LECTURE 9

Option Valuation - Binomial Option Pricing

EC3333 Financial Economics I

Learning Objectives

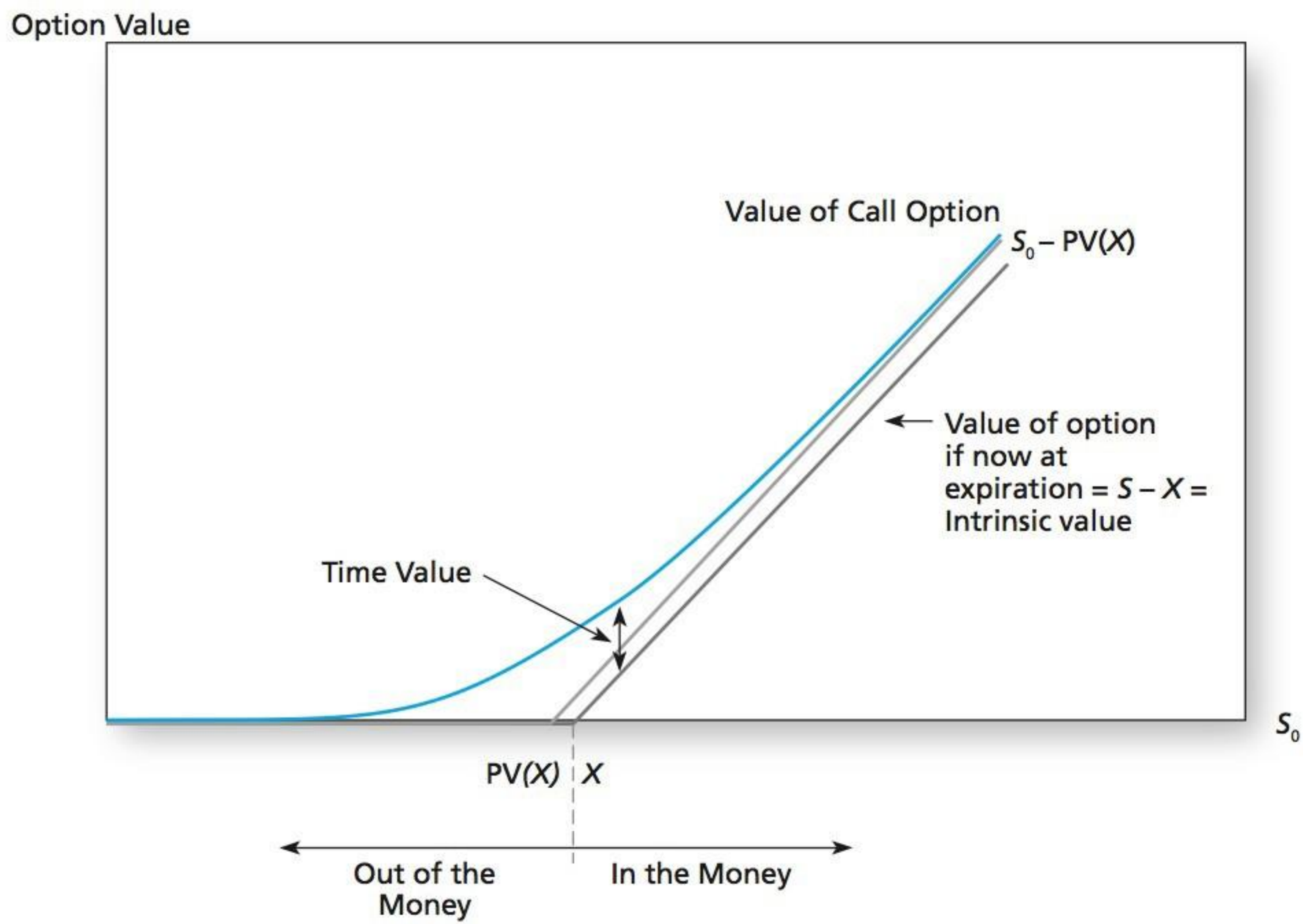
- Distinguish between the intrinsic and time value of an option.
- Examine the factors that affect the option prices.
- Construct a replicating portfolio for Binomial Option Pricing model, and value an option using the Law of One Price.

Intrinsic and Time Values

- **Intrinsic value** – the value the option would have if it expired immediately
  - It is the amount by which the option is currently in-the-money, or zero if the option is out-of-the-money.
  - **In-the-money call:**
    - current stock price - exercise price = $S_0 - X$
  - **In-the-money put:**
    - exercise price – current stock price = $X - S_0$

- **Time value** - the difference between the current option price and its intrinsic value
  - Arises because the option is yet to expire and from "volatility value"

Figure 21.1 Call option value before expiration
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Arbitrage Bounds on Option Prices

- An American option is worth at least as much as its European counterpart.
- A put option cannot be worth more than its strike price.
- A call option cannot be worth more than the stock itself.
- An American option cannot be worth less than its intrinsic value.
- Because an American option cannot be worth less than its intrinsic value, it cannot have a negative time value.

Table 21.1 Determinants of call option values
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

If This Variable Increases . . .	The Value of a Call Option
Stock price, S	Increases
Exercise price, X	Decreases
Volatility, σ	Increases
Time to expiration, T	Increases
Interest rate, r_f	Increases
Dividend payouts	Decreases

The Binomial Option Pricing Model

- **Binomial Option Pricing Model**
  - A technique for pricing options based on the assumption that each period, the stock's return can take on only two values.
    - Up state: u
    - Down State: d
  - We can use this model to value any derivative securities.

- **Binomial Tree**
  - A timeline with two branches at every date representing the possible events that could happen at those times.

A Two-State Single-Period Model

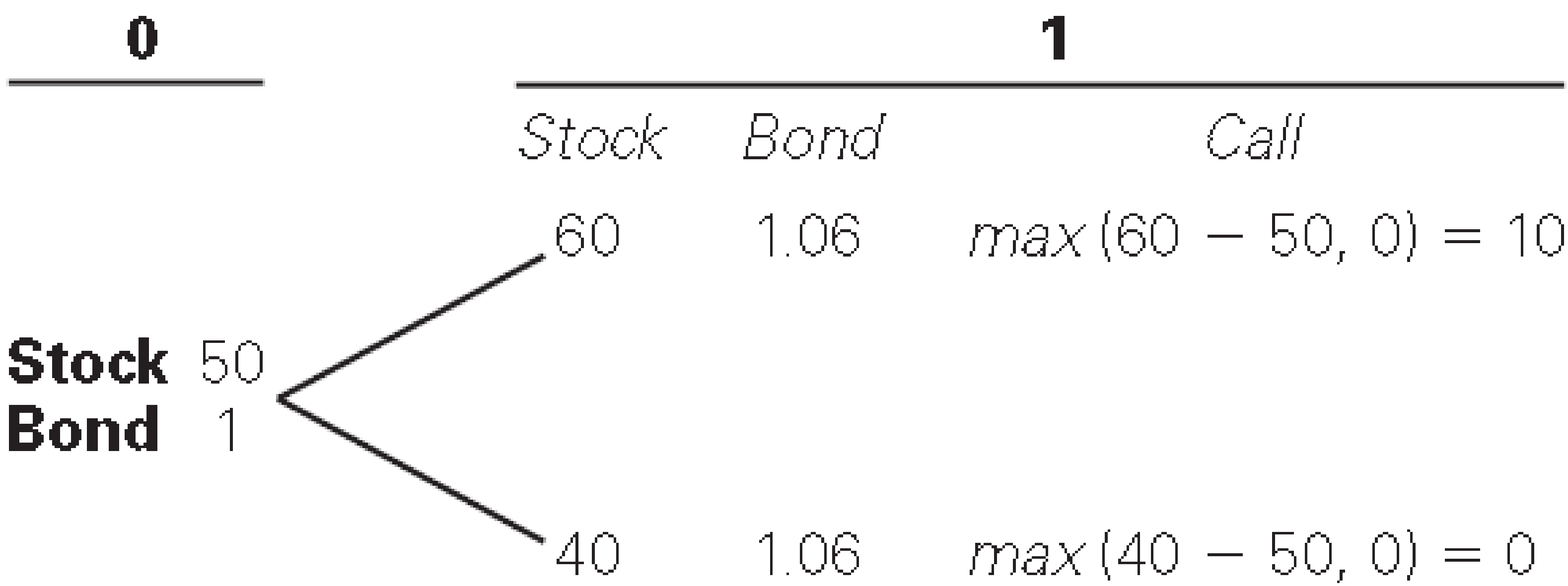
- **Replicating Portfolio**
  - A portfolio consisting of a stock and a risk-free bond that has the same value and payoffs in one period as an option written on the same stock.
  - The Law of One Price implies that the current value of the call and the replicating portfolio must be equal.

A Two-State Single-Period Model

- Assume the following:
  - A European call option expires in one period and has an exercise price of $50.
  - The stock price today is equal to $50 and the stock pays no dividends.
  - In one period, the stock price will either rise by $10 or fall by $10.
  - The one-period risk-free rate is 6%.
  - The price of bond is $1 today.

A Two-State Single-Period Model

- The payoffs can be summarized in a binomial tree.



Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 796

A Two-State Single-Period Model

- Notations:
  - Δ = the number of shares of stock purchased
  - B = the initial investment in bonds

- To create a call option using the stock and the bond, the value of the portfolio consisting of the stock and bond must match the value of the option in every possible state.

- The number of shares needed to replicate one call (Δ) is called the hedge ratio.

A Two-State Single-Period Model

- In the up state, the value of the portfolio must be \$10

$$60\Delta + 1.06B = 10$$

- In the down state, the value of the portfolio must be \$0

$$40\Delta + 1.06B = 0$$

- Two simultaneous equations with two unknowns Δ and B. Solving, we get

$$\Delta = 0.5$$

$$\mathrm{B} = -18.8679$$

A Two-State Single-Period Model

- A portfolio that is
  - long 0.5 share of stock and
  - short approximately \$18.87 worth of bonds (i.e., borrow \$18.87 at 6%)

- will have a value in one period that exactly matches the value of the call.

$$60 \times 0.5 - 1.06 \times 18.87 = 10$$

$$40 \times 0.5 - 1.06 \times 18.87 = 0$$

A Two-State Single-Period Model

- By the Law of One Price, the price of the call option today must equal the current market value of the replicating portfolio.

- The value of the portfolio today is the value of 0.5 shares at the current share price of $50, less the amount borrowed.

$$50\Delta + B = 50(0.5) - 18.87 = 6.13$$

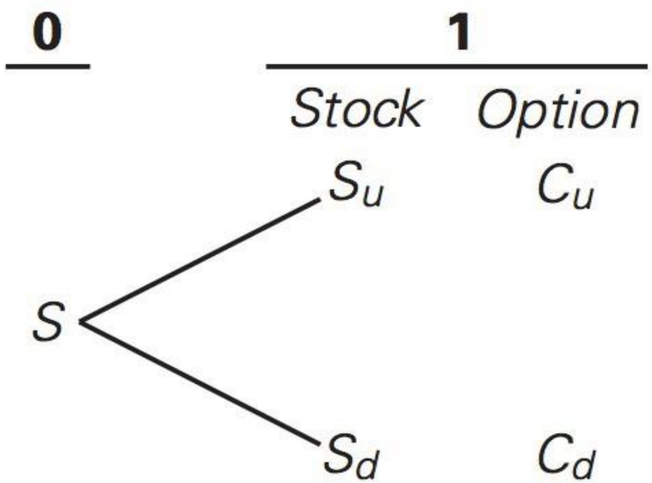
- Note that by using **the Law of One Price and a replicating portfolio**, we are able to solve for the price of the option **without knowing the probabilities of the states in the binomial tree**.

The Binomial Pricing Formula

- Assume:
  - S is the current stock price
  - S will either go up to S_u or go down to S_d next period.
  - The risk-free interest rate is r_f.
  - C_u is the value of the call option if the stock goes up
  - C_d is the value of the call option if the stock goes down

The Binomial Pricing Formula

- Given the above assumptions, the binomial tree would look like:



Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 798

- The payoffs of the replicating portfolios could be written as:

$$S_u\Delta + (1 + r_f)B = C_u, \text{ and}$$
$$S_d\Delta + (1 + r_f)B = C_d$$

The Binomial Pricing Formula

- Solving the two replicating portfolio equations for the two unknowns Δ and B, we get:
- Replicating Portfolio in the Binomial Model

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \qquad B = \frac{C_d - S_d\Delta}{1 + r_f}$$

$$B = \frac{C_u - S_u\Delta}{1 + r_f}$$

- Option Price in the Binomial Model

$$C = S\Delta + B$$

The Binomial Pricing Formula

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{\text{Spread of possible option prices}}{\text{Spread of possible asset price}}$$

- Δ = the sensitivity of the option's value to changes in the stock price
- You can replicate an investment in the call option by a levered investment in the underlying asset.
- Thus, if the option is not traded, you can DIY a homemade option by a replicating strategy.

Valuing a Put Option

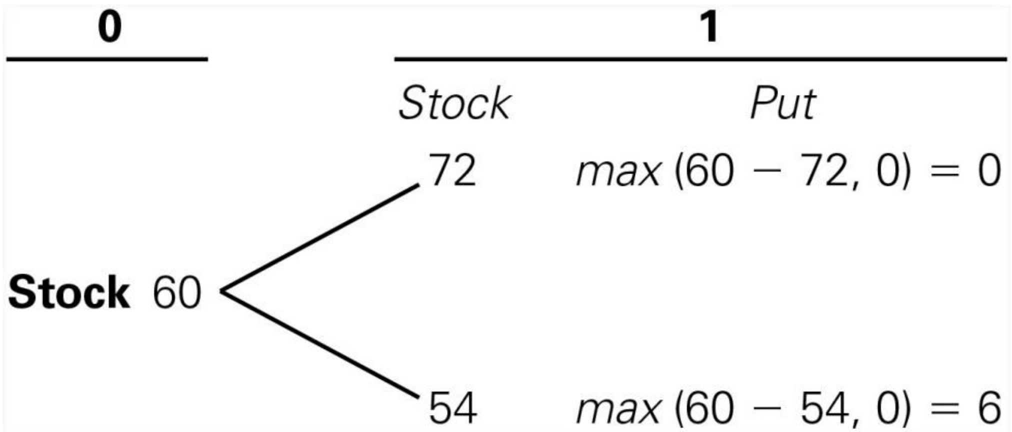
Example 21.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Problem**
  - Suppose a stock is currently trading for $60, and in one period will either go up by 20% or fall by 10%. If the one-period risk-free rate is 3%, what is the price of a European put option that expires in one period and has an exercise price of $60?

Valuing a Put Option

Example 21.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - We begin by constructing a binomial tree:



  - Solve for the value of the put by using $C_u = 0$ (the value of the put when the stock goes up) and $C_d = 6$ (the value of the put when the stock goes down).
  - Therefore,

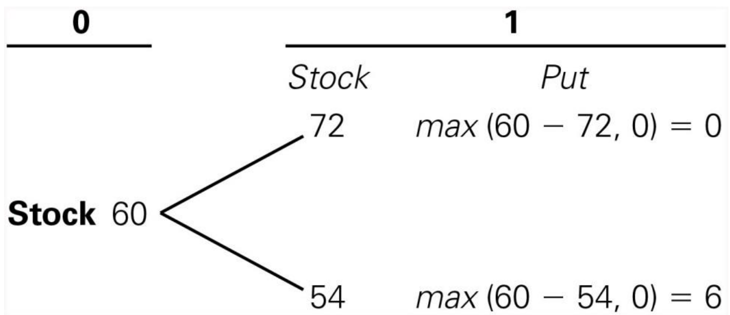
$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{0-6}{72-54} = -0.3333, \text{ and}$$

$$B = \frac{C_d - S_d\Delta}{1+r_f} = \frac{6 - 54(-0.3333)}{1.03} = 23.30$$

Valuing a Put Option

Example 21.1 from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

- **Solution**
  - This portfolio is short 0.3333 shares of the stock, and has \$23.30 invested in the risk-free bond. Let's check that it replicates the put if the stock goes up or down:



$$72(0 - 0.3333) + 1.03(23.30) = 0, \text{ and}$$
$$54(-0.3333) + 1.03(23.30) = 6$$

  - Thus, the value of the put is the initial cost of this portfolio.

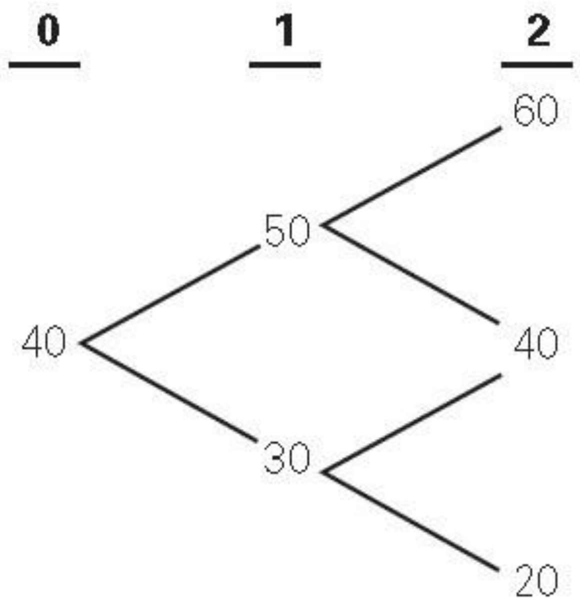
$$\text{Put value } = C = S\Delta + B = 60(-0.3333) + (23.30) = \$3.30$$

A Multiperiod Model: Two-Period Example

- Assume
  - A European call option expires in two periods and has an exercise price of $50
  - The stock price today is equal to $40 and the stock pays no dividends
  - In each period, the stock price will either rise by $10 or fall by $10
  - The one-period risk-free rate is 6%.

A Multiperiod Model

- Consider a two-period binomial tree for the stock price

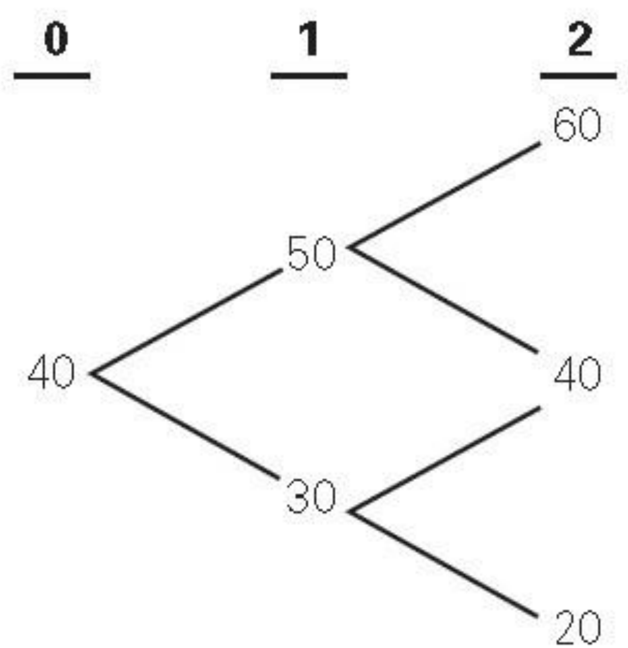


Source: adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e, p. 800

- The multiperiod tree is sometimes referred to as a binomial lattice
- Solve this backwards, starting at the end of the tree with t = 2, then t = 1, and finally t = 0, and focus on one sub-tree with only two possible states each time.

A Multiperiod Model ($t = 2$)

- To calculate the value of an option in a multiperiod binomial tree, start at the end of the tree and work backward.
- At time 2, the option expires, so its value is equal to its intrinsic value.



$$C = \max(S_T - X, 0)$$

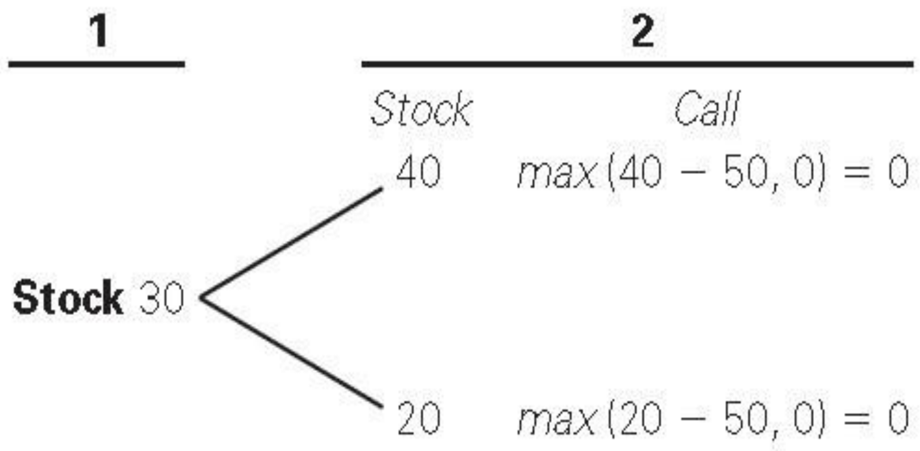
$$\max(60 - 50, 0) = 10$$

$$\max(40 - 50, 0) = 0$$

$$\max(20 - 50, 0) = 0$$

A Multiperiod Model ($t = 1$)

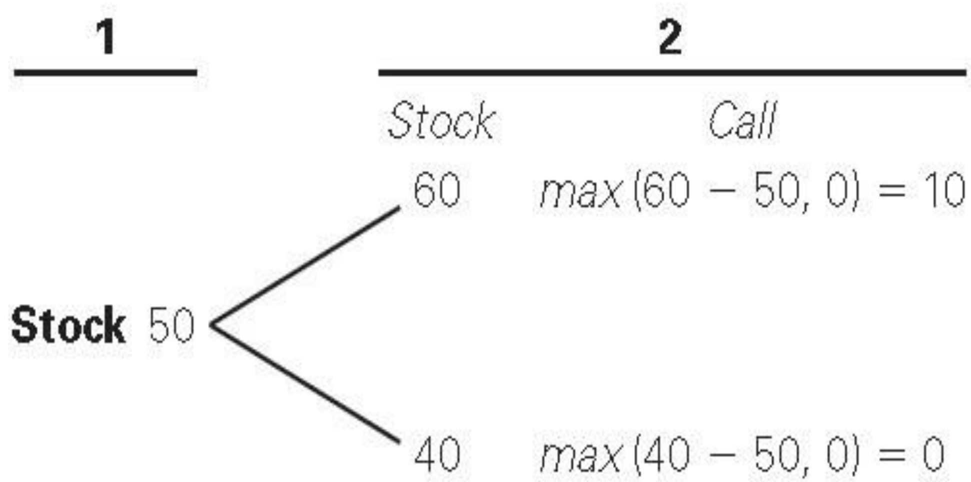
- The next step is to determine the value of the option in each possible state at time 1, focusing on one sub-tree (with only two possible states) at any one time.
- If the stock price has **gone down** to $30 at time 1 (the down state), the binomial tree in the down state will look like this:



- The value of the option will be $0 since it is worth $0 in either state.

A Multiperiod Model (t = 1)

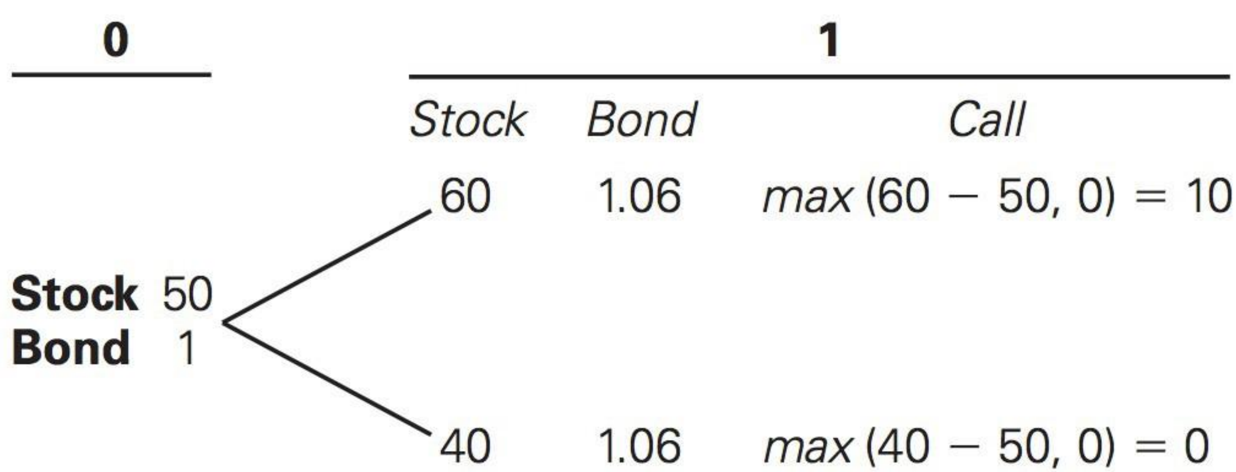
- The next step is to determine the value of the option in each possible state at time 1, focusing on one sub-tree (with only two possible states) at any one time.
- If the stock price has **gone up** to $50 at time 1 (the up state), the binomial tree in the up state will look like this:



- The value of the option = $6.13
- We solved this in the single period example in lecture (next 3 slides)

A Multiperiod Model ($t = 1$) like A Two-State Single-Period Model

- But this was exactly our single-period example in lecture!
- The next 2 slides were from “A Two-State Single-Period Model”
    - Comparing the slides, the only changes are:
        - Time 0 below is time 1 now (in the previous slide)
        - Time 1 below is time 2 now (in the previous slide)
        - But everything else is the same and so the solution is the same



A Two-State Single-Period Model

- In the up state, the value of the portfolio must be \$10

$$60\Delta + 1.06B = 10$$

- In the down state, the value of the portfolio must be \$0

$$40\Delta + 1.06B = 0$$

- Two simultaneous equations with two unknowns Δ and B. Solving, we get

$$\Delta = 0.5$$

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A Two-State Single-Period Model

- By the Law of One Price, the price of the call option today must equal the current market value of the replicating portfolio.

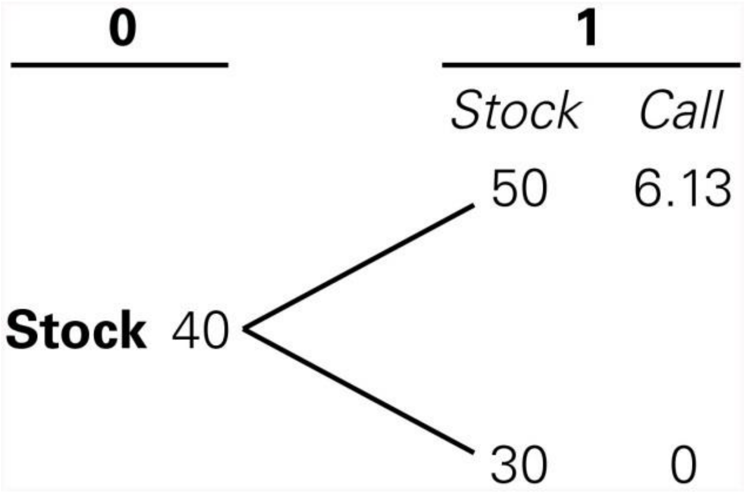
- The value of the portfolio today is the value of 0.5 shares at the current share price of $50, less the amount borrowed.

$$50\Delta + B = 50(0.5) - 18.87 = 6.13$$

- .

A Multiperiod Model ($t = 0$)

- The final step is to determine the value of the option in each possible state at time 0.



- Solving for Δ and B:

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{6.13 - 0}{50 - 30} = 0.3065 \text{ and}$$

$$B = \frac{C_d - S_d\Delta}{1 + r_f} = \frac{0 - 30(0.3065)}{1.06} = -8.67$$

A Multiperiod Model (t = 0)

- The final step is to determine the value of the option in each possible state at time 0.
- Thus, the initial option value is:

$$C = S\Delta + B = 40(0.3065) - 8.67 = \$3.59$$

A Multiperiod Model

- Dynamic Hedging Strategy
  - A replication strategy based on the idea that an option payoff can be replicated by dynamically trading in a portfolio of the underling stock and a risk-free bond.

  - In the two-period example, the replicating portfolio will need to be adjusted at the end of each period.

A Multiperiod Model

- Dynamic Hedging Strategy
- The portfolio starts off long 0.3065 shares of stock and borrowing $8.67.
    - **If the stock price drops to $30**, the shares are worth $9.20, and the debt has grown to $9.20.

      $\$30 \times 0.3065 = \$9.20$ and $\$8.67 \times 1.06 = \$9.20$
    - The net value of the portfolio is worthless, and the portfolio is liquidated.

A Multiperiod Model

- Dynamic Hedging Strategy
- The portfolio starts off long 0.3065 shares of stock and borrowing $8.67.
    - **If the stock price rises to $50**, the net value of the portfolio rises to $6.13.
    - The **new** Δ of the replicating portfolio is 0.5.
    - Therefore 0.1935 more shares must be purchased.

      $0.50 - 0.3065 = 0.1935$
    - The purchase will be financed by additional borrowing.

      $0.1935 \times \$50 = \$9.67$
    - At the end the total debt will be $18.87.

      $\$8.67 \times 1.06 + 9.67 = \$18.87$
    - This matches the value calculated previously.

Making the Model Realistic

- **Binomial model** is an option-valuation model predicated on the assumption that stock prices can move to only two values over any short time period.
- By decreasing the length of each period, and increasing the number of periods in the stock price tree, a realistic model for the stock price and option price can be constructed.
- How to calibrate the up or down movements for each period?

Making the Model Realistic

- Using 900 periods during a year, and a random return of +1/-1% each period.

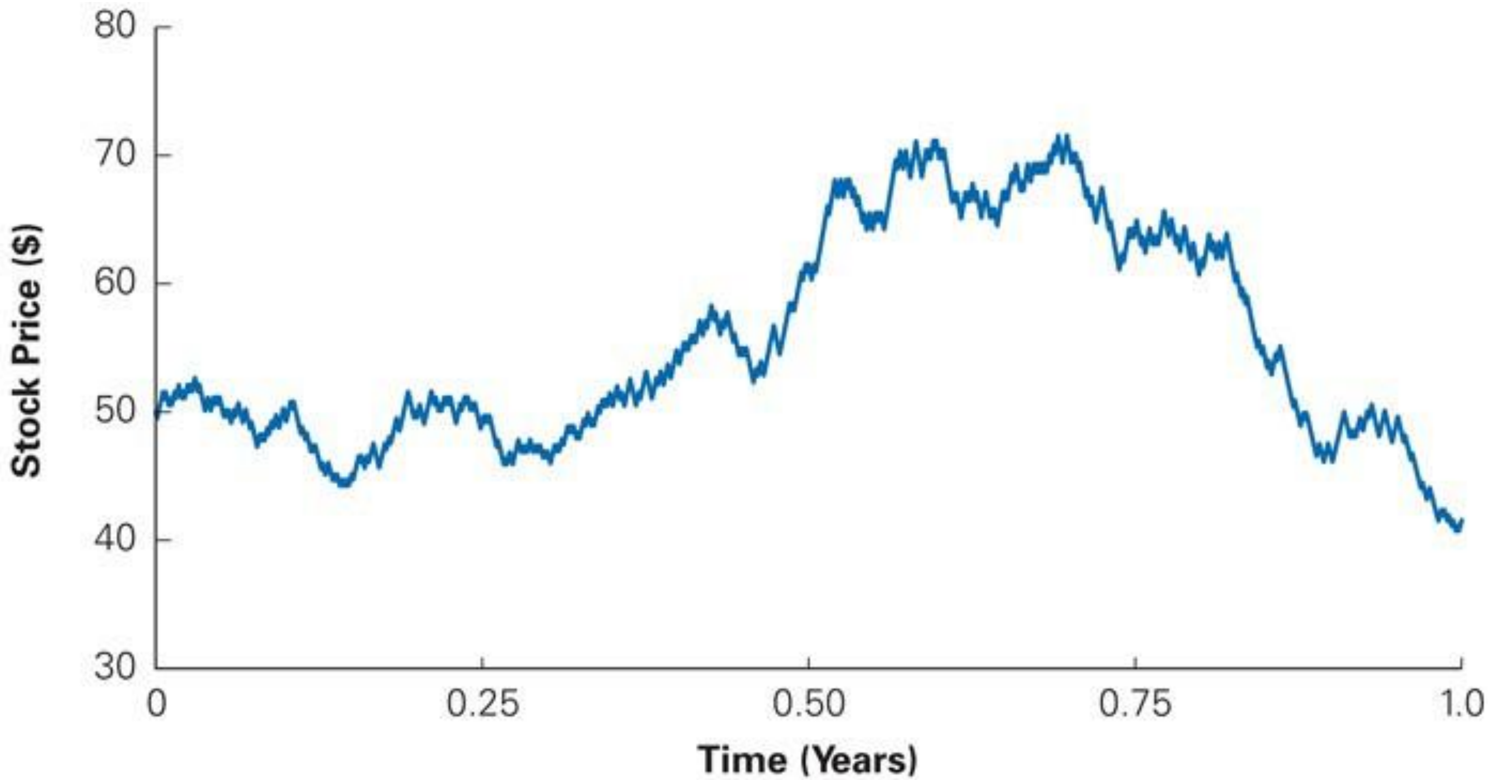


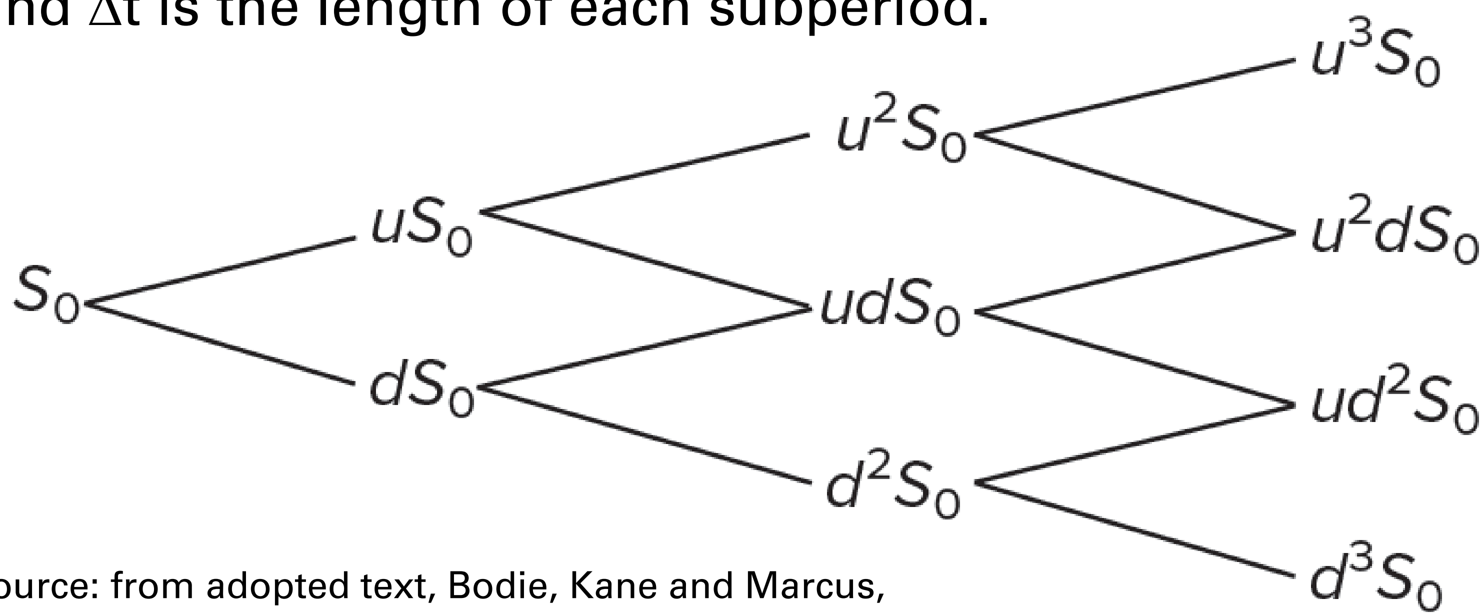
Figure 21.2 A binomial stock price path
from adopted text, Berk and DeMarzo, Corporate Finance, Pearson, 5e

Making the Model Realistic

- Spread between up and down movements in price of the stock reflects the volatility of its rate of return, so *u* and *d* should depend on that volatility.
- **One standard approach** is to set:

$$u = e^{(\sigma \times \sqrt{\Delta t})} \qquad d = e^{(-\sigma \times \sqrt{\Delta t})}$$

- Where σ = the stock's return volatility or standard deviation and Δt is the length of each subperiod.



Source: from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e, p 712.

Figure 21.5 Probability distributions for final stock price.
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

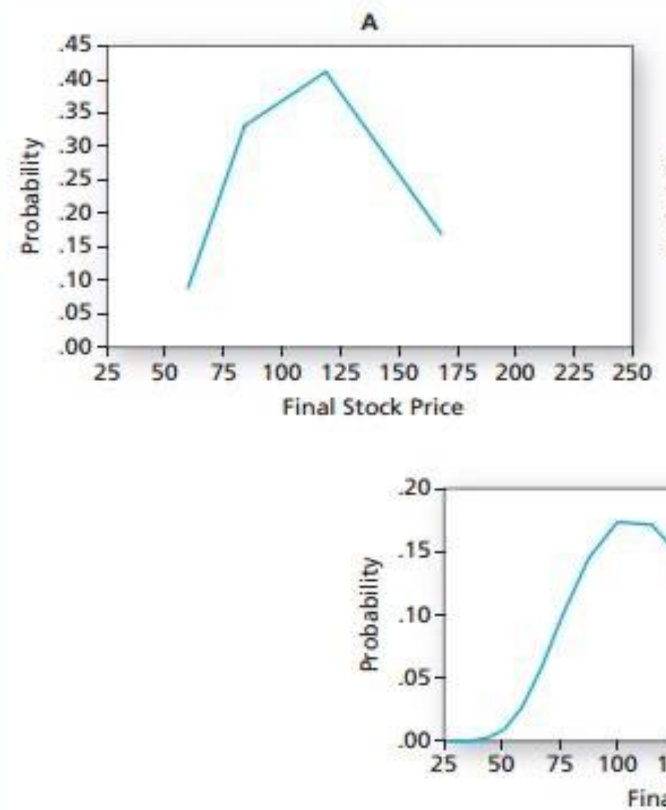


Figure 21.5 Probability distributions for final stock price. Possible outcomes and associated probabilities. In each panel, the stock's annualized, continuously compounded expected rate of return is 10% and its standard deviation is 30%. ***Panel A.*** Three subintervals. In each subinterval, the stock can increase by 18.9% or fall by 15.9%. ***Panel B.*** Six subintervals. In each subinterval, the stock can increase by 13.0% or fall by 11.5%. ***Panel C.*** Twenty subintervals. In each subinterval, the stock can increase by 6.9% or fall by 6.5%.

Binominal Option Pricing

- As the number of subperiods increases, the distribution approaches the skewed log-normal distribution (longer right tail).
- Even if the stock price were to decline in each subinterval, it can never drop below 0.
- But there is no corresponding upper bound on its potential upward performance.