NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 8**

1. Apply Gram-Schmidt Process to convert

   (a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ into an orthonormal basis for $\mathbb{R}^4$.

   (b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ into an orthonormal set. Is the set obtained an orthonormal basis? Why?

2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ -1 \\ 1 \end{pmatrix}$.

   (a) Is the linear system $\mathbf{Ax} = \mathbf{b}$ inconsistent?

   (b) Find a least squares solution to the system. Is the solution unique?

   (c) Use your answer in (b), compute the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$. Is the solution unique?

3. **(Application)** A line

$$p(x) = a_1 x + a_0$$

is said to be the *least squares approximating line* for a given a set of data points (x_1, y_1), (x_2, y_2), ..., (x_m, y_m) if the sum

$$S = [y_1 - p(x_1)]^2 + [y_2 - p(x_2)]^2 + \cdots + [y_m - p(x_m)]^2$$

is minimized. Writing

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \text{ and } p(\mathbf{x}) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_m) \end{pmatrix} = \begin{pmatrix} a_1 x_1 + a_0 \\ a_1 x_2 + a_0 \\ \vdots \\ a_1 x_m + a_0 \end{pmatrix}$$

the problem is now rephrased as finding a_0, a_1 such that

$$S = ||\mathbf{y} - p(\mathbf{x})||^2$$

is minimized. Observe that if we let

$$\mathbf{N} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix},$$

then $\mathbf{Na} = p(\mathbf{x})$. And so our aim is to find $\mathbf{a}$ that minimizes $||\mathbf{y} - \mathbf{Na}||^2$.

It is known the equation representing the dependency of the resistance of a cylindrically shaped conductor (a wire) at 20^oC is given by

$$R = \rho \frac{L}{A},$$

where R is the resistance measured in Ohms Ω, L is the length of the material in meters m, A is the cross-sectional area of the material in meter squared m^2, and ρ is the resistivity of the material in Ohm meters Ωm. A student wants to measure the resistivity of a certain material. Keeping the cross-sectional area constant at $0.002m^2$, he connected the power sources along the material at varies length and measured the resistance and obtained the following data.

L	0.01	0.012	0.015	0.02
R	2.75×10^{-4}	3.31×10^{-4}	3.92×10^{-4}	4.95×10^{-4}

It is known that the Ohm meter might not be calibrated. Taking that into account, the student wants to find a linear graph $R = \frac{\rho}{0.002}L + R_0$ from the data obtained to compute the resistivity of the material.

(a) Relabeling, we let $R = y$, $\frac{\rho}{0.002} = a_1$ and $R_0 = a_0$. Is it possible to find a graph $y = a_1x + a_0$ satisfying the points?

(b) Find the least square approximating line for the data points and hence find the resistivity of the material. Would this material make a good wire?

4. **(Application)** Suppose the equation governing the relation between data pairs is not known. We may want to then find a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

of degree n, $n \leq m - 1$, that best approximates the data pairs (x_1, y_1), (x_2, y_2), ..., (x_m, y_m). A *least square approximating polynomial* of degree n is such that

$$||\mathbf{y} - p(\mathbf{x})||^2$$

is minimized. If we write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix},$$

then $p(\mathbf{x}) = \mathbf{N}\mathbf{a}$, and the task is to find $\mathbf{a}$ such that $||\mathbf{y} - \mathbf{N}\mathbf{a}||^2$ is minimized. Observe that $\mathbf{N}$ is a matrix minor of the Vandermonde matrix. If at least $n+1$ of the x-values $x_1, x_2, ..., x_m$ are distinct, the columns of $\mathbf{N}$ are linearly independent, and thus $\mathbf{a}$ is uniquely determined by

$$\mathbf{a} = (\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T\mathbf{y}.$$

We shall now find a quartic polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

that is a least square approximating polynomial for the following data points

x	4	4.5	5	5.5	6	6.5	7	8	8.5
y	0.8651	0.4828	2.590	-4.389	-7.858	3.103	7.456	0.0965	4.326

Enter the data points.

```
>> x=[4 4.5 5 5.5 6 6.5 7 8 8.5]';
```

```
>> y=[0.8651 0.4828 2.590 -4.389 -7.858 3.103 7.456 0.0965 4.326]';
```

Next, we will generate the 10×10 Vandermonde matrix.

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>> N=fliplr(vander(x));
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We only want the matrix minor up to the 4-th power, that is, up to the the 5-th column,

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>> N=N(:,1:5);
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Use this to find the least square approximating polynomial of degree 4.

5. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

   (a) Find a QR factorization of $\mathbf{A}$.

   (b) Use your answer in (a) to find the least square solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Extra problems

1. (a) Let S be the set of least square solutions to $\mathbf{Ax} = \mathbf{b}$. Show that there exists a $\mathbf{b}'$ such that $\mathbf{Av} = \mathbf{b}'$ for all $\mathbf{v} \in S$. This proves that the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$ is unique even though the least square solutions may not unique.

   (b) Suppose a linear system $\mathbf{Ax} = \mathbf{b}$ is consistent. Show that the solution set of $\mathbf{Ax} = \mathbf{b}$ is equal to the solution set of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.

2. (Uniqueness of orthogonal projection)
   Let V be a subspace of $\mathbb{R}^n$ and $\mathbf{u}$ a vector in $\mathbb{R}^n$. Show that $\mathbf{u}$ can be written uniquely as
   $$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_n,$$
   such that $\mathbf{u}_n$ is a vector orthogonal to V and $\mathbf{u}_p$ is a vector in V. Remark: By the Gram-Schmidt process, $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_n$. We need to prove show that $\mathbf{u}_p$ and $\mathbf{u}_n$ are unique. Hint: if $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_n = \mathbf{u}'_p + \mathbf{u}'_n$, where $\mathbf{u}_n, \mathbf{u}'_n$ are orthogonal to V and $\mathbf{u}_p, \mathbf{u}'_p \in V$, then $\mathbf{u}_n = \mathbf{u}'_n$ and $\mathbf{u}_p = \mathbf{u}'_p$).

3. Let $S = \{\mathbf{w}_1, ..., \mathbf{w}_k\}$ be an orthonormal basis for a subspace V in $\mathbb{R}^n$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V.

   (a) Prove that $\mathbf{u} \cdot \mathbf{v} = [\mathbf{u}]_S \cdot [\mathbf{v}]_S$.

   (b) Prove that $\|[\mathbf{u}]_S\| = \|\mathbf{u}\|$.

   (c) Prove that the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $[\mathbf{u}]_S$ and $[\mathbf{v}]_S$.