REVIEW OF TWO-PERIOD MODEL

CHAPTER 1

(Modern Macroeconomics - Sanjay K. Chugh)

UTILITY FUNCTIONS

- Describe how much “happiness” or “satisfaction” an individual experiences from “consuming” goods – the benefit of consumption

- Marginal Utility
  - The extra total utility resulting from consumption of a small/incremental extra unit of a good
  - Mathematically, the (partial) slope of utility with respect to that good

UTILITY FUNCTIONS

Alternative notation: ***du/dc*** **OR** ***u'*(*c*) OR $u_c(c)$ OR $u_1(c)$**

- One-good case: $u(c)$, with $du/dc > 0$ and $d^2u/dc^2 < 0$
  - Recall interpretation: strictly increasing at a strictly decreasing rate
  - Diminishing marginal utility

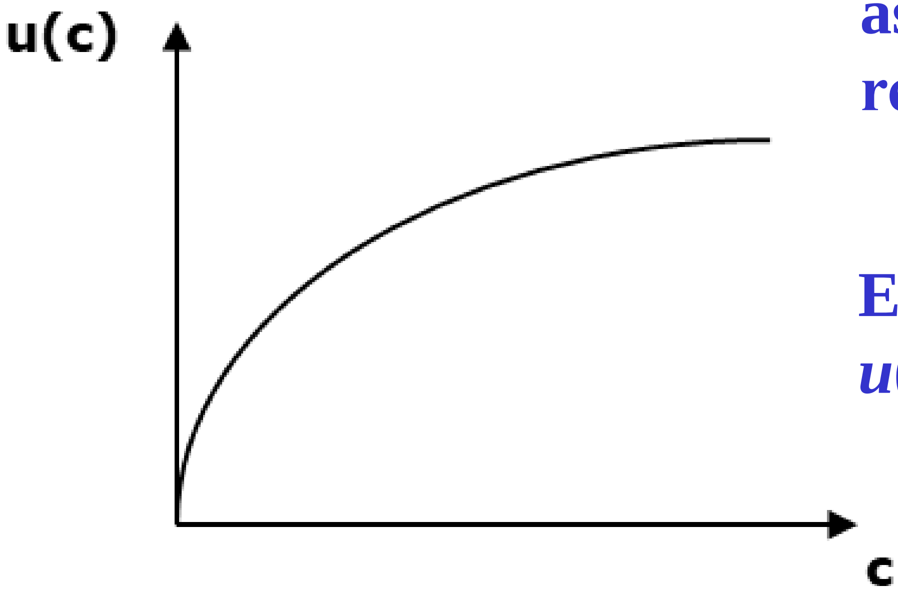
UTILITY FUNCTIONS

- Two-good case: $u(c_1, c_2)$, with $u_i(c_1, c_2) > 0$ and $u_{ii}(c_1, c_2) < 0$ for each of $i = 1,2$
  - Utility strictly increasing in each good individually (partial)
  - Diminishing marginal utility in each good individually

Easily extends to *N*-good case: $u(c_1, c_2, c_3, c_4, \ldots, c_N)$

Utility Functions

- **One-good case**



Slope (marginal utility) asymptotes to (but never reaches…) zero

Example: $u(c) = \ln c$ or $u(c) = \mathrm{sqrt}(c)$

UTILITY FUNCTIONS

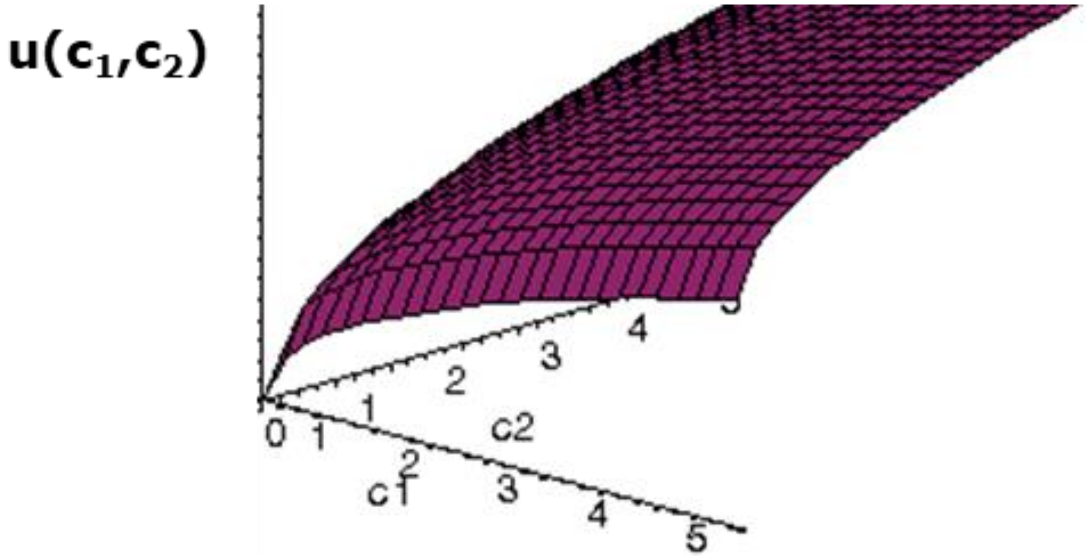
Example:

$$u(c_1, c_2) = \ln c_1 + \ln c_2$$

or

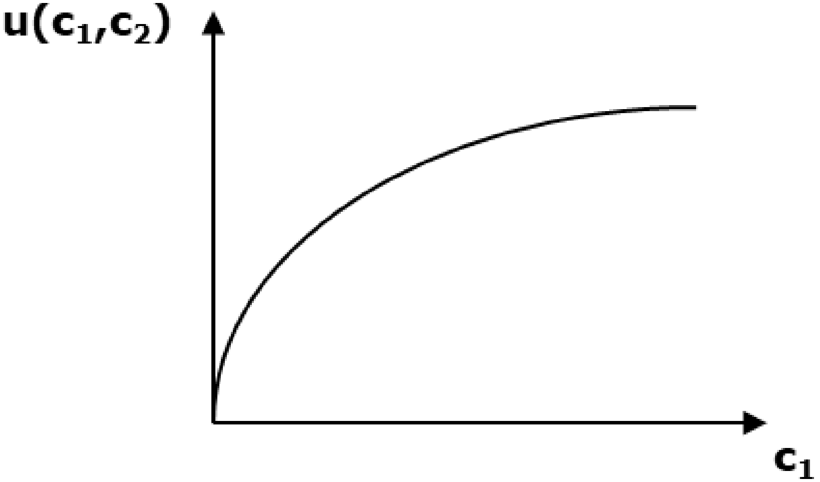
$$u(c_1, c_2) = \sqrt{(c_1)} + \sqrt{(c_2)}$$

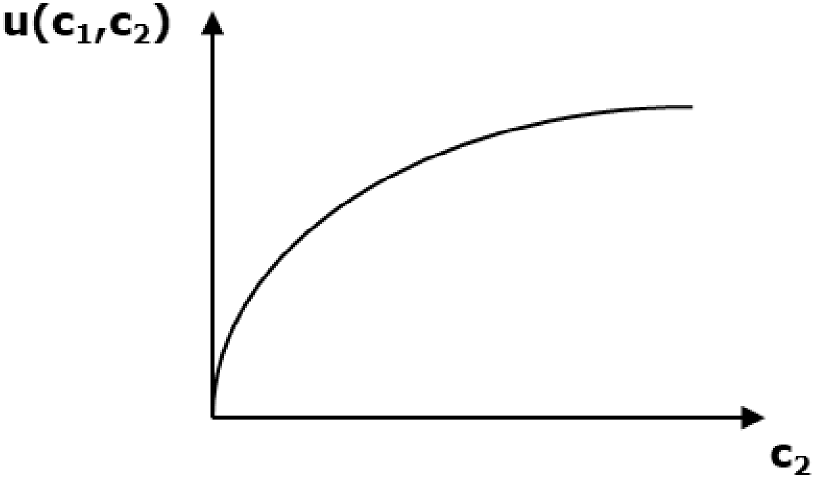
- **Two-good case**



UTILITY FUNCTIONS

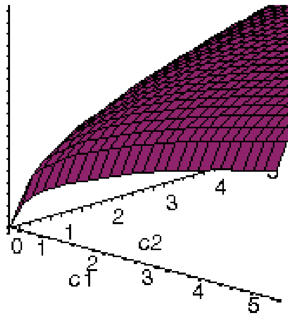
Viewed in good-by-good space



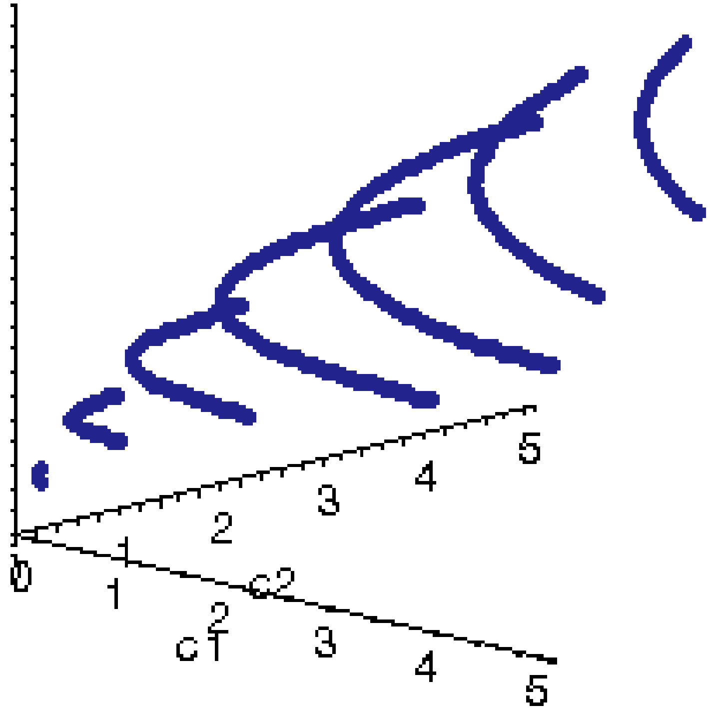


UTILITY FUNCTIONS

Alternative views



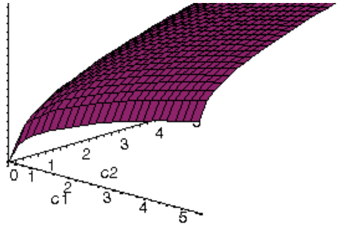
Emphasizing the contours

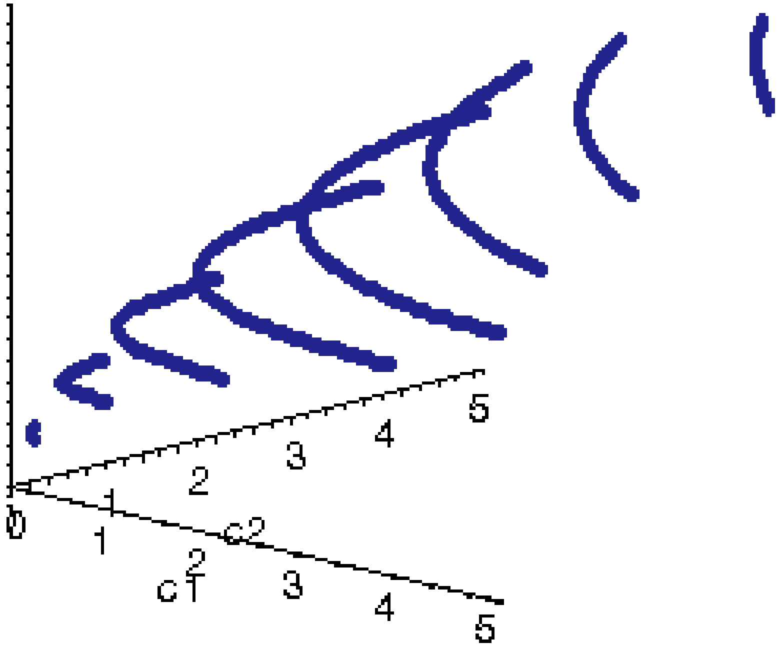


Indifference Curve: the set of all consumption bundles that deliver a particular level of utility/happiness

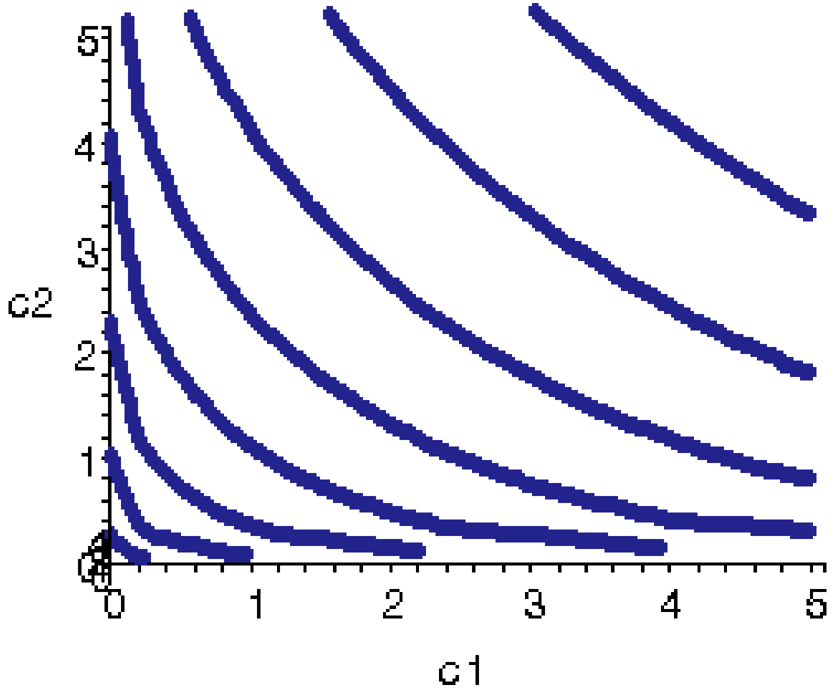
Utility Functions

Alternative views





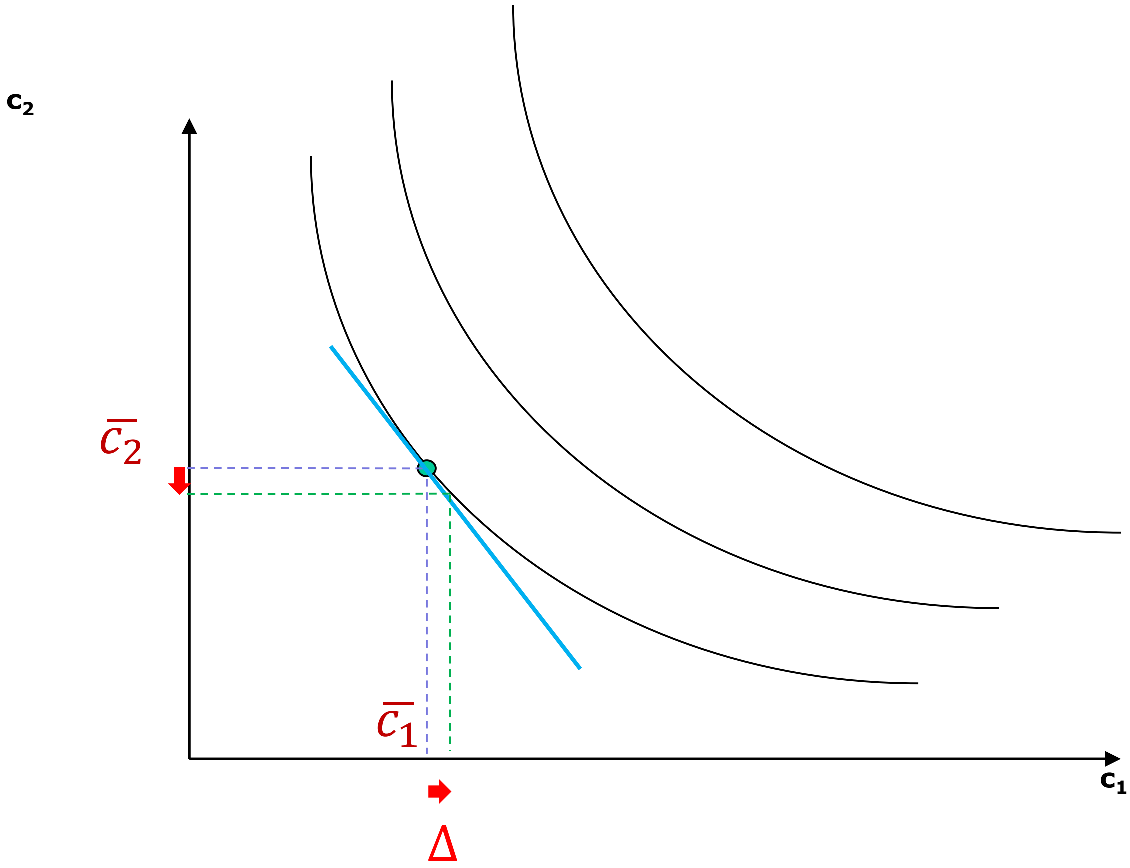
Top-down view of contour



Utility Functions

- Marginal Rate of Substitution (MRS)
  - Maximum quantity of one good consumer is willing to give up to obtain one extra unit of the other good
  - Graphically, the (negative of the) slope of an indifference curve
  - MRS is itself a function of c_1 and c_2 (i.e., $MRS(c_1, c_2)$)

Utility Functions



Utility Functions

- MRS equals ratio of marginal utilities

  - $$MRS(c_1, c_2) = \frac{u_1(c_1, c_2)}{u_2(c_1, c_2)}$$

  - Using Implicit Function Theorem

BUDGET CONSTRAINTS

- Describe the cost side of consumption

- One-good case (trivial): $Pc = Y$
  - Assume income Y is taken as given by consumer for now…

- Two-good case (interesting): $P_1 c_1 + P_2 c_2 = Y$
  - Assume income Y is taken as given by consumer for now…

Budget Constraints

Isolate c_2 to graph the budget constraint

$$P_1 c_1 + P_2 c_2 = Y$$

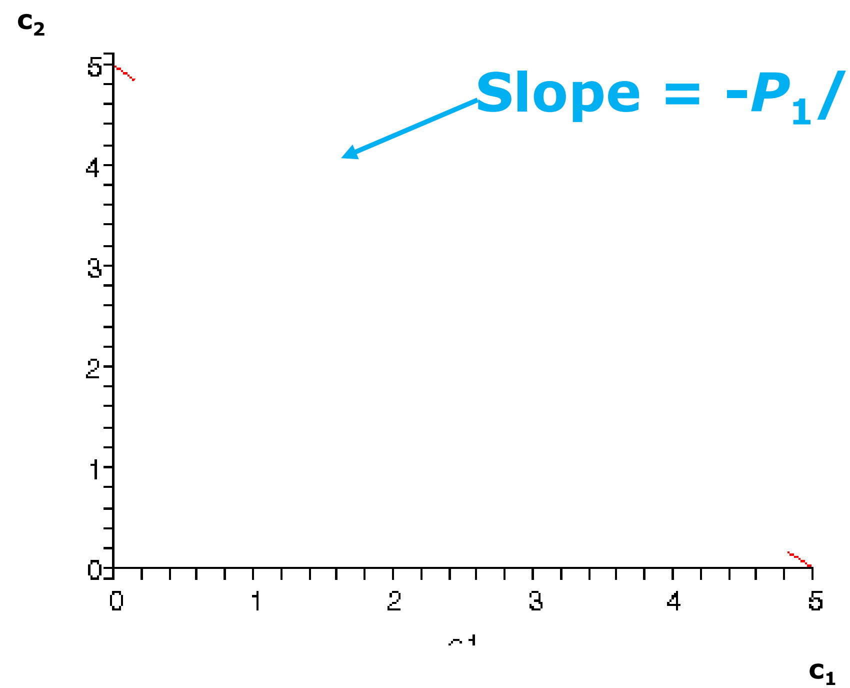
$$\downarrow$$

$$P_2 c_2 = -P_1 c_1 + Y$$

$$\downarrow$$

$$c_2 = -\frac{P_1}{P_2} c_1 + \frac{Y}{P_2}$$

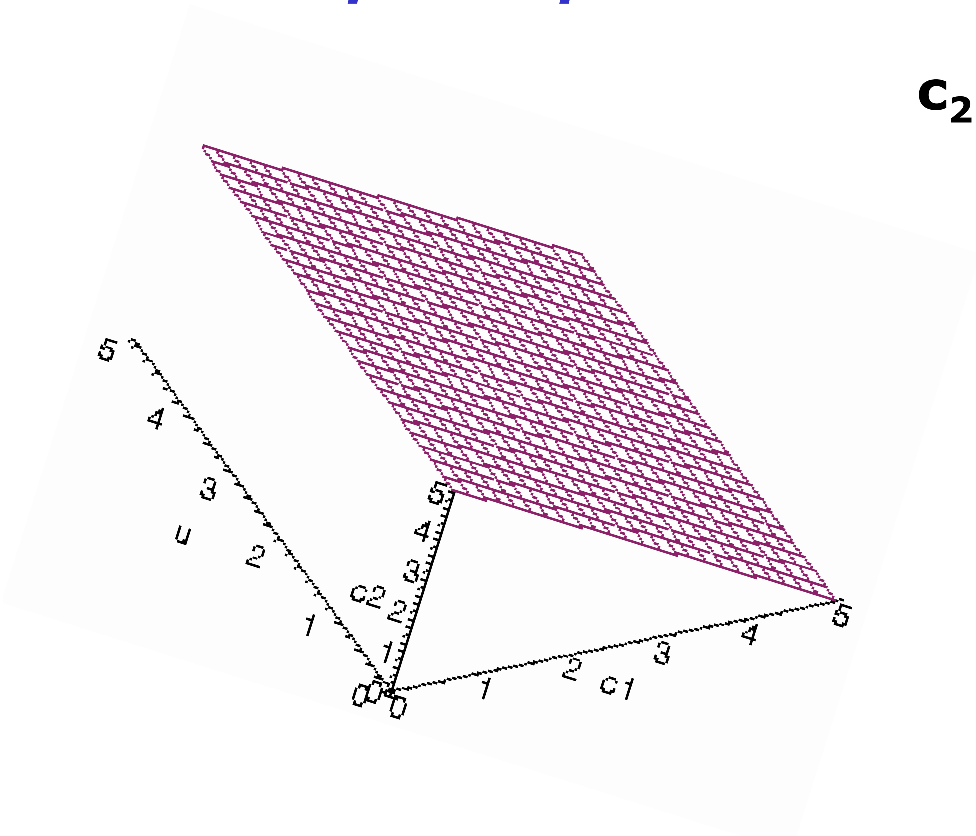
Plotted in 2D-consumption-space



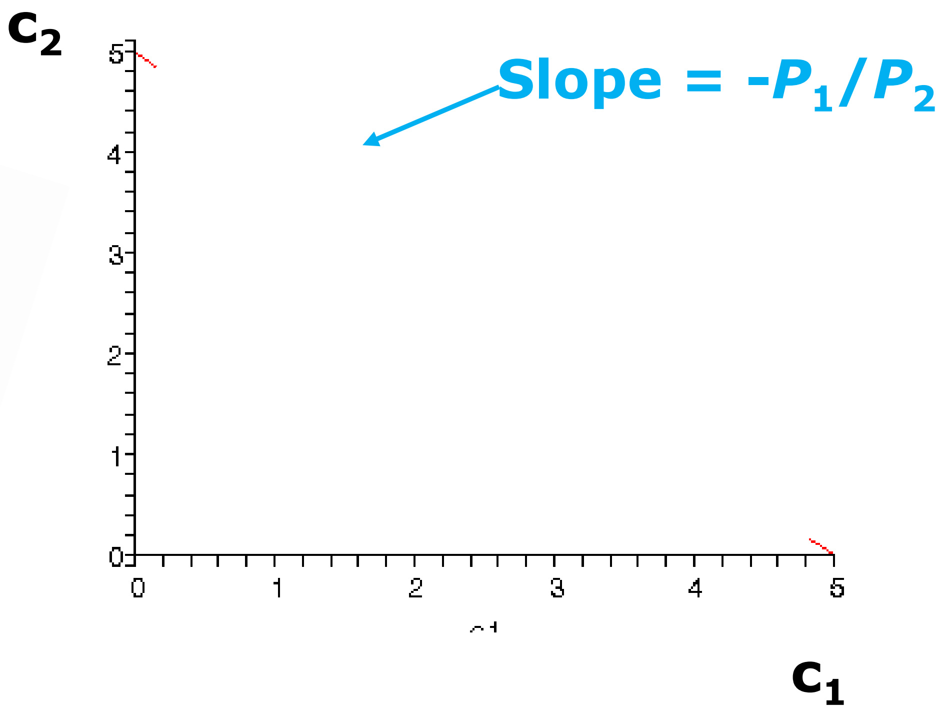
Slope = $-P_1/P_2$

BUDGET CONSTRAINTS

Plotted in 3D-consumption-space



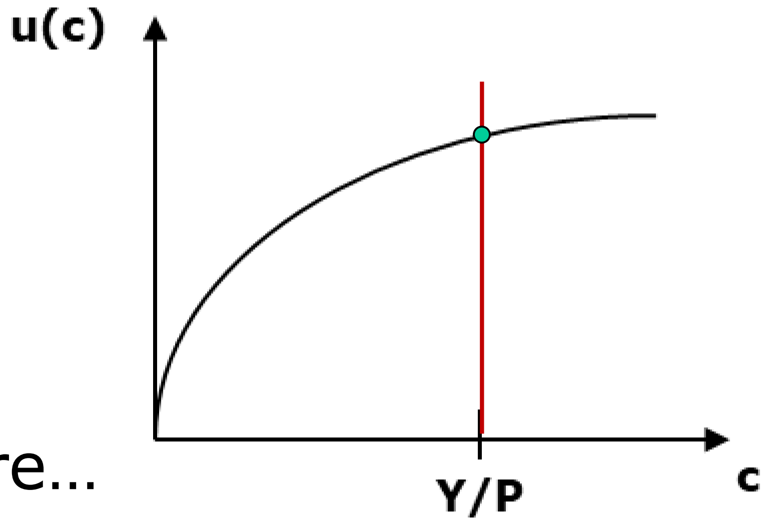
Plotted in 2D-consumption-space



Consumer Optimization

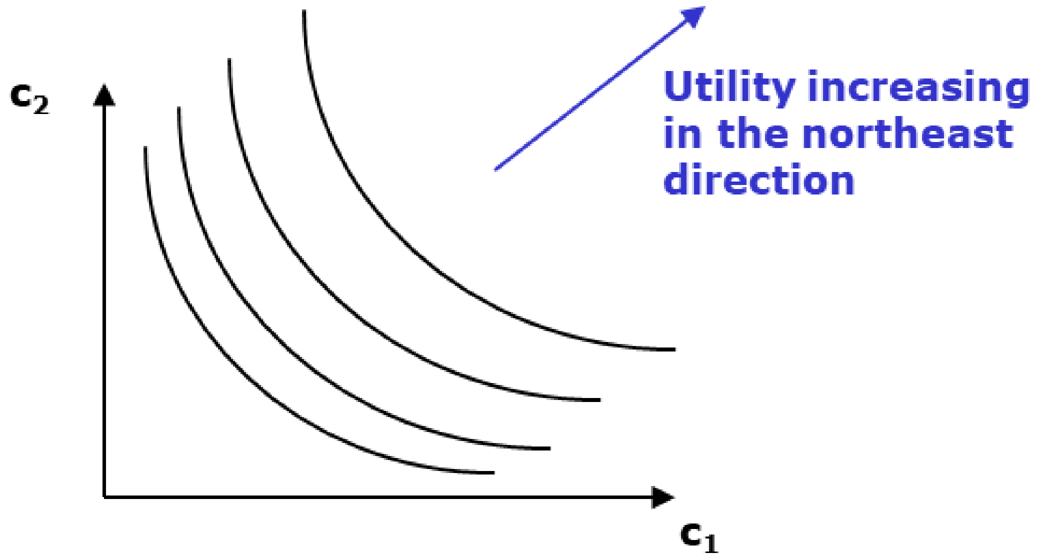
- Consumer's decision problem: maximize utility subject to budget constraint – bring together both cost side and benefit side

- One-good case
  - Trivially, choose $c = Y/P$
  - No decision to make here…



Consumer Optimization

- Two-good case
  - How to optimally allocate Y across the two goods c_1 and c_2?
  - A non-trivial decision problem…



Consumer Optimization

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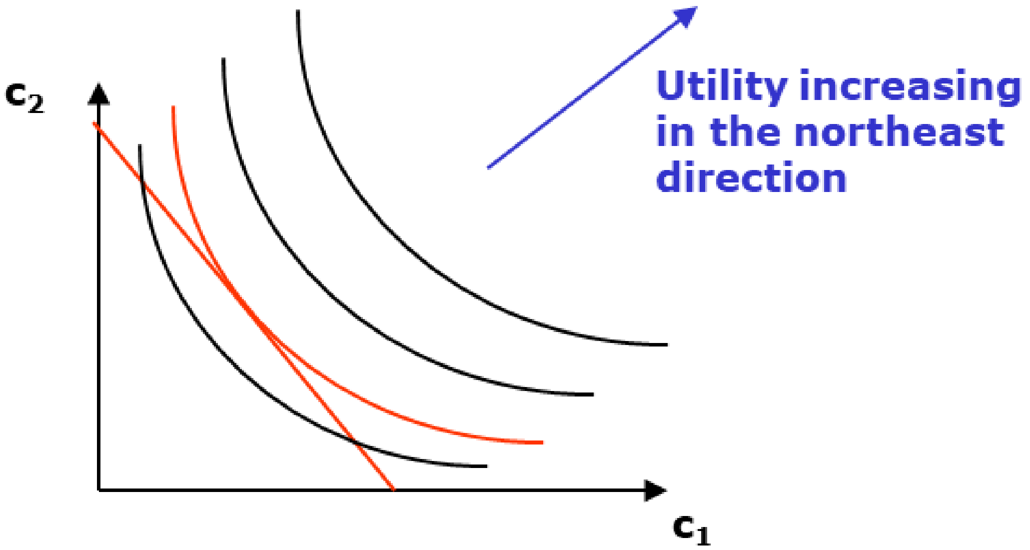
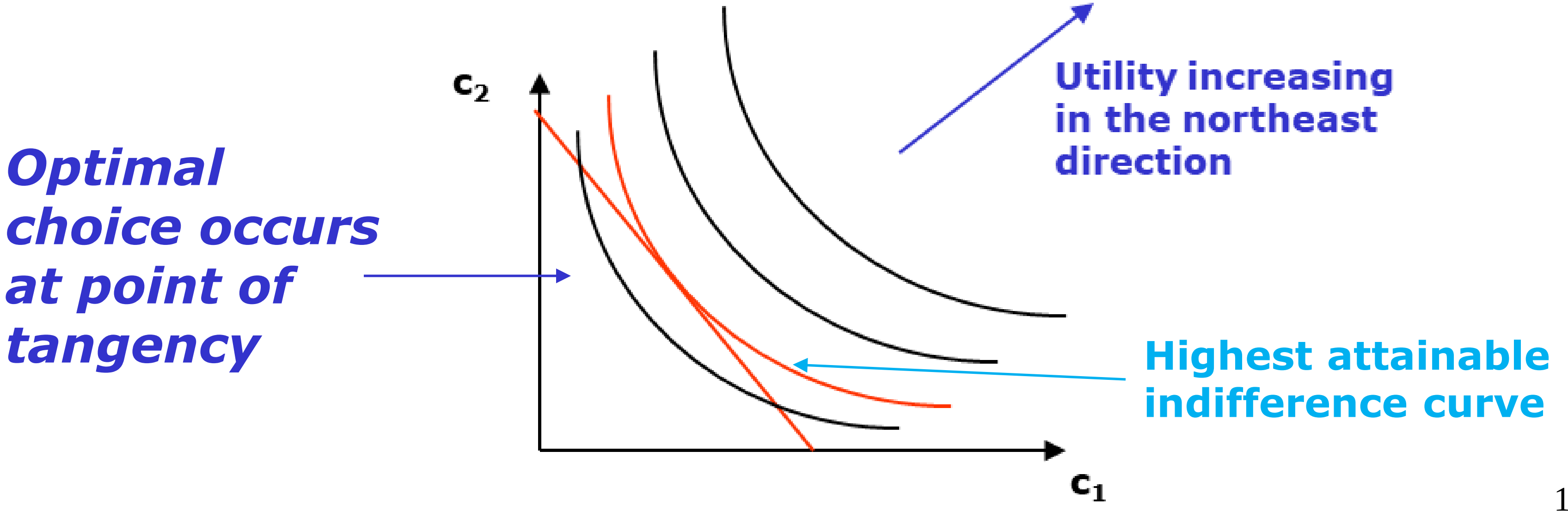


CONSUMER OPTIMIZATION

- Two-good case
  - How to optimally allocate *Y* across the two goods c_1 and c_2?
  - A non-trivial decision problem…



Consumer Optimization

OPTIMALITY CONDITION:

At the optimal choice,

MRS = slope of budget line

ratio of marginal utilities = price ratio

LAGRANGE METHOD

- Consumer optimization a constrained optimization problem
  - Maximize some function (economic application: utility function)…
  - …taking into account some restriction on the objects to be maximized over (economic application: budget constraint)

- Lagrange Method: ***mathematical tool*** to solve constrained optimization problems

Maximizing an OBJECTIVE conditioned on **some** CONSTRAINTS

Original setting

Max Objective Function

$$\text{subject to: } \begin{cases} \mathit{Constraint}\ 1 = 0 \\ \mathit{Constraint}\ 2 = 0 \\ \quad\ldots \\ \mathit{Constraint}\ 3 = 0 \end{cases}$$

Set up another function

$$L(\ldots) = \text{Objective Function} + \lambda_1(\mathit{Constraint}\ 1) + \lambda_2(\mathit{Constraint}\ 2) + \ldots + \lambda_n(\mathit{Constraint}\ n)$$

$\lambda_1, \lambda_2 \ldots \lambda_n$ are the Lagrangian multipliers. They are created as the $L(\ldots)$ function is set up.

Lagrange (the mathematician) proved a very useful result:

The solutions to the
maximization of $L(\ldots)$
are also the solutions
to the original setting.

Example:

Max $f(x, y)$

subject to: $\begin{cases} \boldsymbol{g}(x, y) = 0 \\ \boldsymbol{h}(x, y) = 0 \end{cases}$

Set up the Lagrangian function for this problem, we have:

$$L(x, y, \lambda_1, \lambda_2) = \underbrace{f(x, y)}_{\text{obj. func}} + \lambda_1 \cdot \underbrace{\boldsymbol{g}(x, y)}_{\text{constraint 1}} + \lambda_2 \cdot \underbrace{\boldsymbol{h}(x, y)}_{\text{constraint 2}}$$

Function with 4 variables

We do the usual FOCs:

$$\frac{\partial L(x, y, \lambda_1, \lambda_2)}{\partial x} = 0$$

$$\frac{\partial L(x, y, \lambda_1, \lambda_2)}{\partial y} = 0$$

$$\frac{\partial L(x, y, \lambda_1, \lambda_2)}{\partial \lambda_1} = 0$$

$$\frac{\partial L(x, y, \lambda_1, \lambda_2)}{\partial \lambda_2} = 0$$

$$\boxed{x^\star, y^\star, \lambda_1^\star, \lambda_2^\star}$$

$$\boxed{\begin{gathered}\textit{Solution to} \\ \textbf{Max}\ L(x, y, \lambda_1, \lambda_2)\end{gathered}}$$

$$\boxed{\underbrace{x^{\star}, y^{\star}}, \lambda_1^{\star}, \lambda_2^{\star}}$$

$x^{\star}, y^{\star}$ are also the solution to the original setting.

Which is:

$$\text{Max } f(x, y)$$

$$\text{subject to: } \begin{cases} \boldsymbol{g}(x, y) = 0 \\ \boldsymbol{h}(x, y) = 0 \end{cases}$$

Lagrange Method

- General mathematical formulation
  - Choose (x, y) to maximize a given objective function $f(x,y)$…
  - …subject to the constraint $g(x,y) = 0$ (Note formulation of constraint)
  - Step 1: Construct Lagrange function

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Lagrange multiplier

LAGRANGE ANALYSIS

- Step 2: Compute first-order conditions with respect to *x*, *y*, and *λ*

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Lagrange multiplier

1) $f_x(x, y) + \lambda g_x(x, y) = 0$

2)

3)

Rationale: setting first derivatives to zero isolates maxima (or minima…technically, need to check second-order condition…)

Lagrange Analysis

- Step 3: Solve the system of first-order conditions for *x*, *y*, and *λ*
  - Often most interested in simply eliminating the multiplier…
  - From eqn 1), isolate *λ*: $\lambda = -\dfrac{f_x(x,y)}{g_x(x,y)}$
  - Insert expression for *λ* in eqn 2):

$$f_y(x,y) - \frac{f_x(x,y)}{g_x(x,y)} g_y(x,y) = 0$$

LAGRANGE ANALYSIS

- Step 3:
  - ……
  - Rearrange
  - Optimality condition: at the optimum (x^*, y^*)

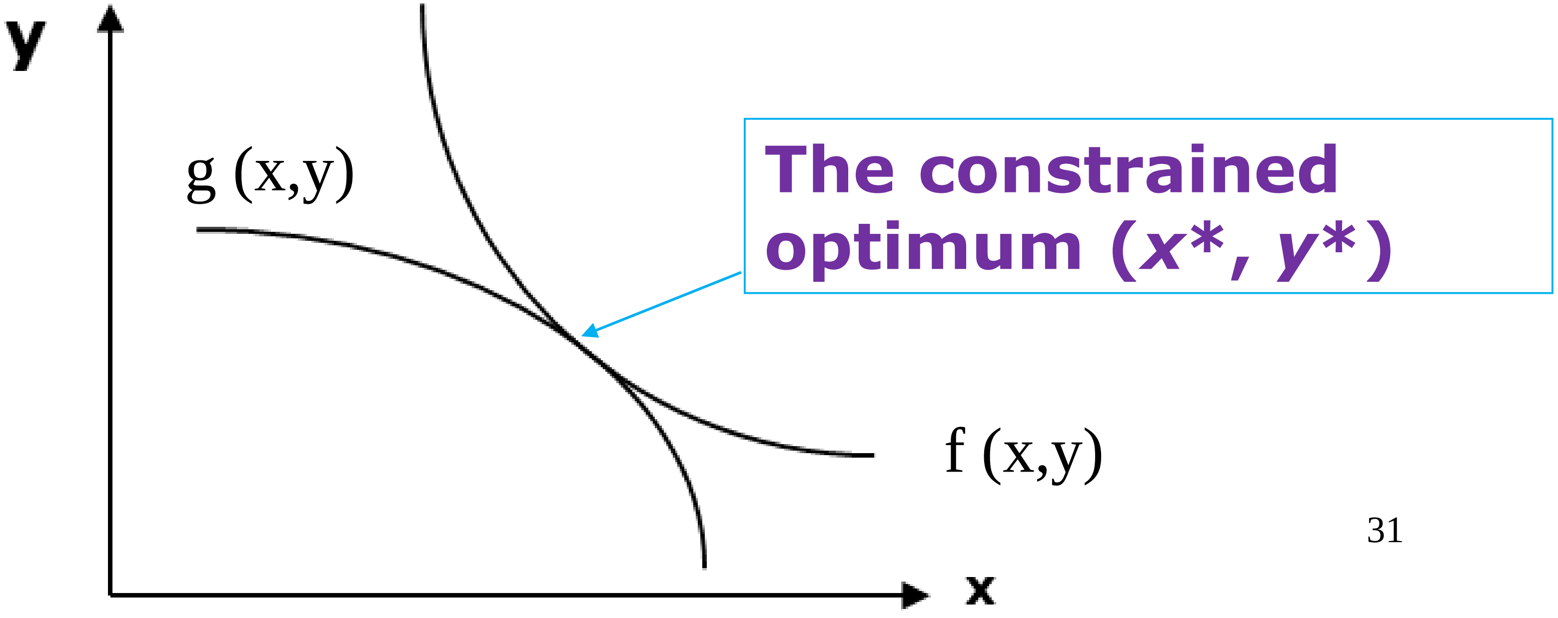
$$\frac{f_x(x^*, y^*)}{f_y(x^*, y^*)} = \frac{g_x(x^*, y^*)}{g_y(x^*, y^*)}$$

Lagrange Analysis

- Step 3:
  - …

$$\frac{f_x(x^*, y^*)}{f_y(x^*, y^*)} = \frac{g_x(x^*, y^*)}{g_y(x^*, y^*)}$$

Graphical interpretation: at the constrained optimum, the function *f*(.) is tangent to the function *g*(.)



LAGRANGE ANALYSIS

- Apply Lagrange tools to consumer optimization
- Objective function: $u(c_1, c_2)$
- Constraint:

$$g(c_1, c_2) = Y - P_1c_1 - P_2c_2 = 0$$

- Step 1: Construct Lagrange function

$$\boxed{L(c_1, c_2, \lambda) = u(c_1, c_2) + \lambda\left[Y - P_1c_1 - P_2c_2\right]}$$

Lagrange Analysis

- Step 2: Compute first-order conditions with respect to c_1, c_2, λ

1) $u_1(c_1, c_2) - \lambda P_1 = 0$

2) $u_2(c_1, c_2) - \lambda P_2 = 0$

3) $Y - P_1 c_1 - P_2 c_2 = 0$

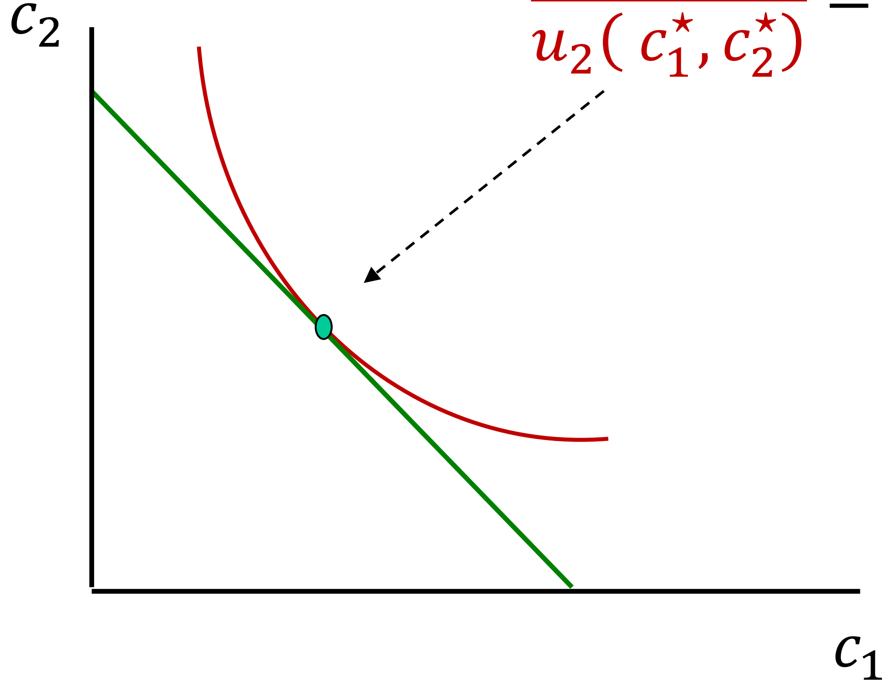
- Step 3: Solve (focus on eliminating multiplier from eqns 1 & 2)

$$\frac{u_1(c_1^\star, c_2^\star)}{u_2(c_1^\star, c_2^\star)} = \frac{P_1}{P_2}$$

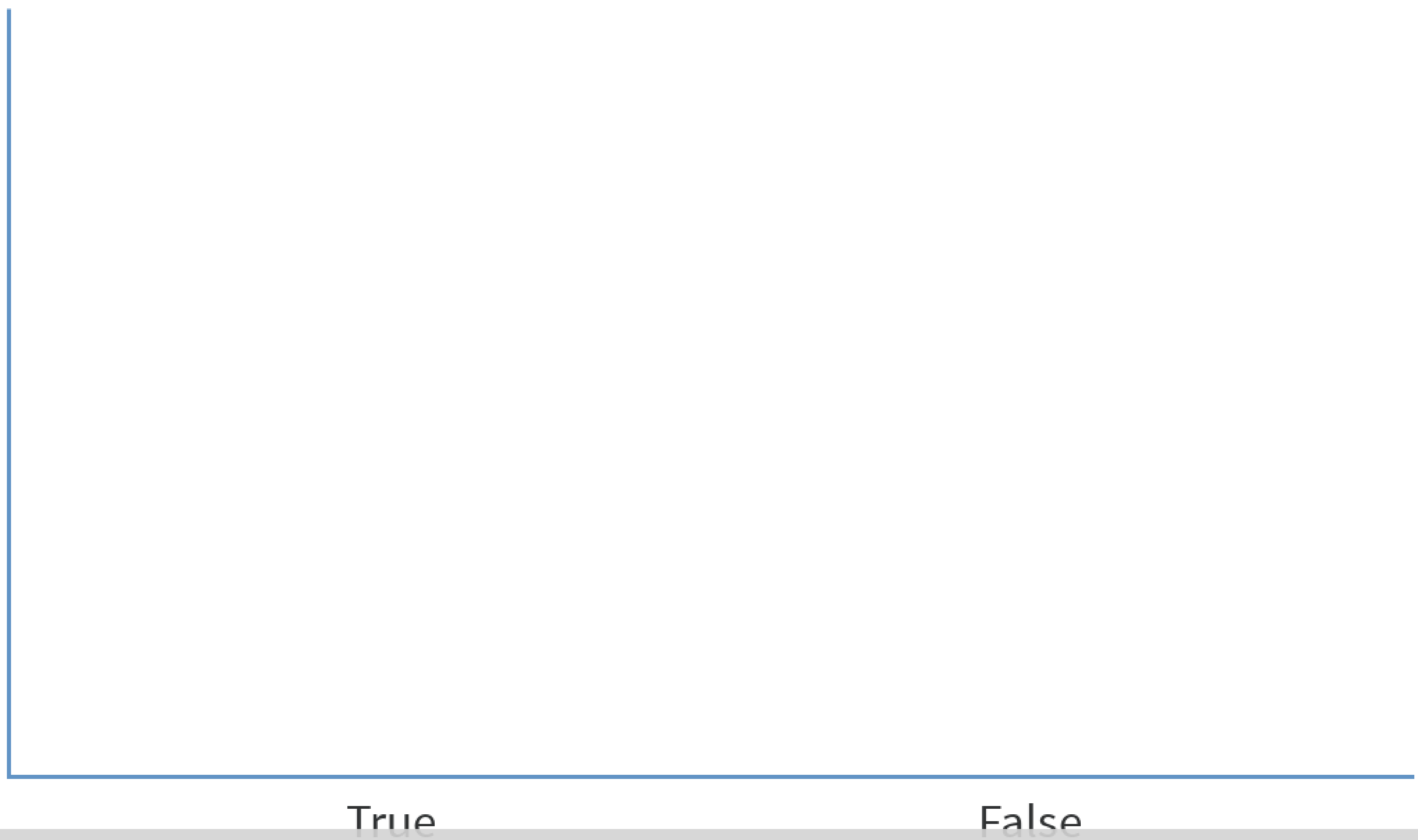
OPTIMALITY CONDITION

i.e., MRS = price ratio

$$\frac{u_1(c_1^\star, c_2^\star)}{u_2(c_1^\star, c_2^\star)} = \frac{P_1}{P_2}$$



At the optimality condition, the trade-off between c1 and c2 in term of utility is equal to the trade-off between c1 and c2 in term of finance (budget). That is, to give up 1 unit of c1, we need to give up same amount of c2 in both aspects.



$$\underbrace{\frac{u_1(c_1^\star, c_2^\star)}{u_2(c_1^\star, c_2^\star)}}_{} = \underbrace{\frac{P_1}{P_2}}_{}$$

MRS: *tradeoff in term of utility*

Price ratio: *trade-off in term of money*

Suppose that I would like to consume one more unit of c_1. I have to spend $\$P_1$ to purchase that unit of c_1.

So that means I have $\$P_1$ less to by $c_2 \Longrightarrow$ how much of c_2 I forewent? It is:

$$\frac{\$P_1}{\$P_2}$$

Consumption-Savings Framework

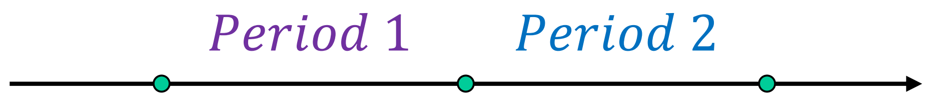
Chapter 3

(Modern Macroeconomics - Sanjay K. Chugh)

THE MACROECONOMICS OF TIME

- Dynamic frameworks the core of modern macroeconomic analysis

Simplest dynamic model:



- Explicit consideration of how economic decisions/behaviors/outcomes unfold over multiple time periods

Eg. Savings/Investments across time

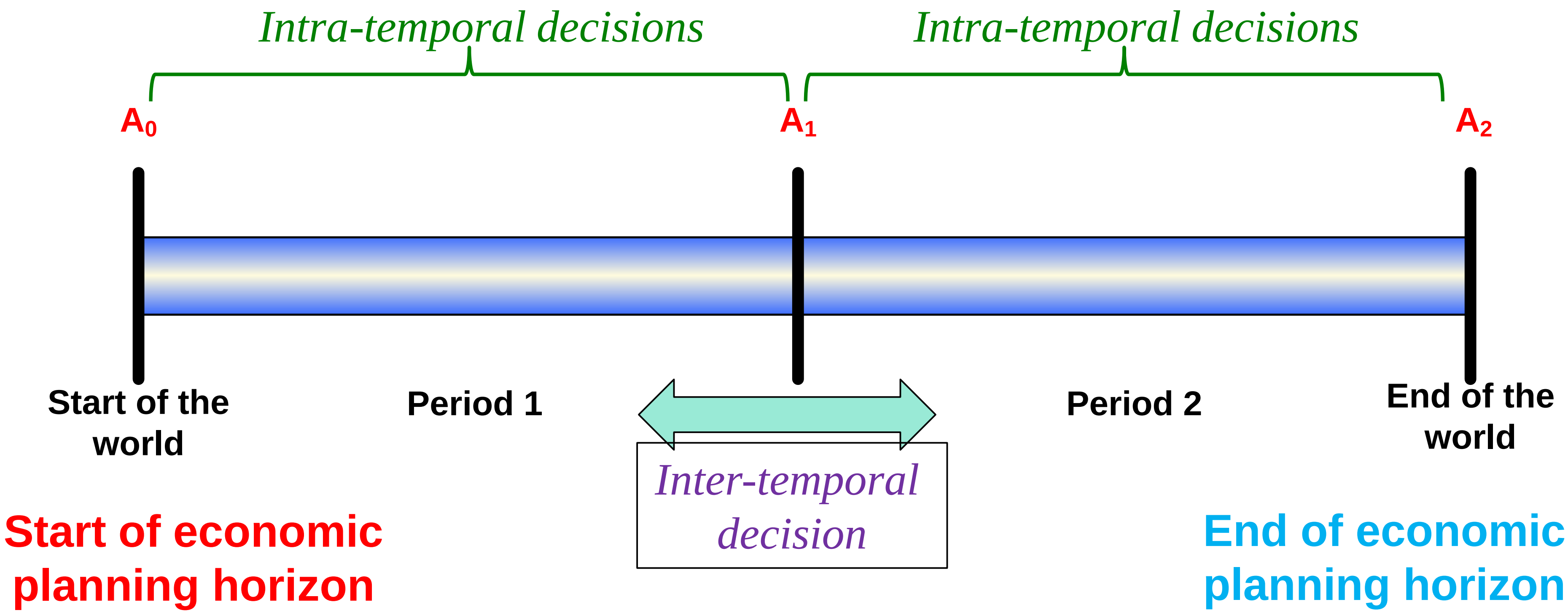
The Macroeconomics of Time

- Two-period framework (Chapters 3 and 4) the simplest possible multi-period framework
  - Will allow us to begin analyzing issues regarding **interest rates and inflation** (phenomena that occur across time)
  - Will allow us to think about credit restrictions and the “credit crunch”

- Infinite-period framework (Chapter 8)
  - Allows a richer quantitative description of the macroeconomy
  - Highlights the role of assets and the intersection between **finance and macroeconomics**

Basics

- Timeline of events



Basics

- Notation
  - c_1: consumption in period 1
  - c_2: consumption in period 2
  - P_1: nominal price of consumption in period 1
  - P_2: nominal price of consumption in period 2

$$\boxed{\pi_2 = \frac{P_2 - P_1}{P_1}\left(= \frac{P_2}{P_1} - 1 \right)}$$

  - Y_1: nominal income in period 1 ("falls from the sky")
  - Y_2: nominal income in period 2 ("falls from the sky")
  - ...

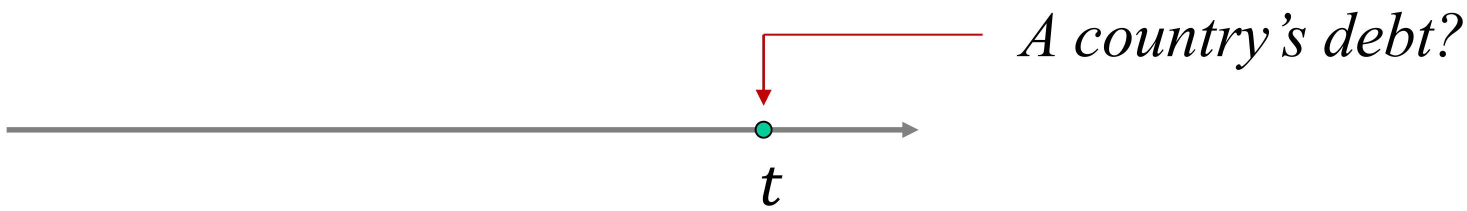
Basics

- Notation
  - …
  - A_0: nominal wealth at the beginning of period 1/end of period 0
  - A_1: nominal wealth at the beginning of period 2/end of period 1
  - A_2: nominal wealth at the beginning of period 3/end of period 2
  - i: nominal interest rate between periods
  - r: real interest rate between periods
  - π_2: net inflation rate between period 1 and period 2
  - y_1: real income in period 1 ($= \frac{Y_1}{P_1}$)
  - y_2: real income in period 2 ($= \frac{Y_2}{P_2}$)

STOCKS VS. FLOWS

- Stock variables, aka accumulation variables
  - Quantity variables whose natural measurement occurs at a particular moment in time
  - Checking account balance
  - Credit card indebtedness
  - Mortgage loan payoff

A is a stock variable



Stocks vs. Flows

- Flow variables
  - Quantity variables whose natural measurement occurs over the course of a given interval of time
  - Income
  - Consumption
  - Savings

- The two broad categories of income
  - Labor income
  - Asset income (generated by interest rate(s) on (components of) wealth)

Basics

- Building blocks of consumption-savings framework

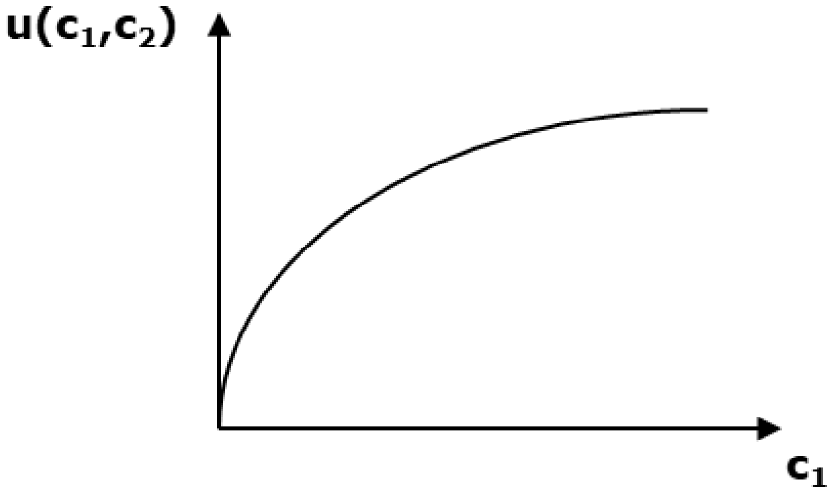
Same building blocks

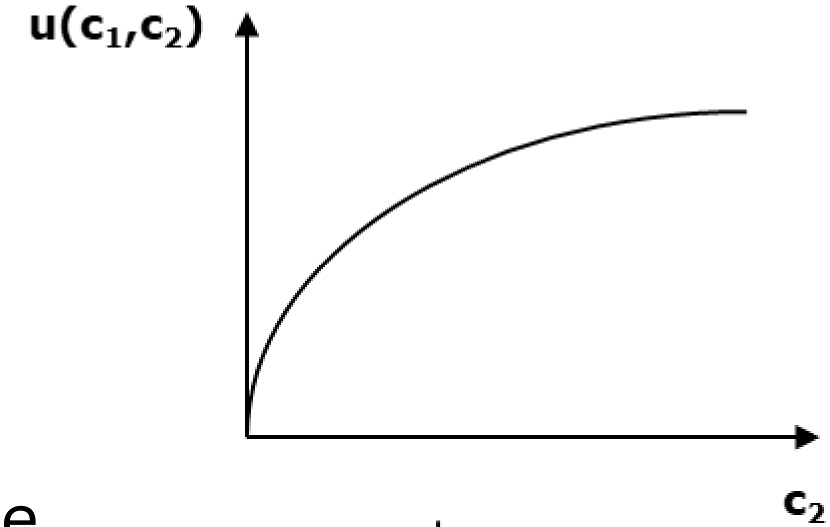
- Utility
  - Describes the benefits of engaging in financial market (and other) activities
- Budget constraint
  - Describes the costs of engaging in financial market (and other) activities

UTILITY

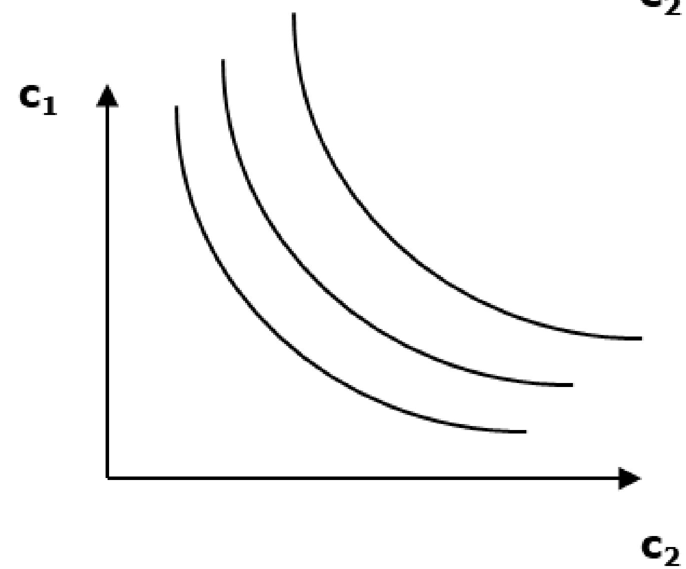
- Preferences $u(c_1, c_2)$ with all the "usual properties"
  - Lifetime utility function
  - Strictly increasing in c_1
  - Strictly increasing in c_2
  - Diminishing marginal utility in c_1
  - Diminishing marginal utility in c_2

UTILITY





- Plotted as indifference curves
- Utility side of consumption-savings framework identical to Chapter 1 framework



BUDGET CONSTRAINT(S)

- Suppose again *Y* "falls from the sky"
  - Y_1 in period 1, Y_2 in period 2
- Need two budget constraints to describe economic opportunities and possibilities
  - One for each period

BUDGET CONSTRAINT(S)

- **Period-1 budget constraint**

Asset income during period 1 (a flow)

Savings during period 1 (a flow)

$$\underbrace{P_1 c_1 + A_1}_{} = \underbrace{Y_1 + (1+i)A_0}_{} \quad \leftrightarrow \quad P_1 c_1 + \overbrace{A_1 - A_0} = Y_1 + iA_0$$

Total expenditure in period 1: period-1 consumption + wealth to *carry into period* 2

Total income in period 1: period-1 *Y* + income from wealth *carried into period* 1 (inclusive of interest)

Rate of return on asset

Initial wealth (e.g. inheritance)

Endowment

$$\underbrace{P_1 c_1 + A_1}_{\text{Consume} + \text{take out of period 1}} = \underbrace{Y_1 + (1+i)A_0}_{\text{Have in period 1}}$$

Consumption 1

Assets brought to period 2

$$A_0 + \underbrace{(A_1 - A_0)}_{\textit{Top-up or sell}} = A_1$$

Interest rate earned from A_0

$$P_1 c_1 + \underbrace{(A_1 - A_0)}_{\text{Amount of new assets bought (or sold)}} = Y_1 + iA_0$$

Amount of new assets bought (or sold)

If new assets purchased:
Purchased by <u>savings</u>

If assets sold → converted into cash to buy c_1 (<u>dissaving</u>)

BUDGET CONSTRAINT(S)

- Period-2 budget constraint

Asset income during period 2 (a flow)

Savings during period 2 (a flow)

$$\underbrace{P_2 c_2 + A_2}_{\text{Total expenditure in period 2}} = \underbrace{Y_2 + (1+i)A_1}_{\text{Total income in period 2}} \leftrightarrow P_2 c_2 + \overbrace{A_2 - A_1}^{\text{Savings during period 2}} = Y_2 + iA_1$$

Total expenditure in period 2: period-2 consumption + wealth to carry into period 3

Total income in period 2: period-2 *Y* + income from wealth *carried into period* 2 (inclusive of interest)

BUDGET CONSTRAINT(S)

- Adopt a lifetime view of the budget constraint(s)
  - All analysis conducted from **perspective of beginning of period 1**
  - Period-1 budget constraint

$$P_1 c_1 + A_1 = Y_1 + (1+i)A_0$$

  - Period-2 budget constraint

$$P_2 c_2 + A_2 = Y_2 + (1+i)A_1$$

Asset position at end of period 1/beginning of period 2 the key link

- will think further about this soon…

Assume = 0 (no bankruptcies + strictly increasing utility)

BUDGET CONSTRAINT(S)

- Combine into lifetime budget constraint (LBC)
  - Solve period-2 budget constraint for A_1…
  - …and substitute into period-1 budget constraint

$$\underbrace{P_1 c_1 + \frac{P_2 c_2}{1+i}}_{\text{Present discounted value (PDV) of all lifetime expenditure}} = \underbrace{Y_1 + \frac{Y_2}{1+i} + (1+i)A_0}_{\text{Present discounted value (PDV) of all lifetime income}}$$

No more asset holding in between

Present discounted value (PDV) of all lifetime expenditure

Present discounted value (PDV) of all lifetime income

For graphical simplicity, will often assume $A_0 = 0$ (i.e., consumer begins planning horizon with zero net wealth).

Note this is a different assumption than $A_2 = 0$.

LIFETIME BUDGET CONSTRAINT

$$P_1 c_1 + \frac{P_2 c_2}{1+i} = Y_1 + \frac{Y_2}{1+i}$$

subtract $P_1 c_1$

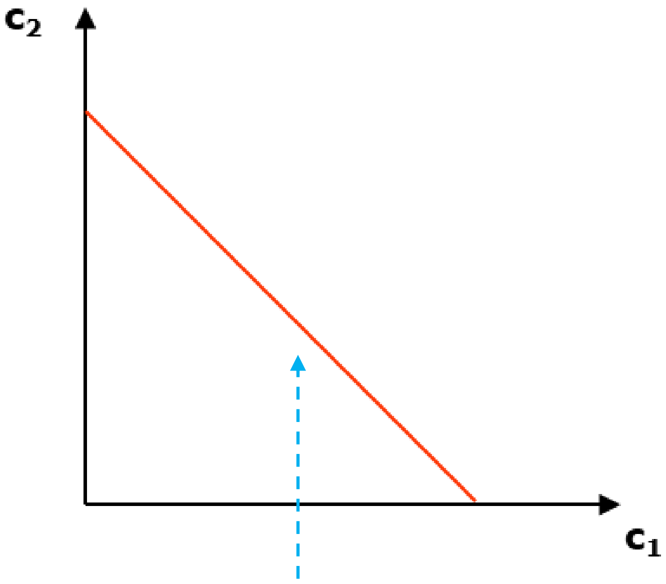
$$\frac{P_2 c_2}{1+i} = -\boldsymbol{P_1 c_1} + Y_1 + \frac{Y_2}{1+i}$$

divide by P_2

$$\frac{c_2}{1+i} = -\left(\frac{P_1}{\boldsymbol{P2}}\right) c_1 + \frac{Y_1}{\boldsymbol{P2}} + \frac{1}{1+i} \cdot \frac{Y_2}{\boldsymbol{P2}}$$

multiply by (1+i)

$$c_2 = -\left(\frac{P_1 \boldsymbol{(1+i)}}{P2}\right) c_1 + \frac{Y_1 \boldsymbol{(1+i)}}{P2} + \frac{Y_2}{P2}$$



$$slope = -\frac{P_1(1+i)}{P_2}$$

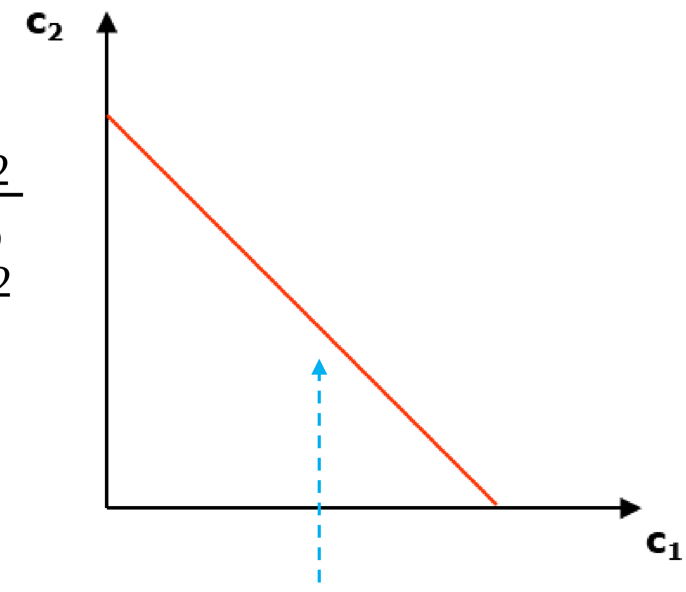
LIFETIME BUDGET CONSTRAINT

$$c_2 = -\underbrace{\left(\frac{P_1(1+i)}{P_2}\right)} c_1 + \left(\frac{1+i}{P_2}\right) Y_1 + \frac{Y_2}{P_2}$$

Rearrange further using definition of inflation:

$$\boxed{1+\pi_2 = \frac{P_2}{P_1} \Rightarrow \frac{1}{1+\pi_2} = \frac{P_1}{P_2}}$$

$$c_2 = -\left(\frac{1+i}{1+\pi_2}\right) c_1 + \left(\frac{1+i}{P_2}\right) Y_1 + \frac{Y_2}{P_2}$$



$slope = -(1+i)/(1+\pi_2)$

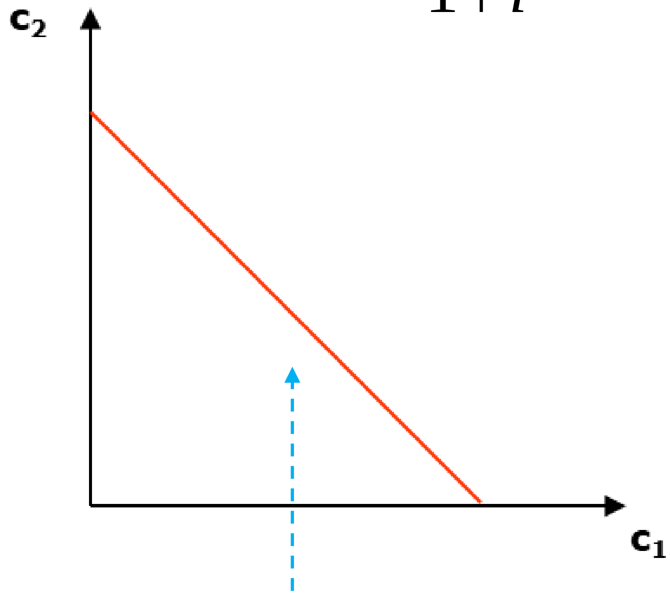
IMPORTANT:
Changes in nominal interest rates (Fed) and/or inflation affect the slope of the LBC

Consumer Optimization

- Consumer's decision problem: maximize lifetime utility subject to lifetime budget constraint – bring together both cost side and benefit side

  - Choose c_1 and c_2 subject to $P_1 c_1 + \frac{P_2 c_2}{1+i} = Y_1 + \frac{Y_2}{1+i}$

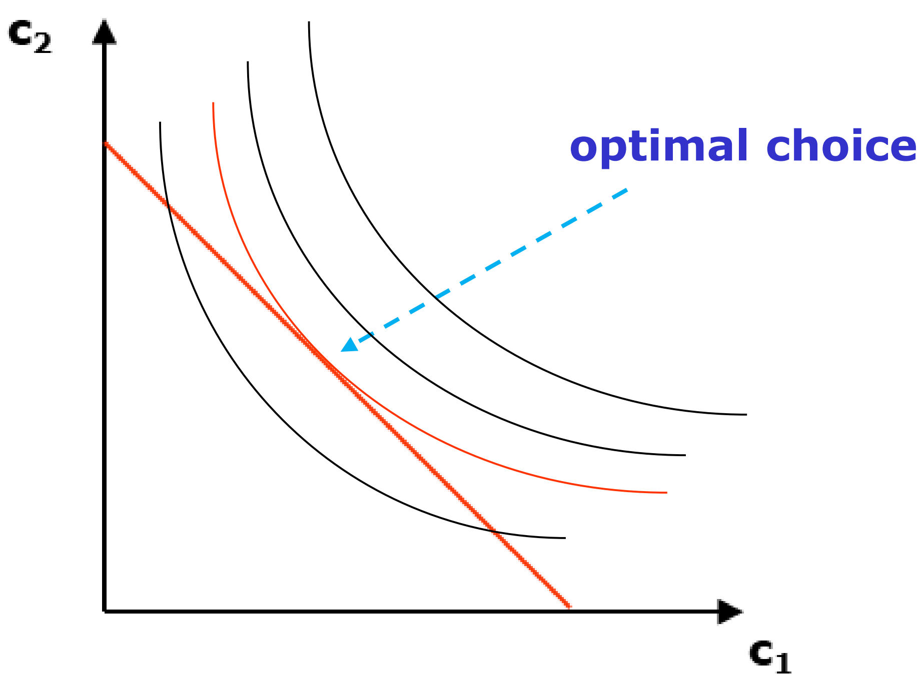
  - Plot budget line



$slope = -(1+i)/(1+\pi_2)$

Consumer Optimization

- Superimpose indifference map



$$\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)} = \frac{1+i}{1+\pi_2}$$

LAGRANGE ANALYSIS

- Apply Lagrange tools to consumption-savings optimization
- Objective function: $u(c_1, c_2)$
- Constraint: $g(c_1, c_2) = Y_1 + \frac{Y_2}{1+i} - P_1 c_1 - \frac{P_2 c_2}{1+i} = 0$
- Step 1: Construct Lagrange function
- Step 2: Compute first-order conditions with respect to c_1, c_2, λ
- Step 3: Combine (1) and (2) (with focus on eliminating multiplier)

LAGRANGE ANALYSIS

- Step 1: Construct Lagrange function

$$L(c_1, c_2, \lambda) = u(c_1, c_2) + \lambda\left[Y_1 + \frac{Y_2}{1+i} - P_1 c_1 - \frac{P_2 c_2}{1+i}\right]$$

- Step 2: Compute first-order conditions with respect to c_1, c_2, λ

(1) $$u_1(c_1, c_2) - \lambda P_1 = 0$$ $\partial L(\ldots)/\partial c_1 = 0$

(2) $$u_2(c_1, c_2) - \frac{\lambda P_2}{1+i} = 0$$ $\partial L(\ldots)/\partial c_2 = 0$

(3) $$Y_1 + \frac{Y_2}{1+i} - P_1 c_1 - \frac{P_2 c_2}{1+i} = 0$$ $\partial L(\ldots)/\partial \lambda = 0$

From 1: $u_1(c_1, c_2) - \lambda P_1 = 0 \quad \Longrightarrow \lambda P_1 = u_1\ (c_1, c_2)$

$$\Longrightarrow \lambda = \frac{u_1(c_1, c_2)}{P_1}$$

From 2: $u_2(c_1, c_2) - \dfrac{\lambda P_2}{1+i} = 0 \quad \Longrightarrow u_2(c_1, c_2) = \dfrac{\lambda P_2}{1+i}$

$$\Longrightarrow u_2(c_1, c_2) = \frac{u_1(c_1, c_2)}{P_1} \frac{P_2}{1+i}$$

$$\Longrightarrow \frac{u_2(c_1, c_2)}{u_1(c_1, c_2)} = \frac{P_2}{P_1} \frac{1}{1+i}$$

$$\Longrightarrow \frac{u_2(c_1, c_2)}{u_1(c_1, c_2)} = \frac{1+\pi_2}{1+i}$$

$$\boxed{\text{or } \frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} = \frac{1+i}{1+\pi_2}}$$

Lagrange Analysis

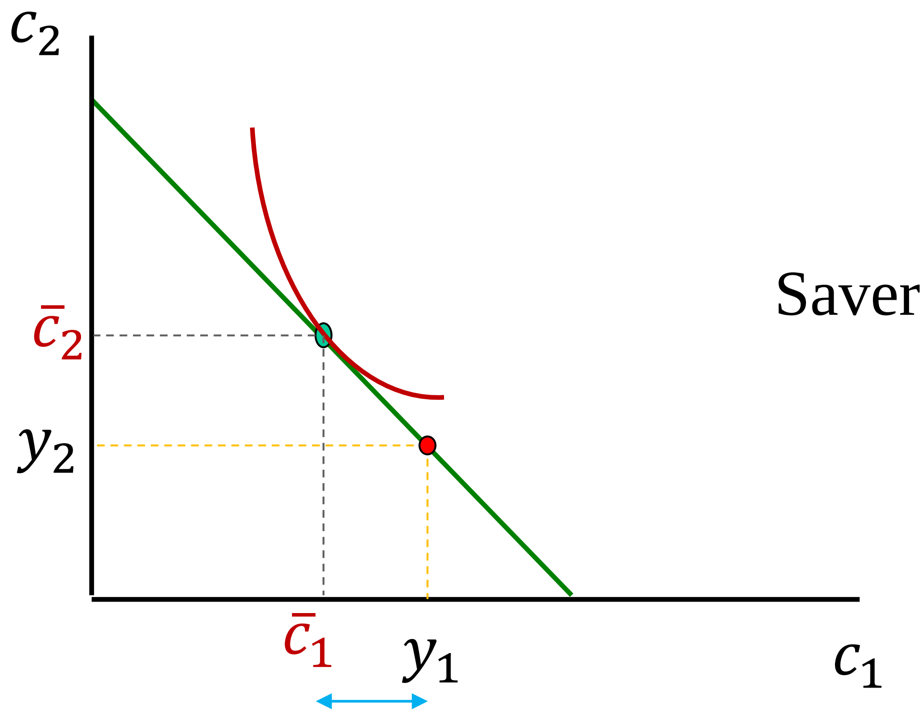
- **Step 3:** Combine (1) and (2) (with focus on eliminating multiplier)

$$\underbrace{\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)}}_{\substack{\text{MRS} \\ \text{(between consumption} \\ \text{in consecutive} \\ \text{time periods)}}} = \underbrace{\frac{1+i}{1+\pi_2}}_{\substack{\text{price ratio} \\ \text{(across consecutive} \\ \text{time periods)}}}$$

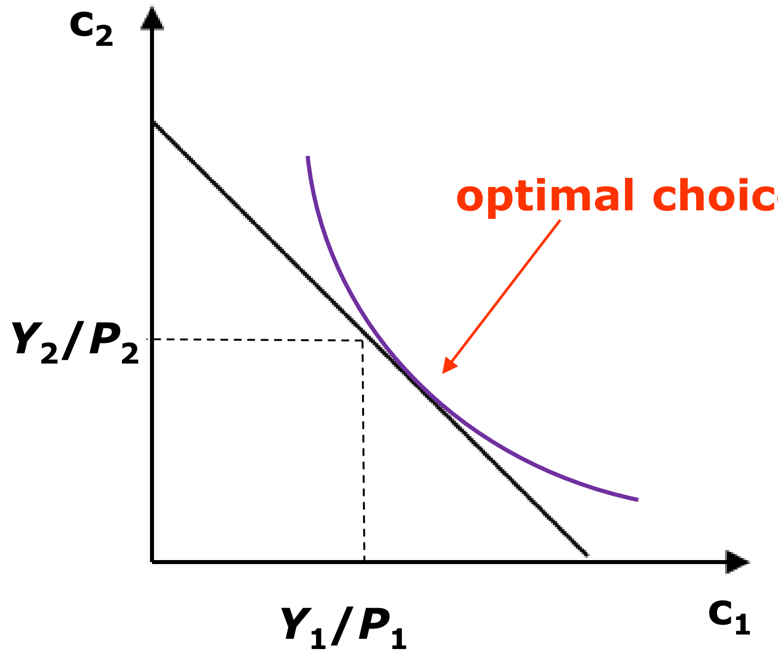
CONSUMPTION-SAVINGS OPTIMALITY CONDITION

$\frac{1+i}{1+\pi_2}$ determines where we should consume and thus savings

$$\boldsymbol{slope = -(1+i)/(1+\pi_2)}$$



(continuing to assume $A_0 = 0$)



"Dissaver"

INFLATION AND INTEREST RATES IN THE CONSUMPTION-SAVINGS FRAMEWORK

CHAPTER 4

(Modern Macroeconomics - Sanjay K. Chugh)

FISHER EQUATION

- Nominal interest rate – measured in dollars
- Real interest rate – measured in goods

- Fisher equation: a link between the nominal interest rate, inflation rate, and real interest rate
  - "Strips out the effect of inflation"
  - Exact Fisher equation

$$1+r=\frac{1+i}{1+\pi}$$

FISHER EQUATION

- **Approximate Fisher equation (intro macro)**

$$(1+r)(1+\pi)=1+i$$

$$\cancel{1}+r+\pi+\underbrace{\cancel{r\pi}}_{\approx\, 0}=\cancel{1}+i$$

In advanced economies, r and π are both generally small → $r\pi \approx 0$

$$\boxed{r=i-\pi}$$

A useful rule of thumb

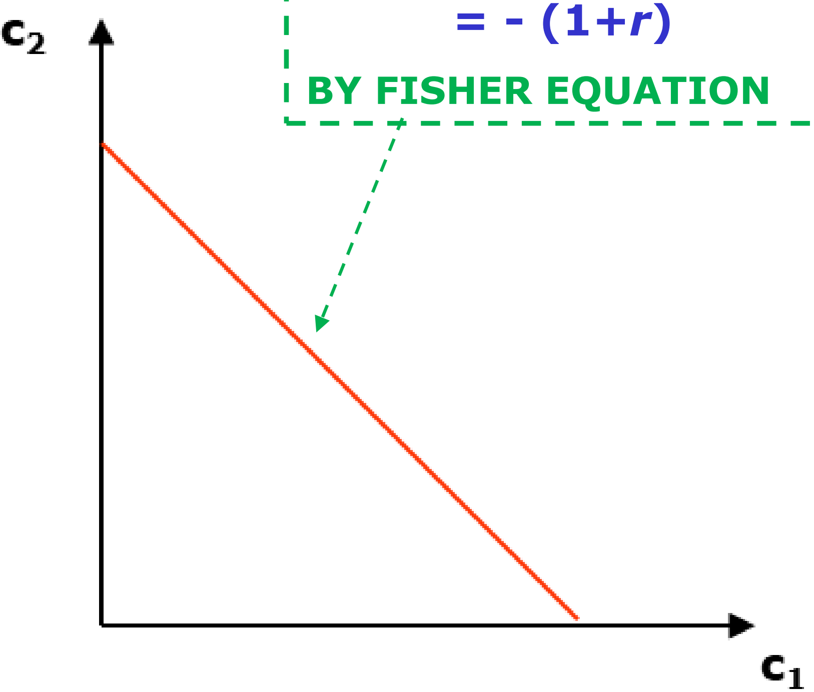
REAL INTEREST RATE

- *r* a key variable for macroeconomic analysis
- Unit Analysis (i.e., analyze algebraic units of variables)

slope = $-(1+i)/(1+\pi_2)$

= - (1+*r*)

BY FISHER EQUATION



Slope measures <u>how much c_2 must be given up in order to obtain one more unit of c_1</u> ("rise over run") when saving or dissaving at market interest rates

1+*r* is the <u>*price*</u> of period-1 consumption in terms of period-2 consumption

More generally: 1+*r* measures the price of current goods in terms of future goods

Real Interest Rate

- Economic decisions depend on *real* interest rates (r), not nominal interest rates (i)
  - Measures the cost of borrowing/lending in terms of goods…
  - …which is presumably what people most care about

TWO-PERIOD FRAMEWORK IN REAL TERMS

- Depending on application, may be useful to work with framework (independent of lifetime vs. sequential approach) in nominal terms or in real terms

LBC in nominal terms (assuming A_0 = 0)

$$P_1 c_1 + \frac{P_2 c_2}{1+i} = Y_1 + \frac{Y_2}{1+i}$$

$\downarrow$ **Divide by P_1**

$$c_1 + \left(\frac{P_2}{P_1(1+i)}\right) c_2 = \frac{Y_1}{P_1} + \frac{Y_2}{P_1(1+i)}$$

Two-Period Framework in Real Terms

LBC in real terms (assuming $A_0 = 0$)

$$c_1 + \left(\frac{P_2}{P_1(1+i)}\right)c_2 = \frac{Y_1}{P_1} + \frac{Y_2}{P_1(1+i)}$$

Multiply *and* divide last term on right-hand-side by P_2

$$c_1 + \left(\frac{P_2}{P_1(1+i)}\right)c_2 = \frac{Y_1}{P_1} + \frac{Y_2}{P_1(1+i)} \cdot \frac{P_2}{P_2}$$

Use definitions: $y_1 = Y_1/P_1$, $y_2 = Y_2/P_2$, and $1+\pi_2 = P_2/P_1$

$$c_1 + \left(\frac{1+\pi_2}{1+i}\right)c_2 = y_1 + \frac{1+\pi_2}{1+i} \cdot y_2$$

Use Fisher equation: $(1+\pi_2)/(1+i) = 1/(1+r)$

$$\boxed{c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}}$$

Real LBC

Consumption-Savings Optimality Condition

- Emphasizing i and π

Working out Lagrange using nominal LBC

$$\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)} = \frac{1+i}{1+\pi}$$

Fisher equation

- Emphasizing r

Working out Lagrange using real LBC

$$\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)} = 1+r$$

- Can also analyze two-period framework sequentially, rather than from a lifetime perspective

$$\frac{1+i}{1+\pi} = (1+i)\frac{P_1}{P_2} = \frac{(\mathbf{1}+\boldsymbol{i})\boldsymbol{P_1}}{P_2}$$

If I refrain from consuming 1 unit of $c_1 \rightarrow$ I save $\$P_1$

Investing $\$P_1$, I get **$(1+i)P_1** in period 2 (nominal)

In period 2, I buy composite goods with **$(1+i)$P_1$**.
The amount is:

$$\frac{\$(\mathbf{1}+\boldsymbol{i})\boldsymbol{P_1}}{\$\boldsymbol{P_2}}$$

Price of consuming 1 unit of c_1

Price of consuming 1 unit of c_2

So: $\dfrac{1+i}{1+\pi} = \dfrac{(\mathbf{1}+\boldsymbol{i})\boldsymbol{P_1}}{P_2}$

$\boxed{\dfrac{\Delta c_2}{\mathbf{1}}}$

The extra units that we can enjoy in period 2 if we <u>did not consume that ONE unit of c_1</u>

$\mathbf{1} + \boldsymbol{r}$

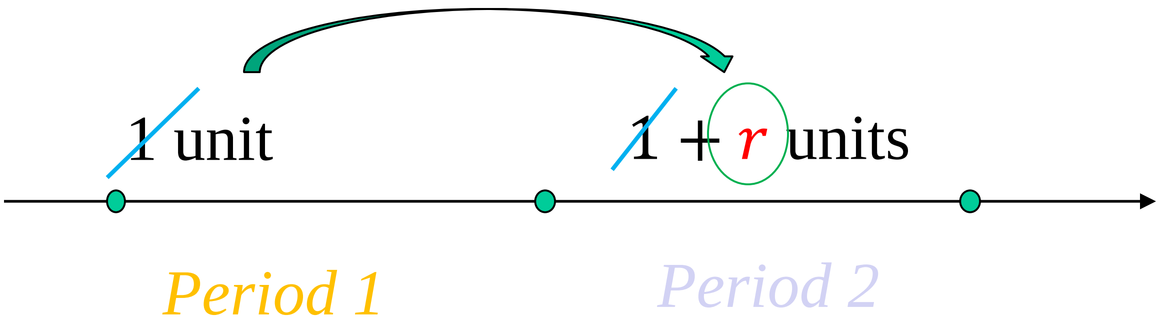
So:

$$\mathbf{1} + \boldsymbol{r} = \frac{\Delta c_2}{\mathbf{1}}$$

1+*r* is the *<u>price</u>* of period-1 consumption in terms of period-2 consumption

More generally: 1+*r* measures the price of current goods in terms of future goods

Sum up:



Lagrange Analysis: Sequential Approach

- Sequential formulation highlights the role of net wealth (A_1) between period 1 and period 2
  - Accords better with the explicit timing of economic events than the lifetime approach…
  - …but yields the same result
  - Advantage: allows us to think about interaction between asset prices and macroeconomic events (intersection of finance theory and macro theory in Chapter 8)

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- Apply Lagrange tools to consumption savings optimization
- Objective function: $u(c_1, c_2)$
- Constraints:
  - Period 1 budget constraint:

$$Y_1 + (1+i)A_0 - P_1 c_1 - A_1 = 0$$

  - Period 2 budget constraint:

$$Y_2 + (1+i)A_1 - P_2 c_2 - A_2 = 0$$

TWO constraints

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- Step 1: Construct Lagrange function

$$L(c_1, c_2, A_1, \lambda_1, \lambda_2) = u(c_1, c_2)$$

$$+\lambda_1 \left[Y_1 + (1+i)A_0 - P_1 c_1 - A_1 \right]$$

$$+\lambda_2 \left[Y_2 + (1+i)A_1 - P_2 c_2 - A_2 \right]$$

Why do I use two periods' constraints instead of collapsing them into one Life-time budget constraint?

Easier to solve

So that A1 appears

$$\boxed{\begin{aligned} L(c_1, c_2, A_1, \lambda_1, \lambda_2) = u(c_1, c_2) + \lambda_1[Y_1 + (1+i)A_0 - P_1 c_1 - A_1] \\ + \lambda_2[Y_2 + (1+i)A_1 - P_2 c_2 - A_2] \end{aligned}}$$

$$\frac{\partial L(c_1, \dots)}{\partial c_1} = \frac{\partial u(c_1, c_2)}{\partial c_1} + \lambda_1 P_1 = 0 \quad \text{or} \quad u_1(c_1, c_2) + \lambda_1 P_1 = 0$$

$$\frac{\partial L(\dots c_2, \dots)}{\partial c_2} = \frac{\partial u(c_1, c_2)}{\partial c_2} + \lambda_2 P_2 = 0 \quad \text{or} \quad u_2(c_1, c_2) + \lambda_2 P_2 = 0$$

$$\frac{\partial L(\dots A_1, \dots)}{\partial A_1} = \lambda_1 + \lambda_2(1+i) = 0 \quad \text{or} \quad \lambda_1 + \lambda_2(1+i) = 0$$

$$\frac{\partial L(\dots \lambda_1, \dots)}{\partial \lambda_1} = Y_1 + (1+i)A_0 - P_1 c_1 - A_1 = 0$$

Period-1 BC

$$\frac{\partial L(\dots \lambda_2, \dots)}{\partial \lambda_2} = Y_2 + (1+i)A_1 - P_2 c_2 - A_2 = 0$$

Period-2 BC

Lagrange Analysis: Sequential Approach

- Step 2: Compute FOCs with respect to c_1, c_2, A_1, λ_1, λ_2

(1) $$u_1(c_1, c_2) - \lambda_1 P_1 = 0$$

(2) $$u_2(c_1, c_2) - \lambda_2 P_2 = 0$$

$$\lambda_1 = u_1(c_1, c_2) / P_1$$

$$\lambda_2 = u_2(c_1, c_2) / P_2$$

into (3)

(3) $$-\lambda_1 + \lambda_2(1+i) = 0$$

$$-\frac{u_1(c_1, c_2)}{P_1} + \frac{u_2(c_1, c_2)}{P_2}(1+\mathrm{i}) = 0$$

$$-\frac{u_1(c_1,c_2)}{P_1}+\frac{\boldsymbol{u_2(c_1,c_2)}}{\boldsymbol{P_2}}(1+\mathrm{i})=0$$

$$\frac{u_1(c_1,c_2)}{P_1}=\frac{\boldsymbol{u_2(c_1,c_2)}}{\boldsymbol{P_2}}(1+\mathrm{i})$$

$$u_1(c_1,c_2)=\frac{u_2(c_1,c_2)P_1}{P_2}(1+\mathrm{i})$$

$$\frac{u_1(c_1,c_2)}{u_2(c_1,c_2)}=\frac{P_1}{P_2}(1+\mathrm{i})$$

$$\boxed{\frac{P_1}{P_2}=\frac{1}{1+\pi_2}}$$

$$\frac{u_1(c_1,c_2)}{u_2(c_1,c_2)}=\frac{1+i}{1+\pi_2}=\boldsymbol{1+r}$$

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- Step 3: Combine (1),(2),(3) (with focus on eliminating multipliers)

using Fisher equation

$$\underbrace{\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)}}_{\text{MRS (between consumption in consecutive time periods)}} = \underbrace{\frac{1+i}{1+\pi_2} = 1+r}_{\text{price ratio (across consecutive time periods)}}$$

- Identical to result of lifetime formulation

CONSUMER BUDGET CONSTRAINT(S)

consumer ***NOMINAL***
savings during period 1

$$P_1 c_1 + \overbrace{A_1 - A_0} = Y_1 + iA_0$$

$$\underbrace{A_1 - A_0}_{\text{Saving}} = Y_1 + iA_0 - P_1 c_1$$

consumer ***NOMINAL***
savings during period 2

$$P_2 c_2 + \overbrace{A_2 - A_1} = Y_2 + iA_1$$

$$\underbrace{A_2 - A_1}_{\text{Saving}} = Y_2 + iA_1 - P_2 c_2$$

Consumer Budget Constraint(s)

$$\underbrace{c_1 + a_1 - a_0}_{\text{consumer } \textit{REAL} \text{ savings during period 1}} = y_1 + ra_0$$

$$\underbrace{c_2 + a_2 - a_1}_{\text{consumer } \textit{REAL} \text{ savings during period 2}} = y_2 + ra_1$$

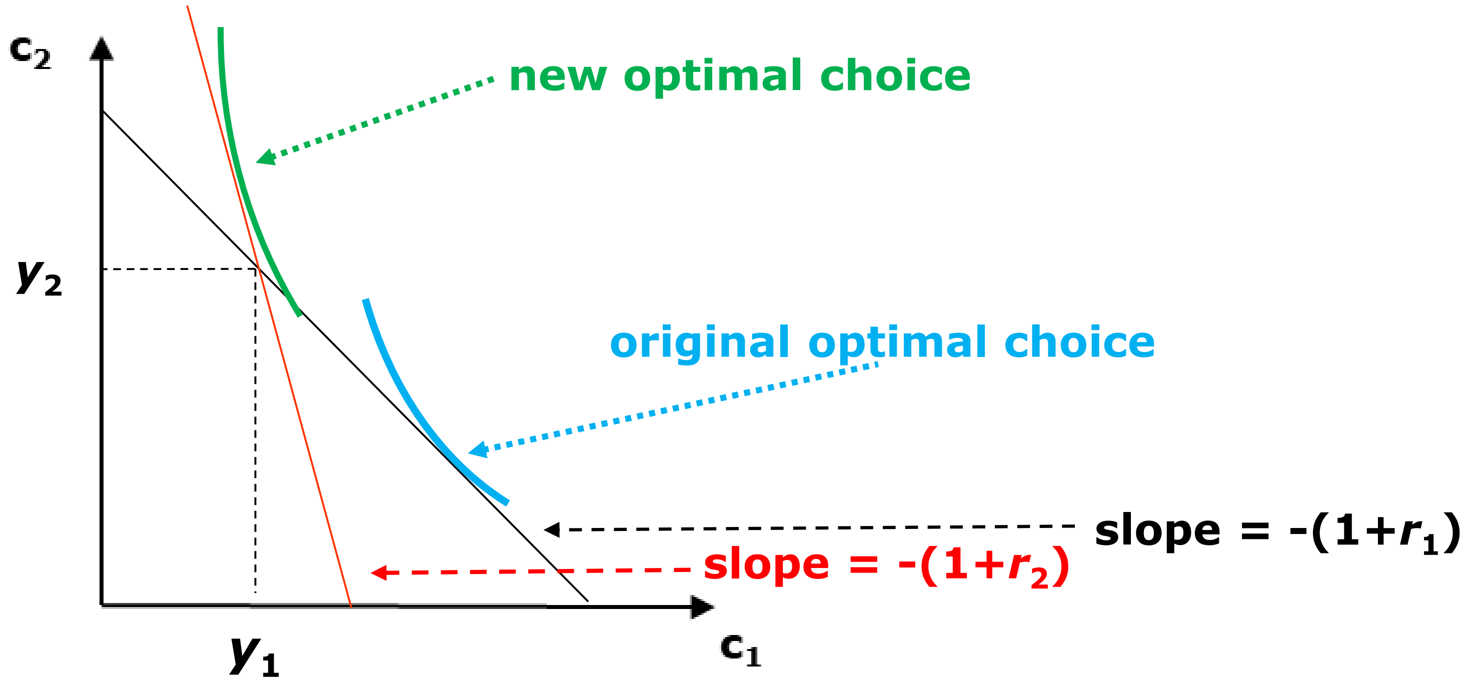
$$a_1 - a_0 = y_1 + ra_0 - c_1$$

$$a_2 - a_1 = y_2 + ra_1 - c_2$$

MICRO-LEVEL SAVINGS

How do micro-level consumption/savings choices change as the real interest rate changes (continue assuming $A_0 = 0$ for simplicity)?

REAL INTEREST RATE: $r_1 < r_2$

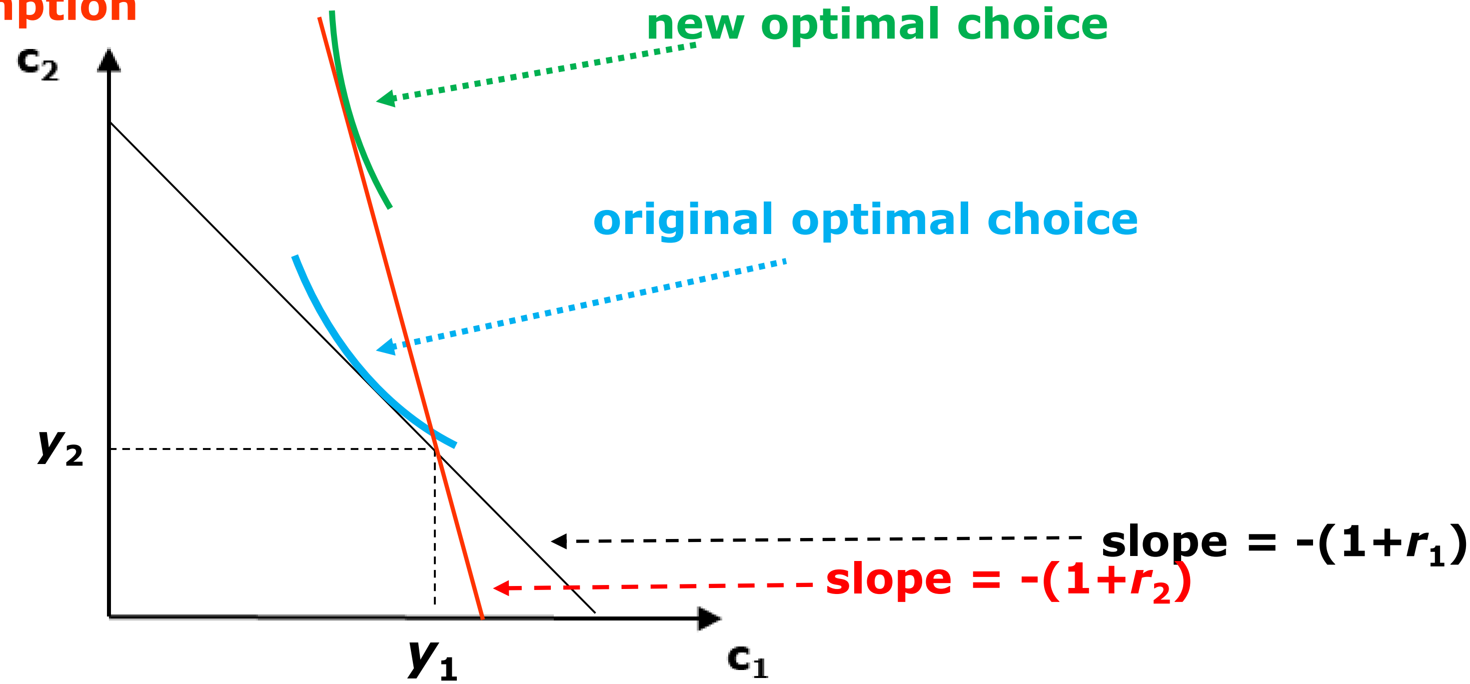


RESULT: optimal choice of c_1 *falls* as r rises → optimal choice of *savings* ($= y_1 - c_1$) *rises* as r rises

MICRO-LEVEL SAVINGS

IMPORTANT: LBC *pivots* around the point (y_1, y_2) because (y_1, y_2) is always a *possible* choice of consumption

RESULT: optimal choice of c_1 *falls* as r rises → optimal choice of *savings* (= y_1 – c_1) *rises* as r rises



MICRO-LEVEL SAVINGS

RESULT: optimal choice of c_1 *falls* as r rises → optimal choice of *savings* ($= y_1 - c_1$) *rises* as r rises

Empirical evidence shows that when r rises, period-1 (i.e., "short-run") *consumption* of all types of consumers falls

implying that when *r* rises, period-1 (i.e., "short-run") *savings* of all types of consumers rises…

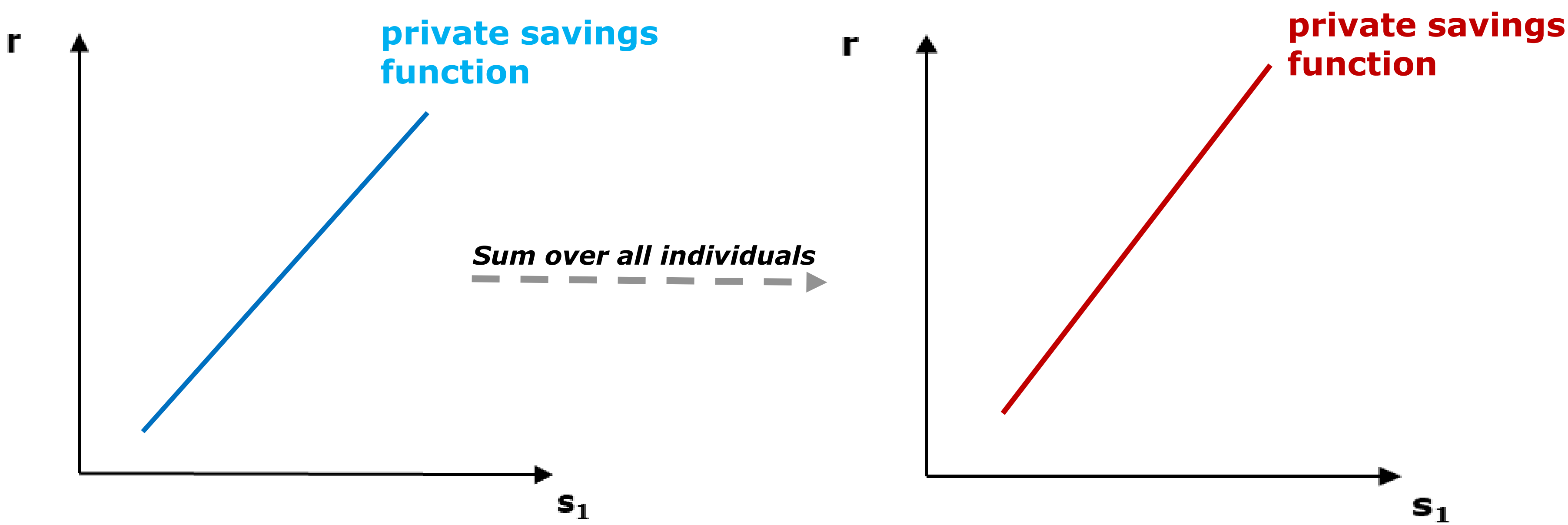
SAVINGS

- ❑ Define private savings function (in period 1) for an individual

Emphasizing functional relationships

(Recall alternative (equivalent) definition: savings is the change in wealth during a period)

$$s_1^{priv}(\underset{+}{r}, y_1, y_2) = y_1 - c_1(\underset{-}{r}, y_1, y_2)$$



Individual-level savings function

Aggregate-level savings function

Using name of the variable

$$\frac{dU(c)}{dc} = U'(c) \text{ or } U_{c}(c) \text{ or } U_{1}(c)$$

Marginal utility

Using position of the variable in the list of variables

$$\frac{\partial\left(\frac{dU(c)}{dc}\right)}{\partial c} = \frac{d^2U(c)}{dc^2} = U''(c) \text{ or } U_{cc}(c) \text{ or } U_{11}(c)$$

How MU changes w.r.t c

Using name of the variable

$$\frac{dU(c_1, c_2)}{dc_1} = \cancel{U^1(c_1, c_2) \text{ or }} U_{c_1}(c_1, c_2) \text{ or } U_1(c_1, c_2) \quad > 0$$

Using position of the variable in the list of variables

$$\frac{dU(c_1, c_2)}{dc_2} = U_{c_2}(c_1, c_2) \text{ or } U_2(c_1, c_2) \quad > 0$$

$$U_i(c_1, c_2) > 0 \text{ for } i = 1,2$$

$$\frac{\partial\left(\frac{\partial U(c_1, c_2)}{\partial c_1}\right)}{\partial c_1} = U_{11}(c_1, c_2) \quad <0$$

$$\frac{\partial\left(\frac{\partial U(c_1, c_2)}{\partial c_2}\right)}{\partial c_2} = U_{22}(c_1, c_2) \quad <0$$

$$U_{ii}(c_1, c_2) < 0 \text{ for } i = 1,2$$

EXAMPLE OF A UTILITY FUNCTION

$$\frac{\partial U(c)}{\partial c} = \frac{\partial \ln c}{\partial c} = \frac{1}{c} > 0$$

$$\frac{\partial\left(\frac{\partial U(c)}{\partial c}\right)}{\partial c} = \frac{\partial\left(\frac{\partial \ln c}{\partial c}\right)}{\partial c} = \frac{\partial\left(\frac{1}{c}\right)}{\partial c} = -\frac{1}{c^2} < 0$$

Implicit function theorem *(not tested)*

$$U(c_1, c_2)$$

Let $G(x, y) = L$ around the point (x_0, y_0)

This means that $x = x_0$ and $y = y_0$

$$G(x_0 + \Delta, y_0 + ?) = L$$ around the point (x_0, y_0)

Δ is small

$$G(x, y(x))$$

Happiness

Attend lecture

Leisure

By chain rule, differentiate $G(x, y) = L$ w.r.t to x at (x_0, y_0)

$$\frac{dG(\dots)}{dx} = \frac{\partial G(x, y(x))}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial G(x, y(x))}{\partial y} \cdot \frac{dy}{dx} = 0$$

Evaluated at (x_0, y_0)

$$\frac{\partial G(x, y(x))}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial G(x, y(x))}{\partial x} \cdot \frac{dx}{dx}$$

Evaluated at (x_0, y_0)

$$\frac{dy}{dx} = -\frac{\dfrac{\partial G\big(\boldsymbol{x}, y(x)\big)}{\partial \boldsymbol{x}}}{\dfrac{\partial G\big(x, \boldsymbol{y}(\boldsymbol{x})\big)}{\partial \boldsymbol{y}}} \Bigg|_{(x_0, y_0)}$$

Evaluated at (x_0, y_0)

Apply to our context, we have:

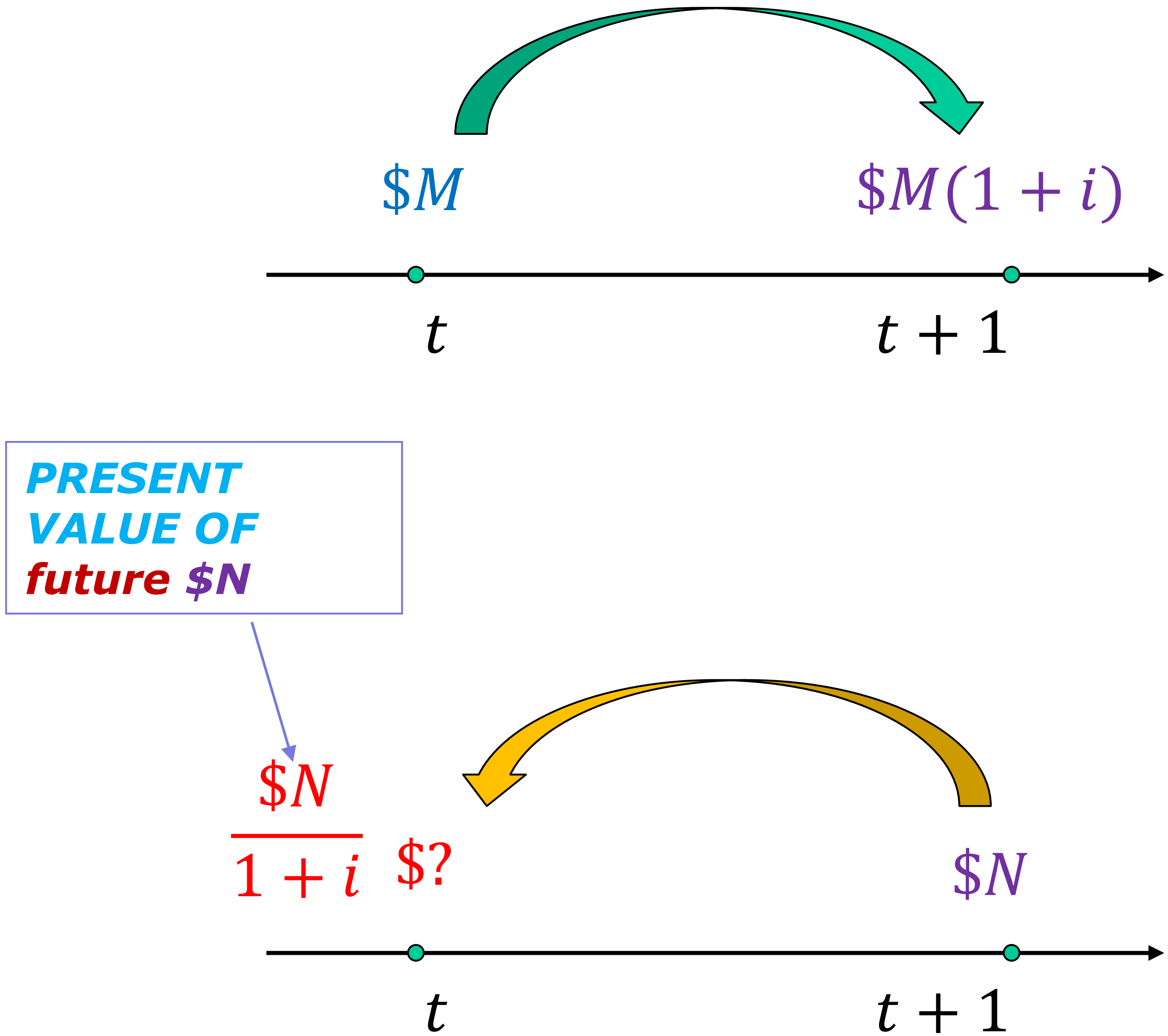
$$\frac{dc_2}{dc_1} = -\frac{\dfrac{\partial U(\boldsymbol{c_1}, c_2(c_1)\ )}{\partial \boldsymbol{c_1}}}{\dfrac{\partial U\big(c_1, \boldsymbol{c_2}(c_1)\big)}{\partial \boldsymbol{c_2}}} \Bigg|_{(\bar{c}_1, \bar{c}_2)}$$

Evaluated at $(\bar{c}_1, \bar{c}_2)$

$$\frac{dc_2}{dc_1} = -\frac{\dfrac{\partial U(\boldsymbol{c_1}, c_2(c_1)\ )}{\partial \boldsymbol{c_1}}}{\dfrac{\partial U\left(c_1, \boldsymbol{c_2}(c_1)\right)}{\partial \boldsymbol{c_2}}} \Bigg|_{(\bar{c}_1, \bar{c}_2)}$$

$$MRS_{c_1,c_2} = -\frac{MU_{c_1}}{MU_{c_2}}$$

Evaluated at $(\bar{c}_1, \bar{c}_2)$


$\$M$
$\$M(1+i)$
t
$t+1$
PRESENT VALUE OF future $N
$\dfrac{\$N}{1+i}$
$\$?$
$\$N$
t
$t+1$

ANOTHER WAY OF INTERPRETING. NOT COVEREd

Real interest rate is REAL – in term of units of good

$\frac{1+i}{1+\pi}$ is the reduced form:

Can interpret: $1+i$ is the amount gotten by investing **$1** in period 1.

Can interpret: $1+\pi$ is the relative price of c_2 w.r.t c_1

RECALL: $\mathbf{1+\pi=\frac{P_2}{P1}}$