Problem Set 1 - Submission

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1. (a) Since $f(n) = O(g(n))$ and $g(n) = O(h(n))$, there are c_1, c_2, n_1, n_2 such that:

$$f(n) \leq c_1 g(n),\ \forall n \geq n_1$$
$$g(n) \leq c_2 h(n),\ \forall n \geq n_2$$
$$\Rightarrow f(n) \leq c_1 c_2 h(n),\ \forall n \geq \max\{n_1, n_2\}$$
$$\Rightarrow f(n) = O(h(n))\blacksquare$$

   (b) Supposed $f(n) = \Theta(g(n))$:

$$f(n) = O(g(n)) \Rightarrow \exists c_1, n_1 \in N \text{ such that } f(n) \leq c_1 g(n),\ \forall n \geq n_1$$
$$\Rightarrow g(n) \geq \frac{1}{c_1} f(n),\ \forall n \geq n_1 \Rightarrow g(n) = \Omega(f(n))$$
$$f(n) = \Omega(g(n)) \Rightarrow \exists c_2, n_2 \in N \text{ such that } f(n) \geq c_2 g(n),\ \forall n \geq n_2$$
$$\Rightarrow g(n) \leq \frac{1}{c_2} f(n),\ \forall n \geq n_2 \Rightarrow g(n) = O(f(n))$$

   Therefore

$$g(n) = \Theta(f(n))\blacksquare$$

2. (a) $f_4(n) \leq f_6(n) \leq f_2(n) \leq f_8(n) \leq f_7(n) \leq f_5(n) \leq f_1(n) \leq f_3(n)$

   (b) i.

$$f_6(n) = 2^{\lg\lg n} = (\lg n)^{\lg 2} < (\lg n)^{0.31} = O(\lg n)$$
$$f_7(n) = 3230n - \lg\lg n = \Theta(n)$$
$$\Rightarrow f_6(n) = O(f_7(n))$$

   ii.

$$f_5(n) = n^2 \log_n n! = n^2(\log_n n + \log_n(n_1) + \cdots + log_n 1)$$

$$\begin{aligned}
\log_n n &= 1 \\
\log_n(n-1) &< 1 \\
&\dots \\
log_n 1 = 0 &< 1 \\
\Rightarrow f_5(n) &< n^3 \\
\Rightarrow f_5(n) &< n^3 + 3n + sin(n) \\
\Rightarrow f_5(n) &= O(f_1(n))
\end{aligned}$$

3.

Algorithm by Alice

(a) Initialization takes $O(n)$ time

(b) Computing distance between n points and P with n calls to $findDist$ takes $nO(n^2) = O(n^3)$ time

(c) $\log n$ iterations of scanning an n-sized array takes $O(n \log n)$ time

This algorithm takes $O(n + n^3 + n \log n) = O(n^3)$ time in total

Algorithm by Bob

(a) $O(n)$ for initialization

(b) $\log n$ iterations, calling $findDist$ n times each iteration takes $O(\log n \times n^2 \times n) = O(n^3 \log n)$

This algorithm runs in $O(n + n^3 \log n) = O(n^3 \log n)$ time

Faster algorithm

Both algorithms are similar in the initialization step and iterating through the points to find the closest one is the same between Alice and Bob. However, Bob recalculates the distance every time he iterates through it, which makes each iteration $0(n^2)$ time slower than Alice's.