NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 9**

1. A father wishes to distribute an amount of money among his three sons Jack, Jim, and John. He wish to distribute such that the following conditions are all satisfied.

   (i) The amount Jack receives plus twice the amount Jim receives is \$300.

   (ii) The amount Jim receives plus the amount John receives is \$300.

   (iii) Jack receives \$300 more than twice of what John receives.

   (a) Is it possible for the following conditions to all be satisfied?

   (b) If it is not possible, find a least square solution. (Make sure that your least square solution is feasible. For example, one cannot give a negative amount of money to anybody.)

2. (a) Suppose $\mathbf{A}$ is a $m \times n$ matrix where $m > n$. Let $\mathbf{A} = \mathbf{QR}$ be a QR factorization of $\mathbf{A}$. Explain how you might use this to write

$$\mathbf{A} = \mathbf{Q}'\mathbf{R}',$$

   where $\mathbf{Q}'$ is an $m \times m$ orthogonal matrix, and $\mathbf{R}'$ a $m \times n$ matrix with $m - n$ zero rows at the bottom. This is known as the *full* QR factorization of $\mathbf{A}$.

   (b) In MATLAB, enter the following.

```
>> A=sym([1 1 0;1 1 0;1 1 1;0 1 1])
```

```
>> [Q R]=qr(A)
```

   What is $\mathbf{Q}$ and $\mathbf{R}$? Compare this with the answer in tutorial 8 question 5(a).

   (c) Explain how you might use the command `qr` in MATLAB to find a QR factorization of a $m \times n$ matrix $\mathbf{A}$?

3. (Cayley-Hamilton theorem)
   Consider

$$p(\mathbf{X}) = \mathbf{X}^3 - 4\mathbf{X}^2 - \mathbf{X} + 4\mathbf{I}.$$

   (a) Compute $p(\mathbf{X})$ for $\mathbf{X} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

   (b) Find the characteristic polynomial of $\mathbf{X}$.

   (c) Show that $\mathbf{X}$ invertible. Express the inverse of $\mathbf{X}$ as a function of $\mathbf{X}$.

This question demonstrated the *Cayley-Hamilton theorem*, which states that if $p(x)$ is the characteristic polynomial of $\mathbf{X}$, then $p(\mathbf{X}) = 0$. This also show that if 0 is not an eigenvalue of $\mathbf{X}$, then the constant term of the characteristic polynomial $p(x)$ is nonzero, and we can use that to compute the inverse of $\mathbf{X}$.

4. For each of the following matrices $\mathbf{A}$, determine if $\mathbf{A}$ is diagonalizable. If $\mathbf{A}$ is diagonalizable, find an invertible $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and determine $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

   (a) $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} 9 & 8 & 6 & 3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

   (c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

   (d) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

   (e) $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ -4 & 2 & 3 \end{pmatrix}$.

5. (a) Show that λ is an eigenvalue of $\mathbf{A}$ if and only if it is an eigenvalue of $\mathbf{A}^T$.

   (b) Suppose $\mathbf{A}$ is diagonalizable. Is $\mathbf{A}^T$ diagonalizable? Justify your answer.

   (c) Suppose $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated to eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any positive integer k.

   (d) If $\mathbf{A}$ is invertible, show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any negative integer k.

   (e) A square matrix is said to be *nilpotent* if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then 0 is the only eigenvalue.

   (f) Let $\mathbf{A}$ be a $n \times n$ matrix with one eigenvalue λ with algebraic multiplicity n. Show that $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ is a scalar matrix, $\mathbf{A} = \lambda\mathbf{I}$.

   (g) Show that the only diagonalizable nilpotent matrix is the zero matrix.

Extra problems

1. Let $\mathbf{A}$ be an orthogonal matrix of order n and let $\mathbf{u}, \mathbf{v}$ be any two vectors in $\mathbb{R}^n$. Show that

   (a) $||\mathbf{u}|| = ||\mathbf{Au}||$;

   (b) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{Au}, \mathbf{Av})$; and

   (c) the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $\mathbf{Au}$ and $\mathbf{Av}$.

2. Let $\mathbf{A}$ be an orthogonal matrix of order n. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ and define $T = \{\mathbf{Au_1}, \mathbf{Au_2}, ..., \mathbf{Au_k}\}$.

   (a) If S is orthogonal, show that T is orthogonal.

   (b) If S is orthonormal, is T orthonormal?

3. (a) Suppose $\mathbf{A}$ and $\mathbf{B}$ are similar matrices, that is, $\mathbf{A} = \mathbf{PBP}^{-1}$ for some invertible matrix $\mathbf{P}$. Then the characteristic polynomial of $\mathbf{A}$ and $\mathbf{B}$ are equal.

   (b) Suppose the characteristic polynomial of $\mathbf{A}$ and $\mathbf{B}$ are equal. Can we conclude that $\mathbf{A}$ and $\mathbf{B}$ are similar? Justify your answer.

   (c) Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices with the same determinant. Is it true that their characteristic polynomials are equal? Justify your answer.

4. (a) Let $\mathbf{A}$ be a 2×2 matrix. Prove that the characteristic polynomial of $\mathbf{A}$ is

   $$x^2 - tr(\mathbf{A})x + \det(\mathbf{A}),$$

   where $tr(\mathbf{A})$ is the sum of the diagonal entries of $\mathbf{A}$.

   (b) Let $\mathbf{A}$ be a $n \times n$ matrix and $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Prove that $a_0 = (-1)^n \det(\mathbf{A})$.