Macroeconomics analysis II, EC3102

Tutorial 6 Solution

Question 1

a.

Suppose we scale both input factors K and L by a factor of $n > 1$, we get:

$$\begin{aligned} F_a(nK, nL) &= (nK)^{\frac{2}{3}}(nL)^{\frac{2}{3}} \\ &= n^{\frac{2}{3}+\frac{2}{3}}K^{\frac{2}{3}}L^{\frac{2}{3}} \\ &= n^{\frac{4}{3}}K^{\frac{2}{3}}L^{\frac{2}{3}} \\ &> nK^{\frac{2}{3}}L^{\frac{2}{3}} \\ &= nF(K, L) \end{aligned} \tag{1}$$

That means that if we scale both input factors by a certain factor, the output would be scaled up by more than that factor. So the production function exhibits increasing returns to scale.

b.

Suppose we scale both input factors $K$and L by a factor $n > 1$, we have:

$$\begin{aligned} F(nK, nL) &= nK + nL \\ &= n(K + L) \\ &= nF(K, L) \end{aligned} \tag{2}$$

That means that if we scale both input factors by a certain factor, the output would be scaled up by that factor *as well*. So the production function exhibits constant returns to scale.

c.

Suppose we scale both input factors $K$and L by a factor $n > 1$, we have:

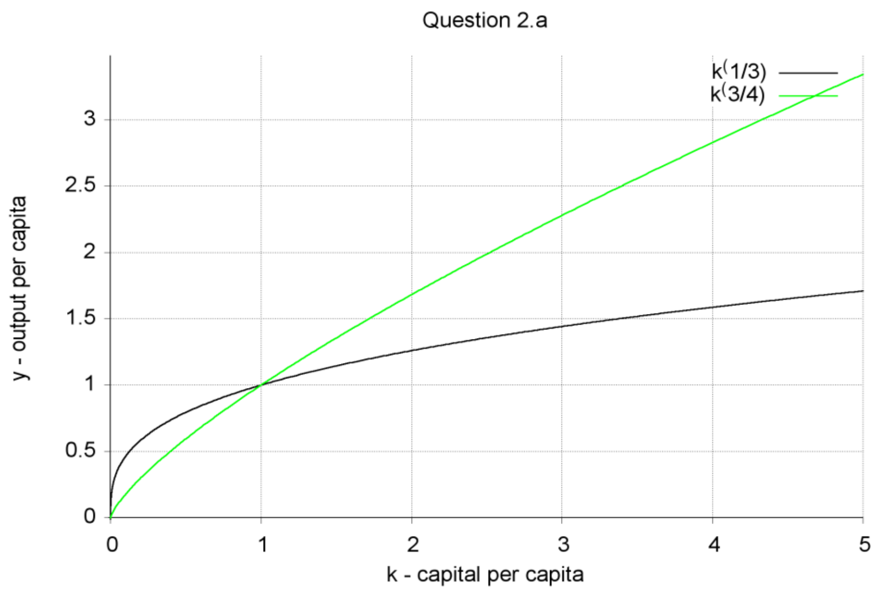
$$\begin{aligned} F(nK, nL) &= nK + (nK)^{\frac{1}{3}}(nL)^{\frac{1}{3}} \\ &= nK + n^{\frac{1}{3}+\frac{1}{3}}K^{\frac{1}{3}}L^{\frac{1}{3}} \\ &= nK + n^{\frac{2}{3}}K^{\frac{1}{3}}L^{\frac{1}{3}} \\ &< nK + nK^{\frac{1}{3}}L^{\frac{1}{3}} \quad \text{since } \mathrm{n}^{\frac{2}{3}} < n \\ &= nF(K, L) \end{aligned} \tag{3}$$

That means that if we scale both input factors by a certain factor, the output would be scaled up by less than that factor. So the production function exhibits decreasing returns to scale.

Question 2

a.

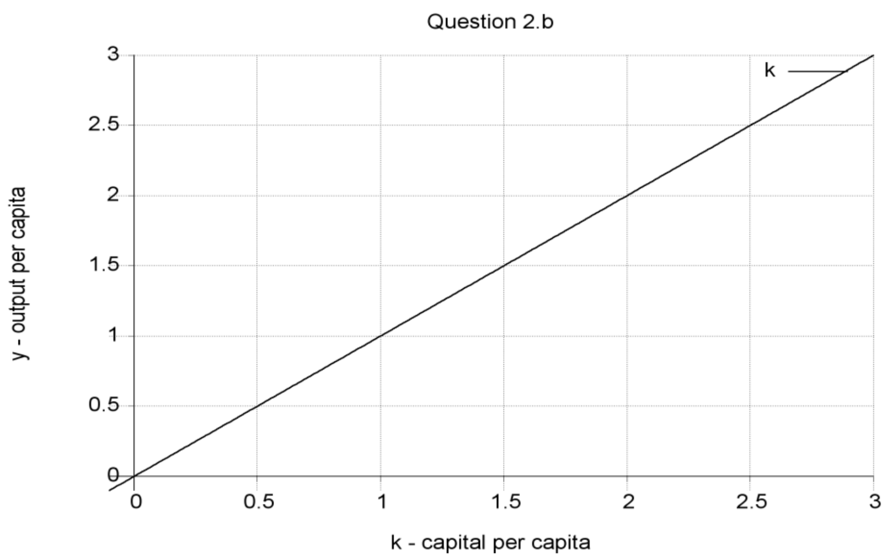
$$
\begin{aligned}
& Y = K^{\alpha} L^{1-\alpha} \\
\Longrightarrow\ & \frac{Y}{L} = \frac{K^{\alpha} L^{1-\alpha}}{L} \\
\Longrightarrow\ & y = \frac{K^{\alpha}}{L^{\alpha}} = k^{\alpha}
\end{aligned}
\tag{4}
$$



The green curve corresponds to $y = k^{\frac{1}{3}}$ and the black, $y = k^{\frac{3}{4}}$

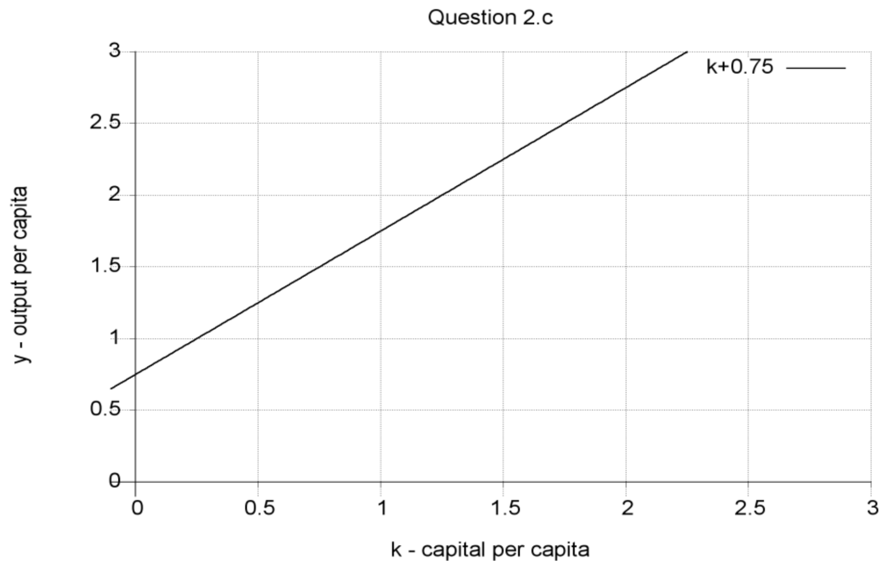
b.

$$
\begin{aligned}
& Y = K \\
\Longrightarrow\ & \frac{Y}{L} = \frac{K}{L} \\
\Longrightarrow\ & y = k
\end{aligned}
\tag{5}
$$



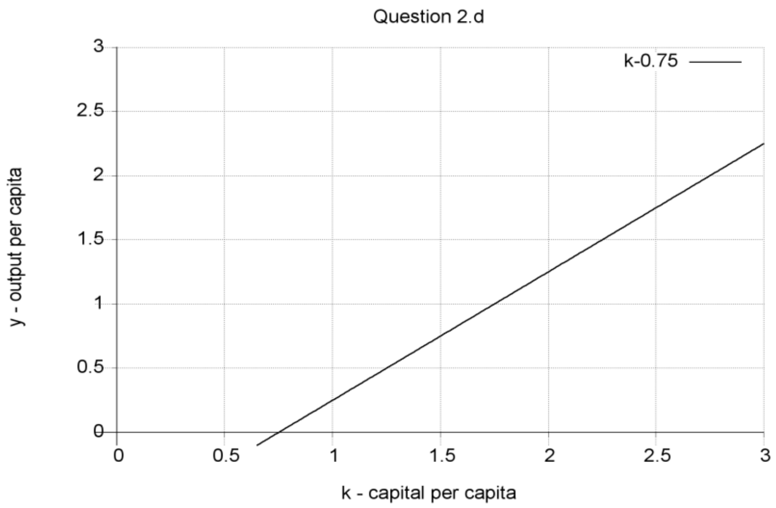
c.

$$
\begin{aligned}
Y &= K + \bar{A}L \\
\Longrightarrow \frac{Y}{L} &= \frac{K}{L} + \bar{A} \\
\Longrightarrow y &= k + \bar{A} \qquad (6)
\end{aligned}
$$



d.

$$
\begin{aligned}
Y &= K - \bar{A}L \\
\Longrightarrow \frac{Y}{L} &= \frac{K}{L} - \bar{A} \\
\Longrightarrow y &= k - \bar{A} \qquad (7)
\end{aligned}
$$



Question 3.

a. For this question, just divide the values in columns 1 by that of US value - 141841 and divide the values in column 2 by the per capita GDP of US (or, 51958). So we get:

	In 2011 USD		Relative to the US values (US = 1)						
	1	2	3	4					
Country	**Capital per capita**	**Per capita GDP**	**Re-scaled capital per capita, k**	**Re-scaled Per capita GDP, y**					
US	*141841*	*51958*	*1.00*	*1.00*					
South Korea	*120472*	*34961*	*0.85*	*0.67*					
Indonesia	*41044*	*9797*	*0.29*	*0.19*					
Ethopia	*3227*	*1505*	*0.02*	*0.03*					

b. For part b, using a simple model in which $A = 1$ or $y = \bar{A}k^{\frac{1}{3}} = k^{\frac{1}{3}}$, we find the outputs for each country based on the capital per capita it has.

Comment: *Or think of it this way: we are given the amount of capital per capita for a country and we have learned some modelling in economics. Naively, we think that all economies have the same level of technology. So we just plug in the value for k in $y = k^{\frac{1}{3}}$. And get the predicted output, $y^{\star}$ (where the $\star$ denotes "predicted").*

	In 2011 USD		Relative to the US values (US = 1)						
	1	2	3	4	5				
Country	**Capital per capita**	**Per capita GDP**	**Re-scaled capital per capita, k**	**Re-scaled Per capita GDP, y**	**Using a simple model where A=1, predicted $y^{\star} = Ak^{\frac{1}{3}} = k^{\frac{1}{3}}$**				
US	*141841*	*51958*	*1.00*	*1.00*	*1.00*				
South Korea	*120472*	*34961*	*0.85*	*0.67*	*0.95*				
Indonesia	*41044*	*9797*	*0.29*	*0.19*	*0.66*				
Ethopia	*3227*	*1505*	*0.02*	*0.03*	*0.28*				

c. We were so naïve in part b, thinking that A=1 for all countries. That is why if we look at column 4 (recording the actual output) and column 5 (recording the predicted output based on our naïve model/assumption), the values are different. This means that our naïve model does not explain all the differences in output per capital between US and other economies. So now, we can relax the assumption that A=1 for all countries, to say that the TFPs (A) are different for different countries. That is:

$$y_{us} = A_{us}k_{us}^{\frac{1}{3}} = A_{us}y_{us}^{\star}$$

$$y_{KR} = A_{KR}k_{KR}^{\frac{1}{3}} = A_{KR}y_{KR}^{\star} \qquad (KR \text{ stands for South Korea})$$

$$y_{ind} = A_{ind}k_{ind}^{\frac{1}{3}} = A_{ind}y_{ind}^{\star} \qquad (ind \text{ stands for Indonesia})$$

$$y_{et} = A_{et}k_{et}^{\frac{1}{3}} = A_{et}y_{et}^{\star} \qquad (et \text{ stands for Ethopia})$$

So to find A of a country, we take y_{KR} divided by $k_{KR}^{\frac{1}{3}}$ or column 4 divided by column 5.

	In 2011 USD		Relative to the US values (US = 1)						
	1	2	3	4	5	6			
Country	**Capital per capita**	**Per capita GDP**	**Re-scaled capital per capita, k**	**Re-scaled Per capita GDP, y**	**Using a simple model where A=1, predicted $y^{\star} = Ak^{\frac{1}{3}} = k^{\frac{1}{3}}$**	**Implied TFP to match data, A (or what explains the difference between column 4 and 5)**			
US	*141841*	*51958*	*1.00*	*1.00*	*1.00*	*1.00*			
South Korea	*120472*	*34961*	*0.85*	*0.67*	*0.95*	*0.71*			
Indonesia	*41044*	*9797*	*0.29*	*0.19*	*0.66*	*0.29*			
Ethopia	*3227*	*1505*	*0.02*	*0.03*	*0.28*	*0.10*			

d. The reason why we take normalized US GDP per capita divided by those of other countries in column 4 is to see how many times US is richer than other countries. That is, we take 1 divided by the other numbers in column 4.

The reason why we take *predicted* US GDP per capita divided by those of other countries in column 5 is to see how many times US is richer than other countries under our *naïve* assumption that A=1 for all economies. That is, we take 1 divided by the other numbers in column 5.

So what we have done so far is:

$$\frac{y_{us}}{y_{KR}}, \frac{y_{us}}{y_{ind}} \; and \; \frac{y_{us}}{y_{et}} \qquad \text{for column 4.}$$

and

$$\frac{k_{us}^{\frac{1}{3}}}{k_{KR}^{\frac{1}{3}}}, \frac{k_{us}^{\frac{1}{3}}}{k_{ind}^{\frac{1}{3}}} \text{ and } \frac{k_{us}^{\frac{1}{3}}}{k_{et}^{\frac{1}{3}}} \quad \text{or in other words,} \quad \frac{y_{us}^{\star}}{y_{KR}^{\star}}, \frac{y_{us}^{\star}}{y_{ind}^{\star}} \text{ and } \frac{y_{us}^{\star}}{y_{et}^{\star}} \text{ for column 5.}$$

Of course column 7 and column 8 will not agree since 7 is the actual and 8 is based on a model with an assumption A=1 for all economies. So what is the difference? In order to see that, let's work on one case:

$$\frac{y_{us}}{y_{KR}} = \frac{A_{us}k_{us}^{\frac{1}{3}}}{A_{KR}k_{KR}^{\frac{1}{3}}} = \frac{A_{us}}{A_{KR}}\frac{k_{us}^{\frac{1}{3}}}{k_{KR}^{\frac{1}{3}}} = \frac{A_{us}}{A_{KR}}\left(\frac{k_{us}}{k_{KR}}\right)^{\frac{1}{3}} = \frac{A_{us}}{A_{KR}}\frac{y_{us}^{\star}}{y_{KR}^{\star}}$$

Note that $\frac{y_{us}}{y_{KR}}$ is in the column 7 while $\frac{y_{us}^{\star}}{y_{KR}^{\star}}$ is in column 8. So the difference between column 7 and 8 is $\frac{A_{us}}{A_{KR}}$. Since A_{us} is normalized to 1 so column 9 is basically equal to 1 divided by the values in column 6. Alternatively, you can find $\frac{A_{us}}{A_{KR}}$ by taking column 7 divided by column 8.

Thus we have the following table:

	In 2011 USD		Relative to the US values (US = 1)						
	1	2	3	4	5	6	7	8	9
Country	**Capital per capita**	**Per capita GDP**	**Re-scaled capital per capita, k**	**Re-scaled Per capita GDP, y**	**Using a simple model where A=1, predicted $y^{\star} = Ak^{\frac{1}{3}} = k^{\frac{1}{3}}$**	**Implied TFP to match data, $A^{\star}$ (or what explains the difference between column 4 and 5)**	**How many times US is actually richer than this country**	**How many times US is richer than this country, using predicted values in column 5?**	**How much does the TFP explains for the difference between column 7 and 8?**
US	*141841*	*51958*	*1.00*	*1.00*	*1.00*	*1.00*	*1.00*	*1.00*	*1.00*
South Korea	*120472*	*34961*	*0.85*	*0.67*	*0.95*	*0.71*	*1.49*	*1.06*	*1.41*
Indonesia	*41044*	*9797*	*0.29*	*0.19*	*0.66*	*0.29*	*5.30*	*1.51*	*3.51*
Ethopia	*3227*	*1505*	*0.02*	*0.03*	*0.28*	*0.10*	*34.52*	*3.53*	*9.78*

In 2011, per capita GDP of US was about 34.52 times bigger than that of Ethopia. Assuming A=1 to take into account only the capital per capita difference, we show that difference in capital per person explain a factor of about 3.53 of this difference (column 8, last row). That means that the difference in TFP explains a factor of 9.78 (column 9, last row). Mathematically, this can be expressed as:

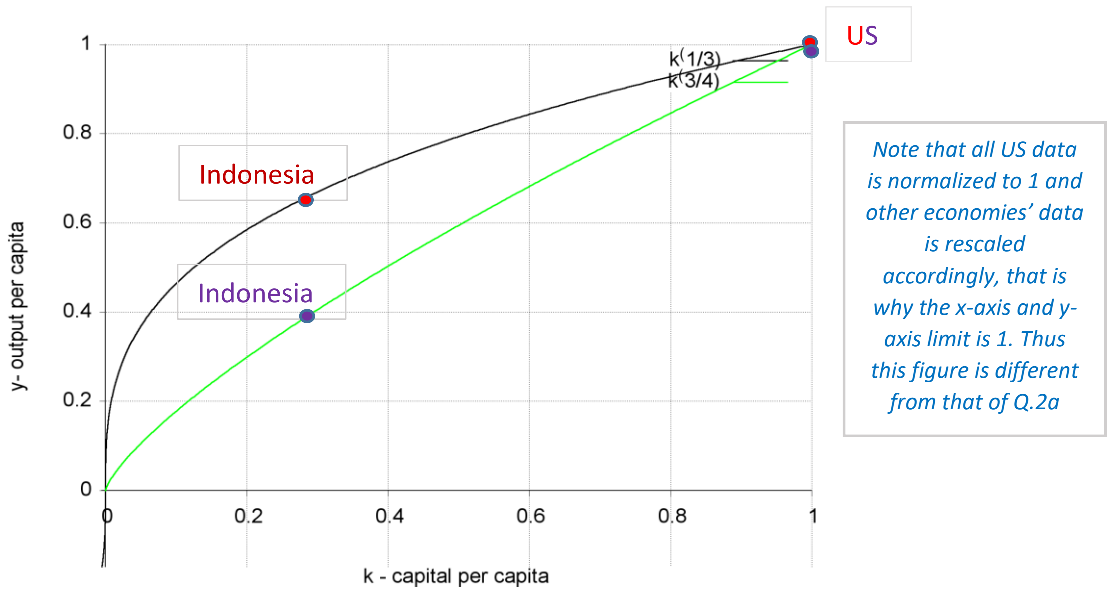
$$\frac{y_{us}}{y_{et}} = \frac{A_{us}}{A_{et}} \times \left(\frac{k_{us}}{k_{et}}\right)^{\frac{1}{3}}$$

$$34.52 = 9.78 \times 3.53$$

Since TFP is roughly 3 times (9.78/3.53) as importance as the capital per person in this accounting exercise (for US and Ethopia), it can be said that TFP explains three quarters of the differences in per capita GDP, while capital per person explains about one quarter.

e. The differences in TFP (in economic literature, it is also called by different terms such as “Technology”, “idea” or “residual”) exert a larger impact than differences in capital per person in driving differences in income per person. The differences in capital per capita, $\frac{K}{L}$, between US and other economies are big but in our model, capital per person runs into diminishing returns quickly (see part f for further explanation), so the difference in capital per capita does not matter that much.

f. With higher exponent on k in $y = Ak^{\alpha}$, the production curve is less curved or in other words, diminishing return on capital is lower. We can observe in the following figure:



In the figure above, comparing the green curve and black curve we can see that, the difference in outputs between US and Indonesia is explained **more** by the difference in capital per capita levels between the two countries using the green curve than the black one. This is because of lower diminishing return effect of the green curve.
Thus with the change in the capital exponent, implied total factor productivity (TFP) difference between US and other countries should be smaller.

Comment: *Please note that $\alpha = \frac{1}{3}$ or $\frac{3}{4}$ is just assumptions, not reality. To know what α is researchers need to empirically calibrate using the real data. What we are discussing here is the effect of the exponent α (capturing the degree of diminishing returns) on the implied TFP.*

Mathematically, we can show:

$$\underbrace{\frac{y_{us}}{y_{ethiopia}}}_{\text{unchanged, this is data}} = \underbrace{\frac{A_{us}}{A_{ethiopia}}}_{\downarrow \text{as } \alpha \downarrow} \times \underbrace{\left(\frac{k_{us}}{k_{ethipia}}\right)^{\alpha}}_{\uparrow \text{as } \alpha \uparrow}$$

So, rework on the table for part b-d with $\alpha = \frac{3}{4}$, we have:

	In 2011 USD		Relative to the US values (US = 1)						
	1	2	3	4	5	6	7	8	9
Country	**Capital per capita**	**Per capita GDP**	**Re-scaled capital per capita, $\boldsymbol{k}$**	**Re-scaled Per capita GDP, $\boldsymbol{y}$**	**Using a simple model where A=1, predicted $y^* = Ak^{\frac{3}{4}} = k^{\frac{3}{4}}$**	**Implied TFP to match data, A^* (or what explains the difference between column 4 and 5)**	**How many times US is actually richer than this country**	**How many times US is richer than this country, using predicted values in column 5?**	**How much does the TFP explains for the difference between column 7 and 8?**
US	*141841*	*51958*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
South Korea	*120472*	*34961*	0.85	0.67	0.88	0.76	1.49	1.13	1.31
Indonesia	*41044*	*9797*	0.29	0.19	0.39	0.48	5.30	2.53	2.09
Ethopia	*3227*	*1505*	0.02	0.03	0.06	0.49	34.52	17.07	2.02

Indeed, look at the column 9, the numbers are smaller, meaning that the difference in TFP between the respective economy with US, according to the new assumption ($\alpha = \frac{3}{4}$) of the model, is smaller.

Comment: *This is interesting because if we assume α wrongly, we might end up with a very different conclusion concerning the technology level or productivity difference across the countries. Clearly we can see that US is much more advanced as compared to that of Ethiopia. So is TFP ratio $\frac{A_{us}}{A_{ethiopia}} = 2.02$ in the last row of column 9 justifiable? I guess not. Deductively, we can say that the assumption $\alpha = \frac{3}{4}$ is not reasonable as well. That is why we need data to support our assumption. So researchers will need to use real data to calibrate α.*

Question 4:

a.

Comment: *when we write X(t), this means that X is changing with time. Mathematically, we say that X is a function of time, t.*
Were you curious why $\frac{\dot{X}(t)}{X(t)}$ is the growth rate of x (g_x)? If you were, very good. But no worries if you have not figured out. Here is why.

We have:

$$\frac{\dot{X}(t)}{X(t)} = \frac{dX(t)}{dt} \times \frac{1}{X(t)}$$

$$= \frac{dX(t)}{X(t)} \times \frac{1}{dt} \quad (6)$$

$$= \frac{\left(\frac{dX(t)}{X(t)}\right)}{dt} \quad (7)$$

Note that in calculus, $\mathrm{dX(t)}$ is a "**continuous** change" in $\mathrm{X(t)}$. Whereas, we use Δ to denote a **discrete** change of something. Thus we can say that:

$$\mathrm{dX(t)} = \Delta\mathrm{X(t)} \quad \text{for } \textbf{infinitesimal small} \text{ change in } \mathrm{X(t)}$$

So if we can interpret $\frac{\mathrm{dX(t)}}{\mathrm{X(t)}}$ as the *percentage change* in $\mathrm{X(t)}$.

And therefore the meaning of

$$\frac{\left(\frac{\mathrm{dX(t)}}{\mathrm{X(t)}}\right)}{\mathrm{dt}}$$

is the percentage change of $\mathrm{X(t)}$ in an infinitesimal small change in time, t; thus, it is the growth rate of X. (*Note: Growth rate of something is the percentage change of that over a period of time*).

Part b.

i. First take $\ln(\quad)$ on both sides:

$$\ln\ Z(t) = \ln X(t) + \ln \mathrm{Y}(t)$$

Differentiate both sides with respect to time, t.

$$\begin{aligned}\frac{d\ln Z(t)}{dt} &= \frac{d\ln X(t)}{dt} + \frac{d\ln Y(t)}{dt}\\ &= \frac{d\ln X(t)}{dX(t)}\cdot\frac{dX(t)}{dt} + \frac{d\ln Y(t)}{dY(t)}\cdot\frac{dY(t)}{dt}\\ &= \frac{1}{X(t)}\cdot\dot{X}(t) + \frac{1}{Y(t)}\cdot\dot{Y}(t)\\ &= \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}\end{aligned}$$

$$\Longrightarrow g_z = g_X + g_Y$$

ii. Similarly,

$$\ln\ Z(t) = \ln\left(\frac{\mathrm{X(t)}}{\mathrm{Y(t)}}\right) = \ln X(t) - \ln \mathrm{Y}(t)$$

Differentiate both sides with respect to time, t.

$$\begin{aligned}\frac{d\ln Z(t)}{dt} &= \frac{d\ln X(t)}{dt} - \frac{d\ln Y(t)}{dt}\\ &= \frac{d\ln X(t)}{dX(t)}\cdot\frac{dX(t)}{dt} - \frac{d\ln Y(t)}{dY(t)}\cdot\frac{dY(t)}{dt}\\ &= \frac{1}{X(t)}\cdot\dot{X}(t) - \frac{1}{Y(t)}\cdot\dot{Y}(t)\\ &= \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}\end{aligned}$$

$$\Longrightarrow g_z = g_X - g_Y$$

Comment: *This result is, to me, very important for growth topic. Suppose that both output $Y(t)$ and capital level $K(t)$ is growing in an economy. The former grows at g_Y while the latter at g_K. So the growth rate of $\frac{Y(t)}{K(t)}$ is:*

$$g_{\frac{Y}{K}} = g_Y - g_K$$

In the long run, it should be that $g_Y = g_K$ because only then $\frac{Y(t)}{K(t)}$ is a constant (or not growing). If not, let's see what happen.

$$\begin{cases} \text{if } g_Y > g_K \text{ then } \dfrac{Y(t)}{K(t)} \text{ is forever increasing} \rightarrow \text{Y(t)} \; will\ be\ one\ day\ infinitely\ bigger\ than\ \text{K(t)} \\ \text{if } g_K > g_Y \text{ then } \dfrac{Y(t)}{K(t)} \text{ is forever decreasing} \rightarrow \text{K(t)} \; will\ be\ one\ day\ infinitely\ bigger\ than\ \text{Y(t)} \end{cases}$$

This kind of model for an economy is not realistic and unstable.

iii. First take $\ln(\quad)$ on both sides,

$$\ln\ Z(t) = \ln\left(\mathrm{X(t)}^{\alpha}\right) = \alpha \ln X(t)$$

Differentiate both sides with respect to time, t:

$$\begin{aligned} \frac{d \ln Z(t)}{dt} &= \alpha \frac{d \ln X(t)}{dt} \\ &= \alpha \frac{d \ln X(t)}{dX(t)} \cdot \frac{dX(t)}{dt} \\ &= \alpha \frac{\dot{X}(t)}{X(t)} \end{aligned}$$

$$\Longrightarrow g_z = \alpha g_X$$