NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 10**

1. A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b. Suppose 100 ants has been placed in compartment a.

   (a) Find the transition probability matrix $\mathbf{A}$. Show that it is a stochastic matrix.

   (b) By diagonalizing $\mathbf{A}$, find the number of ants in each compartment after 3 hours.

   (c) **(MATLAB)** We can use MATLAB to diagonalize the matrix $\mathbf{A}$. Type

   ```
   >> [P D]=eig(sym(A))
   ```

   The matrix $\mathbf{P}$ will be an invertible matrix, and $\mathbf{D}$ will be a diagonal matrix. Compare the answer with what you have obtained in (b).

   (d) In the long run (assuming no ants died), where will the majority of the ants be?

   (e) Suppose initially the numbers of ants in compartments a, b and c are α, β, and γ respectively. What is the population distribution in the long run (assuming no ants died)?

2. By diagonalizing $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

3. For each of the following symmetric matrices $\mathbf{A}$, find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$.

4. (**MATLAB**) Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$.

   (a) Find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$, and compute $\mathbf{P}^T\mathbf{AP}$.

(b) We will use MATLAB to orthogonally diagonalize $\mathbf{A}$. Type

```
>> A=[1 -2 0 0;-2 1 0 0;0 0 1 -2;0 0 -2 1];

>> [P D]=eig(A);

>> sym(P), sym(D)
```

Compare the result with your answer in (a).

5. Find the SVD of the following matrices $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

   (c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

6. (**MATLAB**) Let $\mathbf{A} = \begin{pmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{pmatrix}$.

   (a) Find a SVD of $\mathbf{A}$.

   (b) In MATLAB, type

```
>> [U S V]=svd(A)
```

Compare the result with your answer in (a).

Extra problems

1. Let $\mathbf{A}$ be a stochastic matrix. Prove that $\lambda = 1$ is an eigenvalue of $\mathbf{A}$.

2. Let $\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ associated to the eigenvalue λ_1 and $\mathbf{v}_2$ an eigenvector of $\mathbf{A}^T$ associated to eigenvalue λ_2. Suppose $\lambda_1 \neq \lambda_2$. Show that v_1 and v_2 are orthogonal.

3. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that there exists an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{A}\mathbf{A}^T = \mathbf{Q}^T\mathbf{A}^T\mathbf{A}\mathbf{Q}$$