LECTURE SEVEN

The Solow Growth Model (Part 1)

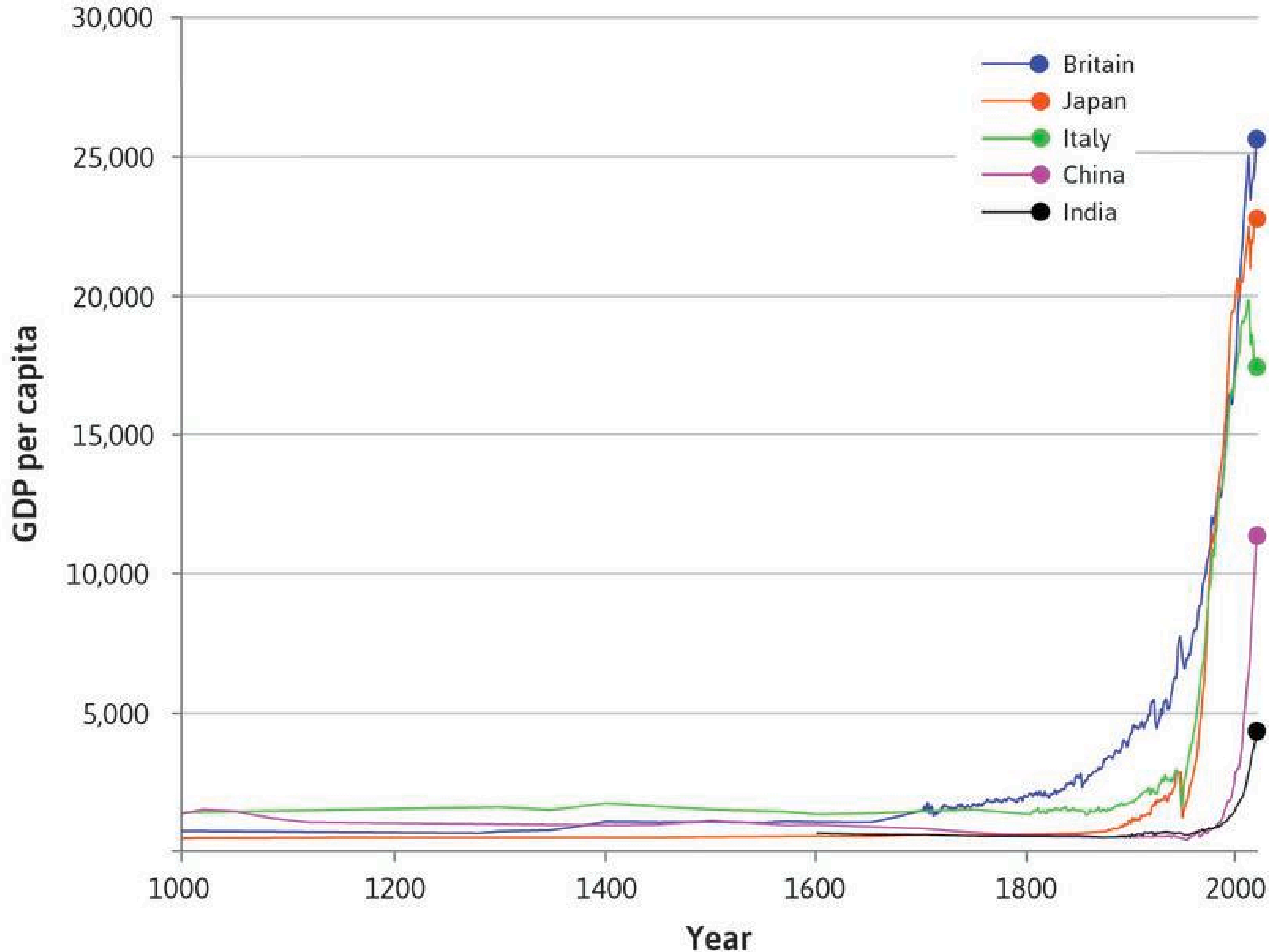
Chapter 5 of Charles Jones' text book

Outline for Lecture 5 part 1

- Introduction (*motivation* and *introduction*)
- Model set-up *(specifications and set-up)*
  - Production
  - Resources
  - Capital Accumulation *(How capital accumulates overtime)*
  - Labour *(Labour market)*
  - Investment *(where investments comes from)*
  - Summarizing the model
    - Assumptions *(recalling or summing up assumptions of the model)*
    - Equations *(how the equations come about and how they are related)*
    - Five equations & Five unknown: *bringing the equations together to see the whole model, preparing to solve.*
- Some questions about the Model *(Discussing some theoretical questions about the model – to tie the losing ends before solving)*
- Solving the model
  - Graphs (Output graph, saving graph, depreciation/break-even graph)
  - Graphical solution: *(How the economy eventually reaches steady state. #transition dynamic, #steady state)*
  - Solving mathematically for the steady state
  - Increase in A *(what happens if TFP increases: graphical analysis)*

5.1 Introduction

Why there is economic growth?



Back to outline

5.1 Introduction

- In this chapter, we learn:
  - how capital accumulates over time,
  - how diminishing MPK explains differences in growth rates across countries,
  - the principle of transition dynamics, and
  - the limitations of capital accumulation.
    - A significant part of economic growth is still unexplained.

Changes in the Model

- The Solow Growth model:
  - Augments the production model with capital accumulation
    - Capital stock is no longer exogenous: *changing with time*
    - Capital stock is now ***endogenized***.

A variable to be solved for in the model

- The accumulation of capital is a possible engine of long-run economic growth.

		South Korea	Philippines
1960	GDP	$1500	$1500
	Population	25 mil (approx.)	25 mil (approx.)
	% college (early 20)	5%	13%
2014	GDP	$35000	$6600
Average Growth		6%	2.4%

- Now, we consider the fact that accumulating capital over time could lead to economic growth.
  - In other words, ***perhaps*** some countries are richer than others because they *invest more in accumulating capital*.

5.2 Model Set-up

5.2 Model Set-up (Production)

- Begin with the previous production model.
  - Add an equation for the accumulation of capital over time.
- The production function:
  - Cobb-Douglas
  - Constant returns to scale in capital and labor
  - Assume exponent of one-third on K ($\alpha = {}^1/_3$)
- Variables are time subscripted (t)

$$Y_t = F(K_t, L_t) = \bar{A} K_t^{\alpha} L_t^{1-\alpha}$$

CRT

Model Set-up (Resources)

- IDEA:
  - How can capital grow? Ans: Investment
  - Where does investment comes from? Ans: Saving
- Output can be used for consumption or investment

$$C_t + I_t = Y_t$$

  - C_t: consumption
  - I_t: investment
- This is called a resource constraint.
  - Assumes no imports or exports

Y_t

C_t

I_t

?

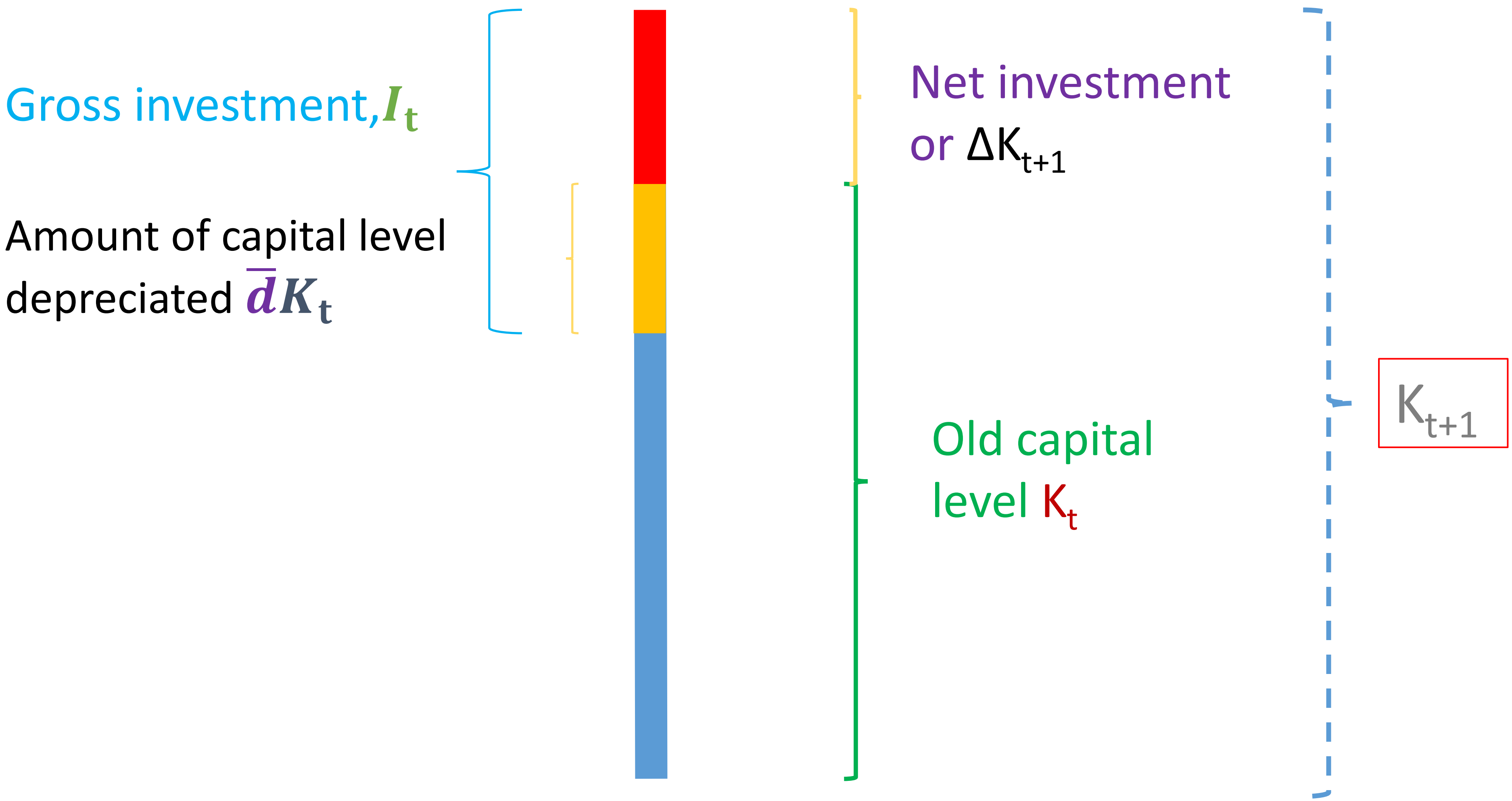
$F(K,L)$

Capital Accumulation—1

- Goods invested for the future determine the accumulation of capital
- Capital accumulation equation:

$$\boldsymbol{K_{t+1}} = \boldsymbol{K_t} + \boldsymbol{I_t} - \underbrace{\boldsymbol{\bar{d}K_t}}_{\textit{Depreciation is linear with } K_t}$$

  - $\boldsymbol{K_{t+1}}$: next year's capital
  - $\boldsymbol{K_t}$: this year's capital
  - $\boldsymbol{I_t}$: this year's investment
  - $\boldsymbol{\bar{d}}$: depreciation rate $(\boldsymbol{0 \le \bar{d} \le 1})$
    - Usually, $\bar{d} = 0.07 \ or \ 0.10$ *(empirical)*


Gross investment, I_t
Amount of capital level depreciated $\bar{d}K_t$
Net investment or ΔK_{t+1}
Old capital level K_t
K_{t+1}

Capital Accumulation—2

- Change in capital stock defined as:

$$\Delta K_{t+1} \equiv K_{t+1} - K_t$$
$$= \boldsymbol{K_t} + \boldsymbol{I_t} - \boldsymbol{\bar{d}K_t} - K_t$$

- Thus:

$$\Delta K_{t+1} = I_t - \bar{d}K_t$$

- Future capital depends on investment today

Case Study: Capital Accumulation

Recall: $K_{t+1} = K_t + I_t - \bar{d}K_t$

- Assume that the economy begins with K_0
- Suppose:
  - The initial amount of capital is 1,000 bushels of corn
  - The depreciation rate is 0.10

Time, t		Capital, K_t	Investment, I_t		Depreciation, $\bar{d}K_t$		Change in capital, ΔK_{t+1}
0		1,000	200	$= 1000 \times 0.1$	100	$= 200 - 100$	100
1	=1000+100	1,100	200	$= 1100 \times 0.1$	110	$= 200 - 110$	90
2	=1100+90	1,190	200	$= 1190 \times 0.1$	119	$= 200 - 11$	81
3		1,271	200		127		73
4		1,344	200		134		66
5		1,410	200		141		59

Model Set-up (Labor)

- For simplicity, labor supply is not included.

- The amount of labor in the economy is given exogenously at a constant level.

$$L_t = \bar{L}$$

This is an assumption. We can remodel to endogenize L.

Model Set-up (Investment)

- The economy consumes a fraction of output and invests the rest

$$I_t = \bar{s} Y_t$$

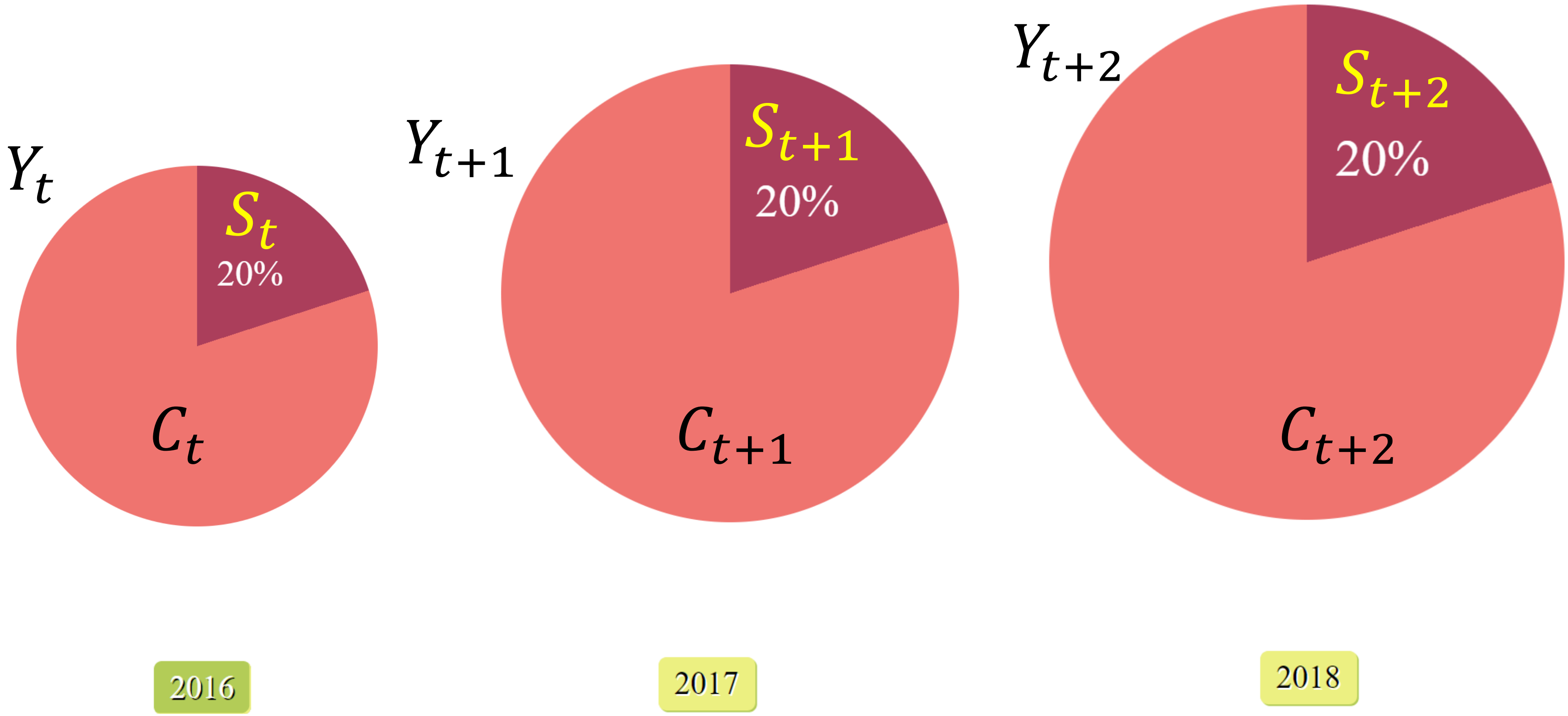
  - $\boldsymbol{I_t}$: investment
  - $\boldsymbol{\bar{s}}$: fraction of total output invested *(also exogenous)*

- Therefore:

$$C_t = (1 - \bar{s}) Y_t$$

  - Consumption is the share of output not invested

If $s = 20\%$



Summary (1): Assumptions

- Assumptions:
  - Constant Labour Force: $\bar{L}$
  - Fixed/unchanged technological level: $\underline{TFP = \bar{A}}$
  - Closed economy *(<u>No</u> import/export);* <u>No</u> government involved *(<u>no</u> taxes or government spending)*, this means: $\mathrm{Y}_t = C_t + I_t + G_t + NX_t$ is simplified to:

$$Y_t = C_t + I_t$$

  - Constant/fixed saving rate: $\bar{s}$,
    - $Savings \; = \; S = \bar{s}\mathrm{Y_t}$
    - $Consumption \; = \; (1 - \bar{s})\mathrm{Y_t}$ *(for now, this is not important)*

Back to outline

Summary (2): Equations

- Capital market clearing condition:

$$Demand\ of\ Capital = Supply\ of\ Capital$$

$$\Rightarrow I_t = S_t = \bar{s}Y_t$$

- Labour market clearing condition:

$$Demand\ of\ labour = Supply\ of\ labour$$

$$L_t = \bar{L}$$

- Capital Accumulation (Law of motion of Capital):

$$\underbrace{K_{t+1} - K_t}_{\text{Change in capital stock (Flow of new capital)}} = \underbrace{I_t - \bar{d}K_t}_{\text{Net investments: Investment after making up for depreciations}}$$

Change in capital stock
(Flow of new capital)

Net investments:
Investment after making up for depreciations

Summary (3): Equations

- Firm's production:

$$Y_t = \bar{A} K_t^{\alpha} L_t^{1-\alpha}$$

$$Output = Function\ of\ inputs$$

$$0 < \alpha \leq 1$$

- Resource constraint:

$$C_t + I_t = Y_t$$

- We solve for:

$$Y_t, K_t, L_t, C_t\ and\ I_t$$

3

This comes from household savings: $I_{t-1} = \bar{s}Y_{t-1}$

4

Resource constraint:

$S_{t-1} = Y_{t-1} - C_{t-1}$

This comes from investment minus depreciation loss:

$I_{t-1} - \bar{d}K_{t-1}$

2

Supply: Old Capital + Flow of new capital $= K_{t-1} + \Delta K_t$

Demand: *Firm demands this amount of capital!*

1

$$Y_t = \bar{A} \underbrace{K_t^{\alpha}}_{} \underbrace{L_t^{1-\alpha}}_{}$$

5

$$L_t = \bar{L}$$

$$(market\ clearing)$$

Demand: *Firm demands this amount of labour!*

Five Equations and Five Unknowns

Endogenous ∵ they are determined within the model

Unknowns/endogenous variables: Y_t, K_t, L_t, C_t, I_t

Production function	$Y_t = \bar{A}K_t^{1/3}L_t^{2/3}$
Capital accumulation	$\Delta K_{t+1} = I_t - \bar{d}K_t$
Labor force	$L_t = \bar{L}$
Resource constraint	$C_t + I_t = Y_t$
Allocation of resources	$I_t = \bar{s}Y_t$

Parameters: $\bar{A}, \bar{s}, \bar{d}, \bar{L}, \bar{K}_0$

Exogenous ∵ they are given

5.3 Some Questions about the Solow Model

5.3 Some Questions about the Solow Model(1)

- Differences between the Solow model and production model:
  - Added dynamics of capital accumulation
  - Omit capital and labor market interaction and their prices
- Why include the investment share but not the consumption share?
  - It would be redundant (we will see as we solve the model)

5.3 Some Questions about the Solow Model(2)

- Why do we not include wage and real interest rate?

  - We can, but adding them will be redundant to our objective of analysis (as you will see).

  - If we add, we would have two more equations, two more unknowns
    - w=MPL and r=MPK

5.4 Solving the Solow Model

5.4 Solving the Solow Model

- The model needs to be solved at every point in time, which cannot be done algebraically.
- Two ways to make progress:
  - Show a graphical solution
  - Solve the model in the long run
- Begin by combining equations algebraically

Solving the Solow Model

- Combine the investment allocation and capital accumulation equation

(1) $I_t = \bar{s}Y_t$

(2) $\Delta K_{t+1} = I_t - \bar{d}K_t$

$$\Delta K_{t+1} = \bar{s}\bar{A}K_t^{\alpha}\bar{L}^{1-\alpha} - \bar{d}K_t$$

$$\underbrace{\Delta K_{t+1}}_{\text{change in capital}} = \underbrace{\bar{s}Y_t - \bar{d}K_t}_{\text{net investment}}.$$

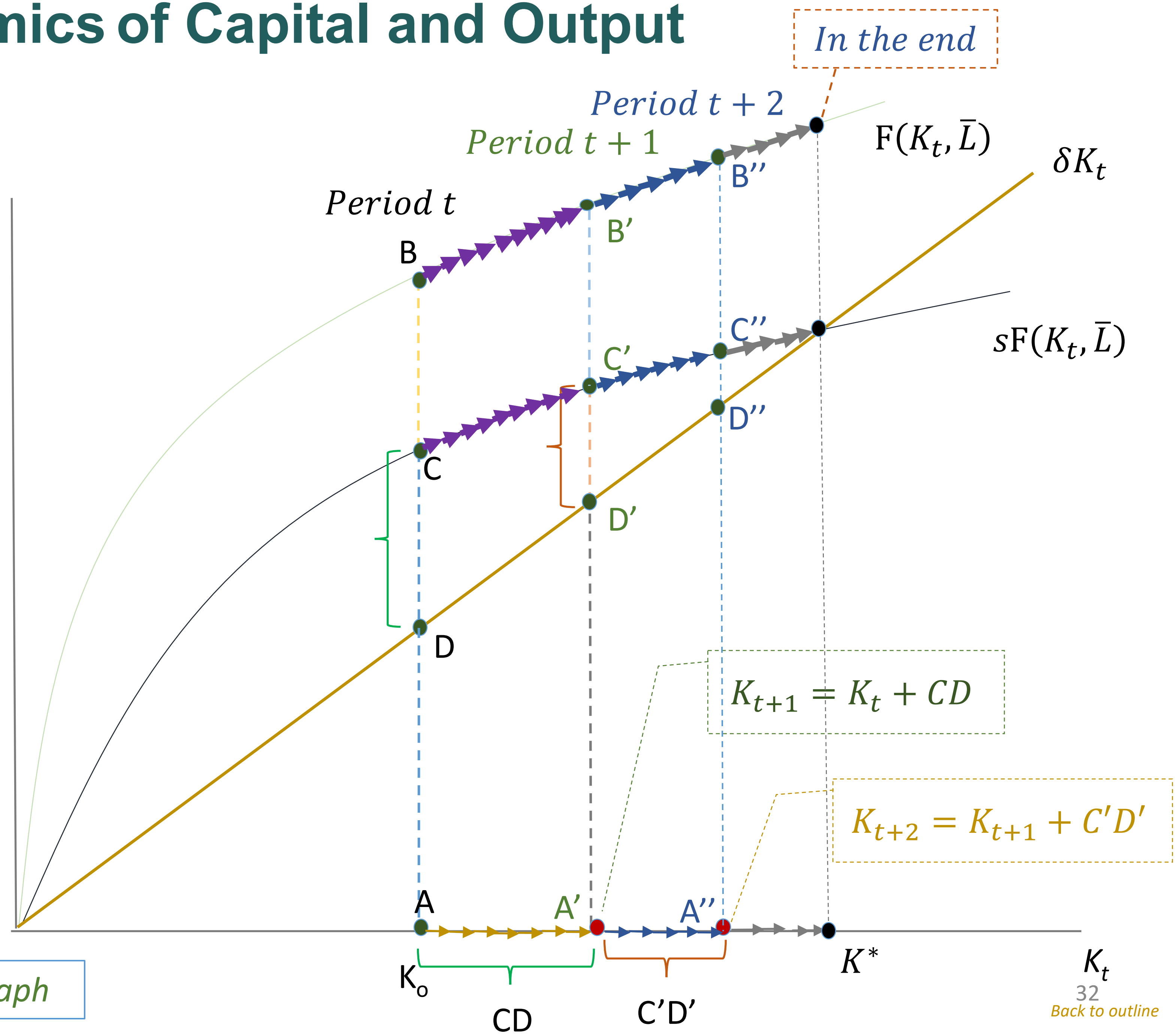
- Substitute the fixed amount of labor into the production function

$$Y_t = \bar{A}K_t^{\alpha}\bar{L}^{1-\alpha}$$

The graphs to be drawn

- There are three graphs that we should draw:
  - Output (as a function of K, $Y_t = AK_t^{\alpha}\bar{L}^{1-\alpha}$)
  - **Saving Curve**, which is also Investment curve since we assumed $S_t = I_t$ (also as a function of K, $\mathrm{S_t} = \mathrm{I}_t = \bar{s}AK_t^{\alpha}\bar{L}^{1-\alpha}$)
  - **Depreciation curve** is linear in K (so, also as a function of K, $Dep_t = \delta K_t$).
    - This curve can also be called the ***break-even curve*** because the curve traces out the level of investment *just enough to make up* for depreciation loss for each level of capital.

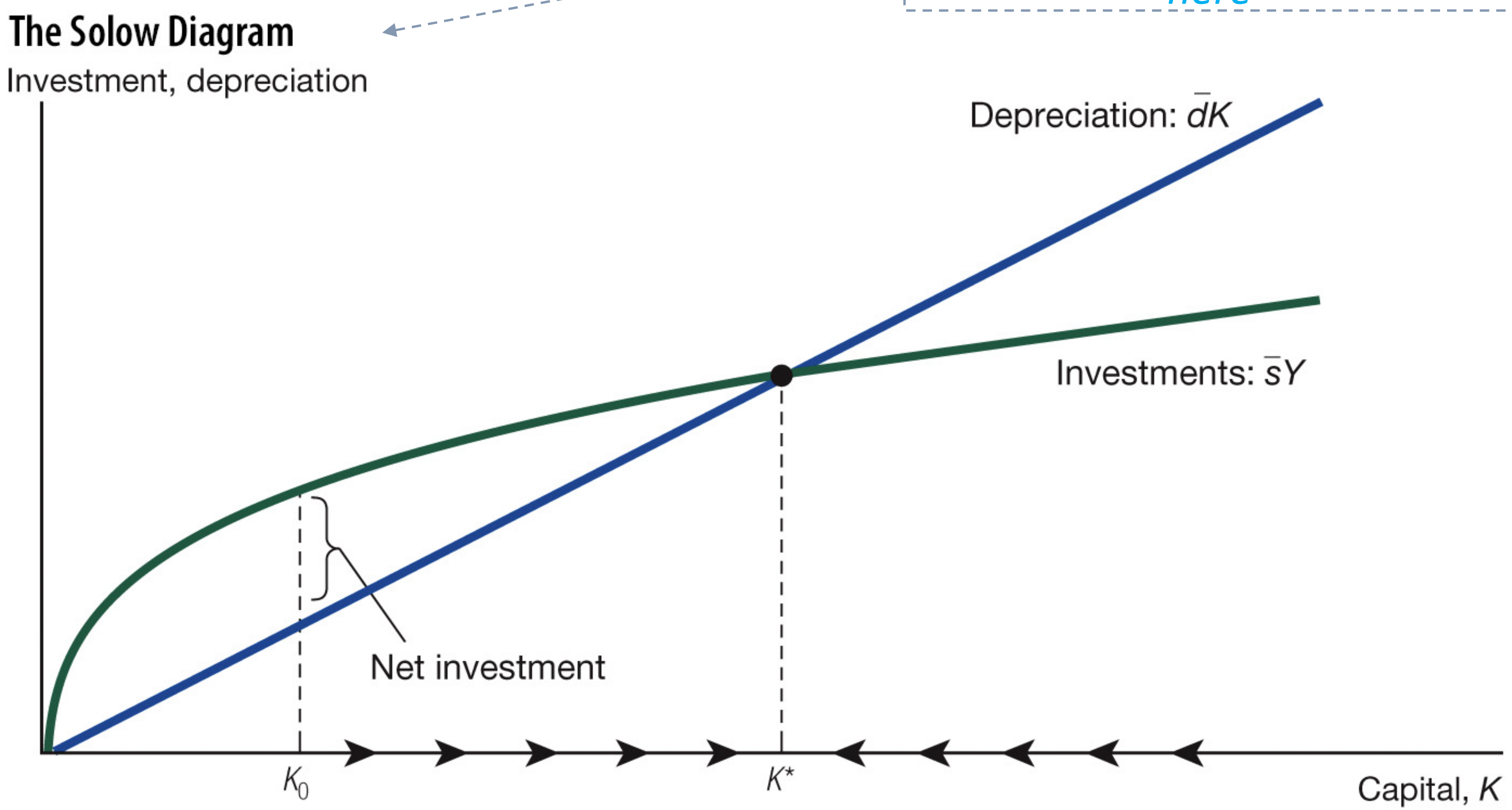
Dynamics of Capital and Output



Animated graph

The Solow Diagram

There are two curves drawn here

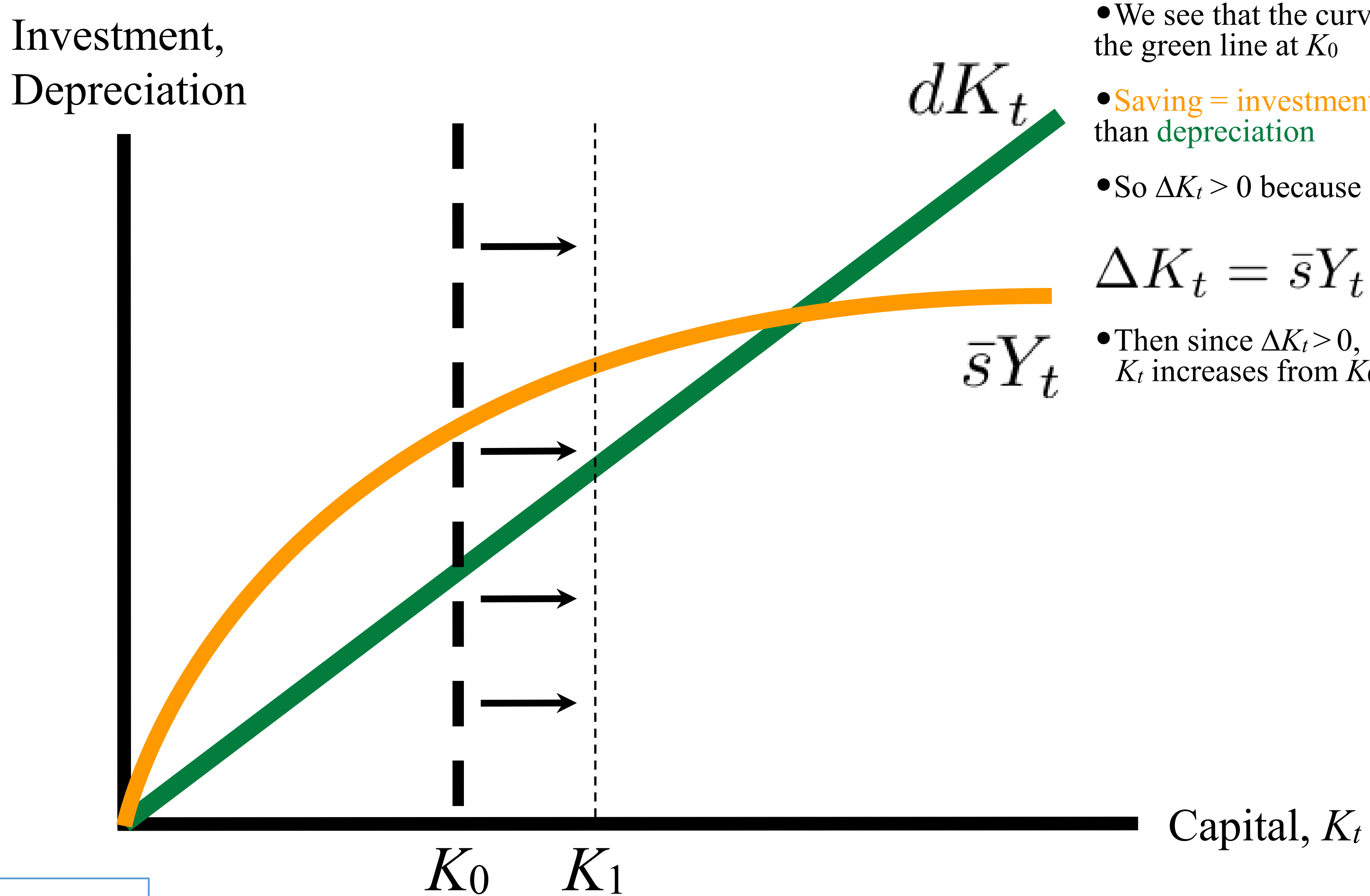


Graph 1.

Using the Solow Diagram

- If the amount of investment > depreciation
  - capital stock will increase until $\bar{s}Y_t = \bar{d}K_t$. (see graph 3 & 4)
    - Here, the change in capital is equal to 0.
    - The capital stock will stay at this value of capital forever.
    - This is called the steady state.
- If depreciation is greater than investment
  - the economy converges to the same steady state as above. (see graph 5)

Suppose the economy starts at this K0:



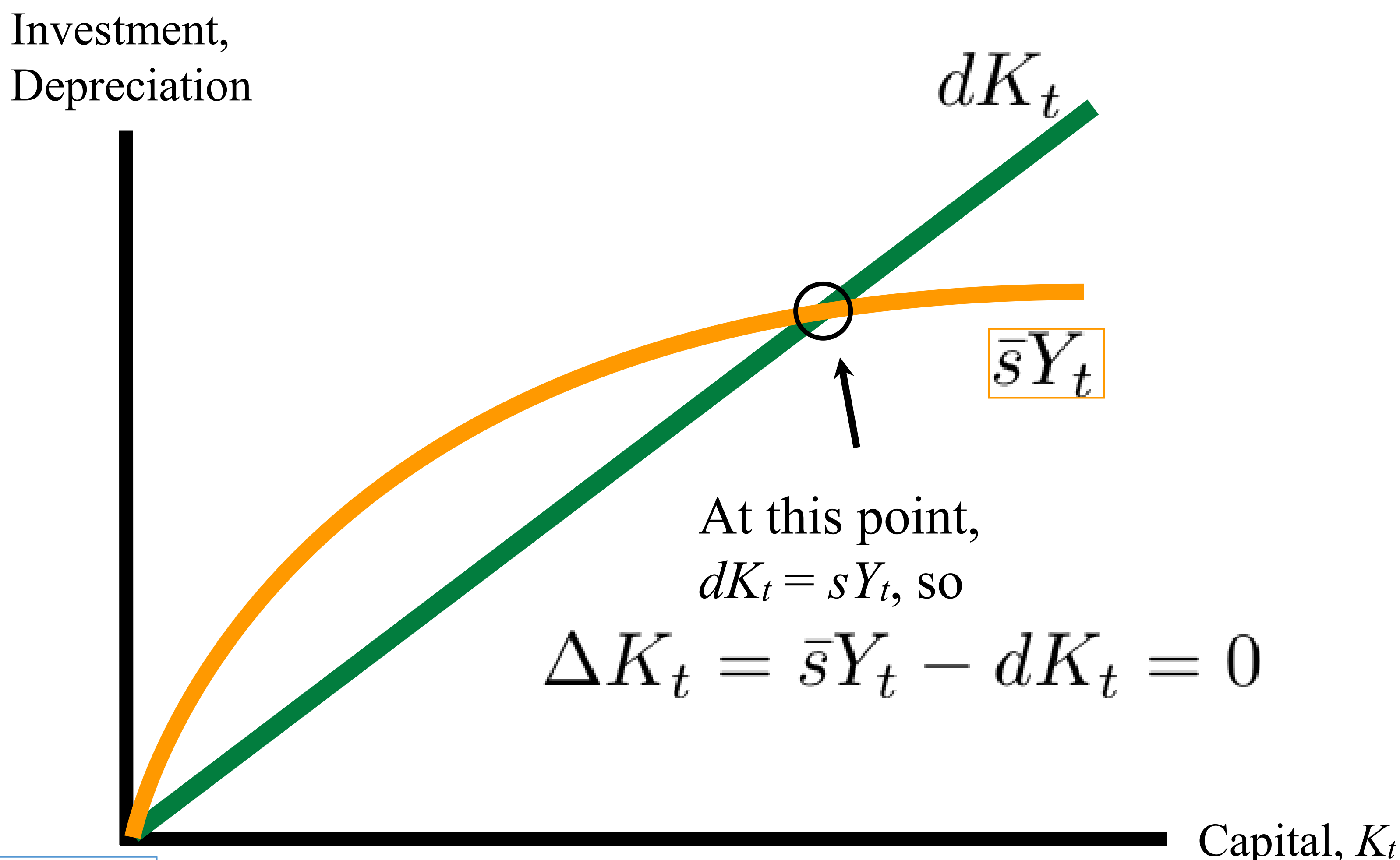
- We see that the curve is above the green line at K_0
- Saving = investment is greater than depreciation
- So $\Delta K_t > 0$ because

$$\Delta K_t = \bar{s}Y_t - dK_t$$

- Then since $\Delta K_t > 0$, K_t increases from K_0 to $K_1 > K_0$

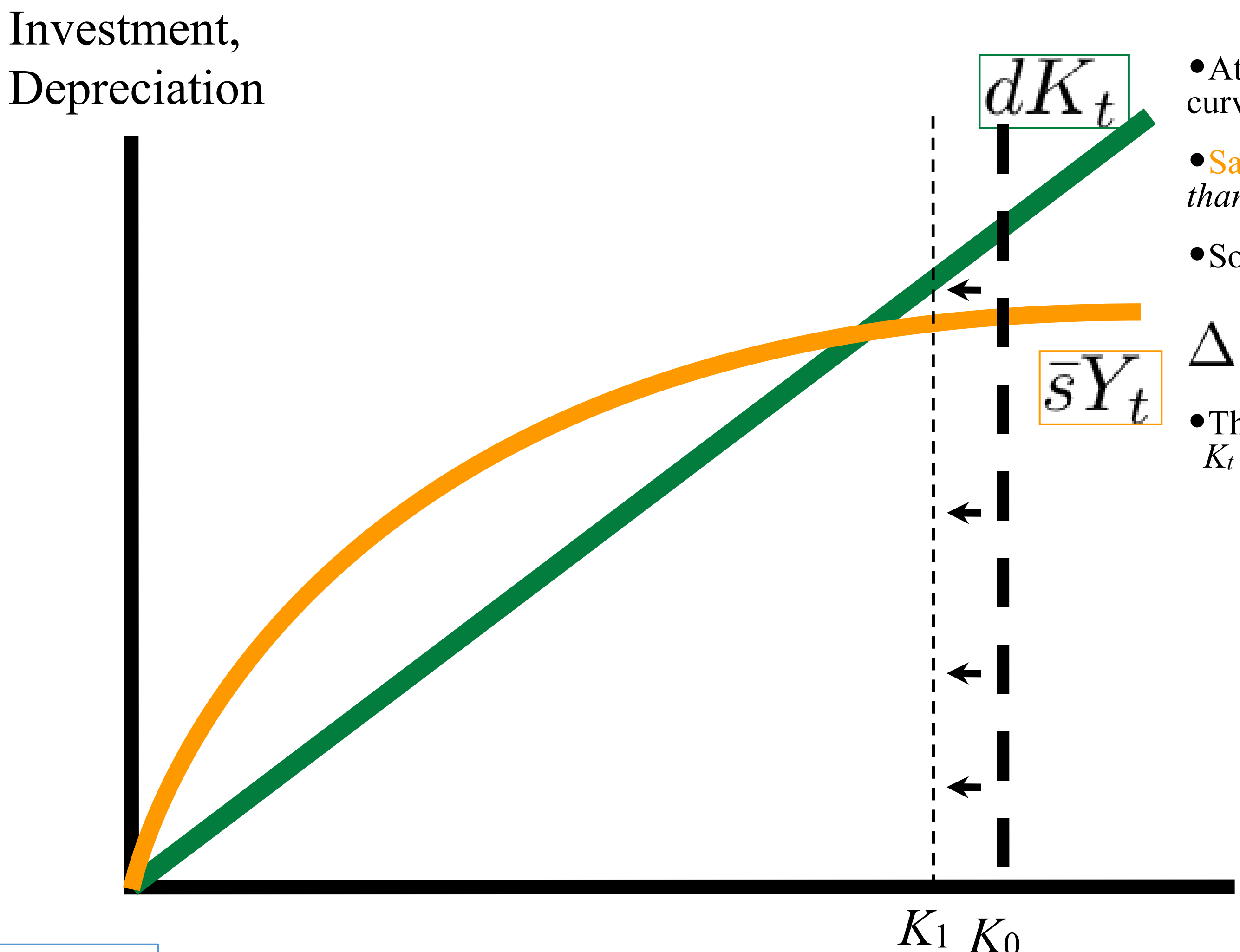
Graph 3.

The Solow Diagram graphs these two pieces together (text book diagrams):



Graph 4.

Now imagine if we start at a K0 here:



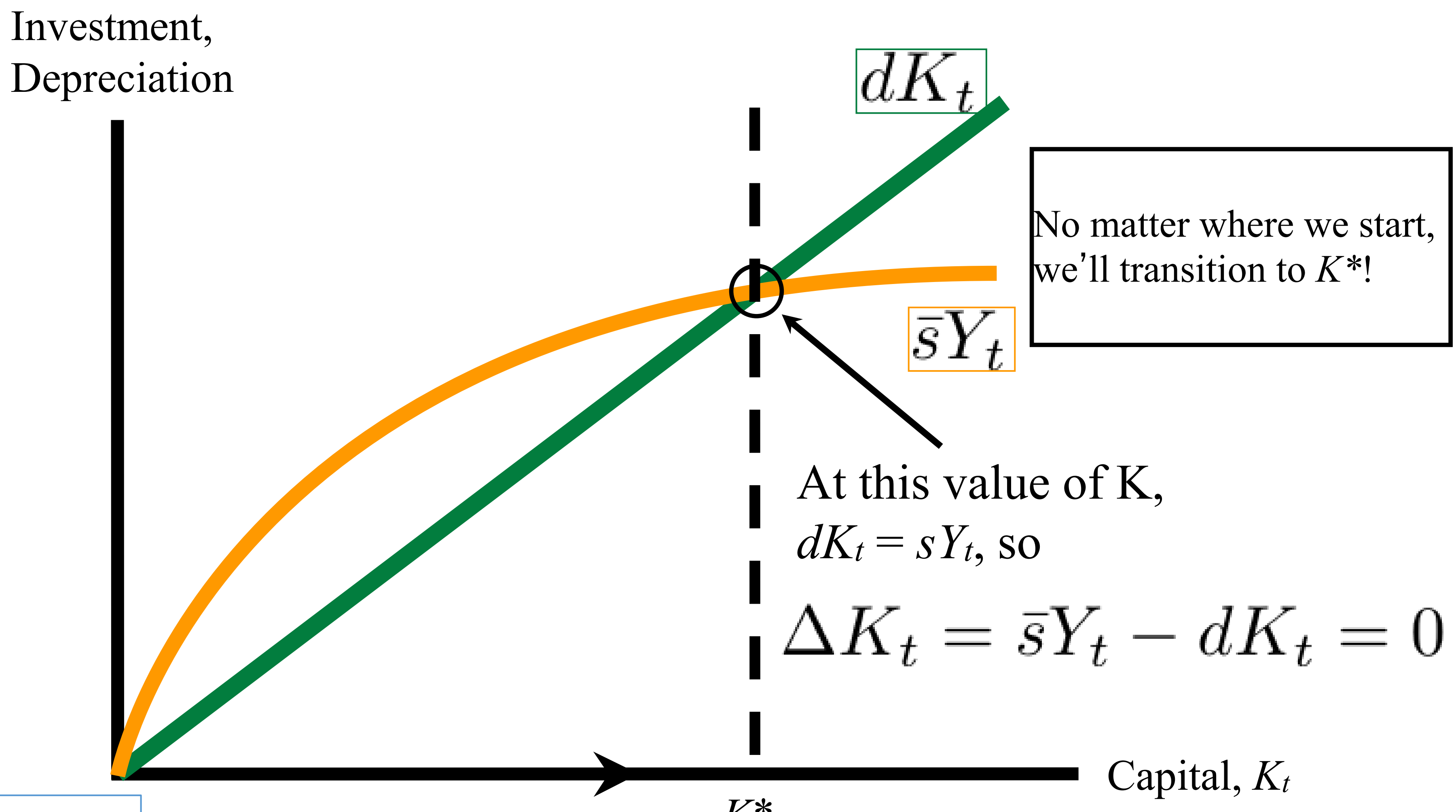
- At K_0, the green line is above the curve
- Saving = investment is now *less than* depreciation
- So $\Delta K_t < 0$ because

$$\Delta K_t = \bar{s}Y_t - dK_t$$

- Then since $\Delta K_t < 0$, K_t **de**creases from K_0 to $K_1 < K_0$

Graph 5.

We call this the process of transition dynamics: Transitioning from any K_t toward the economy's steady state K^*, where $\Delta K_t = 0$

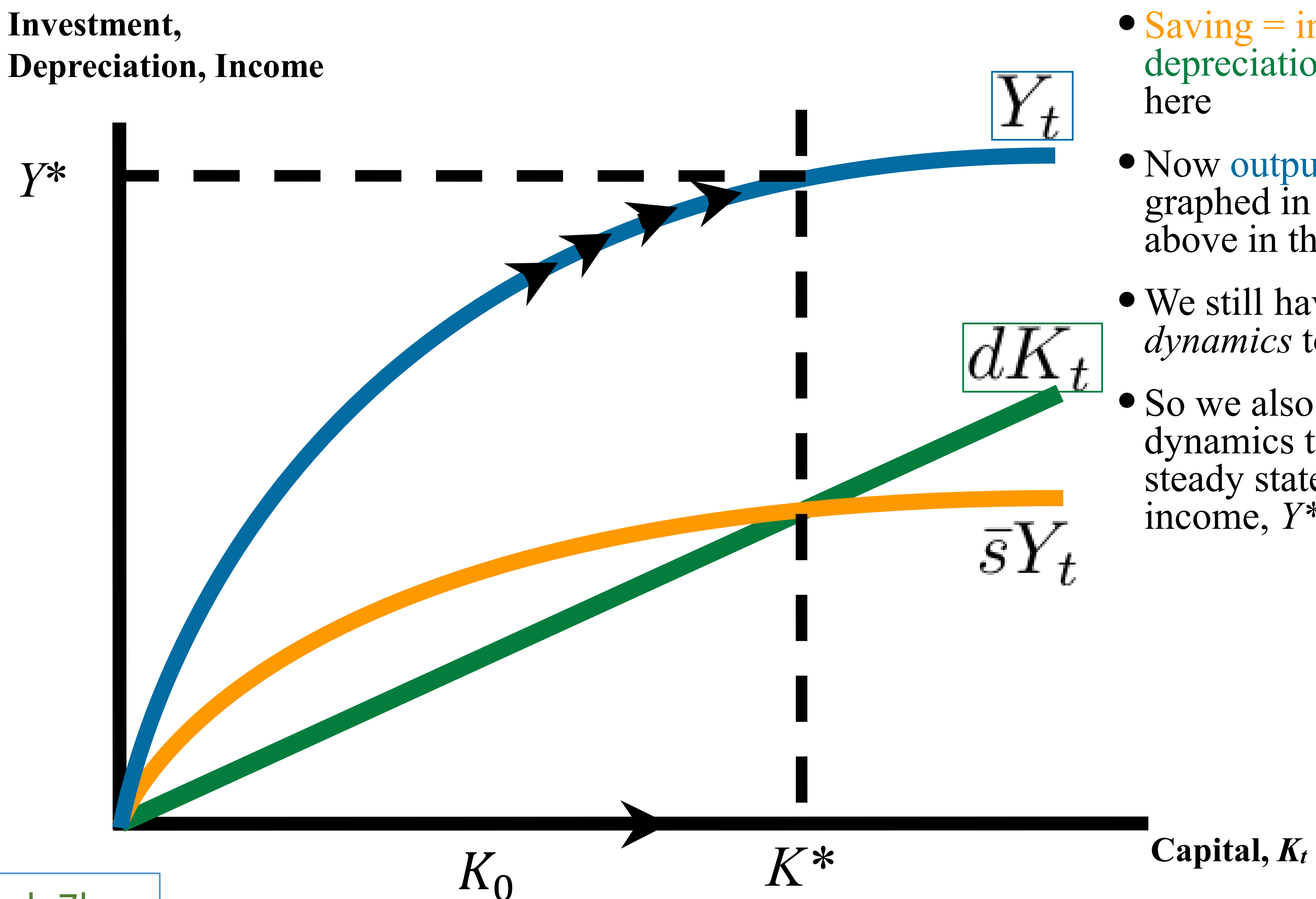


Graph 6.

Model Dynamics

- When not in the steady state,
  - the economy exhibits a change in capital toward the steady state.
- As K moves to its steady state,
  - output will also move to its steady state.
- At the rest point of the economy,
  - all endogenous variables are steady.
- Transition dynamics
  - take the economy from its initial level of capital to the steady state.

We can see what happens to output, Y, and thus to growth if we rescale the vertical axis:

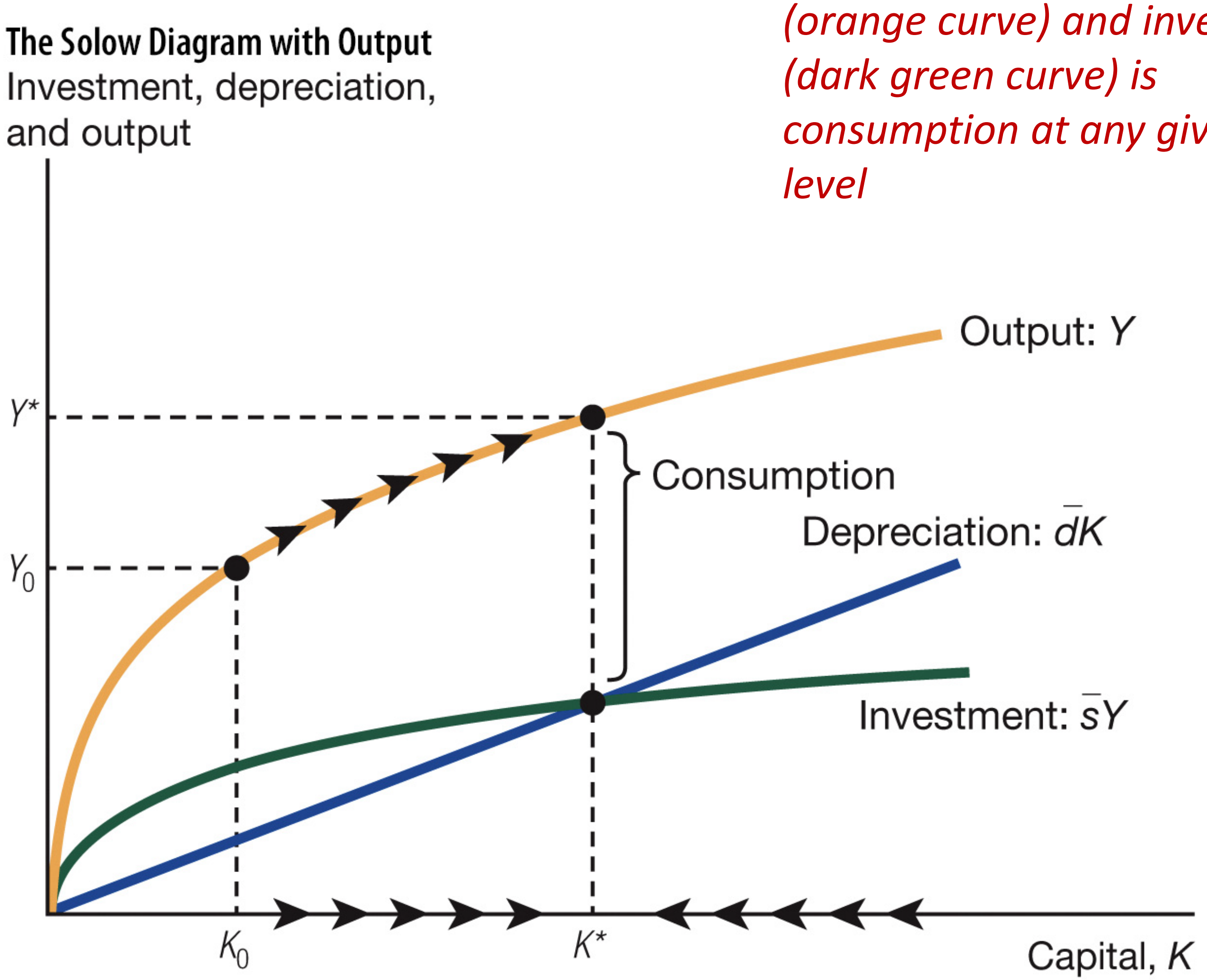


- Saving = investment and depreciation now appear here
- Now output can be graphed in the space above in the graph
- We still have *transition dynamics* toward *K**
- So we also have dynamics toward a steady state level of income, *Y**

Graph 7b.

Steady state consumption

The difference between output (orange curve) and investment (dark green curve) is consumption at any given capital level



Graph 7c.

Solving Mathematically for the Steady State

- We cannot solve for every point in time mathematically. However, we can solve mathematically for the steady state level of capital.
- In the *steady state*, investment equals depreciation.

$$\bar{s}Y^* = \bar{d}K^*$$

- Substitute into the production function:

$$\bar{d}K^* = \bar{s}\bar{A}{K^*}^{\alpha}\bar{L}^{1-\alpha}$$

Mathematic derivation (for next slide)

$$\bar{d}K^* = \bar{s}\bar{A}{K^*}^{\alpha}\bar{L}^{1-\alpha}$$

$$K^* = \frac{\bar{s}\bar{A}{K^*}^{\alpha}\bar{L}^{1-\alpha}}{\bar{d}}$$

$${K^*}^{1-\alpha} = \frac{\bar{s}\bar{A}\bar{L}^{1-\alpha}}{\bar{d}}$$

$$K^* = \left(\frac{\bar{s}\bar{A}\bar{L}^{1-\alpha}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}$$

$$K^* = \bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}$$

Solving for the Steady State—1

- Solve for *K**

$$K^* = \bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}$$

*K** as a function of parameters/exognous variables.

- The steady state level of capital is:
  - Positively related to the
    - investment rate or saving rate, $\bar{s}$
    - the size of the workforce, $\bar{L}$
    - the productivity of the economy, $\bar{A}$ (see graph 8a and 8b)
  - Negatively correlated with
    - the depreciation rate

Mathematic derivation (For next slide)

$$K^* = \bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}$$

$$Y^* = \bar{A}K^{*\alpha}\bar{L}^{1-\alpha}$$

$$Y^* = \bar{A}\left(\bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}\bar{L}^{1-\alpha}$$

Mathematic derivation (For next slide)

$$Y^* = \bar{A}^{\frac{\alpha}{1-\alpha}+1}\left(\left(\frac{\bar{s}}{\bar{\bar{d}}}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}\bar{L}$$

$$Y^* = \bar{A}^{\frac{\alpha}{1-\alpha}+\frac{1-\alpha}{1-\alpha}}\left(\left(\frac{\bar{s}}{\bar{\bar{d}}}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}\bar{L}$$

$$Y^* = \bar{A}^{\frac{1}{1-\alpha}}\left(\frac{\bar{s}}{\bar{\bar{d}}}\right)^{\frac{\alpha}{1-\alpha}}\bar{L}$$

Back to outline

Solving for the Steady State—2

- Plug K^* into the production function to get Y^*

$$Y^* = \bar{A}K^{*\alpha}\bar{L}^{1-\alpha}$$

- Plug in our solved value of K^*

$$Y^* = \bar{A}^{\frac{1}{1-\alpha}}\left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}\bar{L}$$

Y^* as a function of parameters/exognous variables.

- Higher steady state production
  - caused by higher productivity and investment rate
- Lower steady state production
  - caused by faster depreciation

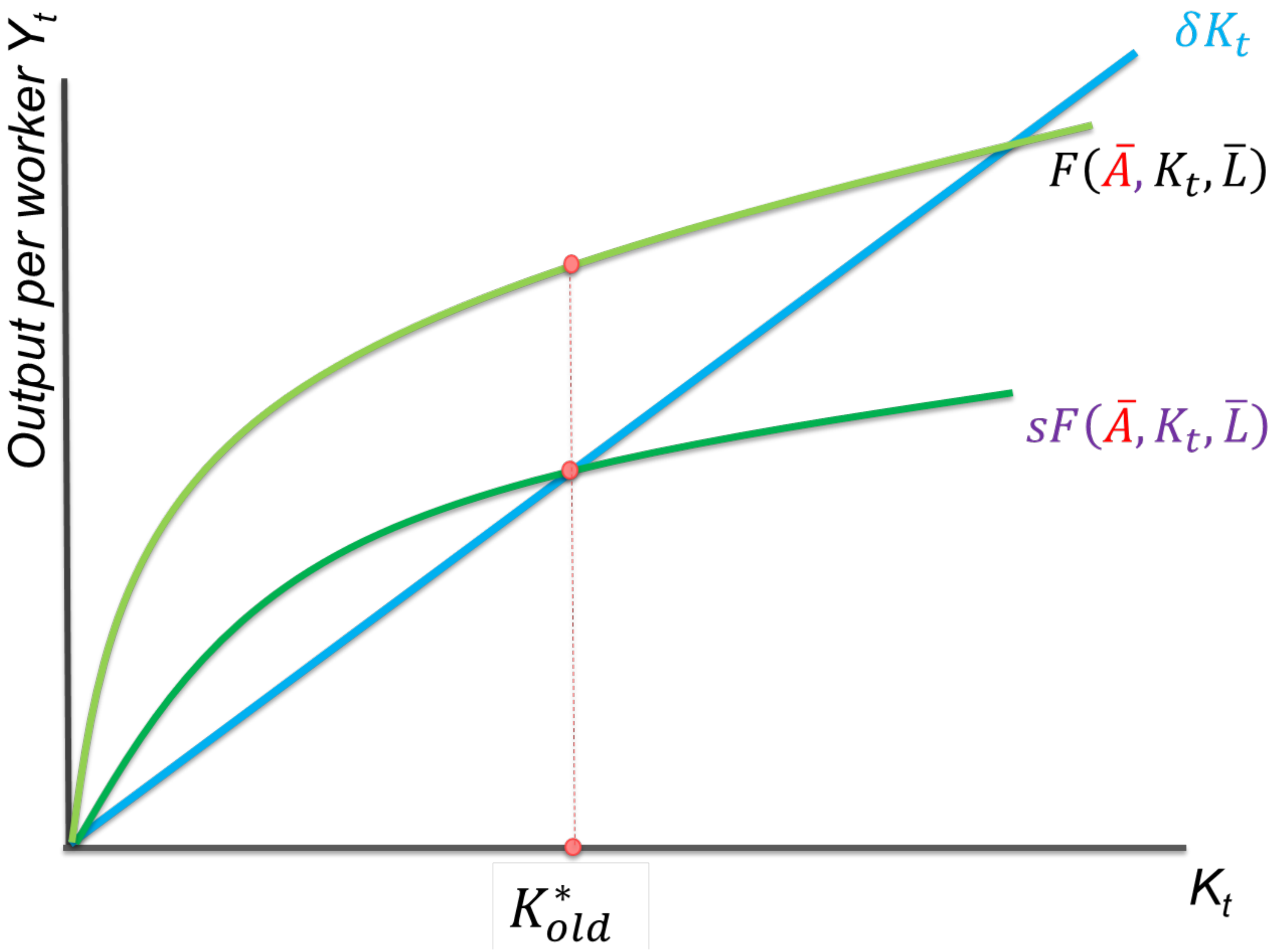
Solving for the Steady State—3

- Divide both sides by labor to get output per person in the steady state:

$$y^* = \frac{Y^*}{\bar{L}} = \bar{A}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}$$

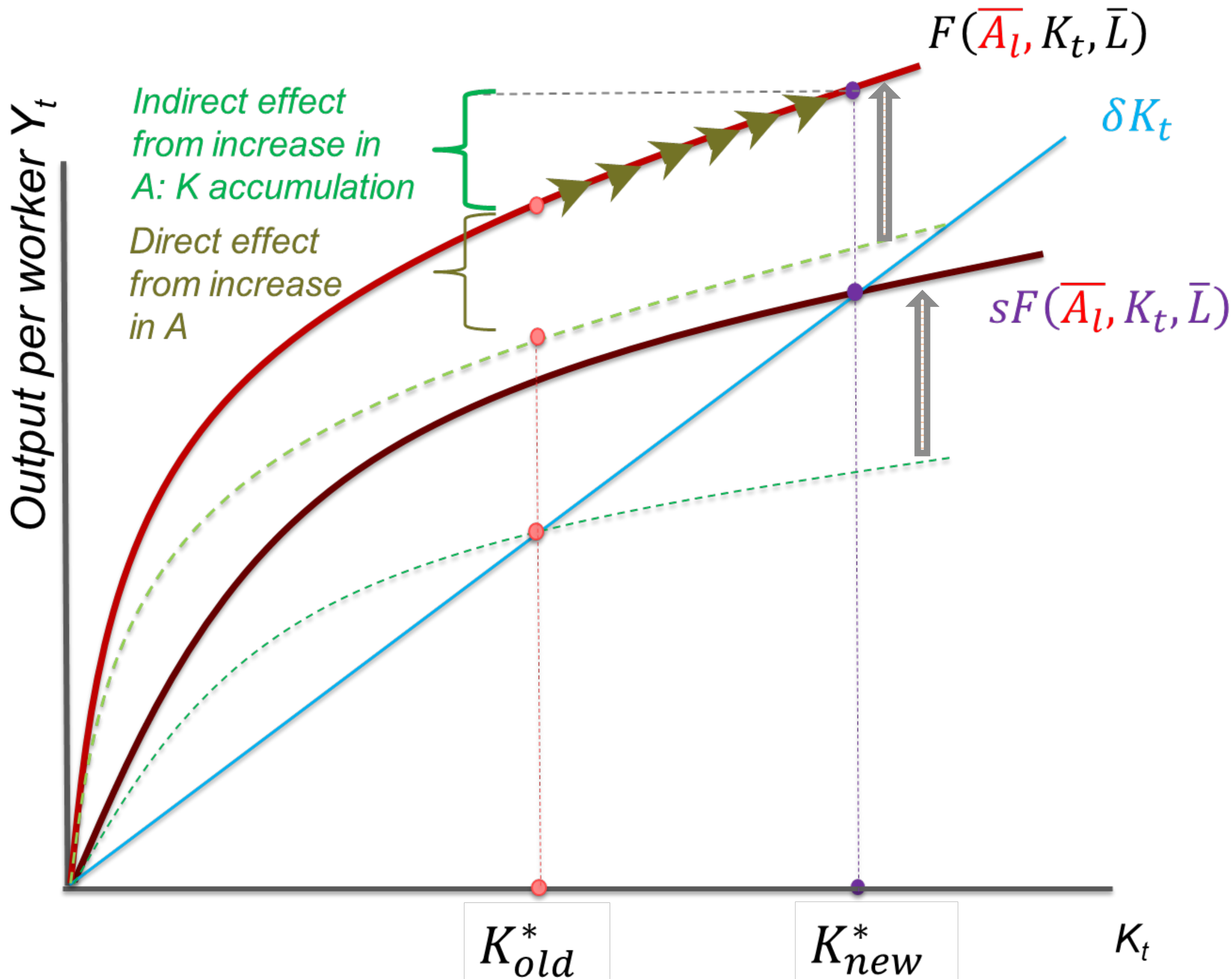
- Note the exponent on productivity is different here than in the production model.
  - Higher productivity has additional effects *(or second effect)* in the Solow model by leading the economy to *accumulate more capital*.

Increase in $\bar{A}$ (*Before*)



Graph 8a

Increase in $\bar{A}$ *(After)*



Graph 8b