Macroeconomics analysis II, EC3102

Tutorial 4 Solution

Question 1:

Part a.

The problem of the representative consumer is:

$$\max_{\{c_t, a_t, B_t, M_t^D\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t, \frac{M_t^D}{P_t}\right)$$

$$\text{subject to} \quad \left\{\underbrace{P_t c_t + S_t a_t + M_t^D + P_t^b B_t}_{\text{outflows}} = \underbrace{Y_t + (S_t + D_t)a_{t-1} + M_{t-1}^D + B_{t-1}}_{\text{inflows}}\right\}_{t=0}^{\infty} \tag{1}$$

The Lagrangian is:

$$\mathcal{L}(\dots) = \sum_{t=0}^{\infty} \beta^t u\left(c_t, \frac{M_t^D}{P_t}\right) - \sum_{t=0}^{\infty} \lambda_t \beta^t \left[P_t c_t + S_t a_t + M_t^D + P_t^b B_t - Y_t - (S_t + D_t)a_{t-1} - M_{t-1}^D - B_{t-1}\right] \tag{2}$$

Or:

$$\begin{aligned}
\mathcal{L}(\dots) = {} & \beta^0 u\left(c_0, \frac{M_0^D}{P_0}\right) + \beta^1 u\left(c_1, \frac{M_1^D}{P_1}\right) + \beta^2 u\left(c_2, \frac{M_2^D}{P_2}\right) \dots + \beta^t u\left(c_t, \frac{M_t^D}{P_t}\right) + \cdots \\
& -\lambda_0 \beta^0 \left[P_0 c_0 + S_0 a_0 + M_0^D + P_0^b B_0 - Y_0 - (S_0 + D_0)a_{0-1} - M_{0-1}^D - B_{0-1}\right] \\
& -\lambda_1 \beta^1 \left[P_1 c_1 + S_1 a_1 + M_1^D + P_1^b B_1 - Y_1 - (S_1 + D_1)a_{1-1} - M_{1-1}^D - B_{1-1}\right] \\
& \dots \\
& -\lambda_t \beta^t \left[P_t c_t + S_t a_t + M_t^D + P_t^b B_t - Y_t - (S_t + D_t)a_{t-1} - M_{t-1}^D - B_{t-1}\right] \\
& -\lambda_{t+1} \beta^{t+1} \left[P_{t+1} c_{t+1} + S_{t+1} a_{t+1} + M_{t+1}^D + P_{t+1}^b B_{t+1} - Y_{t+1} - (S_{t+1} + D_{t+1})a_t - M_t^D - B_t\right] \\
& \dots
\end{aligned}$$

where the ellipsis in $\mathcal{L}(\dots)$ contains $\{c_t, a_t, B_t, M_t^D\}_{t=0}^{\infty}$, which are all the choice variables.

First, let 's work out the FOCs with respect to c_t, a_t, B_t, and M_t^D for **any period t**.

W.r.t c_t

$$\frac{\partial \mathcal{L}(\dots)}{\partial c_t} = \mathcal{L}_{c_t} = \beta^t u_1\left(c_t, \frac{M_t^D}{P_t}\right) - \lambda_t \beta^t P_t = 0$$

$$\Rightarrow u_1\left(c_t, \frac{M_t^D}{P_t}\right) = \lambda_t P_t \tag{3}$$

And w.r.t a_t,

$$\mathcal{L}_{a_t} = -\lambda_t \beta^t S_t + \lambda_{t+1} \beta^{t+1}(S_{t+1} + D_{t+1}) = 0$$

$$\Rightarrow \lambda_t S_t = \lambda_{t+1} \beta (S_{t+1} + D_{t+1}) \tag{4}$$

And w.r.t B_t,

$$\mathcal{L}_{B_t} = -\lambda_t \beta^t P_t^b + \beta^{t+1}\lambda_{t+1} = 0$$
$$\Longrightarrow \lambda_t P_t^b = \beta\lambda_{t+1}$$
$$\Longrightarrow \frac{\lambda_t}{\lambda_{t+1}} = \beta / P_t^b \tag{5}$$

Comment: *Rewriting (5) as:*

$$P_t^b = \frac{\beta\lambda_{t+1}}{\lambda_t}$$

And from the FOC with respect to a_t, we have

$$\Longrightarrow S_t = \frac{\boldsymbol{\beta\lambda_{t+1}}}{\boldsymbol{\lambda_t}}(S_{t+1} + D_{t+1})$$

*As we have discussed in the lecture, $\frac{\boldsymbol{\beta\lambda_{t+1}}}{\boldsymbol{\lambda_t}}$ is the pricing kernel of asset (in this case, stock) pricing. Now including bonds into the model we find that $\frac{\boldsymbol{\beta\lambda_{t+1}}}{\boldsymbol{\lambda_t}}$ is equal to P_t^b which is the price of **risk-less short-term bond**. A student asks me why I have to include both bond, stock and money into the model. Here can you see the answer for half of the question? The reason why we have bond and stock in the model is because we can further understand the pricing kernel, that it is the bond of price itself. This is part of the reason why Singapore government issues bonds. It is not because the government need to borrow money (your government has great surplus). Rather, it is because the bond market will help as a bench mark for debt market. If you know the CAPM model in finance (financial economics), that is the risk-free asset from which we get a point on the y-axis. Do not worry if CAPM is alien to you. It is half alien to me, too.*

And w.r.t M_t^D,

$$\mathcal{L}_{M_t^D} = \frac{u_2\left(c_t, \frac{M_t^D}{P_t}\right)}{P_t} - \lambda_t + \beta\lambda_{t+1} = 0$$
$$\Longrightarrow u_2\left(c_t, \frac{M_t^D}{P_t}\right) = P_t\lambda_t - P_t\beta\lambda_{t+1} \tag{6}$$

Comment: *In this question, **the aim** is to arrive at the real money demand curve using the MRS between consumption and money. Thus, we are going to start with equation (3) and (6). Keep the goal in mind.*

From equation (3) and (6), we have:

$$\frac{u_2\left(c_t, \frac{M_t^D}{P_t}\right)}{u_1\left(c_t, \frac{M_t^D}{P_t}\right)} = \frac{P_t\lambda_t - P_t\beta\lambda_{t+1}}{\lambda_t P_t}$$
$$= 1 - \frac{\boldsymbol{\beta\lambda_{t+1}}}{\boldsymbol{\lambda_t}} \tag{7}$$

Substituting (5) into (7):

$$\frac{u_2\left(c_t, \frac{M_t^D}{P_t}\right)}{u_1\left(c_t, \frac{M_t^D}{P_t}\right)} = 1 - P_t^b$$
$$= 1 - \frac{1}{1+i_t} = \frac{i_t}{1+i_t} \tag{8}$$

where i_t is the interest rate.

Part b)

To find the real money demand, we have to express $\frac{M_t^D}{P_t}$ as a function of c_t and i_t (where i_t is the price of holding money). That is, $\frac{M_t^D}{P_t} = \phi(c_t, i_t)$.

i.

$$u\left(c_t, \frac{M_t^D}{P_t}\right) = \ln c_t + \ln\left(\frac{M_t^D}{P_t}\right)$$

$$\Rightarrow \begin{cases} u_1\left(c_t, \frac{M_t^D}{P_t}\right) = \frac{1}{c_t} \\ u_2\left(c_t, \frac{M_t^D}{P_t}\right) = \frac{1}{\frac{M_t^D}{P_t}} \end{cases} \tag{9}$$

Substituting the above expressions into equation (8), we have:

$$\frac{u_2\left(c_t, \frac{M_t^D}{P_t}\right)}{u_1\left(c_t, \frac{M_t^D}{P_t}\right)} = \frac{\frac{1}{\frac{M_t^D}{P_t}}}{\frac{1}{c_t}} = \frac{i_t}{1+i_t}$$

$$\Rightarrow \frac{P_t c_t}{M_t^D} = \frac{i_t}{1+i_t}$$

$$\Rightarrow \frac{M_t^D}{P_t} = \frac{c_t(1+i_t)}{i_t} = \frac{c_t}{i_t} + c_t \tag{10}$$

So the real money demand function is $\frac{M_t^D}{P_t} = \phi(c_t, i_t) = \frac{c_t}{i_t} + c_t$. We can observe that it is increasing in consumption and decreasing in the price of money which is the interest rate.

ii.

$$u\left(c_t, \frac{M_t^D}{P_t}\right) = 2\sqrt{c_t} + 2\sqrt{\frac{M_t^D}{P_t}}$$

$$\Rightarrow \begin{cases} u_1\left(c_t, \frac{M_t^D}{P_t}\right) = \frac{1}{\sqrt{c_t}} \\ u_2\left(c_t, \frac{M_t^D}{P_t}\right) = \frac{1}{\sqrt{\frac{M_t^D}{P_t}}} \end{cases} \tag{11}$$

Substituting the above expressions into equation (8), we have:

$$\frac{u_2\left(c_t, \frac{M_t^D}{P_t}\right)}{u_1\left(c_t, \frac{M_t^D}{P_t}\right)} = \frac{\frac{1}{\sqrt{\frac{M_t^D}{P_t}}}}{\frac{1}{\sqrt{c_t}}} = \frac{i_t}{1+i_t}$$

$$\Rightarrow \frac{\sqrt{c_t}}{\sqrt{\frac{M_t^D}{P_t}}} = \frac{i_t}{1+i_t}$$

$$\Rightarrow \frac{M_t^D}{P_t} = \frac{c_t(1+i_t)^2}{i_t^2} \tag{12}$$

We can observe that here the real money demand is also increasing in consumption and decreasing in the price of money which is the interest rate.

iii.

This part's utility function is of Cobb-Douglas type.

$$u\left(c_t, \frac{M_t^D}{P_t}\right) = c_t^{\sigma} \cdot \left(\frac{M_t^D}{P_t}\right)^{1-\sigma}$$

$$\Rightarrow \begin{cases} u_1\left(c_t, \frac{M_t^D}{P_t}\right) = \sigma c_t^{\sigma-1}\left(\frac{M_t^D}{P_t}\right)^{1-\sigma} \\ u_2\left(c_t, \frac{M_t^D}{P_t}\right) = (1-\sigma)c_t^{\sigma}\left(\frac{M_t^D}{P_t}\right)^{-\sigma} \end{cases} \tag{13}$$

Substituting the above expressions into equation (8), we have:

$$\frac{u_2\left(c_t, \frac{M_t^D}{P_t}\right)}{u_1\left(c_t, \frac{M_t^D}{P_t}\right)} = \frac{(1-\sigma)c_t^{\sigma}\left(\frac{M_t^D}{P_t}\right)^{-\sigma}}{\sigma c_t^{\sigma-1}\left(\frac{M_t^D}{P_t}\right)^{1-\sigma}} = \frac{i_t}{1+i_t}$$

$$\Rightarrow \frac{1-\sigma}{\sigma} \cdot \frac{1}{c_t^{-1}\left(\frac{M_t^D}{P_t}\right)^1} = \frac{i_t}{1+i_t}$$

$$\Rightarrow \frac{1}{\left(\frac{M_t^D}{P_t}\right)} = \frac{i_t}{1+i_t} \cdot c_t^{-1} \cdot \frac{\sigma}{1-\sigma}$$

$$\Rightarrow \frac{M_t^D}{P_t} = \frac{1-\sigma}{\sigma} \cdot c_t \cdot \frac{1+i_t}{i_t}$$

We can observe that here the real money demand is also increasing in consumption and decreasing in the price of money which is the interest rate.