EC3312: Game Theory & Applications to Economics

Lecture 3: Duopoly

Sam Jindani

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The Cournot model

Two firms compete to sell a homogenous good. They simultaneously choose quantity q_i, where $i = 1, 2$

They face inverse demand function $p(q) = \max\{a - q, 0\}$, where $q = q_1 + q_2$, and constant marginal cost $c < a$.

The firms seek to maximise profit.

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The firms seek to maximise profit.

As a game:

- Players $N = \{1, 2\}$
- Strategy space: $S_i = \mathbb{R}_+$
- Payoffs: $\pi_i(q_i, q_j) = p(q_i + q_j)q_i - cq_i$.

Best responses

Firm i solves

$$\max_{q_i \in \mathbb{R}_+} (a - q_i - q_j - c)q_i.$$

How do we solve this problem?

Best responses

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$$\max_{q_i \in \mathbb{R}_+} (a - q_i - q_j - c)q_i.$$

How do we solve this problem?

First-order condition:

$$\frac{\partial \pi_i}{\partial q_i}(q_i, q_j) = 0 \implies a - 2q_i - q_j - c = 0 \implies q_i = \frac{a - q_j - c}{2}.$$

Second-order condition:

$$\frac{\partial^2 \pi_i}{\partial q_i^2}(q_i, q_j) = -2 \implies \pi_i \text{ is strictly concave in } q_i.$$

So the optimal choice is $q_i = \frac{a - q_j - c}{2}$ provided this is non-negative. Define the *response function*

$$B_i(q_j) = \max\left\{\frac{a - q_j - c}{2}, 0\right\}.$$

Nash equilibrium

A Nash equilibrium (q_1^*, q_2^*) must satisfy $q_1^* = B_1(q_2^*)$ and $q_2^* = B_2(q_1^*)$.

Simultaneously solving

$$\begin{cases} q_1^* = \frac{a - q_2^* - c}{2} \\ q_2^* = \frac{a - q_1^* - c}{2}, \end{cases}$$

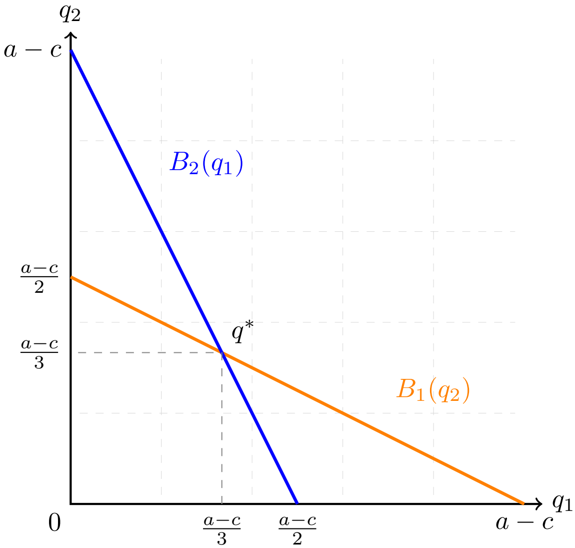
yields

$$q_1^* = q_2^* = \frac{a-c}{3}.$$

Since this satisfies $q_1^* = B_1(q_2^*)$ and $q_2^* = B_2(q_1^*)$, it is a Nash equilibrium.

Are there any other Nash equilibria in pure strategies? Note that $B_i(0) = \frac{a-c}{2} > 0$ so $(0,0)$ is not a Nash equilibrium. Moreover $B_i(\frac{a-c}{2}) = \frac{a-c}{4} \neq 0$. So there are no other Nash equilibria.

The equilibrium graphically



Price is

$$\begin{aligned} p^* &= a - q_1^* - q_2^* \\ &= a - 2\frac{a-c}{3} \\ &= \frac{a+2c}{3} \end{aligned}$$

Profit is

$$\begin{aligned} \pi_i^* &= (p^* - c)q_i^* \\ &= \left(\frac{a+2c}{3} - c\right)\frac{a-c}{3} \\ &= \frac{(a-c)^2}{9} \end{aligned}$$

What if the two firms collude?

Suppose the two firms maximise total profit (and split it equally):

$$\max_{q \in \mathbb{R}_+} (a - q - c)q$$

First-order condition:

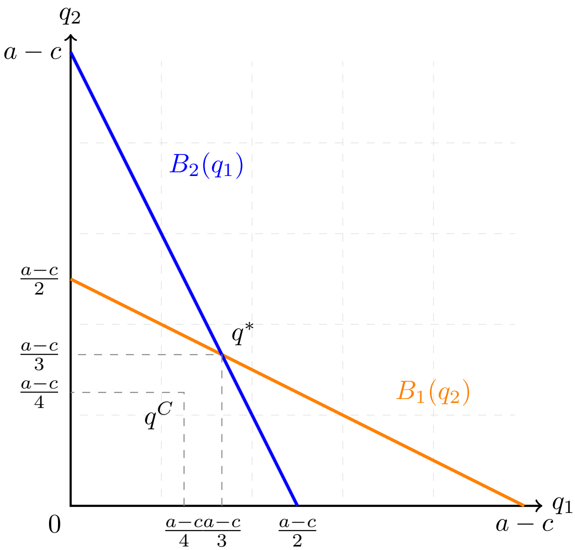
$$\frac{\partial \pi}{\partial q} = 0 \implies a - 2q - c = 0 \implies q = \frac{a-c}{2}.$$

Second-order condition:

$$\frac{\partial^2 \pi}{\partial q^2} = -2 \implies \pi \text{ is strictly concave in } q.$$

So the optimal choice is $q = \frac{a-c}{2} > 0$.

Collusive outcome



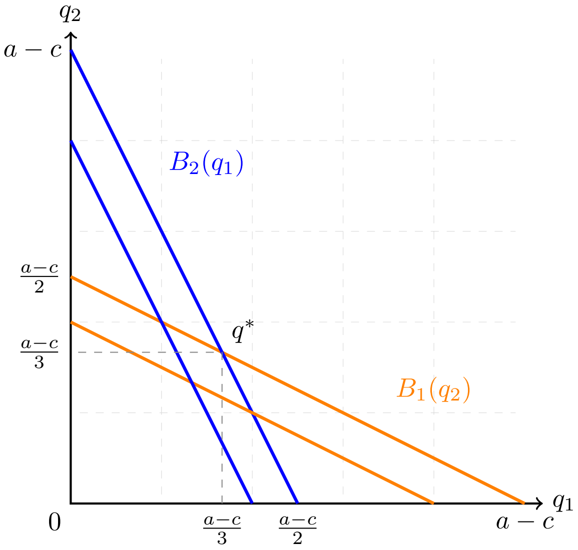
If each firm chooses $q_1^C = q_2^C = \frac{a-c}{4}$, then

$$\begin{aligned} p^C &= a - q_1^C - q_2^C \\ &= a - \frac{a-c}{2} \\ &= \frac{a+c}{2} \\ \pi_i^C &= (\frac{a+c}{2} - c)\frac{a-c}{4} \\ &= \frac{(a-c)^2}{8}. \end{aligned}$$

Since $\pi_i^C > \pi_i^*$, firms have an incentive to collude.

Note that $p^C > p^*$ and $q^C < q^*$.

Comparative statics



If demand increases (a increases), then quantities, price, and profit all increase.

If cost increases (c increases), then quantities and profit decrease, but price increases.

The Bertrand model

Two firms compete to sell a homogenous good. They simultaneously choose price p_i, where $i = 1, 2$.

The entire demand goes to the firm with the lowest price:

$$q_i(p_i, p_j) = \begin{cases} a - p_i & \text{if } p_i < p_j \\ \frac{a-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

Marginal cost is c and players seek to maximise profit.

As a game:

- Players $N = \{1, 2\}$
- Strategy space: $S_i = \mathbb{R}_+$
- Payoffs: $\pi_i(p_i, p_j) = (p_i - c)q_i(p_i, p_j)$.

Nash equilibrium

Intuitively, players have an incentive to undercut their opponent.

Consider any strategy profile (p_1, p_2) such that $p_1, p_2 > c$.

- First, suppose $p_2 > p_1 > c$. Then player 2 faces zero demand and therefore receives zero profit. But by choosing $p_2' \in (c, p_1)$ she could make positive profit. Hence (p_1, p_2) is not a Nash equilibrium.
- Second, suppose $p_2 = p_1 > c$. Then 2 receives payoff

$$\pi_2(p_2, p_1) = (p_2 - c)\frac{a - p_2}{2}$$

Suppose player 2 deviates to $p_2' = p_2 - \varepsilon$, where $\varepsilon > 0$. Then she receives

$$\pi_2(p_2', p_1) = (p_2 - \varepsilon - c)(a - p_2)$$

If ε is small enough, then player 2 has an incentive to deviate.

Nash equilibrium, ctd.

Next, consider any strategy profile (p_1, p_2) such that $p_1 \geq p_2 = c$.

- First, suppose $p_1 > c$. Then player 2 receives profit 0, but could receive a positive profit by deviating to $p_2' \in (c, p_1)$.
- Second, suppose $p_1 = c$. Then each player receives payoff 0. Deviating to $p_i < c$ yields negative profit and deviating to $p_i > c$ yields zero profit. Hence (c, c) is a Nash equilibrium.

Finally, consider any strategy profile (p_1, p_2) such that $p_1 \leq p_2 < c$. Then player 1 is making negative profit and could make zero profit by deviating to $p_1' = c$.

Hence the unique Nash equilibrium in pure strategies is (c, c). Price is $p^* = c$. Each firm produces $q_i^* = \frac{a-c}{2}$ and receives profit $\pi_i^* = 0$.

Cournot versus Bertrand

Cournot (1801–1877)

Bertrand (1822–1900)

Price competition seems more natural.

But Cournot's predictions are intuitive: in a duopoly, we expect firms to have some market power and profit.

Differentiated Bertrand model

Two firms produce differentiated products (e.g., Coca Cola and Pepsi). They simultaneously choose price p_i, where $i = 1, 2$.

The demand for each firm is

$$q_i(p_i, p_j) = a - p_i + bp_j,$$

where $b \in (0, 2)$. Marginal cost is constant as before.

As a game:

- Players $N = \{1, 2\}$
- Strategy space: $S_i = \mathbb{R}_+$
- Payoffs: $\pi_i(p_i, p_j) = (p_i - c)(a - p_i + bp_j)$.

Best responses

Firm i solves

$$\max_{p_i \in \mathbb{R}_+} (p_i - c)(a - p_i + bp_j).$$

First-order condition:

$$\frac{\partial \pi_i}{\partial p_i}(p_i, p_j) = 0 \implies a - 2p_i + bp_j + c = 0 \implies p_i = \frac{a + bp_j + c}{2}.$$

Second-order condition:

$$\frac{\partial^2 \pi_i}{\partial p_i^2}(p_i, p_j) = -2 \implies \pi_i \text{ is strictly concave in } p_i.$$

Note that $\frac{a+bp_j+c}{2} > 0$. So the optimal choice is

$$B_i(p_j) = \frac{a + bp_j + c}{2}.$$

Nash equilibrium

A Nash equilibrium (p_1^*, p_2^*) must satisfy $p_1^* = B_1(p_2^*)$ and $p_2^* = B_2(p_1^*)$.

Simultaneously solving

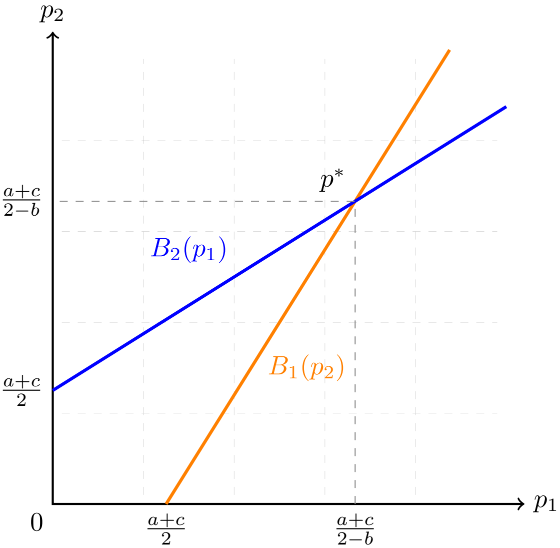
$$\begin{cases} p_1^* = \frac{a+bp_2^*+c}{2} \\ p_2^* = \frac{a+bp_1^*+c}{2}, \end{cases}$$

yields

$$p_1^* = p_2^* = \frac{a+c}{2-b}.$$

This is the unique Nash equilibrium in pure strategies.

The equilibrium graphically



Quantities are

$$\begin{aligned} q_i^* &= a - p_i^* + bp_j^* \\ &= a + (b-1)\frac{a+c}{2-b} \\ &= \frac{a-c+bc}{2-b} \end{aligned}$$

Profit is

$$\begin{aligned} \pi_i^* &= (p^* - c)q_i^* \\ &= \left(\frac{a+c}{2-b} - c\right)\frac{a-c+bc}{2-b} \\ &= \frac{(a-c+bc)^2}{2-b} \end{aligned}$$

What if $b \geq 2$?

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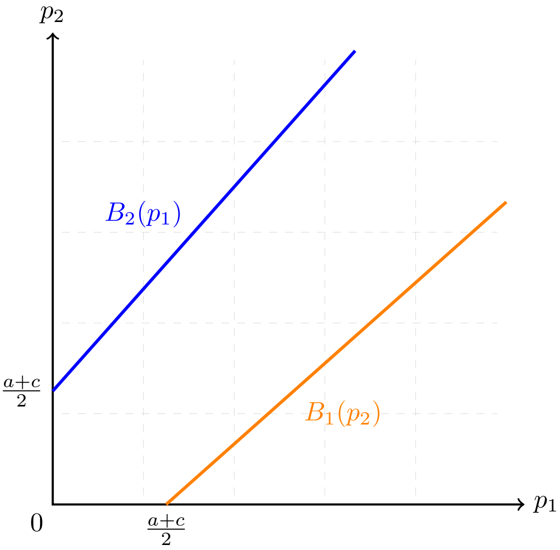
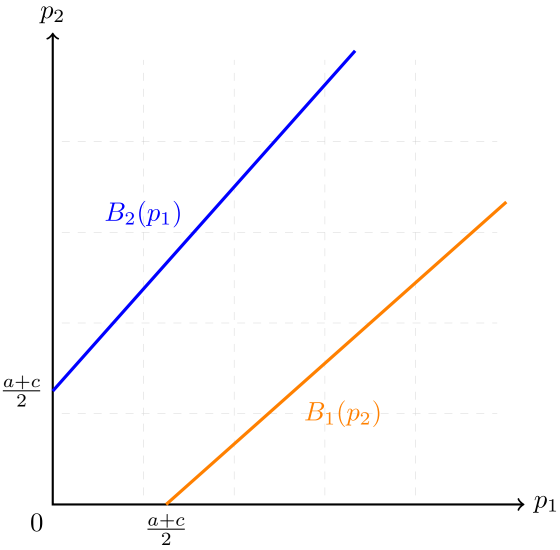


What if $b \geq 2$?



Then there is no Nash equilibrium in pure strategies.

Image credits

Slide 11:
`https://en.wikipedia.org/wiki/File:Antoine_Augustin_Cournot.jpg`
`https://en.wikipedia.org/wiki/File:Bertrand.jpg`