NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 6**

1. (a) Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$. Show that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ forms a basis for $\mathbb{R}^3$.

   (b) Suppose $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the coordinate vector of $\mathbf{w}$ relative to S.

   (c) Let $T = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be another basis for $\mathbb{R}^3$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$. Find the transition matrix from T to S.

   (d) Find the transition matrix from S to T.

   (e) Use the vector $\mathbf{w}$ in Part (b). Find the coordinate vector of $\mathbf{w}$ relative to T.

2. Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for a subspace V. Define $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3,\ \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3 \text{ and } \mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3.$$

   (a) Show that T is a basis for V.

   (b) Find the transition matrix from S to T.

3. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Is $\mathbf{b}$ in the column space of $\mathbf{A}$? If it is, express it as a linear combination of the columns of $\mathbf{A}$.

   (b) Let $\mathbf{A} = \begin{pmatrix} 1 & 9 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5, 1, -1 \end{pmatrix}$. Is $\mathbf{b}$ in the row space of $\mathbf{A}$? If it is, express it as a linear combination of the rows of $\mathbf{A}$.

   (c) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}$. Is the row space and column space of $\mathbf{A}$ the whole $\mathbb{R}^4$?

4. For each of the following matrices $\mathbf{A}$,

   (i) Find a basis for the row space of $\mathbf{A}$.

(ii) Find a basis for the column space of $\mathbf{A}$.

(iii) Find a basis for the nullspace of $\mathbf{A}$.

(iv) Hence determine rank($\mathbf{A}$), nullity($\mathbf{A}$) and verify the dimension theorem for matrices.

(v) Is $\mathbf{A}$ full rank?

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix}.$

5. Let W be a subspace of $\mathbb{R}^5$ spanned by the following vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 \begin{pmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 5 \\ 15 \\ 10 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 2 \\ 1 \\ 15 \\ 8 \\ 6 \end{pmatrix}.$$

(a) Find a basis for W.

(b) What is $\dim(W)$?

(c) Extend the basis W found in (a) to a basis for $\mathbb{R}^5$.

6. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 5 \\ 12 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ 4 \end{pmatrix} \right\}$ and $V = \operatorname{span}(S)$. Find a subset $S' \subseteq S$ such that S' forms a basis for V.

Extra problems

1. Suppose $\mathbf{A}$ and $\mathbf{B}$ are two matrices such that $\mathbf{AB} = \mathbf{0}$. Show that the column space of $\mathbf{B}$ is contained in the nullspace of $\mathbf{A}$.

2. Let $\mathbf{A}$ be a $n \times m$ matrix and $\mathbf{P}$ an $n \times n$ matrix.

   (a) If $\mathbf{P}$ is invertible, show that $\text{rank}(\mathbf{PA}) = \text{rank}(\mathbf{A})$.

   (b) Given an example such that $\text{rank}(\mathbf{PA}) < \text{rank}(\mathbf{A})$.

   (c) If $\text{rank}(\mathbf{PA}) = \text{rank}(\mathbf{A})$. Can we conclude that $\mathbf{P}$ is invertible? Justify your answer.

3. Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$ be row equivalent $m \times n$ matrices, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the i-th column of $\mathbf{A}$ and $\mathbf{B}$, respectively, for $i = 1, ..., n$. Show that for any $c_1, c_2, ..., c_n \in \mathbb{R}$,
$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$
if and only if
$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}.$$

4. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{u}$ be a vector in V and let c be a scalar. Prove the following:

   (a) $[\mathbf{u} + \mathbf{v}]_S = [\mathbf{u}]_S + [\mathbf{v}]_S$.

   (b) $[c\mathbf{u}]_S = c\,[\mathbf{u}]_S$.

   (c) Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are vectors in V. Note that for each $i = 1, 2, \ldots, k$, $[\mathbf{u}_i]_S$ is a vector in $\mathbb{R}^n$. By induction and using (a) and (b), it follows that if $c_1, c_2, \ldots, c_k \in \mathbb{R}$, then
$$[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k]_S = c_1\,[\mathbf{u}_1]_S + c_2\,[\mathbf{u}_2]_S + \ldots + c_k\,[\mathbf{u}_k]_S\,.$$
   Prove that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if $\{[\mathbf{u}_1]_S, [\mathbf{u}_2]_S, \ldots, [\mathbf{u}_k]_S\}$ is linearly independent in $\mathbb{R}^n$.