EC3312: Game Theory & Applications to Economics

Lecture 4: Dynamic games of complete information

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An entry game

A monopolist faces a potential entrant. If the entrant stays out, the monopolist makes \$3m in profit. If the entrant enters, the incumbent can either fight or accommodate. If the incumbent fights, each firm makes losses of \$1m. If the incumbent accommodates, each firm makes \$1m in profit.

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Normal form of the game:

		I	
		A	F
E	In	1, 1	-1, -1
	Out	0, 2	0, 2

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$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{I} \\
 & & A & F \\
\hline
\multirow{2}{*}{E} & In & \begin{matrix} & 1 \\ 1 & \end{matrix} & \begin{matrix} & -1 \\ -1 & \end{matrix} \\
\hline
 & Out & \begin{matrix} & 2 \\ 0 & \end{matrix} & \begin{matrix} & 2 \\ 0 & \end{matrix} \\
\hline
\end{array}
$$

The Nash equilibria in pure strategies are (Out, F) and (In, A).

Credibility

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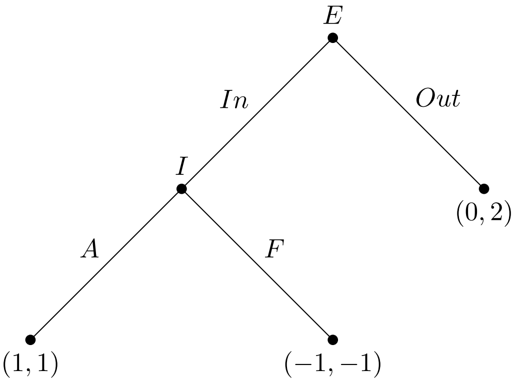
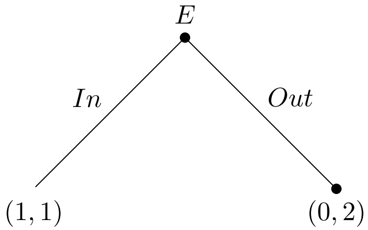
Note that this issue arises because the game is sequential (aka dynamic).

Tree representation



Backward induction

At the incumbent's decision node, F is not a best response. Given that the incumbent will choose A, the entrant's decision simplifies to:



Hence the outcome should be $(In, A) \implies$ Solution by backward induction.

The centipede game (Rosenthal 1981)

There are two jars on a table. One contains \$1 and the other \$4. Two players take turns. At each player's turn, they can either take one of the jars or pass. If they take a jar, they get the contents of that jar, the other player gets the contents of the other jar, and the game ends. If they pass, then the money in each jar is doubled and the turn passes to the other player. After each player has played three turns, the game ends and they split the contents of the jars equally.

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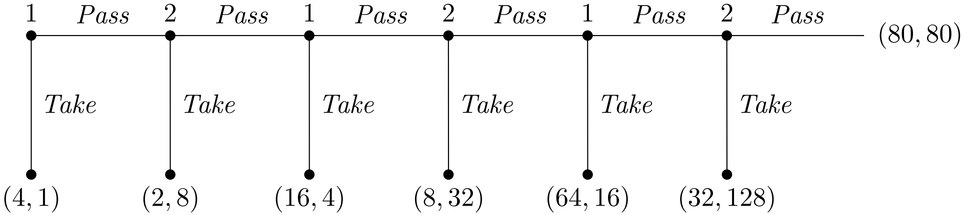


Backward induction

In the last period, it is rational for player 2 to take the largest jar. Knowing that, it is rational for player 1 to take in period 5. Knowing that, it is rational for player 2 to take in period 4...

The only reasonable outcome, assuming common knowledge of rationality, is that player 1 takes in the first period.

Backward induction can help us solve quite complicated games.

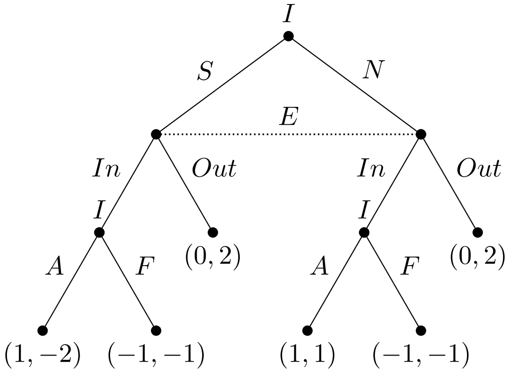
Entry game with inventory

Before the entrant's decision, the incumbent can decide to stockpile inventory. This means her production costs are sunk, and she only cares about revenue. As a result, the market will be oversupplied and prices will be lower. Accommodating would be particularly bad for the incumbent because some of her inventory would go unsold.

The payoffs are as before, except that if the incumbent stockpiles, the entrant enters, and the incumbent accommodates, the entrant gets 1 and the incumbent gets -2.

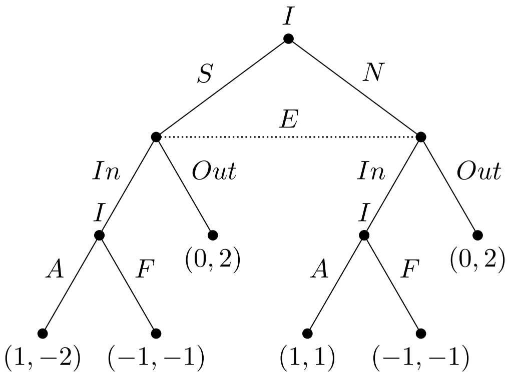
Game tree with information sets

How can we write the game tree?



Game tree with information sets

How can we write the game tree?



Can this game be solved using backward induction? At the entrant's information set, the entrant doesn't know what the incumbent did.

Extensive-form games

An extensive-form game is defined by

- A set of players N
- A set of nodes
- At each node that isn't terminal, a player and action set are specified
- Nodes can belong to information sets.
  - The set of information sets is a partition of the set of nodes
  - All the nodes in an information set share the same player and action set
  - The interpretation is that the player doesn't know which node of an information set has been reached.
- At each terminal node, a payoff is assigned to each player
- A strategy for a player specifies an action at each of her information sets.
  $\implies$ complete contingent plan

Subgame-perfect equilibrium

4.1. Definition. A subgame is the continuation of the game after a given node, such that no information set is broken up.

4.2. Definition. A subgame-perfect equilibrium (Selten 1965) is a Nash equilibrium such that the continuation strategy profiles in each subgame constitute a Nash equilibrium.

SPE in the entry game

Consider a strategy profile (s_E, s_I), where $s_E \in \{In, Out\}$ and $s_I \in \{A, F\}$

There are two subgames: the whole game and the subgame following *In*.

In the subgame following *In*, the incumbent's unique best response is A.

Given that the incumbent's strategy is $s_I = A$, the entrant's unique best response in the whole game is *In*.

$\implies$ The unique SPE is (In, A).

SPE in the centipede game

There are six subgames: one for each decision node. A strategy for player i is a triple $s_i \in \{T, P\}^3$ which specifies whether to take or pass at each of her decision nodes.

In the final subgame, 2's unique best response is T. Given this, in the second-to-last subgame, 1's unique best response is T.

Continuing in this way, we see that the unique SPE is $((T, T, T), (T, T, T))$.

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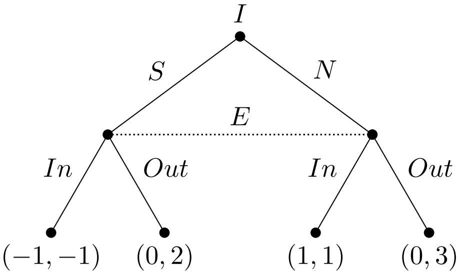
More formally, we can argue by contradiction: Suppose that s is an SPE but involves at least one player choosing P at some node. Let $k \in \{1, 2, \ldots, 6\}$ be the last node such that a player chooses P and consider the subgame starting at node k. Since the game ends after this node, the player gets 2^{k+1} from taking and 2^k from passing. Hence it is not a Nash equilibrium for the player to choose P in this subgame, and therefore s is not an SPE.

SPE in the entry game with inventory

In the subgame following (N, In), it is optimal for I to choose F. In the subgame following (N, In), it is optimal for I to choose A. The reduced game tree is:



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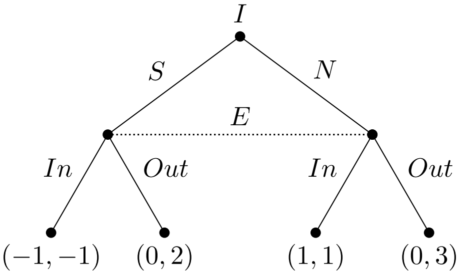

Now there is only one subgame.

SPE in the entry game with inventory, ctd.

The normal form of the reduced game is:

		I: S	I: N
E	In	-1, -1	1, 1
	Out	0, 2	0, 2

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The normal form of the reduced game is:

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There are two Nash equilibria in pure strategies: (In, N) and (Out, S). Any $(Out, (q, 1-q))$ with $q \in [0, 1]$ is a mixed equilibrium.

The SPE in the whole game are $(In, (N, F, A))$ and $(Out, ((q, 1-q), F, A))$ for any $q \in [0, 1]$.

Normal-form games in extensive form

Using information sets, we can write normal-form games in extensive form.

Battle of the sexes:

		2	
		A	B
1	A	2, 1	0, 0
	B	1, 0	0, 2

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Battle of the sexes:

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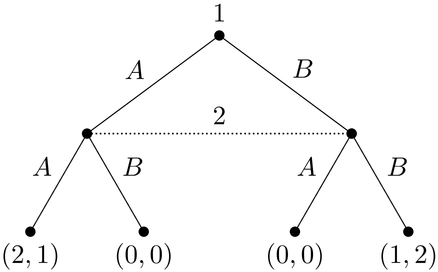

Stackelberg competition

As in the Cournot model, two firms compete by setting quantities, facing inverse market demand $p(q) = a - q$ and constant marginal cost $c < a$. Now, however, firm 1 chooses q_1 first. After observing q_1, firm 2 chooses q_2.

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$N = \{1, 2\}$ and $S_1 = \mathbb{R}_+$. But firm 2's strategy must specify an action for each possible action of firm 1; that is, firm 2's strategy is a function $q_2 : \mathbb{R}_+ \to \mathbb{R}_+$.

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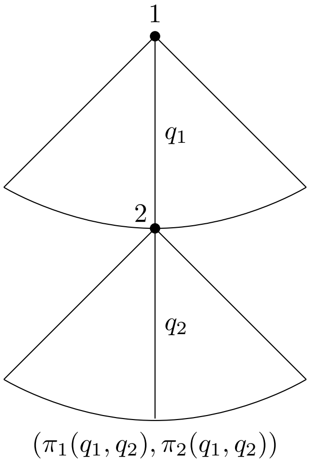

Firm 2's problem

Suppose firm 1 has chosen q_1. Then firm 2 solves

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This is the same problem as in the Cournot model, so

$$q_2^*(q_1) = \frac{a - q_1 - c}{2}.$$

Firm 1's problem

We use backward induction: Firm 1 deduces that firm 2 will choose $q_2^*(q_1)$ and therefore solves

$$\max_{q_1 \in \mathbb{R}_+} \left(a - q_1 - \frac{a - q_1 - c}{2} - c \right) q_1,$$

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First-order condition:

$$\frac{\partial \pi_1}{\partial q_1}(q_1) = 0 \implies \frac{a-c}{2} - q_1 = 0 \implies q_1 = \frac{a-c}{2}.$$

Second-order condition:

$$\frac{\partial^2 \pi_1}{\partial q_1^2}(q_1) = -1 \implies \pi_1 \text{ is strictly concave in } q_1 .$$

So the optimal choice is

$$q_1^* = \frac{a-c}{2}.$$

Equilibrium

The equilibrium price is

$$p^* = \frac{a+3c}{4}$$

Equilibrium profit is

$$\pi_1^* = \frac{(a-c)^2}{8} \qquad \pi_2^* = \frac{(a-c)^2}{16}$$

Cournot vs Stakelberg vs Collusion

	Cournot	Stackelberg	Collusion
Price	$\frac{a+2c}{3}$	$\frac{a+3c}{4}$	$\frac{a+c}{2}$
Quantity per firm	$\frac{a-c}{3}$	$\left(\frac{a-c}{2}, \frac{a-c}{4}\right)$	$\frac{a-c}{4}$
Profit per firm	$\frac{(a-c)^2}{9}$	$\left(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{16}\right)$	$\frac{(a-c)^2}{8}$
Total profit	$\frac{2(a-c)^2}{9}$	$\frac{3(a-c)^2}{16}$	$\frac{(a-c)^2}{4}$

Firm 1 has a first-mover advantage. It achieves its collusive profit. Firm 2 receives less than under Cournot.

Price is lower than under Cournot and therefore total profit is lower.