MA1522 Linear Algebra for computing
Chapter 1b
Linear Equations in Linear Algebra

August 21, 2023

1.4 Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition

Let A be an m by n matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.
If $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$ is

$$A\mathbf{x} = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

Remark. The product $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

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Example 1

For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^n$, write the linear combination

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$$

as a matrix times a vector.

Solution

Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5 and 7 into a vector $\mathbf{x}$.

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} = A\mathbf{x}. \quad \square$$

We write the system of linear equations as a vector equation involving a linear combination of vectors.

Example. The system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 4 \\ & - & 5x_2 & + & 3x_3 & = & 1 \end{array}$$

is equivalent to

$$x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

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The left side the last equation is a matrix times a vector, so the equation becomes

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \tag{*}$$

Equation (*) above has the form

$$A\mathbf{x} = \mathbf{b}.$$

Such an equation is called a **matrix equation**.

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Theorem

Let A be an m by n matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and let $\mathbf{b}$ be in $\mathbb{R}^m$. The following solution sets are the **same**.

(i) *The solution set of the matrix equation $A\mathbf{x} = \mathbf{b}$.*

(ii) *The solution set of the vector equation*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}.$$

(iii) *The solution set of the system of linear equations whose augmented matrix is*

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \\ | & | & \cdots & | & | \end{array}\right).$$

Sketch of proof

The left hand sides of (i) and (ii) are the same because

$$A\mathbf{x} = x_1\mathbf{a}_1 \ + \ x_2\mathbf{x}_2 \ + \ \ldots \ + \ x_n\mathbf{a}_n.$$

Hence (i) and (ii) are equivalent.

We have proved in the previous sections that the solution sets of (i) and (iii) are the same.

Try filling in the details. □

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Try filling in the details. $\square$

Existence of solutions

The next theorem addresses whether the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Theorem

Let A be an m by n matrix.

Then the following statements are logically equivalent.

(a) *For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*

(b) *Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*

(c) *The columns of A span $\mathbb{R}^m$.*

(d) *The matrix A has a pivot position in every row.*

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(b) *Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*

(c) *The columns of A span $\mathbb{R}^m$.*

(d) *The matrix A has a pivot position in every row.*

Remark

We explain an potential misunderstanding of the theorem.

Suppose we arbitrarily choose and fix an m by n matrix A.

(I) For certain n by 1 column vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ <u>has</u> a solution.

(II) For other n by 1 column vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ <u>does not</u> have a solution.

Part (a) says that no matter what $\mathbf{b}$ we pick, Scenario (II) does not happen. This is a very strong requirement on the matrix A.

Proof

Statements (a), (b), and (c) are logically equivalent. (Why?)

Hence it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or both false.

Let U be an echelon form of A.

Given $\mathbf{b}$ in $\mathbb{R}^n$, we can row reduce the augmented matrix $(A \mid \mathbf{b})$ to an augmented matrix $(U \mid \mathbf{d})$ for some $\mathbf{d}$ in $\mathbb{R}^n$:

$$(A \mid \mathbf{b}) \sim \sim \ \cdots \ \sim (U \mid \mathbf{d})$$

Proof of (d) implies (a).

If statement (d) is true, then each row of U contains a pivot position.

Then $(U \mid \mathbf{d})$ does not contain a row of the form

$$(0\,0\,\ldots\,0\,0 \mid b)$$

for some nonzero b.

By a previous theorem, the linear system with augmented matrix $(A \mid \mathbf{b})$ is consistent, i.e. it has a solution.

Hence the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true.

Proof of (a) implies (d)

We will show that if (d) is false, then (a) is false.

Suppose (d) is false. Then the last row of U is all zeros.

Let $\mathbf{d} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ be a vector with a 1 in its last entry.

Proof of (a) implies (d)

We will show that if (d) is false, then (a) is false.

Suppose (d) is false. Then the last row of U is all zeros.

Let $\mathbf{d} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ be a vector with a 1 in its last entry.

Then

$$(U \mid \mathbf{d}) = \left(\begin{array}{cccc|c} * & * & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

represents an inconsistent system, i.e. it does not have a solution.

Since row operations are reversible, $(U \mid \mathbf{d})$ can be transformed into the form $(A \mid \mathbf{b})$.

For this particular $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false.

(Here we need to use the fact in (a) that we are allowed to choose any $\mathbf{b}$ in $A\mathbf{x} = \mathbf{b}$.)

Example 2

Compute $A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Solution

From the definition,

$$\begin{pmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{pmatrix} + \begin{pmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{pmatrix}.$$

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Remark

The first entry in $A\mathbf{x}$ is the dot product of the first row of A and the vector $\mathbf{x}$.

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The second entry in $A\mathbf{x}$ is the dot product of the second row of A and the vector $\mathbf{x}$.

$$\begin{pmatrix} & & \\ -1 & 5 & -3 \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \end{pmatrix}.$$

Likewise, the third entry in $A\mathbf{x}$ is the dot product of the third row of A and the vector $\mathbf{x}$.

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Likewise, the third entry in $A\mathbf{x}$ is the dot product of the third row of A and the vector $\mathbf{x}$.

More generally, for an m by n matrix A and a column vector $\mathbf{x}$ in $\mathbb{R}^n$,

$$A\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

is a column vector in $\mathbb{R}^m$.

The i-th entry c_i in $A\mathbf{x}$ is the dot product of the i-th row of A and the vector $\mathbf{x}$.

Identity matrix

The matrix with 1's on the diagonal and 0's elsewhere is called an *identity matrix* and is denoted by I.

Example. The 3 by 3 identity matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem

If A is an m by n matrix, $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^n$*, and c is a scalar, then*

1. (i) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$,
2. (ii) $A(c\mathbf{x}) = c(A\mathbf{x})$.

Proof

We will prove the special case when A is a 3 by 3 matrix.

The proof for the general case is similar.

We write

$$A = \begin{pmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{pmatrix} \text{ and } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

in $\mathbb{R}^3$.

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(i)

$$
\begin{aligned}
A(\mathbf{u}+\mathbf{v}) &= \begin{pmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} \\
&= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + (u_3+v_3)\mathbf{a}_3 \\
&= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\
&= A\mathbf{u} + A\mathbf{v}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
A(c\mathbf{u}) &= \begin{pmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix} \\
&= (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 \\
&= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3) \\
&= c(A\mathbf{u}). \quad \square
\end{aligned}
$$

1.5 Solution Sets of Linear Systems

Definition

A system of linear equations is said to be *homogeneous* if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

Here A is an m by n matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Remarks

1. The system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$).
   This zero solution is usually called the *trivial solution*.

2. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a *nontrivial* solution (i.e. a nonzero solution) if and only if the equation has at least one free variable. (See Example 1 below.)

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2. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a *nontrivial* solution (i.e. a nonzero solution) if and only if the equation has at least one free variable. (See Example 1 below.)

Example 1

Determine if the following homogeneous system has a nontrivial solution.

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0. \end{array}$$

Describe the solution set.

Solution

We could write the linear system as $A\mathbf{x} = \mathbf{0}$.

The augmented matrix is

$$(A \mid \mathbf{0}) = \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right).$$

We row reduce the augmented matrix to echelon form:

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

In particular it has a nontrivial (i.e. nonzero) solution.

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Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

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We further reduce to the reduced echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Leftrightarrow \quad \begin{array}{rcl} x_1 - \frac{4}{3}x_3 & = & 0 \\ x_2 & = & 0 \\ 0 & = & 0 \end{array}.$$

Solving for the basic variables x_1 and x_2, we get

$$x_1 = \frac{4}{3}x_3, \;\; x_2 = 0 \;\text{ and }\; x_3 \text{ free.}$$

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$$x_1 = \frac{4}{3}x_3, \;\; x_2 = 0 \;\text{ and }\; x_3 \text{ free.}$$

As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form given below.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} = x_3 \mathbf{v}$$

where

$$\mathbf{v} = \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}. \quad \square$$

Remarks

1. Here x_3 is factored out of the expression for the general solution vector.
2. Every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v}$.
3. The trivial solution $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is obtained by choosing $x_3 = 0$.

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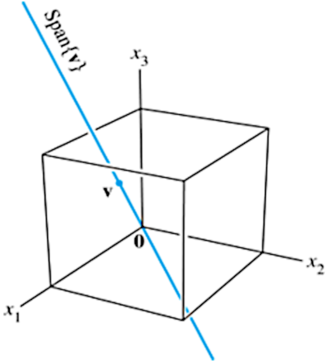
4. Geometrically, the solution set is a line through $\mathbf{0}$ in $\mathbb{R}^3$.



The equation

$$\mathbf{x} = t\mathbf{v} \quad \text{for } t \text{ in } \mathbb{R} \tag{*}$$

is a *parametric vector equation of a line*.

Here we had substituted x_3 with t.

The Equation (*) is the solution of Example 1.

We say that solution (*) is in *parametric vector form*.

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Parmetric vector form

The equation of the form

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad \text{for } s, t \text{ in } \mathbb{R}$$

is called a *parametric vector equation of a plane*.

Solutions of non-homogeneous systems

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

We will see how this works in the next example.

Example 2

Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}.$$

Remark. The matrix A here is the same matrix appearing in Example 1.

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Remark. The matrix A here is the same matrix appearing in Example 1.

Solution

We apply row operations on $(A \ \ \mathbf{b})$ to produce

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{array}{rcrcr} x_1 & - & \frac{4}{3}x_3 & = & -1 \\ & & x_2 & = & 2 \\ & & 0 & = & 0 \end{array}.$$

Thus

$$x_1 = -1 + \frac{4}{3}x_3, \quad x_2 = 2 \quad \text{and} \quad x_3 \text{ is free.}$$

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As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form given below.

$$\begin{aligned}
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} \\
&= \mathbf{p} + x_3 \mathbf{v}
\end{aligned}$$

where $\mathbf{p} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$.

We verify that the solution is correct:

$$A\left(\begin{pmatrix}-1\\2\\0\end{pmatrix}+x_3\begin{pmatrix}\frac{4}{3}\\0\\1\end{pmatrix}\right)=A\begin{pmatrix}-1\\2\\0\end{pmatrix}+x_3A\begin{pmatrix}\frac{4}{3}\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}3&5&-4\\-3&-2&4\\6&1&-8\end{pmatrix}\begin{pmatrix}-1\\2\\0\end{pmatrix}+x_3\begin{pmatrix}3&5&-4\\-3&-2&4\\6&1&-8\end{pmatrix}\begin{pmatrix}\frac{4}{3}\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}7\\-1\\-4\end{pmatrix}+x_3\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}7\\-1\\-4\end{pmatrix}. \quad \square$$

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Remarks

1. The solution set of $A\mathbf{x} = \mathbf{b}$ in Example 2 has the parametric vector equation
$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}). \tag{1}$$
2. The solution set of $A\mathbf{x} = \mathbf{0}$ in Example 1 has the parametric vector equation
$$\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{2}$$
(with the same $\mathbf{v}$ that appears in Equation (1)).

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3. **Important conclusion.**

$$\boxed{\begin{array}{c}(\text{Solutions of } A\mathbf{x} = \mathbf{b}) \\ \text{is equal to} \\ \mathbf{p} + (\text{Solutions of } A\mathbf{x} = \mathbf{0}).\end{array}}$$

4. The vector $\mathbf{p}$ itself is one particular solution of $A\mathbf{x} = \mathbf{b}$ (corresponding to $t = 0$ in Equation (1)).

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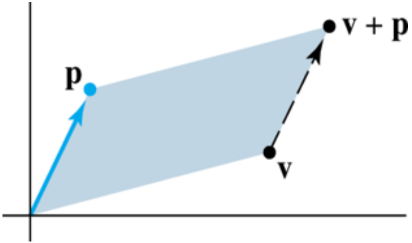
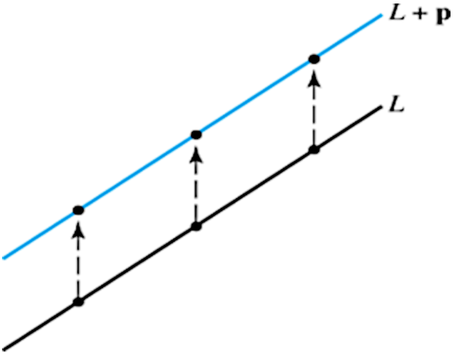
4. The vector $\mathbf{p}$ itself is one particular solution of $A\mathbf{x} = \mathbf{b}$ (corresponding to $t = 0$ in Equation (1)).

5. In order to describe the solution of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a translation.

6. Given $\mathbf{v}$ and $\mathbf{p}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to move $\mathbf{v}$ in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$.
   We say that $\mathbf{v}$ is *translated* by $\mathbf{p}$ to $\mathbf{v}+\mathbf{p}$.



7. If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector $\mathbf{p}$, the result is a line parallel to L.



- Suppose L is the line through $\mathbf{0}$ and $\mathbf{v}$, described by Equation (2).
- Adding $\mathbf{p}$ to each point on L produces the translated line described by Equation (1).
- We call Equation (1) the **equation of the line through p parallel to v**.

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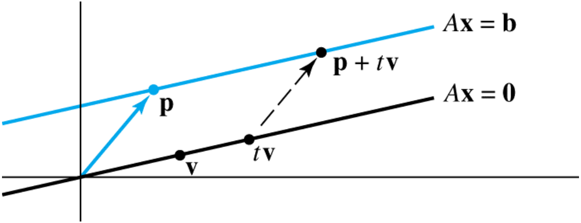
- Thus the solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.



- The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ shown in the figure above generalizes to any consistent equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables.

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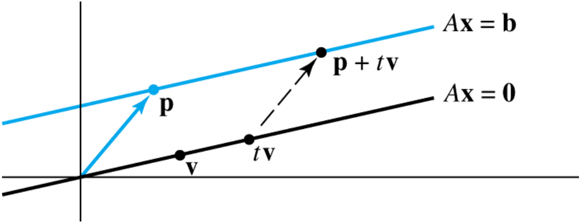

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Solutions of non-homogeneous systems

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$.

Let $\mathbf{p}$ be a solution.

Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h$$

where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

The theorem says that if $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$ by $\mathbf{p}$.

Writing a solution set in parametric vector form

Suppose we are given a consistent linear system $A\mathbf{x} = \mathbf{b}$.

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.

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3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.

4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

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