the market value of the obligations. Therefore, the face value must be:

$$\boxed{\$11.57 \text{ million} \times (1.10)^{1.8572} = \$13.81 \text{ million}}$$

4. An insurance company must make payments to a customer of $10 million in one year and $4 million in five years. The yield curve is flat at 10%.

a. If it wants to fully fund and immunize its obligation to this customer with a single issue of a zero-coupon bond, what maturity bond must it purchase?

b. What must be the face value and market value of that zero-coupon bond?

a.

$$D = \sum_{t=1}^{T} \frac{PV(C_t)}{P} \times t$$

$$PV(C_t) = \frac{C_t}{(1+y)^t} \text{ and } P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

(1) Time until Payment (Years)	(2) Cash Flow	(3) PV of CF (Discount Rate = 10%)	(4) Weight	(5) Column (1) × Column (4)
1	$10 million	$ 9.09 million	0.7857	0.7857
5	4 million	2.48 million	0.2143	1.0715
Column sums		$11.57 million	1.0000	1.8572

D = 1.8572 years = required maturity of zero coupon bond.

b.

The market value of the zero must be $11.57 million, the same as the market value of the obligations. Therefore, the face value must be:

$$\$11.57 \text{ million} \times (1.10)^{1.8572} = \$13.81 \text{ million}$$

Q&A

- As for Q4 b, I am not sure where the formula (highlighted with the red box in the screenshot) comes from either.

b.

The market value of the zero must be \$11.57 million, the same as the market value of the obligations. Therefore, the face value must be:

$$\boxed{\$11.57 \text{ million} \times (1.10)^{1.8572} = \$13.81 \text{ million}}$$

You need to match the present value for duration matching to immunize the portfolio.
From part (a), the present value of the liabilities (cash outflows) is \$11.57 million
Thus, you would need to match that with assets of the same present value and duration, which is \$11.57 million.
What is the face value of the zero coupon bond that you need to buy (future value of the current \$11.57 million)?
Given in the question that the yield curve is flat at 10%, YTM of a zero coupon bond that matures in 1.8572 years is 10%.
Thus, the face value = PV*(1.10)^1.8572

Duration-Based Hedging

- Immunization is not a once and for all strategy

- As interest rates change and the durations of assets and/or liabilities change (due merely to the passage of time), the portfolio must be rebalanced continuously to remain duration-neutral

Duration-Based Hedging

- A duration-neutral portfolio is only protected against parallel shifts in the yield curve when interest rate changes affect all yields identically.

- Even if assets have similar maturities, they may have different credit risks.

L8: Learning Objectives

- Calculate potential profits resulting from various option trading strategies.
- Formulate portfolio management strategies to modify the risk-return profiles of the portfolio.
- Understand and apply the put-call parity relationship.

Call Option

- A call option gives its holder the **right to buy** an asset:
  - At the exercise price (X) or strike price (K)
  - On or before the expiration date
- It's a right. Not an obligation
- So exercise the option only if it is profitable to do so
- **Exercise** the option to buy the underlying asset
  - **if market value > strike price**

Put Option

- A put option gives its holder the **right to sell** an asset:
  - At the exercise price (X) or strike price (K)
  - On or before the expiration date
- It's a right. Not an obligation
- So exercise the option only if it is profitable to do so
- **Exercise** the option to sell the underlying asset
  - **if market value < strike price**

American vs. European Options

- American - the option can be exercised at any time before expiration or maturity

- European - the option can only be exercised on the expiration or maturity date

Accounting for Stock Split and Dividend Payout

- To account for stock split, the exercise price is reduced by a factor of the split, and the number of options held is increased by that factor

- A similar adjustment is made for stock dividends of more than 10%

- Cash dividends do not affect the terms of an option contract
  - Call option values are lower for high-dividend payout policies because they slow the stock price increase
  - Vice versa for put option

- Anticipated dividend payments are factored into the option price

Values of Options at Expiration – Call Holder

- The value of the call option at expiration equals

$$\textbf{Payoff to call holder} = \begin{cases} S_T - X & \text{if } S_T > X \\ 0 & \text{if } S_T \leq X \end{cases}$$

- Where:
  - S_T is the value of the stock at expiration
  - X is the exercise price

- Call is exercised only if $S_T > X$

- Profit to Call Holder = Payoff – Premium

- If you **hold** a call option, you have a **long** call position

Figure 20.2
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

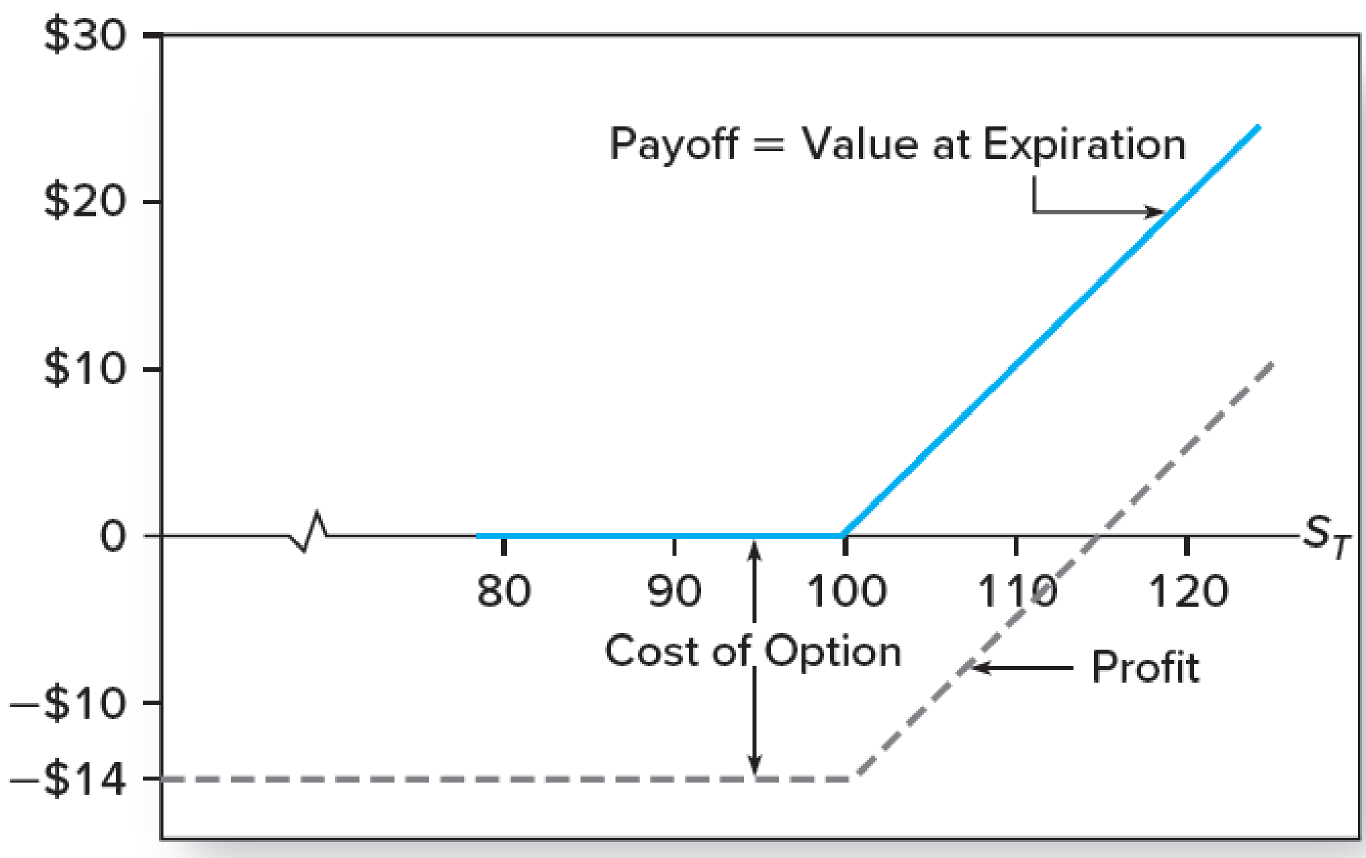


Figure 20.2 Payoff and profit to call option at expiration

Implications on the various strategies

- An option **offers leverage**
  - E.g. for Portfolio B, values respond more than proportionately to changes in stock price.

- Options can be used **as an insurance.**
  - E.g. Portfolio C, T-bill-plus-option strategy
  - Value of portfolio C cannot be worth less than $9,270 at the end of the year
  - Trade-off: Some return potential is sacrificed to limit downside risk; hence Portfolio C underperforms A by 9.33% when share price rises

Protective Put

- Puts can be used as **insurance** against stock price declines
  - **E.g. Protective Put: Invest in stock and buy a put option on the stock**
  - Protective puts lock in a minimum portfolio value
  - The cost of the insurance is the put premium

- That is, options can be used for risk management, not just for speculation
  - Some return potential is sacrificed to limit downside risk
  - The absolute limitation on downside risk is a novel and attractive feature of this strategy

Table 20.1
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Table 20.1

Value of a protective put portfolio at option expiration

	$S_T \leq X$	$S_T > X$
Stock	S_T	S_T
+ Put	$X - S_T$	0
Total	X	S_T

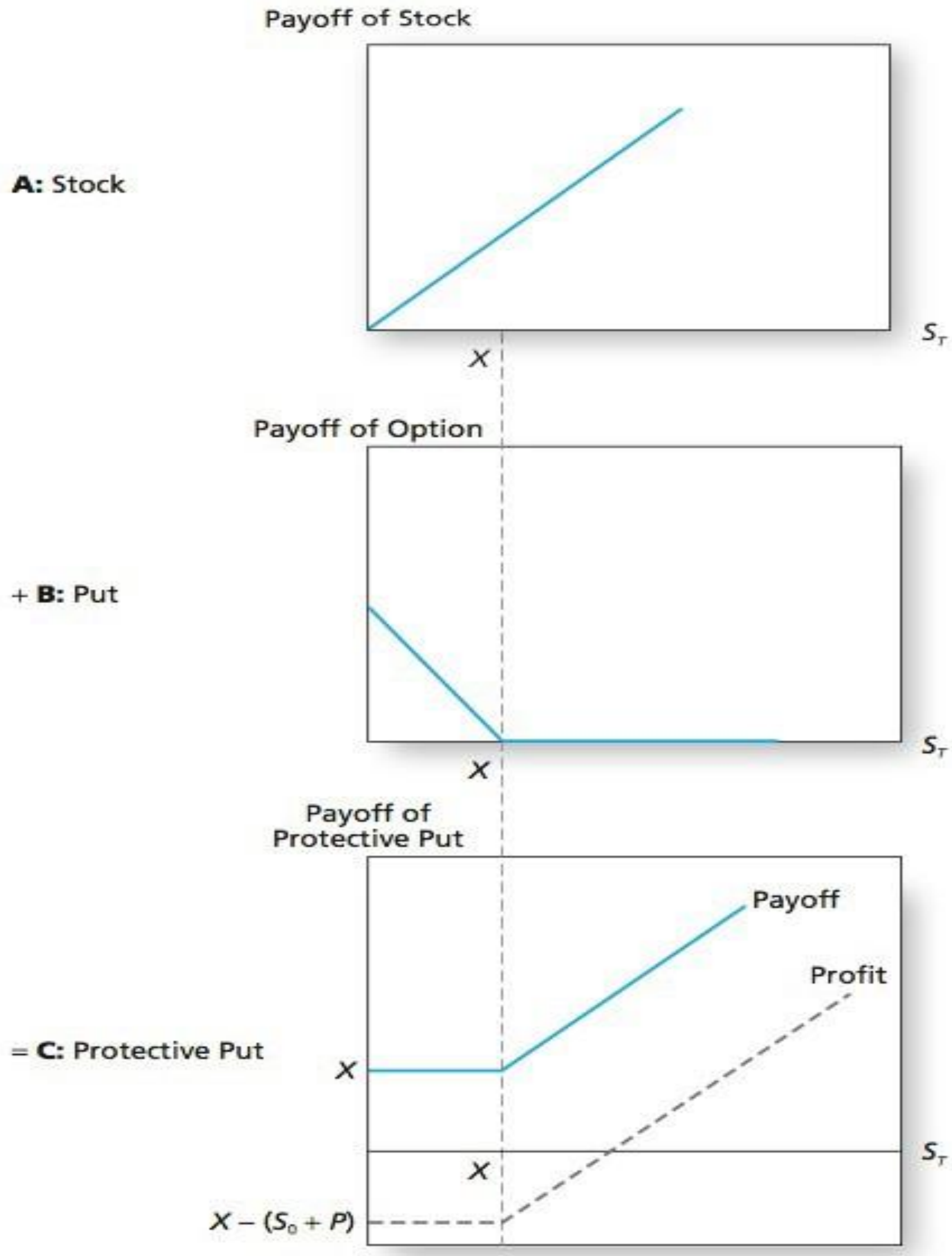


Figure 20.6 Value of a protective put position at expiration (from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Figure 20.7
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

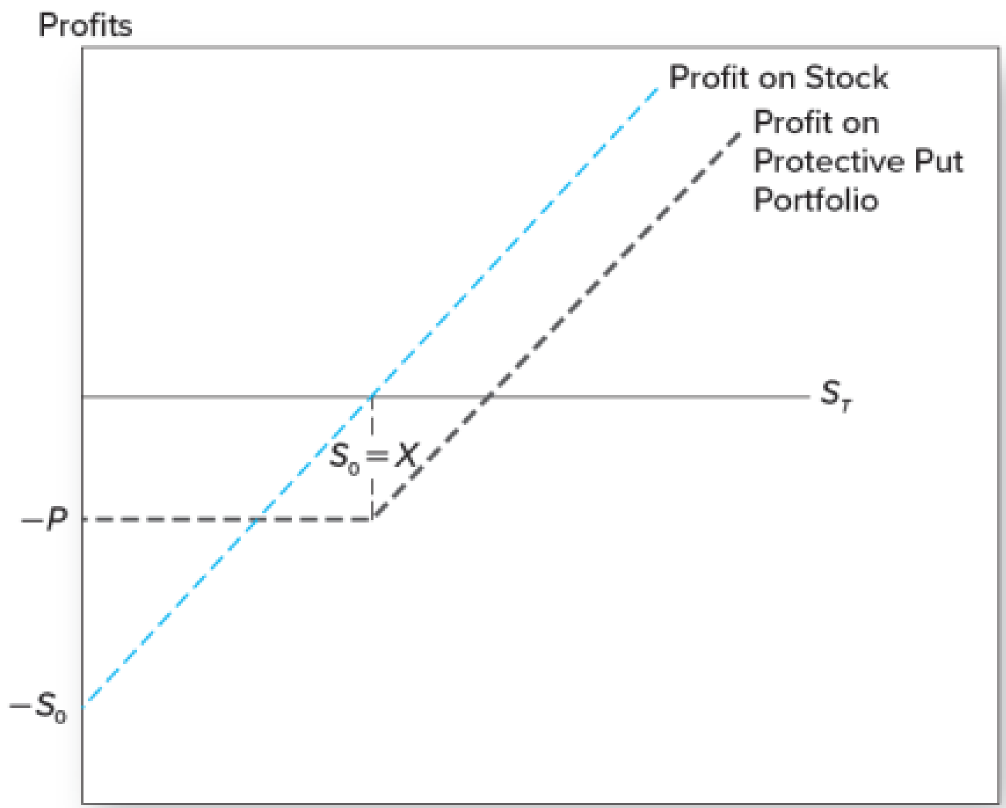


Figure 20.7 Protective put versus stock investment (at-the-money option)

Covered Calls

- The purchase of a **share of stock coupled with a sale of a call option on that stock**
- Payoff of a covered call position equals the stock value minus the value of the written call
- Trade-off: Investors forfeit potential capital gains, should the stock price rise above the exercise price

Value of a Covered Call Position at Option Expiration

Table 20.2
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Table 20.2

Value of a covered call position at option expiration

	$S_T \leq X$	$S_T > X$
Payoff of stock	S_T	S_T
+ Payoff of written call	-0	$-(S_T - X)$
Total	S_T	X

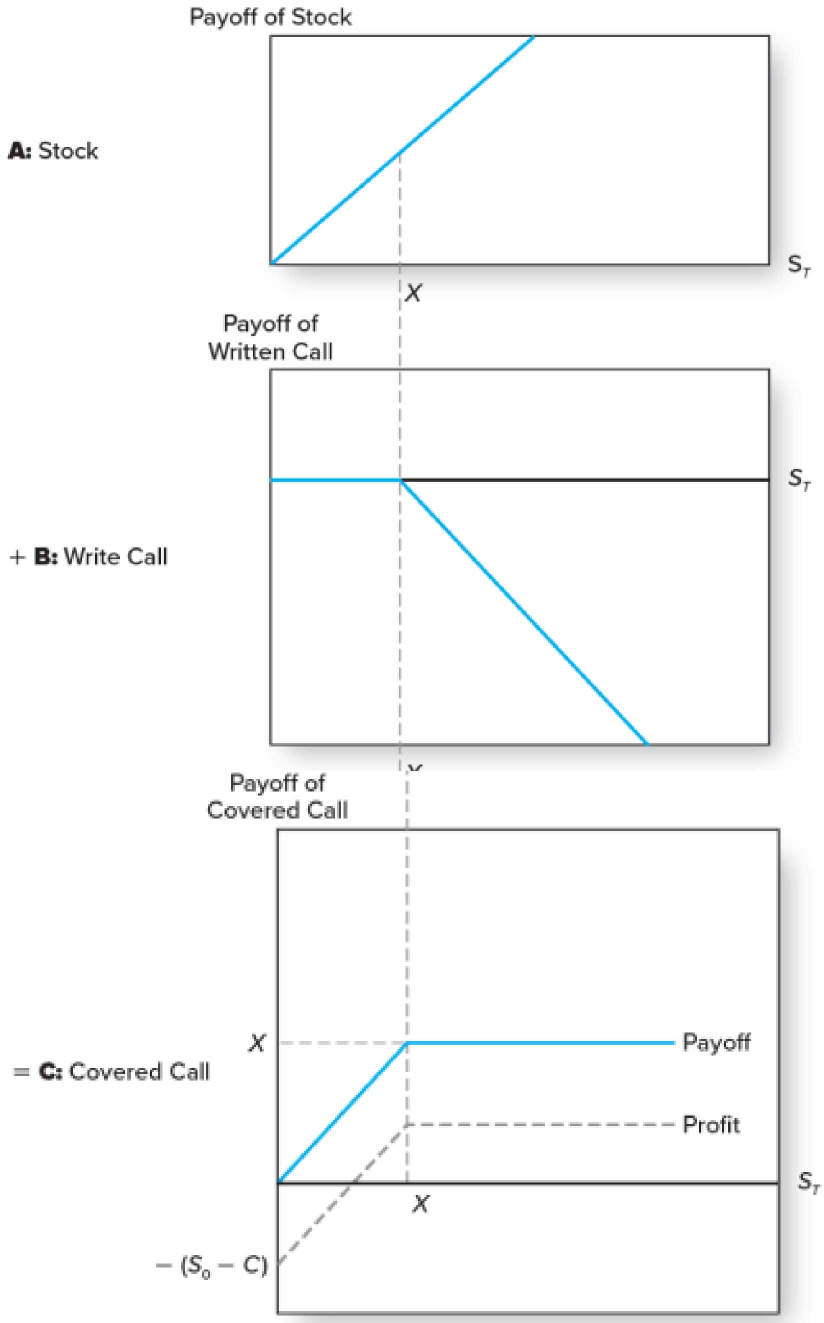


Figure 20.8 Value of a covered call position at expiration (from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Put-Call Parity

- A protective put portfolio provides a payoff with a guaranteed minimum value, but with unlimited upside potential
- Table 20.1
- Figure 20.6
- But a call-plus-bills portfolio can also provide similar payoff
- Buy a call option and buy zero-coupon Treasury bills with face value equal to the exercise price X of the call and with maturity date equal to the expiration date of the option

Value of a Call-plus-Bills Portfolio at Expiration

Source: adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e, page 678

	$S_T \le X$	$S_T > X$
Value of call option	0	$S_T - X$
Value of riskless bond	X	X
TOTAL	X	S_T

Table 20.1

Value of a protective put portfolio at option expiration

	$S_T \le X$	$S_T > X$
Stock	S_T	S_T
+ Put	$X - S_T$	0
Total	X	S_T

Put-Call Parity

- Since both portfolios provide the same exact payoffs, the call-plus-bond portfolio (on left) must cost the same as the stock-plus-put portfolio (on right):

$$C + \frac{X}{(1 + r_f)^T} = S_0 + P$$

- This equation, also known as the Put-call parity theorem depicts the relation between put and call prices
- Violation of parity implies the existence of arbitrage opportunities

Put-Call Parity - Disequilibrium Example with Arbitrage

Call Price = 17

Risk Free = 5%

X = 105

Stock Price = 110

Put Price = 5

Maturity = 1 yr

$$C + \frac{X}{(1+r_f)^T} >=<? S_0 + P$$

$$117 > 115$$

- Since the protective put is less expensive, acquire the low cost alternative and sell the high cost alternative

Put-Call Parity - Disequilibrium Example with Arbitrage

Table 20.5 Arbitrage strategy
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Table 20.5

Arbitrage strategy

Position	Immediate Cash Flow	Cash Flow in 1 Year	
		$S_T < 105$	$S_T \geq 105$
Buy stock	−110	S_T	S_T
Borrow \$105/1.05 = \$100	+100	−105	−105
Sell call	+17	0	$-(S_T - 105)$
Buy put	−5	$105 - S_T$	0
Total	2	0	0

L9: Learning Objectives

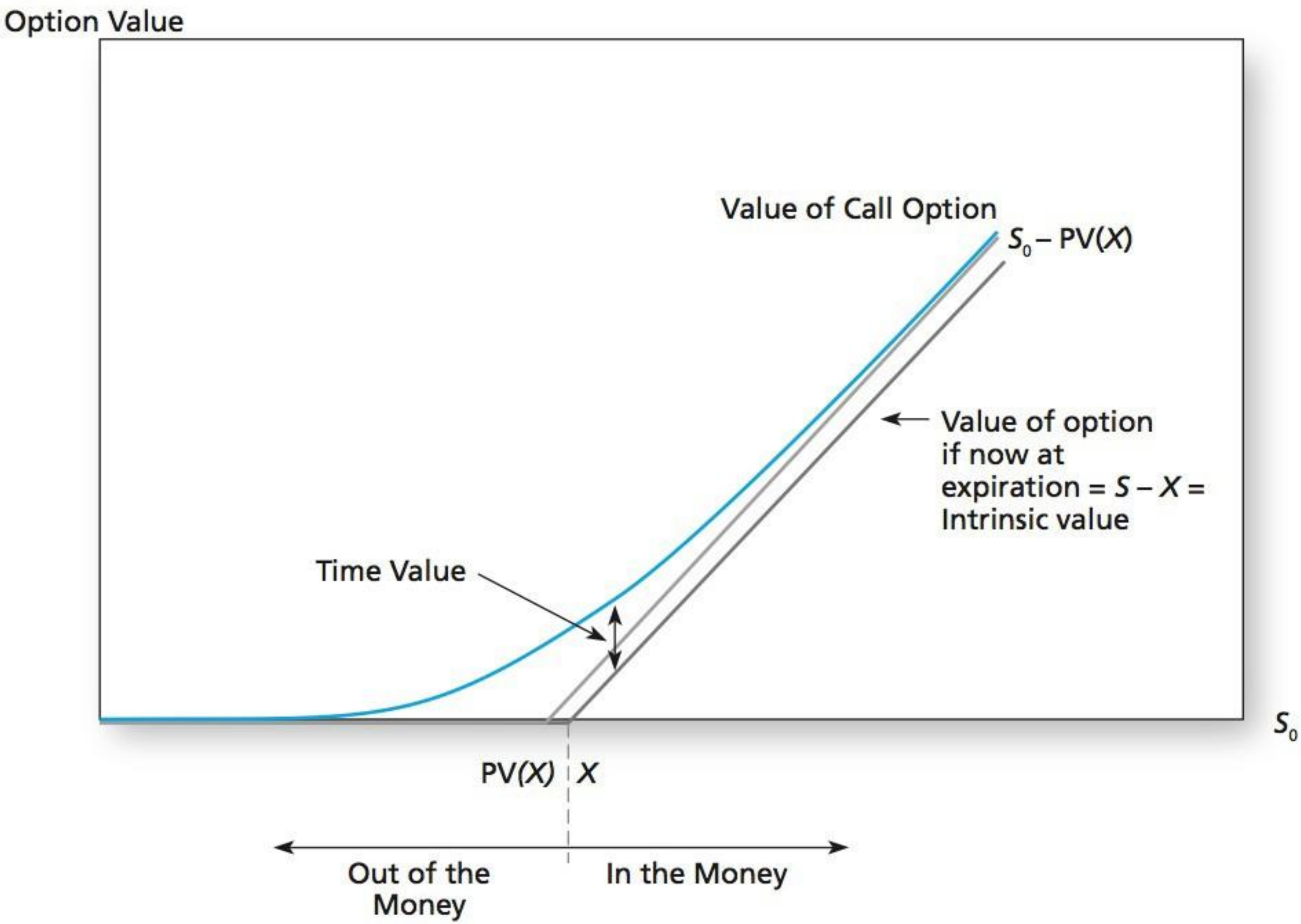
- Distinguish between the intrinsic and time value of an option.
- Examine the factors that affect the option prices.
- Construct a replicating portfolio for Binomial Option Pricing model, and value an option using the Law of One Price.

Intrinsic and Time Values

- **Intrinsic value** – the value the option would have if it expired immediately
  - It is the amount by which the option is currently in-the-money, or zero if the option is out-of-the-money.
  - **In-the-money call:**
    - current stock price - exercise price = $S_0 – X$
  - **In-the-money put:**
    - exercise price – current stock price = $X – S_0$

- **Time value** - the difference between the current option price and its intrinsic value
  - Arises because the option is yet to expire and from “volatility value”

Figure 21.1 Call option value before expiration
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Arbitrage Bounds on Option Prices

- A put option cannot be worth more than its strike price.
  - The maximum payoff for a put is when the stock becomes worthless, in which case the payoff is equal to the strike price. This is the highest payoff, so it cannot be worth more than the strike price.
- A call option cannot be worth more than the stock itself.
  - The lower the strike price, the more valuable the call option. If the call option had a strike price of zero, the holder would always exercise the option and receive the stock at zero cost. (Payoff of the call on a stock is maximum when the strike is zero.)

Arbitrage Bounds on Option Prices

- An American option is worth at least as much as its European counterpart.
  - Option price cannot be negative
  - American option carries the same rights and privileges as an equivalent European option, in addition to an early exercise feature absent in the European option

- An American option cannot be worth less than its intrinsic value.
  - If an American option is worth less than its intrinsic value, you could make arbitrage profits by purchasing the option and immediately exercising it.

- Because an American option cannot be worth less than its intrinsic value, it cannot have a negative time value.

Table 21.1 Determinants of call option values
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

If This Variable Increases . . .	The Value of a Call Option
Stock price, S	Increases
Exercise price, X	Decreases
Volatility, σ	Increases
Time to expiration, T	Increases
Interest rate, r_f	Increases
Dividend payouts	Decreases

The Binomial Option Pricing Model

- **Binomial Option Pricing Model**
  - A technique for pricing options based on the assumption that each period, the stock's return can take on only two values.
    - Up state: u
    - Down State: d
  - We can use this model to value any derivative securities.

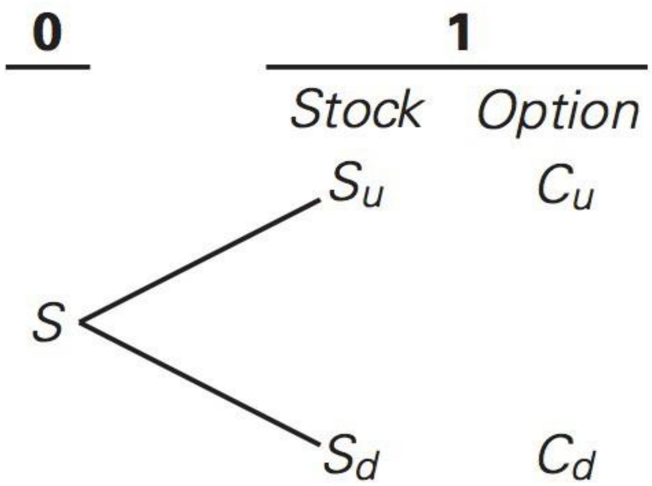
- **Binomial Tree**
  - A timeline with two branches at every date representing the possible events that could happen at those times.

A Two-State Single-Period Model

- **Replicating Portfolio**
  - A portfolio consisting of a stock and a risk-free bond that has the same value and payoffs in one period as an option written on the same stock.
  - The Law of One Price implies that the current value of the call and the replicating portfolio must be equal.

The Binomial Pricing Formula

- Given the above assumptions, the binomial tree would look like:



Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 798

- The payoffs of the replicating portfolios could be written as:

$$S_u\Delta + (1+r_f)B = C_u, \text{ and}$$
$$S_d\Delta + (1+r_f)B = C_d$$

The Binomial Pricing Formula

- Solving the two replicating portfolio equations for the two unknowns Δ and B, we get:
- Replicating Portfolio in the Binomial Model

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \qquad B = \frac{C_d - S_d\Delta}{1 + r_f}$$

$$B = \frac{C_u - S_u\Delta}{1 + r_f}$$

- Option Price in the Binomial Model

$$C = S\Delta + B$$

The Binomial Pricing Formula

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{\text{Spread of possible option prices}}{\text{Spread of possible asset price}}$$

- Δ = the sensitivity of the option's value to changes in the stock price
- You can replicate an investment in the call option by a levered investment in the underlying asset.
- Thus, if the option is not traded, you can DIY a homemade option by a replicating strategy.

A Multiperiod Model

- Dynamic Hedging Strategy
  - A replication strategy based on the idea that an option payoff can be replicated by dynamically trading in a portfolio of the underling stock and a risk-free bond.

  - In the two-period example, the replicating portfolio will need to be adjusted at the end of each period.

Binominal Option Pricing

- As the number of subperiods increases, the distribution approaches the skewed log-normal distribution (longer right tail).
- Even if the stock price were to decline in each subinterval, it can never drop below 0.
- But there is no corresponding upper bound on its potential upward performance.

L10: Learning Objectives

- Apply the Black-Scholes Option Pricing formula to value a European option.
- Construct Black-Scholes replicating portfolio for a European option.
- Explain the idea behind dynamic hedging, option delta and gamma.

Black-Scholes Option Pricing Model

- A technique for pricing European-style options when the stock can be traded continuously.
- It can be derived from the Binomial Option Pricing Model by:
  - allowing the length of each period to shrink to zero and
  - letting the number of periods grow infinitely large.
- All techniques used in financial engineering to price financial securities can be traced to the Black-Scholes formula.
- It uses the Law of One Price, without the need to model preferences.

Black-Scholes Formula for a Call

$$C_0 = S_0 \times N(d_1) - Xe^{-rT} \times N(d_2)$$

- Where: $$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

C_o = Current call option value

S_o = Current stock price

$N(d)$ = The cumulative normal distribution = $Pr(z \leq d)$
i.e., the probability that a random draw from a standard normal distribution will be less than d

X = Exercise price

e = 2.71828, the base of the natural log

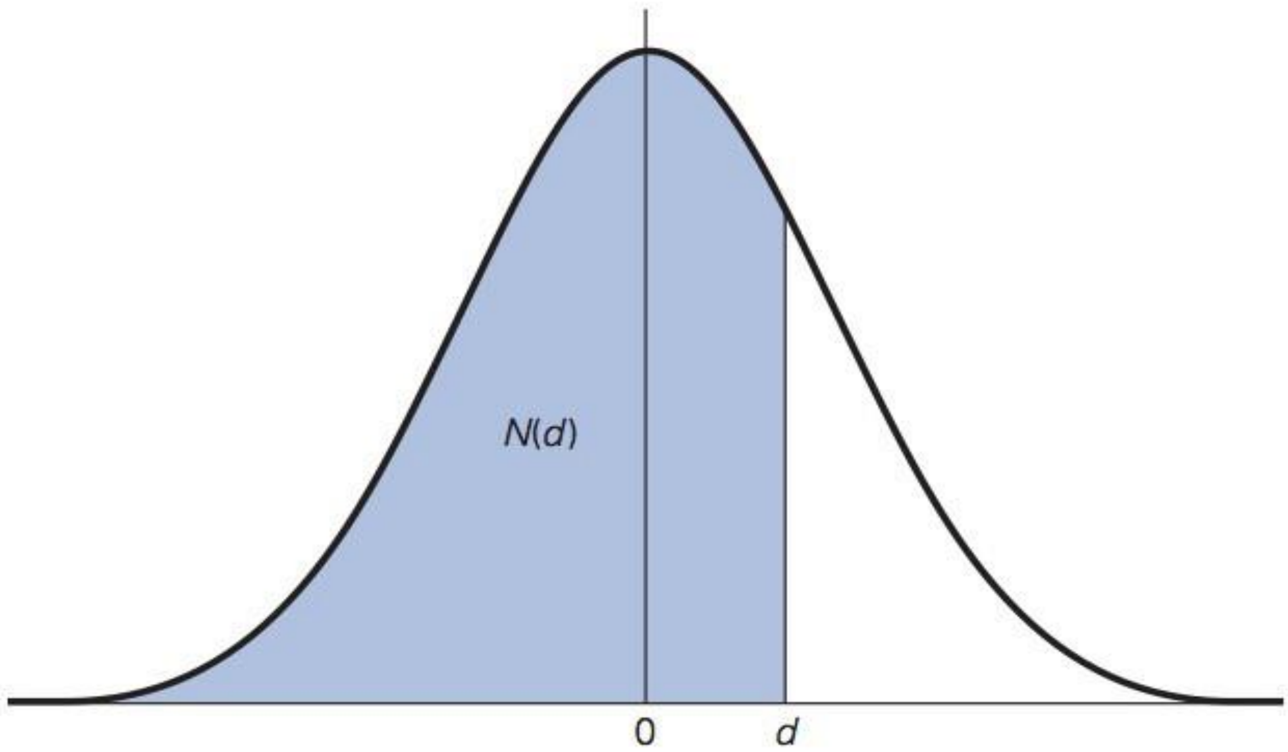
r = Risk-free interest rate

T = time to expiration of the option in years

ln = Natural log function

σ = Annual volatility (standard deviation) of the stock's return

Figure 21.3 A Standard Normal Distribution
from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e



Black-Scholes Model with Dividends

- The previous formula applies to stocks that do not pay dividends
- What if dividends ARE paid?
- Replace stock price with a dividend adjusted stock price
- Replace S_0 with S_0 – PV(Dividends)
  - Because a European call option is the right to buy the stock without the dividends paid during the life of the option
  - Because stock price tends to drop by the amount of the dividend when the stock goes ex-dividend

The Black-Scholes Formula for a Put

- Use the put-call parity theorem

$$P = C + Xe^{-rT} - S_0$$

- The value of a European put option is:

$$P = Xe^{-rT}[1 - N(d_2)] - S_0[1 - N(d_1)]$$

- Using Example 21.4 data:
- S_0 = 100, r = 0.10, X = 95, σ = 0.5, and T = 0.25

$$P = \$95\mathrm{e}^{-0.10\times0.25}(1 - 0.5714) - \$100(1 - 0.6664)$$
$$P = \$6.35$$

Recall from Tutorial 9

1. The common stock of the ABC Corporation has been trading in a narrow price range for the past month, and you are convinced it is going to break far out of that range in the next three months. You do not know whether it will go up or down, however. The current price of the stock is \$100 per share, and the price of a 3-month call option at an exercise price of \$100 is \$10.
   a. If the risk-free interest rate is 10% per year, what must be the price of a 3-month put option on ABC stock at an exercise price of \$100? (The stock pays no dividends.)
   b. What would be a simple options strategy to exploit your conviction about the stock price's future movements? How far would it have to move in either direction for you to make a profit on your initial investment?

a.
From put-call parity:

$$P = C - S_0 + \frac{X}{(1+r_f)^T} = 10 - 100 + \frac{100}{(1.10)^{0.25}} = \$7.65$$

Put Option: Using Put-Call Parity

$$
\begin{aligned}
P &= C + PV(X) - S_0 \\
&= C + Xe^{-rT} - S_0
\end{aligned}
$$

- Using the data in the example,

$$
P = 13.70 + 95e^{-0.10 \times 0.25} - 100
$$
$$
P = \$6.35
$$

The Option Delta Δ

- From the B-S formula, it can be proved that

$$\frac{\partial C}{\partial S} = N(d_1)$$

- Delta is the number of shares in the replicating portfolio for the option.
- When the option value is plotted as a function of the stock value, it is the slope of the value curve evaluated at the current stock price.
- Option delta changes with the stock price → Dynamic Hedging.

Figure 21.9 Call option value and hedge ratio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

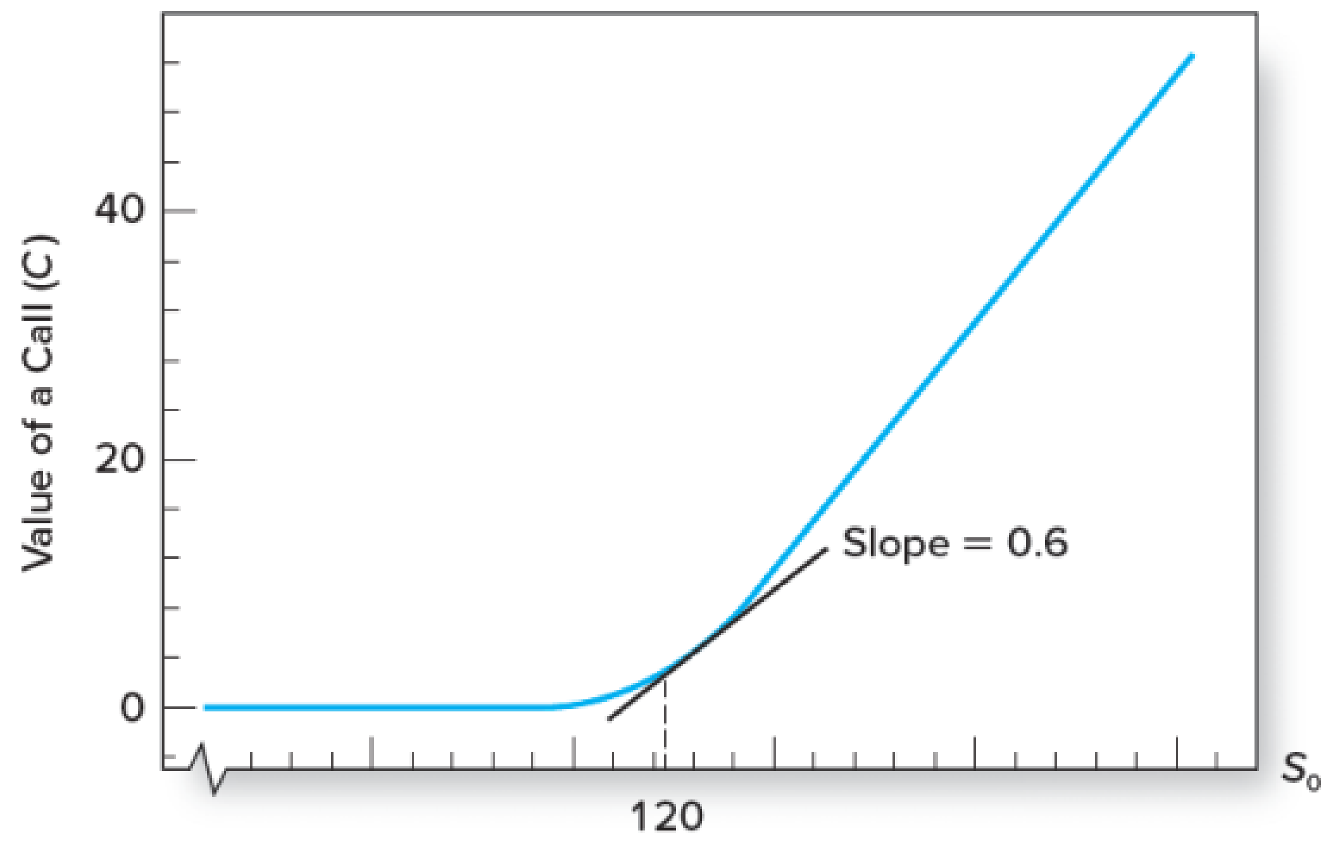


Figure 21.9 Call option value and hedge ratio

The Option Delta Δ and the Replicating Portfolio

- Recall
  - Option Price in the Binomial Model

$$C = S\Delta + B$$

- Then
  - Black-Scholes Replicating Portfolio of a Call Option

$$\Delta = N(d_1)$$
$$B = -PV(X)N(d_2)$$

  - Black-Scholes Replicating Portfolio of a Put Option

$$\Delta = -[1 - N(d_1)]$$
$$B = PV(X)[1 - N(d_2)]$$

The Option Delta Δ and the Replicating Portfolio

- Option Delta (Δ) is the change in the price of an option given a $1 change in the price of the stock. The number of shares in the replicating portfolio for the option.

- For a call option, Δ is always less than 1; the change in the call price is always less than the change in the stock price.

- The replicating portfolio of a call option always consists of a long position in the stock and a short position in the bond.

- The replicating portfolio of a put option always consists of a long position in the bond and a short position in the stock.

Computing the Replicating Portfolio

Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
- PNA Systems pays no dividends and has a current stock price of $10 per share. If its returns have a volatility of 40% and the risk-free rate is 5%, what portfolio would you hold today to replicate a one-year at-the-money call option on the stock?

Computing the Replicating Portfolio

Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- We can apply the Black-Scholes formula with $S_0 = 10$,

$$PV(X) = \frac{10}{1.05} = 9.524, \text{ and}$$

$$d_1 = \frac{\ln\left[\frac{S_0}{PV(X)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = \frac{\ln\left(\frac{10}{9.524}\right)}{40\%} + \frac{40\%}{2} = 0.322$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.322 - 0.40 = -0.078$$

c.f. Black-Scholes Formula for a Call

$$C_0 = S_0 \times N(d_1) - Xe^{-rT} \times N(d_2)$$

- Where: $$d_1 = \frac{\ln(S_0/X) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

C_o = Current call option value

S_o = Current stock price

$N(d)$ = The cumulative normal distribution = $Pr(z \leq d)$
i.e., the probability that a random draw from a standard normal distribution will be less than d

X = Exercise price

e = 2.71828, the base of the natural log

r = Risk-free interest rate

T = time to expiration of the option in years

ln = Natural log function

σ = Annual volatility (standard deviation) of the stock's return

Computing the Replicating Portfolio

Example 21.7 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - Black-Scholes Replicating Portfolio of a Call Option

$$\Delta = N(d_1) = N(0.322) = 0.626$$
$$B = -PV(X)N(d_2) = -9.524 \times N(-0.078) = -4.47$$

  - That is, we should buy 0.626 shares of the PNA stock, and borrow \$4.47, for a total cost of \$10(0.626) − 4.47 = \$1.79, which is the Black-Scholes value of the call option.

Hedging and Delta

- As the stock price changes, so do the deltas used to calculate the hedge ratio.
- **Gamma** = sensitivity of the delta to the stock price
- Gamma is similar to bond convexity.
- The hedge ratio will change with market conditions.
- Rebalancing is necessary.

Figure 21.11 Deltas change as stock price fluctuates
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

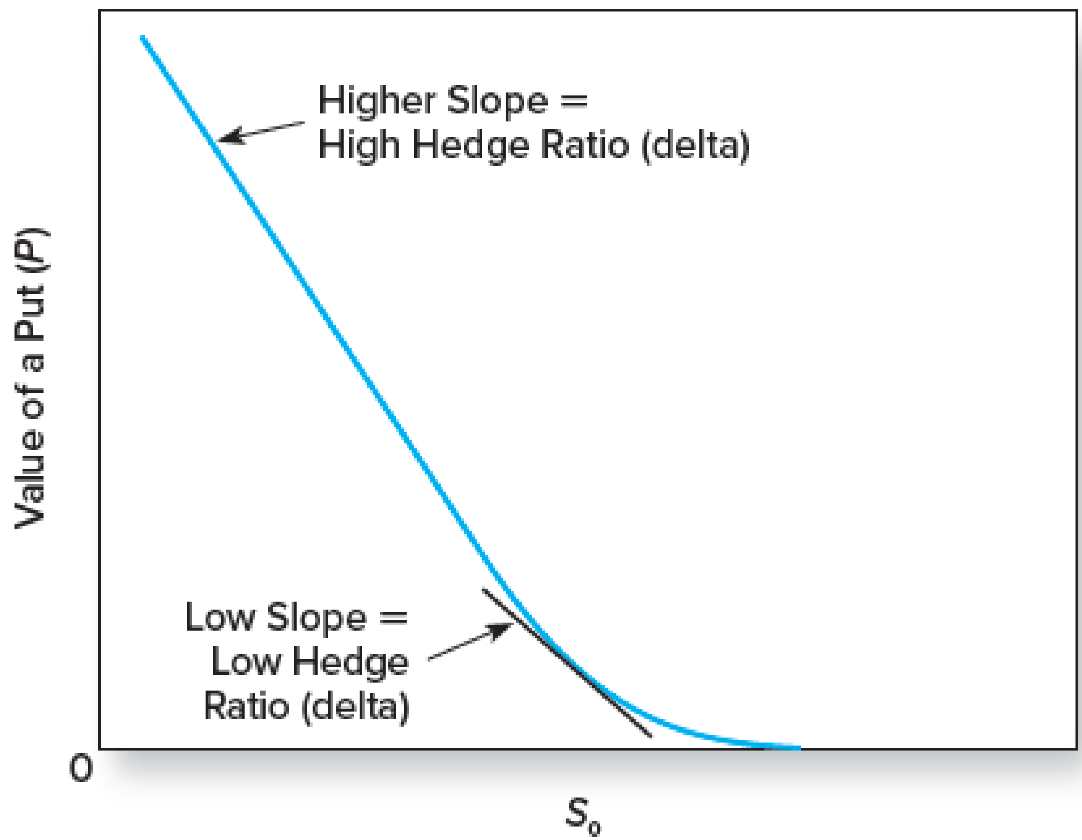


Figure 21.11 Deltas change as the stock price fluctuates

Delta Neutral

- When you establish a position in stocks and options that is hedged with respect to fluctuations in the price of the underlying asset, your portfolio is said to be delta neutral.

- In other words, a delta-neutral portfolio has no tendency to either increase or decrease in value when the stock price fluctuates.

Black-Scholes Formula for a Call

- Only 5 inputs are needed for the Black-Scholes formula.
  1. Stock price
  2. Strike price
  3. Exercise date
  4. Risk-free rate
  5. Volatility of the stock

- We do not need to know the expected return of the stock.
  - Expected return is already reflected in the current stock price (which is the present value of its future payoffs).

Implied Volatility

- Of the five required inputs in the Black-Scholes formula, only σ is not observable directly.
- The volatility of an asset's returns that is consistent with the quoted price of an option on the asset.
- Using the Black-Scholes formula and the actual price of the option, solve for volatility implied by the price observed in the market.

L11: Learning Objectives

- Discuss what is meant by risk-neutral probabilities.
- Show how these probabilities can be used to price an option.

Risk-Neutral Probabilities

- Imagine a risk-neutral economy, that is, an economy in which all investors are risk-neutral.
- This hypothetical economy must value options the same as the real world because risk aversion cannot affect the valuation formula.
- In a risk-neutral world, investors would not demand risk premiums to compensate for risk.
- Hence, all financial assets (including options) must have the same cost of capital, i.e., the risk-free rate of interest.
- We can therefore value all assets by discounting expected payoffs at the risk-free rate of interest.

A Risk-Neutral Two-State Model

- Probabilites: ρ and (1 − ρ) are risk-neutral probabilities.
    - The probability of future states that are consistent with current prices of securities assuming all investors are risk neutral.

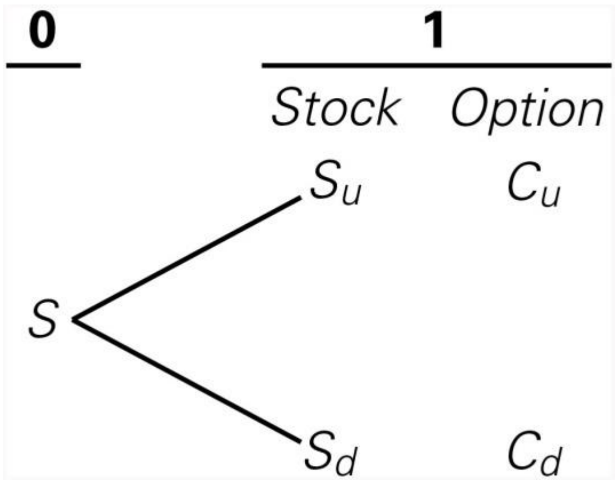
- In other words, ρ is not the actual probability of the stock price increasing.
    - It represents how the actual probability would have to be adjusted to keep the stock price the same in a risk-neutral world.
    - The risk-neutral probability that makes the stock's expected return equal to the risk-free interest rate.

Implications of the Risk-Neutral World

- To ensure that all assets in the risk-neutral world have an expected return equal to the risk-free rate, relative to the true probabilities, the risk-neutral probabilities overweight the bad states and underweight the good states.

Option Pricing with Risk-Neutral Probabilities

- Consider again the general binomial stock price tree.



Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 815

- Compute the risk-neutral probability that makes the stock's expected return equal to the risk-free interest rate:

$$\frac{\rho S_u + (1-\rho) S_d}{S} - 1 = r_f$$

Option Pricing with Risk-Neutral Probabilities

- Solving, the risk-neutral probability *ρ:*

$$\rho = \frac{(1+r_f)S - S_d}{S_u - S_d} = \frac{(1+r_f) - d}{u - d}$$

- Define *u* and *d* as follows:

$$S_d = dS \text{ and } S_u = uS$$

- E.g.,
  - $u = 1.2$
  - $d = 0.9$
- The value of the option can be calculated by computing its expected payoff using the risk-neutral probabilities, and discount the expected payoff at the risk-free interest rate.

Option Pricing with Risk-Neutral Probabilities

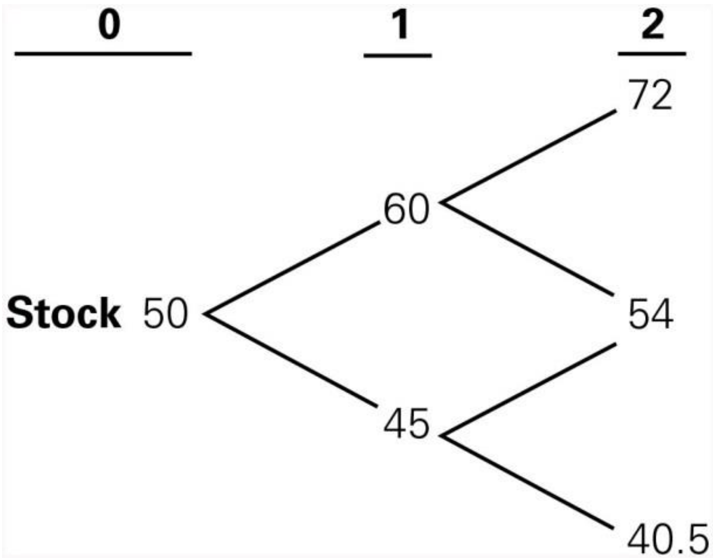
Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
- Suppose the current price of the stock is $50 per share. In each of the next two years, the stock price will either increase by 20% or decrease by 10%. The 3% one-year risk-free rate of interest will remain constant.
- Imagine all investors are risk neutral and calculate the probability of every state in the next two years. Use these probabilities to calculate the price of a two-year call option on this stock with a strike price $60.

Option Pricing with Risk-Neutral Probabilities

Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- The binomial tree in the three-state example is:



- First, compute the risk-neutral probability that the stock price will increase.
- At time 0, we have:

$$\rho = \frac{(1+r_f)S - S_d}{S_u - S_d} = \frac{(1.03)50 - 45}{60 - 45} = 0.433$$

Option Pricing with Risk-Neutral Probabilities

Adapted from Example 21.8 of adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
- Because the stock has the same returns(up 20% or down 10%) at each date, we can check that the risk neutral probability is the same at each date as well.
- Consider the call option with a strike price of $60. This call pays $12 if the stock goes up twice, and zero otherwise.
- The risk-neutral probability that the stock will go up twice is 0.433 × 0.433, so the call option has an expected payoff of:

$$0.433 \times 0.433 \times \$12 = \$2.25$$

- We compute the current price of the call option by discounting this expected payoff at the risk-free rate:

$$C = \frac{\$2.25}{1.03^2} = \$2.12$$

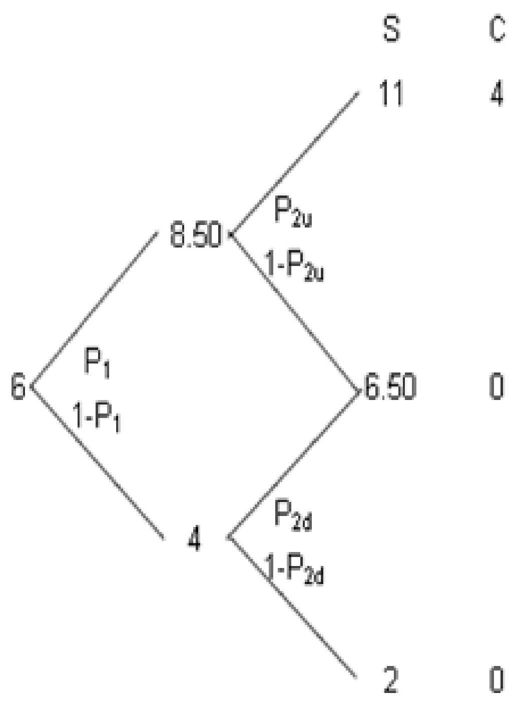
Risk-Neutral Probabilities and Option Pricing

- Derivative Security
  - A security whose cash flows depend solely on the prices of other marketed assets.

- The probabilities in the risk-neutral world can be used to price any derivative security.

Recall Tutorial 10: Risk-Neutral Probabilities and Option Pricing

- The current price of the ABC Corporation stock is $6. In each of the next two years, this stock price can either go up by $2.50 or go down by $2. The stock pays no dividends. The one-year risk-free interest rate is 3% and will remain constant. Use the Binomial Option Pricing Model to find the replicating portfolio and the value of a two-year European call option on the ABC stock with a strike price of $7.



Recall L2: Expected Return

- Expected (Mean) Return
  - Calculated as a weighted average of the possible returns, where the weights correspond to the probabilities.

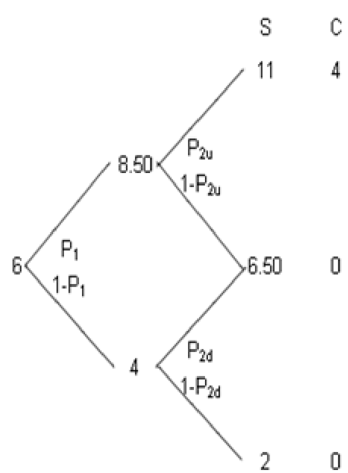
$$\textbf{Expected Return} = E[R] = \sum_{R} P_R \times R$$

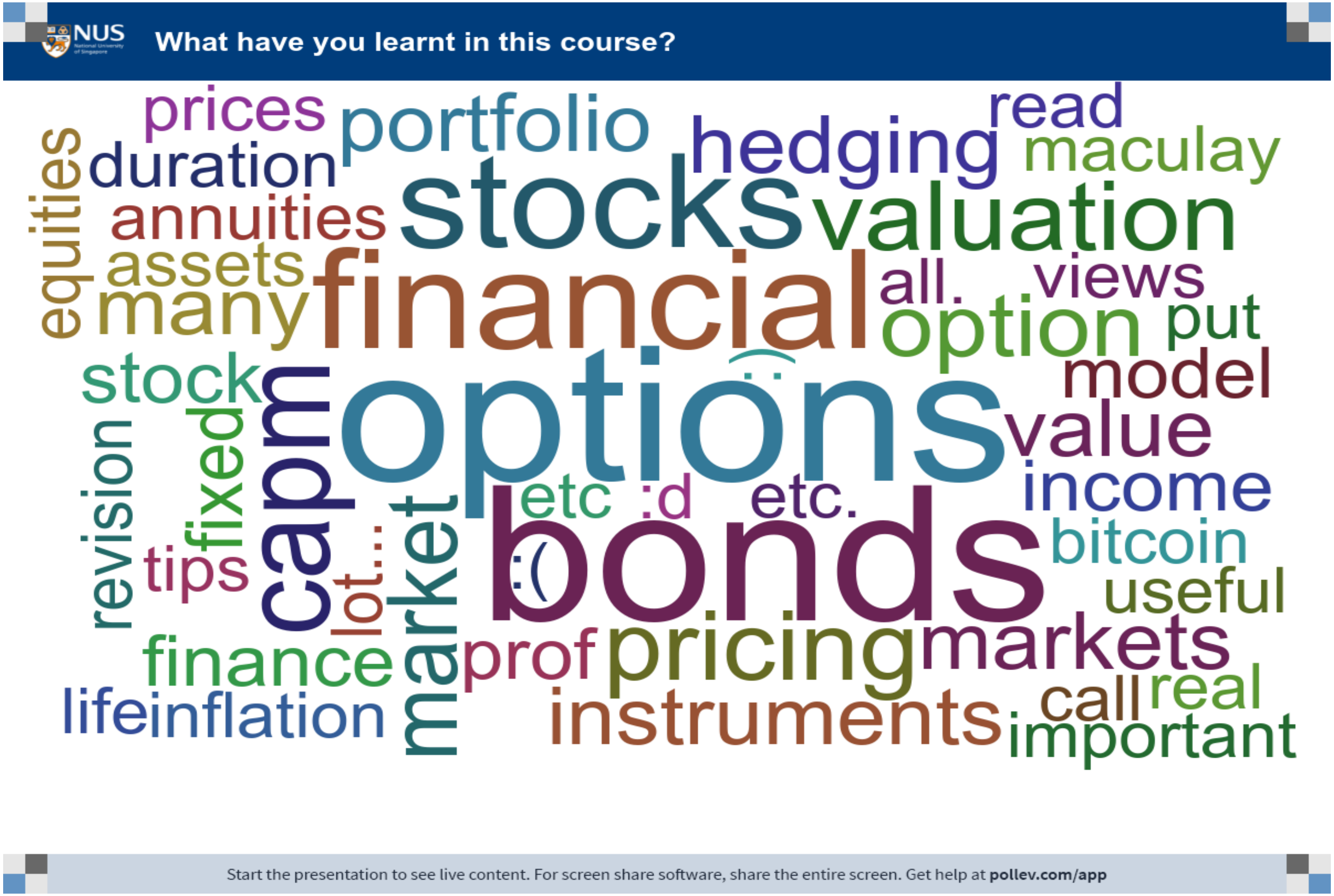
Risk-Neutral Probabilities and Option Pricing

- The risk neutral probabilities are
- $p_1 = \frac{(1.03)6-4}{8.5-4} = 48.44\%$
- $p_{2u} = \frac{(1.03)8.5-6.5}{11-6.5} = 50.11\%$
- $p_{2d} = \frac{(1.03)4-2}{6.5-2} = 47.11\%.$

(Using the present value of expected returns of the option in the risk neutral world)

- The value of the call is therefore
- $\frac{1}{1.03^2}\left(4p_1p_{2u} + 0\left(p_1(1-p_{2u}) + (1-p_1)(p_{2d})\right) + 0(1-p_1)(1-p_{2d})\right)$
- $4(0.4844)(0.5011)$ = $0.9153.





Thank you. Wishing you the very best in your academic journey and in life.