Homework 1 - Submission

A0219739N - Le Van Minh

September 7, 2023

1. (i)

$$\left(\begin{array}{ccc|c} 1 & 1 & 3-a & 2 \\ 3 & 4 & 2 & b \\ 2 & 3 & -1 & 1 \end{array}\right)$$

(ii)

$$\left(\begin{array}{ccc|c} 1 & 1 & 3-a & 2 \\ 3 & 4 & 2 & b \\ 2 & 3 & -1 & 1 \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3-a & 2 \\ 0 & 1 & 3a-7 & b-6 \\ 2 & 3 & -1 & 1 \end{array}\right) \xrightarrow{R_3-2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3-a & 2 \\ 0 & 1 & 3a-7 & b-6 \\ 0 & 1 & 2a-7 & -3 \end{array}\right)$$

$$\xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3-a & 2 \\ 0 & 1 & 3a-7 & b-6 \\ 0 & 0 & -a & 3-b \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 10-4a & 8-b \\ 0 & 1 & 3a-7 & b-6 \\ 0 & 0 & -a & 3-b \end{array}\right)$$

For the system to have no solution, the last row can be made so that all cells on the LHS are zeros, but the cell in the RHS is non-zero. One such instance is:

$$\begin{pmatrix} -a \\ 3-b \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(iii) We can get a system with infinite solutions by making the last row a zero row.

$$\begin{pmatrix} -a \\ 3-b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

The augmented matrix became:

$$\left(\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -7 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = t$:

$$\begin{cases} x + 10z = 5 \\ y - 7z = -3 \\ z = t \end{cases} \Rightarrow \begin{cases} x = 5 - 10t \\ y = -3 + 7t \\ z = t \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}$$

(iv) The system has unique solution when $-a \neq 0$. One possible parameter is:

$$\begin{pmatrix} -a \\ 3-b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The last row translates to:

$$z = 0$$

which is the value of z of the unique solution.

2. (i)

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 6 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 4 & 1 & 6 \end{pmatrix} \xrightarrow{R_3 - 4R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & -2 \end{pmatrix}$$
$$\xrightarrow{-R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{-\frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) The reduced echelon form of B is:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii)

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

(iv) From (i), (ii), A can be transformed into $R = I_3$ through a series of elementary row operation, i.e. A is row equivalent to I_3. Therefore, by Theorem (7), the series of elementary row operation that transform A into I_3 also transforms I_3 into A^{-1}, and A^{-1} is the product of those respective elementary matrices.

(v)

$$A^{-1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}$$
$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -4 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} I_3$$
$$= \begin{pmatrix}-3 & \frac{2}{3} & \frac{2}{3}\\ 0 & -\frac{2}{3} & \frac{1}{3}\\ 2 & -\frac{1}{3} & -\frac{1}{3}\end{pmatrix}$$

3. Consider the (1,1) cell in each of these products:

$$A_{21}B_{11} = \begin{pmatrix}a_{31}b_{11} + a_{32}b_{21} & *\\ * & *\end{pmatrix}$$
$$A_{22}B_{21} = \begin{pmatrix}a_{33}b_{31} + a_{34}b_{41} & *\\ * & *\end{pmatrix}$$
$$A_{23}B_{31} = \begin{pmatrix}a_{35}b_{51} + a_{36}b_{61} & *\\ * & *\end{pmatrix}$$

$$(a_{31}b_{11} + a_{32}b_{12}) + (a_{33}b_{13} + a_{34}b_{14}) + (a_{35}b_{15} + a_{36}b_{16}) = \sum_{1\le r\le 6} a_{3r}b_{r1}$$

Similarly, we can obtain the values of the other cells:

$$A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} = \begin{pmatrix}\sum_{1\le r\le 6} a_{3r}b_{r1} & \sum_{1\le r\le 6} a_{3r}b_{r2}\\ \sum_{1\le r\le 6} a_{4r}b_{r1} & \sum_{1\le r\le 6} a_{4r}b_{r2}\end{pmatrix} = \begin{pmatrix}c_{31} & c_{32}\\ c_{41} & c_{42}\end{pmatrix} = C_{21}$$