INTRODUCTION TO MATLAB

1. INSTALLATION

The National University of Singapore has a Total Academic Headcount licence for MATLAB. Students may use it for academic, research, and learning. The license allows students to install MATLAB on personally-owned computers.

1. If you are NOT using NUS network, you are required to use nVPN:

- `https://webvpn.nus.edu.sg/`

Please sign in with your NUSNET ID in the format: **nusstu\nusnetid** and password.

If you are using NUS network, please proceed to Step 2.

2. Click the link `https://sm05.stf.nus.edu.sg/studentmatlab/` to download MATLAB. It is required to sign in with your NUSNET ID in the format: **nusstu\nusnetid** and password. Then follow the instructions.

3. Create a MathWorks account using your NUS email address.

(i) `https://www.mathworks.com/mwaccount/register`
- ◇ Email address: Your NUS email account (e.g., e0012345@u.nus.edu)
- ◇ Location: Singapore
- ◇ How will you use MathWorks software? Student use
- ◇ Are you at least 13 years or older? Yes

(ii) You will receive an email from `service@mathworks.com` with title "Verify Email Address". Click the link in the email to verify your account.

(iii) Finish creating your profile. Then you should be able to see the following information:
- Your account has been created and license 40707750 has been associated with your account.

4. Click the **Download** bottom to download and run the installer.

(i) When prompted, log in with your MathWorks Account (your NUS email account).

(ii) Select your licence (40707750, Student, Academic — Total Headcount).

(iii) Choose installation folder.

(iv) Select products to install.

2. Basic Operations

First of all, we learn some basic operations in MATLAB.

MATLAB environment behaves like a super-complex calculator. You can enter the commands at the >> command prompt. The answer appears by pressing Enter.

We can use the following arithmetic operators: addition `+`, subtraction `-`, multiplication `*`, division `/`, exponentiation `^`.

For example, to add two numbers 123 and 321, we simply type

```
>> 123 + 321
```

and then press Enter:

```
ans = 444
```

Similarly, we can apply other operators to use MATLAB as an ordinary calculator:

```
>> 123 - 321
ans = -198
>> 123 * 321
ans = 39483
>> 123 / 321
ans = 0.3832
>> 123 ^ 3
ans = 1860867
```

You may add a semicolon `;` at the end of the statement; then MATLAB will hide the output. For example,

```
>> a = 3;
>> a ^ 2
ans = 9
```

Note that the symbol `a` is now defined as 3. We may remove it from the memory by using `clear a` or remove all variables from the memory by `clear`. If we want to clear the command window, we can use `clc`.

3. Basic Functions

The function `sqrt(x)` computes the principle square root of the number x. For example,

```
>> sqrt(2)
```

```
ans =  1.4142
```

By default, MATLAB displays four decimal digits to its answers. But we can change the format for numeric display. (The percent symbol `%` is used for indicating a comment line.)

```
>>  format long    % 16 decimal digits
>>  sqrt(2)
ans =  1.414213562373095
>>  format rat    % rational approximation
>>  sqrt(2)
ans =  1393/985
>>  format short    % four decimal digits (default)
>>  sqrt(2)
ans =  1.4142
```

We can also use the following functions in MATLAB:

(i) Trigonometric functions: `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`, `sec(x)`, `csc(x)`.
(ii) Inverse trigonometric functions: `asin(x)`, `acos(x)`, `atan(x)`, `acot(x)`.
(iii) Exponential and logarithmic functions: `exp(x)`, `log(x)` (base e), `log10(x)` (base 10).

For trigonometric and inverse trigonometric functions, the angles are measured in radian (π radian $= 180°$), and the constant $\pi = 3.1415926\cdots$ is defined by `pi` (MATLAB is case-sensitive).

```
>>  sin(pi/4)
ans =  0.7071
>>  atan(1)
ans =  0.7854
```

4. Solving Basic Algebraic Equations

The command `solve` is used for solving algebraic equations. We shall

(i) Use `syms` to declare the variables.
(ii) Use `solve(equations, variables)` to solve the specific equations.

Note that `=` is used to assign a value to a variable; and `==` shall be used for equality.

4.1. Single Variable Equations.

For example, $x^2 + x - 1 = 0$.

```
>>  syms x;
>>  solve(x^2 + x - 1 == 0, x)
```

```
ans =  -5^(1/2)/2 - 1/2
        5^(1/2)/2 - 1/2
```

We may use `vpa(x)` or `vpa(x,d)` to evaluate each element of the symbolic input to d digits (the default is 32 digits).

```
>>  vpa(ans,10)
ans =  -1.618033989
        0.6180339887
```

Sometimes the exact solutions cannot be specifically displayed. For example,

$$x^4 - 7x^3 + 3x^2 - 5x + 9 = 0.$$

```
>>  solve(x^4 - 7*x^3 + 3*x^2 - 5*x + 9 == 0, x)
ans =  root(z^4 - 7*z^3 + 3*z^2 - 5*z + 9, z, 1)
       root(z^4 - 7*z^3 + 3*z^2 - 5*z + 9, z, 2)
       root(z^4 - 7*z^3 + 3*z^2 - 5*z + 9, z, 3)
       root(z^4 - 7*z^3 + 3*z^2 - 5*z + 9, z, 4)
```

Then evaluate using floating points to obtain the decimal expressions of the four roots:

```
>>  vpa(ans,10)
ans =  1.059780463
       - 0.3450883978 - 1.077836295i
       - 0.3450883978 + 1.077836295i
       6.630396332
```

Here the symbol `i` refers to the *imaginary unit* i where $i = e^{\pi i/2}$ such that $i^2 = -1$.

Alternatively, one may use `vpasolve` to get the decimal expression directly:

```
>>  vpasolve(x^4 - 7*x^3 + 3*x^2 - 5*x + 9 == 0, x)
ans =  1.0597804633025896291682772499885
       6.630396332390718431485053218985
       - 0.34508839784665403032666523448675 - 1.0778362954630176596831109269793i
       - 0.34508839784665403032666523448675 + 1.0778362954630176596831109269793i
```

MATLAB can also be used to solve symbolic equations. For example, $ax^2 + bx + c = 0$.

```
>>  syms a b c x;
>>  solve(a*x^2 + b*x + c == 0, x)
ans =  -(b + (b^2 - 4*a*c)^(1/2))/(2*a)
```

```
-(b - (b^2 - 4*a*c)^(1/2))/(2*a)
```

4.2. **Multi-Variable Equations.** MATLAB can also solve multi-variable equations, for which the equations shall be put together in square brackets `[ ]`, so do the variables. Moreover, since the output consists multiple variables, we shall assign the solution as a vector. For example,

$$3x + y = 10, \qquad x + y = 20.$$

```
>> syms x y;
>> [Sx,Sy] = solve([3*x+y==10, x+y==20], [x,y])
Sx = -5
Sy = 25
```

The solution of the a general linear system in variables x and y

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

can be obtained as follows:

```
>> syms a b c d e f x y;
>> [Sx,Sy] = [Sx, Sy] = solve([a*x+b*y == e, c*x+d*y == f], [x,y])
Sx = -(b*f - d*e)/(a*d - b*c)
Sy = (a*f - c*e)/(a*d - b*c)
```

5. Sums

If we need to find the sum of a sequence, use the command `symsum`. The format is

```
>> symsum(expression, variable, [min, max])
```

For example, for the arithmetic sequence $\{a_n\}$ given by $a_n = a + (n-1)d$, the sum of its first n terms is $(2a + (n-1)d)n/2$.

```
>> syms a d k n;
>> symsum(a+(k-1)*d, k, [1, n])
ans = a*n - d*(n - (n*(n + 1))/2)
>> simplify(ans)    % simplify the previous answer
ans = (n*(2*a - d + d*n))/2
```

For the geometric sequence $\{a_n\}$ given by $a_n = ar^{n-1}$, the sum of its first n terms is

$$S_n = \begin{cases} a(r^n - 1)/(r-1) & \text{if } r \neq 1, \\ an & \text{if } r = 1. \end{cases}$$

```
>> syms a r k n;
>> symsum(a*r^(k-1), k, [1, n])
ans =  piecewise(r == 1, a*n, r ~= 1, (a*(r^n - 1))/(r - 1))
```

Moreover, we can find the sum to infinity, which is defined by `inf` in MATLAB. Recall that for the geometric sequence $\{a_n\}$ given by $a_n = ar^{n-1}$, the sum to infinity is

$$S_\infty = \begin{cases} a/(1-r) & \text{if } |r| < 1, \\ \text{does not exist} & \text{if } |r| \geq 1. \end{cases}$$

```
>> syms a r k;
>> assume(abs(r)<1);
>> symsum(a*r^(k-1), k, [1, inf])
ans =  -a/(r - 1)
```

6. Vector

A vector $\boldsymbol{a i} + \boldsymbol{b j} + \boldsymbol{c k}$ or (a, b, c) can be defined as `[a, b, c]` or simply `[a b c]`.

The addition, subtraction and multiplication with numbers can be evaluated using `+`, `-` and `+` respectively.

For example, let $\boldsymbol{u} = (1, 2, 3)$ and $\boldsymbol{v} = (4, 5, 6)$.

```
>> u = [1 2 3];
>> v = [4 5 6];
>> u + v
ans =  5   7   9
>> u - v
ans =  -3  -3  -3
>> 3*u
ans =  3   6   9
```

The dot product $\boldsymbol{u} \bullet \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by `dot(u,v)`.

```
>> dot(u,v)
ans =  32
```

We can use the formula $|\boldsymbol{v}| = \sqrt{\boldsymbol{v} \bullet \boldsymbol{v}}$ to find the norm of $\boldsymbol{v}$:

```
>> sqrt(dot(v,v))
ans =  8.7750
```

Alternatively, MATLAB provides a command `norm` for the norm of a vector.

```
>> norm(v)
ans =  8.7750
```

The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is defined by `cross(u,v)`.

```
>> cross(u,v)
ans =  -3  6  -3
```

7. Function

7.1. Standard Function. To define a function, we shall (i) give the name of the function, (ii) use `@` to declare the name of the variable, (iii) provide the expression of the function. For example, define $f(x) = x^2$:

```
>> f = @(x) x^2;
```

Then $f(2)$ can be evaluated by

```
>> f(2);
ans =  4
```

Another example: $g(x) = \sin(x^3)/x^2$

```
>> g = @(x) sin(x^3)/x^2;
```

Multi-variable functions can be defined similarly by declaring more variables. For example, $h(x, y) = \sqrt{x^2 + y^2}$:

```
>> h = @(x,y) sqrt(x^2 + y^2);
```

To evaluate $h(5,12)$:

```
>> h(5,12)
ans =  13
```

7.2. Piecewise Function. The absolute value function $f(x) = |x|$ is a piecewise function:

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases}$$

which can be defined in MATLAB:

```
>> syms x
>> f = piecewise(x>=0, x, x<0, -x);
```

The general format is to use the `piecewise` as

```
>> f = piecewise(condition 1, value 1, ..., condition N, value N);
```

In the example above, the first condition is `x>=0` and this means "x is greater than or equal to 0". The value `x` is the expression of the function when the condition `x>=0` is satisfied. It defines $f(x) = x$ when $x \geq 0$. The second condition `x<0` and the second value `-x` defines $f(x) = -x$ when $x < 0$.

In order to evaluate the value of the function at given point, for example, $f(-2)$, we use

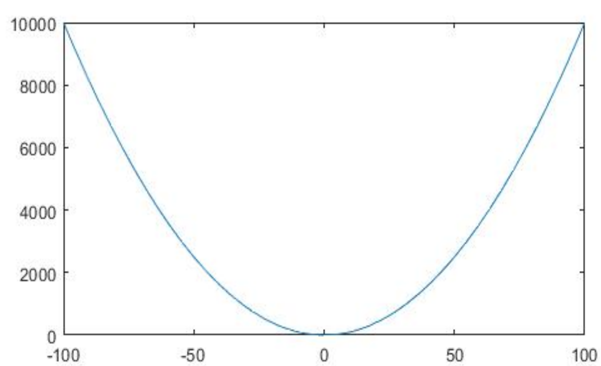
```
>> subs(f, x, -2)
ans =  2
```

8. Curve Plotting

8.1. Standard Function. Once a function is defined, `fplot` can be used to plot its graph over a specified interval in the form `[xmin, xmin]`.

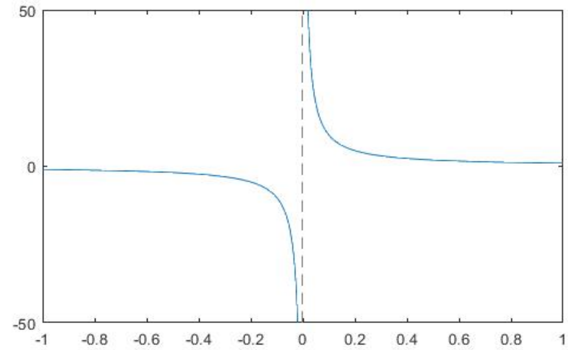
For example, we plot $f(x) = x^2$ on the interval $(-100, 100)$:

```
>> f = @(x) x^2;
>> fplot(f, [-100,100]);
```



The command can be used even if the function is undefined somewhere on the specific interval. For example, $g(x) = 1/x$ on $(-1, 1)$ is defined at $x = 0$.

```
>> g = @(x) 1/x;
>> fplot(g, [-1,1]);
```

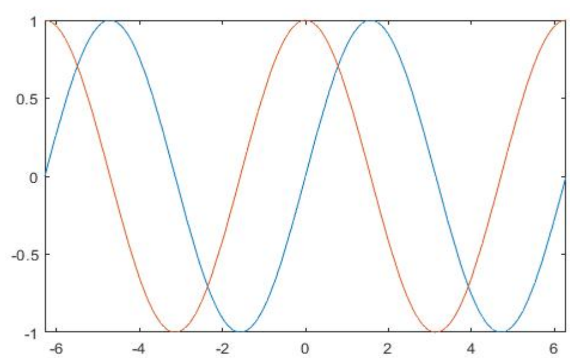


8.2. **Multiple Curves.** In order to plot more curves in the same coordinate system, we use the commands `on` and `off`. For example,

$$f(x) = \sin x \quad \text{and} \quad g(x) = \cos t, \qquad -2\pi < x < 2\pi.$$

(We can use Shift + Enter to start a new line without breaking the command.)

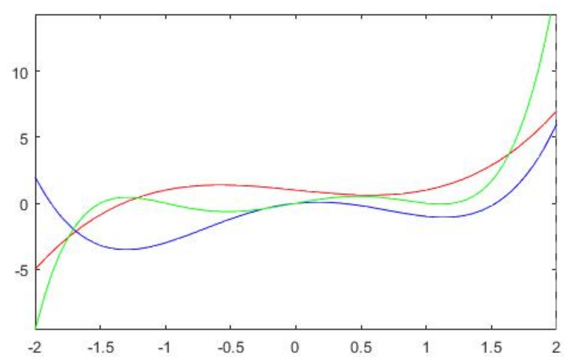
```
>> f = @(x) sin(x);
>> g = @(x) cos(x);
>> fplot(f, [-2*pi, 2*pi]);
   hold on
   fplot(g, [-2*pi, 2*pi]);
   hold off
```



We can draw multiple graphs on the same plot. MATLAB provides some basic colour options: white `w`, black `b`, blue `b`, red `r`, cyan `c`, green `g`, magenta `m`, yellow `y` to distinguish different graphs. For example,

$$f(x) = x^3 - x + 1,$$
$$g(x) = x^4 - 3x^2 + x,$$
$$h(x) = x^5 + 0.3x^4 - 2.8x^3 - 0.3x^2 + 1.8x.$$

```
>> f = @(x) x^3 - x + 1;
>> g = @(x) x^4 - 3*x^2 + x;
>> h = @(x) x^5 + 0.3*x^4 - 2.8*x^3 - 0.3*x^2 + 1.8*x;
>> fplot(f, [-2,2], 'r');
   hold on
   fplot(g, [-2,2], 'b');
   fplot(h, [-2,2], 'g');
   hold off;
```



8.3. **Parametric Equations.** A curve may be defined using a pair of parametric equations:

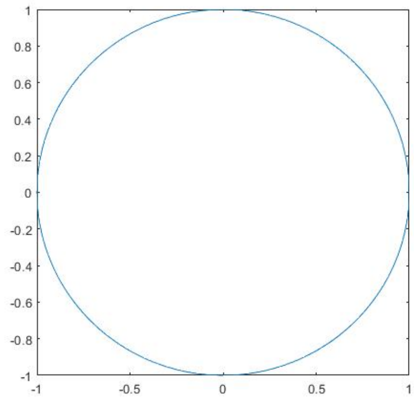
$$x = x(t) \qquad \text{and} \qquad y = y(t).$$

In order to use `fplot`, we shall first define the functions for the x- and y-coordinates.

Recall that a unit circle can be parameterized by

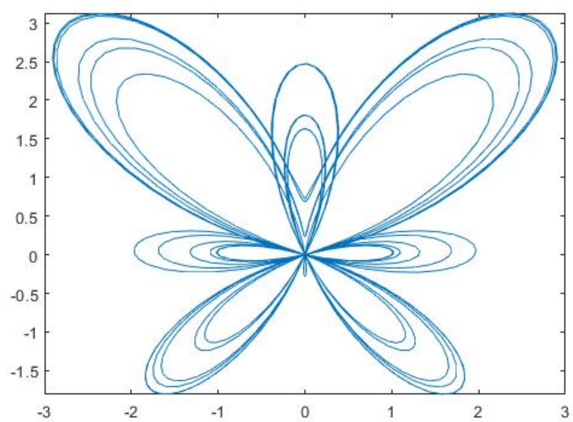
$$x = \cos t, \quad y = \sin t, \qquad 0 \le t \le 2\pi.$$

```
>> f = @(t) cos(t);
>> g = @(t) sin(t);
>> fplot(f,g, [0,2*pi]);
```



MATLAB can draw a butterfly:

```
>> f = @(t) sin(t)*(exp(cos(t))-2*cos(4*t) - (sin(t/12))^5);
>> g = @(t) cos(t)*(exp(cos(t))-2*cos(4*t) - (sin(t/12))^5);
>> fplot(f,g, [0,12*pi]);
```

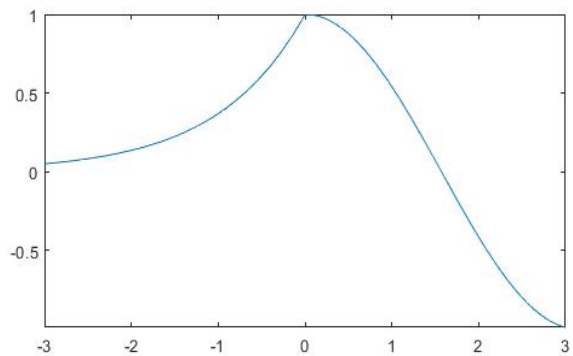


8.4. **Piecewise Function.** There are two ways to plot a piecewise function.

Once a piecewise function is already defined, `fplot` can directly plot its graph:

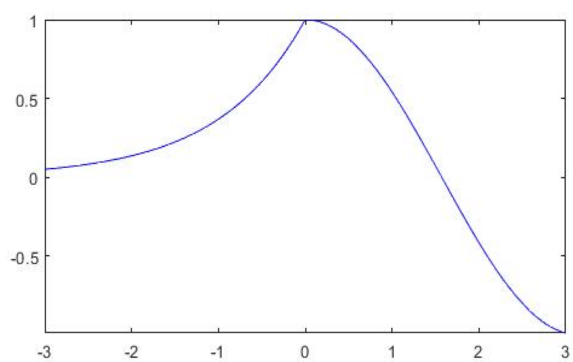
$$f(x) = \begin{cases} e^x & \text{if } x < 0, \\ \cos x & \text{if } x \geq 0. \end{cases}$$

```
>> syms x;
>> f = piecewise(x<0, exp(x), x>=0, cos(x));
>> fplot(f, [-3,3]);
```



Alternatively, we can plot different branches on different intervals using the same colour.

```
>> f1 = @(x) exp(x);
>> f2 = @(x) cos(x);
>> fplot(f1, [-3,0], 'b');
   hold on
   fplot(f2, [0,3], 'b');
   hold off;
```

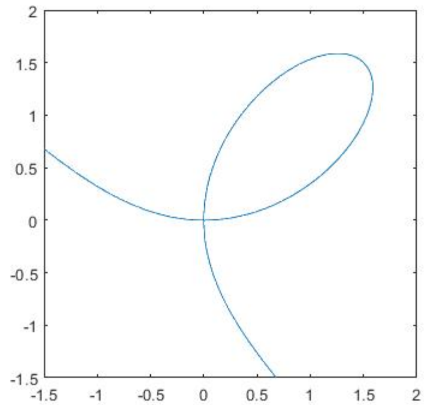


8.5. **Implicit Function.** The command `fimplicit` can be used to plot graphs defined by implicit functions, i.e., $f(x, y) = 0$. The format is:

```
>> fimplicit(function in two variables, [xmin, xmax, ymin, ymax])
```

For example, to plot $x^3 + y^3 = 3xy$, we shall first define the function $f(x, y) = x^3 + y^3 - 3xy$:

```
>> f = @(x,y) x^3 + y^3 - 3*x*y;
>> fimplicit(f, [-1.5, 2, -1.5, 2]);
```



9. LIMITS

The limit of a function describes the behavior of a function near a point or at infinity. The limit $\lim_{x\to a} f(x)$ can be easily evaluated with the command `limit`:

```
>> limit(f(x), x, a)
```

For example, $\lim_{x\to 0} \frac{x}{\sqrt{1+3x}-1}$:

```
>> syms x;
>> limit(x/(sqrt(1+3*x)-1), x, 0)
ans =  2/3
```

The one-sided limits can be evaluated by specifying the direction `left` or `right`. For example, let the floor function $\lfloor x \rfloor$ denote the greatest integer smaller or equal to x. Then

$$\lim_{x\to 2^-} \lfloor x \rfloor = 1, \quad \lim_{x\to 2^+} \lfloor x \rfloor x = 2, \quad \text{and} \quad \lim_{x\to 2} \lfloor x \rfloor \text{ does not exist.}$$

```
>> syms x;
>> limit(floor(x), x, 2, 'left')
ans =  1
>> limit(floor(x), x, 2, 'right')
ans =  2
>> limit(floor(x), x, 2)
ans =  limit(floor(x), x, 2)
```

We can find the limit at infinity `inf` or negative infinity `-inf`. For example, $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x$:

```
>> syms a x;
>> limit((1+a/x)^x, x, inf)
```

```
ans =  exp(a)
```

Sometimes the limit depends on the value of the parameters. For example,

$$\lim_{x\to\infty} 2^{ax} = \begin{cases} 0 & \text{if } a < 0, \\ 1 & \text{if } a = 0, \\ \infty & \text{if } a > 0. \end{cases}$$

We may use `assume` to declare the value of the parameter.

```
>>  syms a x;
>>  assume(a<0)
>>  limit(2^(a*x), x, inf)
ans =  0
>>  assume(a==0)
>>  limit(2^(a*x), x, inf)
ans =  1
>>  assume(a>0)
>>  limit(2^(a*x), x, inf)
ans =  Inf
```

10. Derivative

The derivative of a function f at point a is defined as the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

For example, if $f(x) = x^2$, then $f'(x) = 2x$:

```
>>  syms x h;
>>  f = @(x) x^2;
>>  limit((f(x+h)-f(x))/h, h, 0)
ans =  2*x
```

On the other hand, MATLAB includes the command `diff` for differentiation. For example, find the derivative of $f(x) = \cos(x^2)$ with respect to x:

```
>>  syms x;
>>  f = @(x) cos(x^2);
>>  diff(f, x)
ans =  -2*x*sin(x^2)
```

In order to differentiate $g(x) = \dfrac{x - 3x\sqrt{x}}{\sqrt{x}}$ with respect to x:

```
>> syms x;
>> g = @(x) (x-3*x*sqrt(x))/sqrt(x);
>> diff(g, x)
ans = - (x - 3*x^(3/2))/(2*x^(3/2)) - ((9*x^(1/2))/2 - 1)/x^(1/2)
```

The answer seems complicated. Fortunately MATLAB provides the command `simplify` which may help in this situation:

```
>> simplify(ans)
ans = -(6*x^(1/2) - 1)/(2*x^(1/2))
```

Suppose we are looking for $f^{(4)}(x)$, the 4^{th} order derivative of $f(x)$ with respect to 4, of course we may use

```
>> diff(diff(diff(diff(f, x), x), x), x)
ans = 16*x^4*cos(x^2) - 12*cos(x^2) + 48*x^2*sin(x^2)
```

However, it may be a bit cumbersome. The following shorter commands have the same effect:

```
>> diff(f, x,x,x,x)
ans = 16*x^4*cos(x^2) - 12*cos(x^2) + 48*x^2*sin(x^2)

>> diff(f, x, 4)
ans = 16*x^4*cos(x^2) - 12*cos(x^2) + 48*x^2*sin(x^2)
```

For example, we can find the local extreme values of $f(x) = x^3 - 3x^2 + x - 2$.

```
>> syms x;
>> f = @(x) x^3 - 3*x^2 + x -2;
>> g = diff(f, x);    % define g(x) to be f'(x)
>> h = diff(f, x, x);    % define h(x) to be f''(x)
```

(i) Solve $f'(x) = 0$ to obtain the stationary points.

```
>> solve(g(x) == 0, x)
ans = 1 - 6^(1/2)/3
      6^(1/2)/3 + 1
```

(ii) Check the sign of $f''(x)$ at the stationary points. (Note that you can use Copy and Paste in the scripts.)

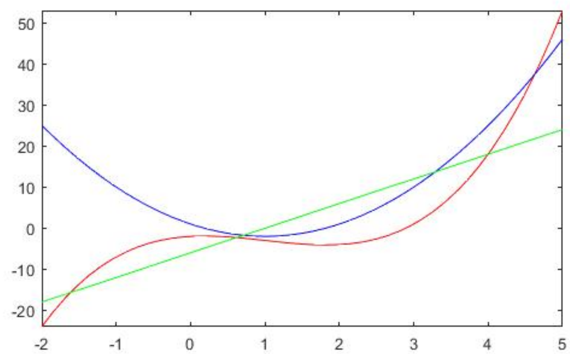
```
>> subs(h, x, 1-6^(1/2)/3)
ans = -22063042185692343/4503599627370496
```

```
>> subs(h, x, 6^(1/2)/3+1)
ans =  5515760546423085/1125899906842624
```

Hence, f has a local maximum at $1-\sqrt{6}/3$ and a local minimum at $1+\sqrt{6}/3$.

We can plot $f(x)$, $f'(x)$ and $f''(x)$ in the same coordinate system.

```
>> fplot(f, [-2,5], 'r');
   hold on
   fplot(g, [-2,5], 'b');
   fplot(h, [-2,5], 'g');
   hold off
```



11. Integral

11.1. **Indefinite Integral.** For a function $f(x)$, its indefinite integral is a function $F(x)$ such that $F'(x) = f(x)$, denoted by

$$F(x) = \int f(x)\,dx.$$

All indefinite integrals differ by a constant only. So we also use

$$F(x) = \int f(x)\,dx + C$$

to represent the entire family of indefinite integrals of $f(x)$, where C is an arbitrary constant.

In MATLAB, we can use the command `int` to find an indefinite integral.

For example, $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C.$

```
>> syms x;
>> f = @(x) 1/(1+x^2)^2;
>> int(f, x)
ans =  atan(x)/2 + x/(2*(x^2 + 1))
```

Note that the constant C is dropped in the answer. Another example:

$$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x + C.$$

```
>> syms x;
>> g = @(x) sin(x)^6;
>> int(g, x)
ans = (5*x)/16 - (15*sin(2*x))/64 + (3*sin(4*x))/64 - sin(6*x)/192
```

11.2. **Definite Integral.** The definite integral $\int_a^b f(x)\,dx$ represents the net area bounded between the graph of $y = f(x)$ and the x-axis from $x = a$ to $x = b$. The command `int` can also be used to find definite integral by specifying the lower limit a and upper limit b in the form

```
>> int(expression, variable, [lower limit, upper limit])
```

For example, $\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{8} + \frac{1}{4}$.

```
>> syms x;
>> f = @(x) 1/(1+x^2)^2;
>> int(f, x, [0, 1])
ans = pi/8 + 1/4
```

Another example:

$$\int_0^{\pi} \sin^6 x\,dx = \frac{5}{16}\pi.$$

```
>> syms x;
>> g = @(x) sin(x)^6;
>> int(g, x, [0, pi])
ans = (5*pi)/16
```

We can verify the fundamental theorem of calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dx = f(x).$$

```
>> syms a t x f(t);    % use f(t) to declare that f is a function in t
>> diff(int(f(t), t, [a, x]), x)
ans = f(x)
```

It is also possible to integrate over an infinite interval by specifying the lower limit and/or the upper limit as `-inf` or `inf`. For example,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

```
>> int(exp(-x^2), x, [-inf, inf])
ans =  pi^(1/2)
```

12. Differential Equation

12.1. **General Solution.** In order to solve an ordinary differential equation of the form $\dfrac{dy}{dx} = f(x, y)$, we shall first declare that y is a function in x, and then use `dsolve`.

For example, $\dfrac{dy}{dx} = 1 + x + y$. Note that the equal sign is represented by `==`.

```
>> syms y(x);    % declare that y is a function in x
>> dsolve(diff(y, x) == 1+x+y)
ans =  C1*exp(x) - x -2
```

Here `C1` refers to an undermined constant. So the general solution of the given equation is $y = Ce^x - x - 2$, where C is an arbitrary constant.

Ordinary differential equations with higher order can be solved similarly.

For example, $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = e^x$.

```
>> syms y(x);
>> dsolve(diff(y,x,x) + 2*diff(y,x) + 2*y == 0)
ans =  C2*exp(-x)*cos(x) - C3*exp(-x)*sin(x)
```

The general solution is $y = C_2e^{-x}\sin x + C_3e^{-x}\cos x$, where C_2, C_3 are arbitrary constants.

12.2. **Particular Solution.** Suppose an initial condition is given. We can use

```
>> dsolve(ode, condition)
```

For example, $\dfrac{dy}{dx} = 1 + x + y$ such that $y = 1$ at $x = 0$.

```
>> syms y(x);
>> dsolve(diff(y, x) == 1+x+y, y(0)==1)
ans =  3*exp(x) - x-2
```

For higher order differential equations, we need more initial conditions. We use

```
>> dsolve(ode, [condition 1, ..., condition N])
```

The condition $y'(a) = b$ shall be defined as `Dy(a)=b` if we set

```
>> Dy = diff(y,x);
```

For example, $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} - 9y = 0$, where $y(1) = 2$ and $y'(1) = 0$.

```
>> syms y(x);
```

```
>> Dy = diff(y,x);
>> dsolve(diff(y,x,x) + 8*diff(y,x) - 9*y == 0, [y(1) == 2, Dy(1) == 0])
ans =  (exp(-9*x)*exp(9))/5 + (9*exp(-1)*exp(x))/5
>> simplify(ans)
ans =  (9*exp(x - 1))/5 + exp(9 - 9*x)/5
```

12.3. **System of Differential Equations.** The command `dsolve` can also be used for a system of differential equations. Note that the output consists multiple functions. So the output must be in vector form:

```
>> [var 1, ..., var N] = dsolve([eqn1, ..., eqn M], [cond 1, ..., cond k])
```

For example, the following linear system consists of two functions $x(t)$ and $y(t)$ with two equations and two initial conditions:

$$\begin{cases} x'(t) = 3x + 4y \\ y'(t) = -4x + 3y \end{cases} \qquad \text{where} \quad x(0) = 2 \quad \text{and} \quad y(0) = 3.$$

```
>> syms x(t) y(t);
>> [Sx, Sy] = dsolve([diff(x,t) == 3*x + 4*y, diff(y,t) == -4*x + 3*y], [x(0)
== 2, y(0) == 3])
Sx =  2*cos(4*t)*exp(3*t) + 3*sin(4*t)*exp(3*t)
Sy =  3*cos(4*t)*exp(3*t) - 2*sin(4*t)*exp(3*t)
```

13. Plotting in the Space

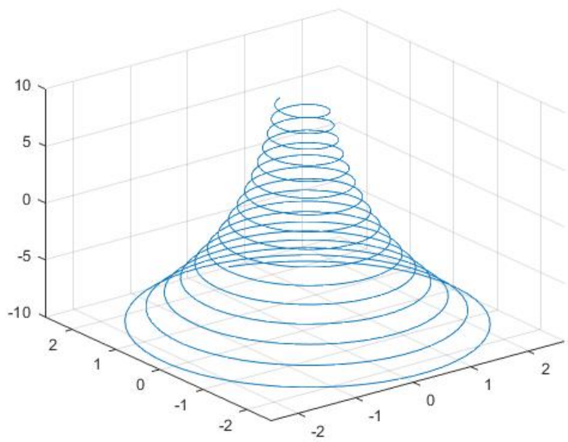
13.1. **Curve in the Space.** A curve in the xyz-space can be parametrized as

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, \qquad a \le t \le b.$$

Plotting a space curve in the xyz-space is similar to plotting a parametrized curve in the xy-plane. The only difference is to replace `fplot` by `fplot3`. For example,

$$\boldsymbol{r}(t) = (e^{-t/10} \sin 5t)\,\boldsymbol{i} + (e^{-t/10} \cos 5t)\,\boldsymbol{j} + t\boldsymbol{k}.$$

```
>> x = @(t) exp(-t/10)*sin(5*t);
>> y = @(t) exp(-t/10)*cos(5*t);
>> z = @(t) t;
>> fplot3(x,y,z, [-10,10])
```



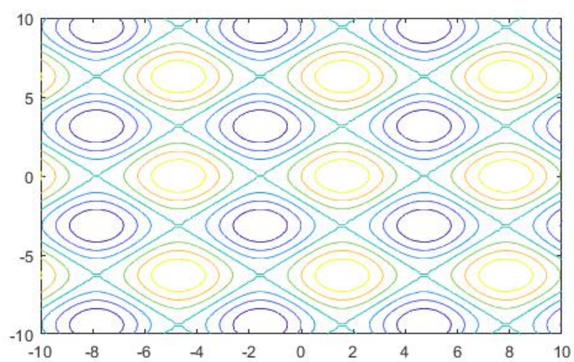
13.2. **Level Curves.** A surface is defined as a two-variable function $z = f(x, y)$. Its level curve at $z = c$ is the intersection of the surface with the horizontal plane at $z = c$; that is, $f(x, y) = c$.

The command `fcontour` plots the level curve of a two-variable function.

```
>> fcontour(function, [xmin, xmax, ymin, ymax])
```

For example, $f(x, y) = \sin x + \cos y$:

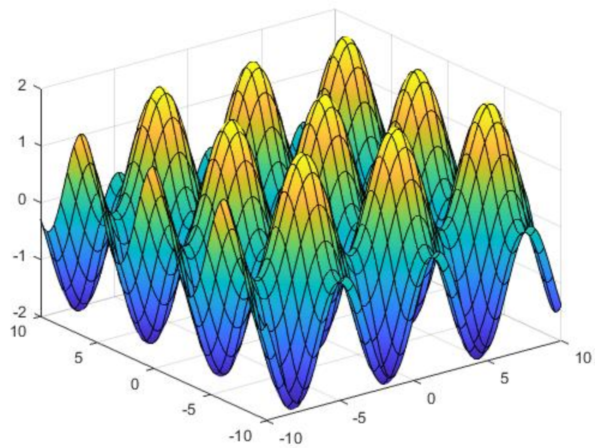
```
>> f = @(x,y) sin(x) + cos(y);
>> fcontour(f, [-10,10, -10,10])
```



It is also possible plot the level curves of two surfaces using `hold on` and `hold off`.

13.3. **Surface.** A surface $z = f(x, y)$ in two variables can be plotted in xyz-space using `fsurf`. For example, $f(x, y) = \sin x + \cos y$.

```
>> f = @(x,y) sin(x) + cos(y);
>> fsurf(f, [-10,10, -10,10])
```



13.4. **Parametrized Surface.** A surface can be parametrized by two parameters u and v:

$$\boldsymbol{r}(u,v) = x(u,v)\,\boldsymbol{i} + y(u,v)\,\boldsymbol{j} + z(u,v)\,\boldsymbol{k}.$$

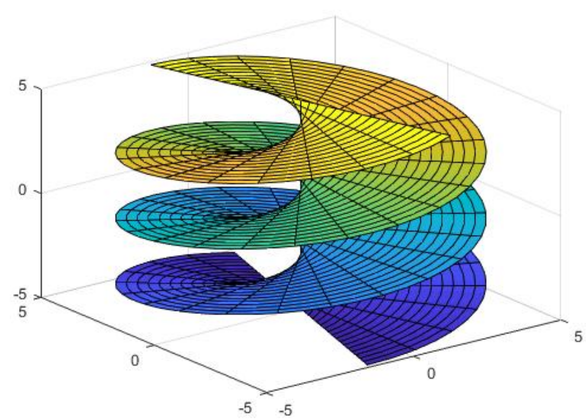
We can still use `fsurf` to plot parametrized surface in the form

```
>> fsurf(x, y, z, [umin, umax, vmin, vmax])
```

For example,

$$\boldsymbol{r}(u,v) = u\cos v\,\boldsymbol{i} + u\sin v\,\boldsymbol{j} + v\,\boldsymbol{k}.$$

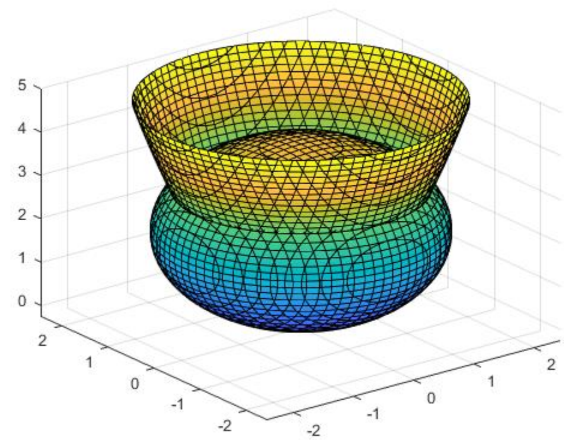
```
>> x = @(u,v) u*cos(v);
>> y = @(u,v) u*sin(v);
>> z = @(u,v) v;
>> fsurf(x,y,z, [-5,5, -5,5])
```



13.5. **Implicit Function.** A surface can also be defined by an implicit function, i.e., $f(x,y,z)=0$. It is almost the same as plotting a curve defined by implicit function in the xy-plane. Simply use `fimplicit3` instead of `fimplicit`.

For example, plot the sphere $x^2+y^2+z^2=4z$ and the paraboloid $z=x^2+y^2$.

```
>> f = @(x,y,z) x^2 + y^2 + z^2 - 4*z;
>> g = @(x,y,z) x^2 + y^2 - z;
>> fimplicit3(f, [-3,3, -3,3, -1,5]);
   hold on
   fimplicit3(g, [-3,3, -3,3, -1,5]);
   hold off
```



14. Multi-Variable Derivatives

14.1. **Partial Derivatives.** For a multi-variable function, we can find its partial derivatives by specifying the variables to be differentiated with respect to properly.

For example, for $f(x,y,z)=x\ln(xy^2z^3)$, its partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are given by

```
>> syms x y z;
>> f = @(x,y,z) x*log(x*y^2*z^3);
>> diff(f, x)
ans =  log(x*y^2*z^3)+1
>> diff(f, y)
ans =  (2*x)/y
>> diff(f, z)
```

```
ans =  (3*x)/z
```

The command `gradient` provides the gradient vector ∇f of a function f so that we can obtain all the derivatives at once (the output is a column vector).

```
>>  gradient(f, [x,y,z])
ans =  log(x*y^2*z^3)+1
       (2*x)/y
       (3*x)/z
```

We may verify the mixed derivative theorem: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$:

```
>>  diff(f, x, y)
ans =  2/y
>>  diff(f, y, x)
ans =  2/y
```

Recall that the directional derivative of $f(x, y, z)$ along a vector $\boldsymbol{v}$ at (a, b, c) is given by

$$\nabla f(a, b, c) \bullet \boldsymbol{v}/|\boldsymbol{v}|.$$

For example, at $(1,2,3)$, the directional derivative of f along $\boldsymbol{v} = (4,5,6)$ is

```
>>  v = [4 5 6];
>>  subs(gradient(f, [x,y,z]), {x,y,z}, {1,2,3});
>>  dot(ans, v)/norm(v)
ans =  (77^(1/2)*(4*log(108) + 15))/77
```

14.2. **Derivative of a Curve.** For a curve C defined by

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k},$$

its derivative vector is $\boldsymbol{r}'(t) = x'(t)\boldsymbol{i} + y'(t)\boldsymbol{j} + z'(t)\boldsymbol{k}$.

One may define the x-, y- and z-components separately. Alternatively, we can define $\boldsymbol{r}$ as a vector function in t. For example, $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}$.

```
>>  r = @(t) [cos(t) sin(t) t];
```

Then its derivative vector is

```
>>  syms t;
>>  diff(r, t)
ans =  [ -sin(t), cos(t), 1]
```

15. Integral for Multi-Variable Functions

15.1. Double Integral. Suppose that a region D in the xy-plane is parametrized by

$$a \le x \le b, \quad \alpha(x) \le y \le \beta(x).$$

Then for any function $f(x,y)$, the double integral

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\,dx,$$

is converted to two integrals.

Let $D = \{(x,y) \mid 0 \le x \le 2, x^2 \le y \le 2x\}$, then $\iint_D (x^2+y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$.

```
>> syms x y;
>> f = @(x,y) x^2 + y^2;
```

We first find $\int_{x^2}^{2x} f(x,y)\,dy$:

```
>> int(f, y, [x^2,2*x])
ans = -(x^3*(x^3 + 3*x - 14))/3
```

then integrate the result with respect to x on $[0,2]$:

```
>> int(ans, x, [0,2])
ans = 216/35
```

Note that D can also be parametrized by $D = \{(x,y) \mid 0 \le y \le 4, y/2 \le x \le \sqrt{y}\}$. So

$$\iint_D (x^2+y^2)\,dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy.$$

```
>> int(f, x, [y/2, sqrt(y)])
ans = y^(3/2)*(y - (13*y^(3/2))/24 + 1/3)
>> int(ans, y, [0,4])
ans = 216/35
```

15.2. Line Integral of a Function. Let C a curve parametrized by

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, \qquad a \le t \le b.$$

Its length is the line integral

$$\int_C 1\,ds = \int_a^b |\boldsymbol{r}'(t)|\,dt.$$

For example, $\boldsymbol{r} = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}$, $0 \le t \le 2\pi$.

```
>> r = @(t) [cos(t) sin(t) t];
>> syms t;
```

```
>> norm(diff(f,t))
ans =  (abs(cos(t))^2 + abs(sin(t))^2 + 1)^(1/2)
>> int(ans, [0, 2*pi])
ans =  2*pi*2^(1/2)
```

In general, the line integral of a function f along C is

$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\,|\boldsymbol{r}'(t)|\,dt.$$

Since there is no simple way to evaluate $f(\boldsymbol{r}(t))$ directly, we may need to define all the components of $\boldsymbol{r}$ as functions.

For example, $\boldsymbol{r} = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}$, $0 \le t \le 2\pi$, and $f(x, y, z) = x^2 + yz$.

```
>> x = @(t) cos(t);
>> y = @(t) sin(t);
>> z = @(t) t;
>> r = @(t) [x(t) y(t) z(t)];    % define the curve
>> f = @(x,y,z) x^2 + y*z;
>> syms t;
>> dot(f(x(t),y(t),z(t)), norm(diff(r,t)));
>> int(ans, t, [0, 2*pi])
ans =  -pi*2^(1/2)
```

15.3. **Line Integral of a Vector Field.** A vector field $\boldsymbol{F}$ is a vector valued function in multi-variables:

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}.$$

Its line integral along the curve C defined by $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, $a \le t \le b$, is

$$\int_C \boldsymbol{F} \bullet d\boldsymbol{r} = \int_a^b \boldsymbol{F}(x(t), y(t), z(t)) \bullet \boldsymbol{r}'(t)\,dt.$$

For example, $\boldsymbol{r} = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}$, $0 \le t \le 2\pi$, and $\boldsymbol{F}(x, y, z) = x^2\boldsymbol{i} + y^2$.

```
>> x = @(t) cos(t);
>> y = @(t) sin(t);
>> z = @(t) t;
>> r = @(t) [x(t) y(t) z(t)];
>> F = @(x,y,z) x^2 + y^2 + z^2;
>> syms t;
```

```
>> dot(F(x(t),y(t),z(t)), diff(r,t));
>> int(ans, t, [0, 2*pi])
ans =  (8*pi^3)/3
```

16. Series

16.1. Taylor Series.

The Taylor series of a function $f(x)$ at $x = a$ is of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In particular, if $a = 0$, it is called the Maclaurin series of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

MATLAB has the command `taylor` to produce the Taylor series of a function up to certain order:

```
>> taylor(function, variable, point, 'Order', number)
```

For example, find the Taylor series of $f(x) = \dfrac{x^2+3}{x+5}$ at $x = 3$ of order < 6:

```
>> taylor((x^2+3)/(x+5), x, 3, 'Order', 6)
ans =  (9*x)/16 + (7*(x - 3)^2)/128 - (7*(x - 3)^3)/1024 + (7*(x - 3)^4)/8192
- (7*(x - 3)^5)/65536 - 3/16
```

16.2. Fourier Series.

The fourier series of a periodic function $f(x)$ with periodic $2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}\,dx \quad \text{and} \quad b_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}\,dx.$$

There is no simple command in MATLAB to produce the coefficients for fourier series. But we can define by ourself.

For example, let $f(x) = \begin{cases} 0 & \text{if } -2 \le x < 0, \\ x & \text{if } 0 \le x < 2, \end{cases}$ and $f(x) = f(x+4)$ for all x; so $L = 2$.

```
>> syms x
>> f = piecewise(-2<=x<0, 0, 0<=x<2, x);
>> L = 2;
>> a = @(n) 2/L*int(f*cos(pi*n*x/L), x, [-L, L]);
>> b = @(n) 2/L*int(f*sin(pi*n*x/L), x, [-L, L]);
```

We may find $a_0, \ldots, a_6$ and $b_1, \ldots, b_6$ using simple commands:

```
>>  a(0:6)
ans =  [ 2, -8/pi^2, 0, -8/(9*pi^2), 0, -8/(25*pi^2), 0]
>>  b(1:6)
ans =  [ 0, 4/pi, -2/pi, 4/(3*pi), -1/pi, 4/(5*pi), -2/(3*pi)]
```

17. Operations on Matrices

17.1. **Basic Operations.** The entries of a matrix shall be entered row by row, while the entries in each row are separated by a space and the rows are separated by a semi-colon `;`. For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

```
>>  A = [1 2 3; 4 5 6]
A =  1  2  3
     4  5  6
```

The matrix addition, subtraction and multiplication with scalar can be evaluated using `+`, `-` and `*` respectively. For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}.$$

```
>>  A = [1 2; 3 4];
>>  B = [4 1; 2 5];
>>  A + B
ans =  5  3
       5  9
>>  A - B
ans =  -3  1
       1  -1
>>  3*A
ans =  3  6
       9  12
```

We illustrate more operations using the previously defined $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$.

(i) Transpose $\boldsymbol{A}^{\mathrm{T}}$:

```
>>  A'
ans =  1  3
       2  4
```

(ii) Rank rank($\boldsymbol{A}$):

```
>>  rank(A)
       2
```

(iii) Determinant det($\boldsymbol{A}$), provided that $\boldsymbol{A}$ is a square matrix.

```
>>  det(A)
ans =  -2
```

(iv) Powers $\boldsymbol{A}^n$, provided that $\boldsymbol{A}$ is a square matrix. If $n < 0$, $\boldsymbol{A}$ needs to be invertible (that is, non-singular).

```
>>  A^2
ans =  7  10
       15  22
```

(v) If $\boldsymbol{A}$ is invertible, its inverse can be evaluated using either `A^(-1)` or `inv(A)`.

```
>>  inv(A)
ans =  -2.0000  1.0000
       1.5000  -0.5000
```

(vi) The adjoint **adj**($\boldsymbol{A}$), provided that $\boldsymbol{A}$ is a square matrix.

```
>>  adjoint(A)
ans =  4.0000  -2.0000
       -3.0000  1.0000
```

(vii) Matrix product $\boldsymbol{AB}$, provided that the sizes are matched.

```
>>  A * B
ans =  8  11
       20  23
```

Moreover, we can generate special matrices using the following command:

(i) Zero matrix $\mathbf{0}_{m\times n}$ of size $m \times n$: `zeros(m,n)`.

```
>>  zeros(2,3)
ans =  0  0  0
```

```
        0   0   0
```

(ii) Identity matrix $\boldsymbol{I}_n$ of order n: `eye(n)`.

```
>>  eye(2)
ans =   1   0
        0   1
```

(iii) Diagonal matrix with diagonal entries $a_1, \ldots, a_n$: `diag([a1 ... an])`.

```
>>  diag(2,3)
ans =   2   0
        0   3
```

17.2. **Row and Column Operations.** Let $\boldsymbol{A}$ be a matrix. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

```
>>  A = [1 2 3 4 5; 2 3 4 5 6; 3 4 5 6 7; 4 5 6 7 8];
```

(i) The size of $\boldsymbol{A}$ is given by

```
>>  size(A)
ans =   4   5
```

(ii) The (i, j) entry of $\boldsymbol{A}$ is simply `A(i,j)`. For example,

```
>>  A(2,5)
ans =   6
```

(iii) The ith row of $\boldsymbol{A}$: `A(i,:)`. For example,

```
>>  A(4,:)
ans =   4   5   6   7   8
```

If we need more rows, indicate the indices in square brackets. For example,

```
>>  A([2,4], :)
ans =   2   3   4   5   6
        4   5   6   7   8
```

(iv) The ith column of $\boldsymbol{A}$ is given by `A(:,i)`.

```
>>  A(:,3)
ans =   3
```

```
        4
        5
        6
```

(v) If we need more columns, indicate the indices in square brackets.

```
>> A(:, [3,4])
ans =   3   4
        4   5
        5   6
        6   7
```

(vi) Submatrix of $\boldsymbol{A}$. Simply indicate the required rows and columns in square brackets.

```
>> A([1,2], [3,4])
ans =   3   4
        4   5
```

We can perform the elementary row operations as follows:

(i) Multiplying the ith row by a constant c: `A(i,:) = c*A(i,:)`.

```
>> A(1,:) = -2*A(1,:);
A =   -2   -4   -6   -8   -10
       2    3    4    5    6
       3    4    5    6    7
       4    5    6    7    8
```

(ii) Interchanging the ith and jth rows: `A([i,j],:) = A([j,i],:)`.

```
>> A([2,3],:) = A([3,2],:);
A =   -2   -4   -6   -8   -10
       3    4    5    6    7
       2    3    4    5    6
       4    5    6    7    8
```

(iii) Adding c times of the ith row to the jth row: `A(j,:) = A(j,:) + c*A(i,:)`.

```
>> A(4,:) = A(4,:) + 2*A(1,:)
A =   -2   -4   -6   -8   -10
       3    4    5    6    7
       2    3    4    5    6
```

```
40   -3   -6   -9   -12
```

(iv) The reduced row echelon form of $\boldsymbol{A}$ is given by

```
>> rref(A)
ans =  1   0   -1   -2   -3
       0   1   2   3   4
       0   0   0   0   0
       0   0   0   0   0
```

Similarly, we can perform the elementary column operations as follows:

(i) Multiplying the ith column by a constant c: `A(:,i) = c*A(:,i)`.

(ii) Interchanging the ith and jth columns: `A(:,[i,j]) = A(:,[j,i])`.

(iii) Adding c times of the ith column to the jth column: `A(:,j) = A(:,j) + c*A(:,i)`.

18. Linear System

18.1. **Homogeneous Linear System.** Let $\boldsymbol{A}$ be a matrix. The solution set of the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ can be obtained by back-substitution from its reduced row echelon form.

For example, let $\boldsymbol{A} = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{pmatrix}$.

```
>> A = [0 3 -6 6 4; 3 -7 8 -5 8; 3 -9 12 -9 6];
>> rref(A)
ans =  1   0   -2   3   0
       0   1   -2   2   0
       0   0   0   0   1
```

Alternatively, we note that the solution set of $\boldsymbol{Ax} = \boldsymbol{0}$ is precisely the nullspace of $\boldsymbol{A}$. The command `null(A)` finds a basis for the nullspace of $\boldsymbol{A}$ (using column vectors).

```
>> null(A)
ans =  -0.7952   0.1467
       -0.3797   0.5633
       0.2256,   0.6983
       0.4155   0.4166
       -0.0000   -0.0000
```

Let $\boldsymbol{v}_1 = (-0.7952, 0.3797, 0.2256, 0.4155, -0.0000)^{\mathrm{T}}$ and $\boldsymbol{v}_2 = (0.1467, 0.5633, 0.6983, 0.4166, -0.0000)^{\mathrm{T}}$. Then $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$.

18.2. **Non-Homogeneous Linear System.** Let $\boldsymbol{A}$ be an $m\times n$ matrix and $\boldsymbol{b}$ an $m\times 1$ vector. Recall that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ can be solved as follows:

(i) Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$ (if the system is consistent), say $\boldsymbol{x}_p$.

(ii) Find the general solution to the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$, say $\boldsymbol{x}_h$.

Then the general solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_h$.

One may use the back-substitution method to solve the system from the reduced row echelon form of the augment matrix $(\boldsymbol{A}\mid\boldsymbol{b})$.

For example, let $\boldsymbol{A}$ be the matrix previously defined, and $\boldsymbol{b} = \begin{pmatrix} -5 \\ 9 \\ 15 \end{pmatrix}$.

```
>> b = [-5; 9; -15];
>> rref([A b])
ans =  1  0  -2  3  0  -24
       0  1  -2  2  0  -7
       0  0  0  0  1  4
```

Alternatively, both `linsolve(A,b)` and `A\b` gives a particular solution to the system $\boldsymbol{Ax} = \boldsymbol{b}$, provided that the system is consistent.

```
>> linsolve(A,b)
ans =  -17.000
       0
       3.5000
       0
       4.0000
```

So the system $\boldsymbol{Ax} = \boldsymbol{b}$ has a particular solution $\boldsymbol{v} = (-17, 0, 3.5, 0, 4)^{\mathrm{T}}$.

Recall that $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$. Then $\boldsymbol{Ax} = \boldsymbol{b}$ has a general solution $\boldsymbol{x} = \boldsymbol{v} + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$, where c_1, c_2 are arbitrary parameters.

18.3. **Least Squares Solution.** Sometimes the system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent, we prefer to get the least squares solution, i.e., the solution to $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$. A least squares solution can also be obtained by `linsolve(A,b)` or `A\b`.

For example, $\boldsymbol{A} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$.

One checks that the system is inconsistent. But we have least squares solution $\boldsymbol{x} = (1,2)^{\mathrm{T}}$.

```
>>  A = [4 0; 0 2; 1 1];
>>  b = [2; 0; 11];
>>  linsolve(A,b)
ans =   1.0000
        2.0000
```

19. More Matrix Operations

19.1. **Orthonormal Basis.** Let V be a vector space spanned by vectors $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$. Using Gram-Schmidt process, one obtains an orthonormal basis for V. In MATLAB, `orth` can be used to generate orthonormal basis for V.

More precisely, `orth(A)` gives an orthonormal basis for the column space of the matrix $\boldsymbol{A}$.

For example, let V be the vector space spanned by

$$\boldsymbol{v}_1 = (-10,2,-6,16,2), \quad \boldsymbol{v}_2 = (13,1,3,-16,1), \quad \boldsymbol{v}_3 = (7,-5,13,-2,-5), \quad \boldsymbol{v}_4 = (11,3,-3,5,-7).$$

We shall first define a matrix $\boldsymbol{A}$ whose columns are $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4$. Since it is easier to input row vectors in MATLAB, we may view each $\boldsymbol{v}_i$ as a row vector and take transpose:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -10 & 13 & 7 & 11 \\ 2 & 1 & -5 & 3 \\ -6 & 3 & 13 & -3 \\ 16 & -16 & -2 & 5 \\ 2 & 1 & -5 & -7 \end{pmatrix}.$$

```
>>  A = [-10 2 -6 16 2; 13 1 3 -16 1; 7 -5 13 -2 -5; 11 3 -3 5 -7]';
>>  orth(A)
ans =   -0.6140   -0.5638   0.3459   0.3489
        0.0720   -0.0389   0.4409   0.2483
        -0.3406   -0.0986   -0.8114   0.3016
        0.7011   -0.6159   -0.1476   0.3117
        0.1016   0.5399   0.0764   0.7928
```

The columns of the resulting matrix is thus an orthonormal basis for V.

19.2. **Eigenvalues.** Let $\boldsymbol{A}$ be a square matrix of order n. Its characteristic polynomial is $\det(x\boldsymbol{I}_n - \boldsymbol{A})$. Alternatively, we can use the command `charpoly`.

For example, let $\boldsymbol{A} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$.

```
>> A = [4 -1 6; 2 1 6; 2 -1 8];
```

We shall first declare the variable of the polynomial, e.g., x:

```
>> syms x;
```

Then either of the following commands evaluates the characteristic polynomial of $\boldsymbol{A}$:

```
>> det(x*eye(3)-A)
ans =  x^3 - 13*x^2 + 40*x - 36
>> charpoly(A,x)
ans =  x^3 - 13*x^2 + 40*x - 36
```

The roots of the characteristic equation of $\boldsymbol{A}$ are precisely all the eigenvalues of $\boldsymbol{A}$:

```
>> solve(charpoly(A,x) == 0, x)
ans =  2
       2
       9
```

Alternatively, MATLAB can evaluate the eigenvalues using `eig`.

```
>> eig(A)
ans =  9.0000
       2.0000
       2.0000
```

The command `[matrix1, matrix2] = eig(A)` can return the corresponding eigenvectors. Here `matrix2` is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$ (counting multiplicities) and `matrix1` is a matrix whose columns are the corresponding (unit) eigenvectors of $\boldsymbol{A}$.

```
>> [P, D] = eig(A)
P =  -0.5774  -0.6122  0.3205
     -0.5774  -0.7873  -0.9112
     -0.5774  0.0728  -0.2587
D =  9.0000     0     0
```

```
0      2.0000     0
0      0     2.0000
```

Note that $\boldsymbol{A}$ is diagonalizable if and only if `matrix2` is invertible.

In this problem, one verifies that $\boldsymbol{P}$ is invertible and $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$:

```
>>  inv(P)*A*P
ans =   9.0000   -0.0000   -0.0000
        0.0000   2.0000   0
        0.0000   -0.0000   2.0000
```

19.3. **Differential Equation in Matrix Form.** A linear differential equation may be written in matrix form. For example,

$$\begin{cases} \dfrac{dx}{dt} = x + 2y + 1, \\ \dfrac{dy}{dt} = -x + y + t. \end{cases}$$

It can be expressed in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Let $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ t \end{pmatrix}$. Then the differential equation is simply $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$.

The command `dsolve` can also be used for differential equation in matrix form:

```
>>  syms x(t) y(t);
>>  X = [x; y];
>>  A = [3 -4; 4 -7];
>>  b = [1; t];
>>  [Sx, Sy] = dsolve(diff(X,t) == A*X + b)
Sx =  (exp(-5*t)*(C1 + (2*exp(5*t)*(10*t - 7))/75))/2 + 2*exp(t)*(C2 + (exp(-t)*(t
- 1))/3)
Sy =  exp(-5*t)*(C1 + (2*exp(5*t)*(10*t - 7))/75) + exp(t)*(C2 + (exp(-t)*(t
- 1))/3)
```