EC3333 Tutorial 3 Suggested Answers

1. A pension fund manager is considering three mutual funds. The first is a stock fund *(S)*, the second is a long-term government and corporate bond fund *(B)*, and the third is a T-bill money market fund that provides a risk-free rate of return of 8%. The characteristics of the risky funds are as follows:

	Expected Return	Standard Deviation
Stock fund (*S*)	20%	30%
Bond fund (*B*)	12%	15%

The correlation between the fund returns is 0.10.

a. What are the investment proportions in the minimum-variance portfolio of the two risky funds, and what is the expected value and standard deviation of its rate of return?

The parameters of the opportunity set are:

$E(r_S)$ = 20%, $E(r_B)$ = 12%, σ_S = 30%, σ_B = 15%, ρ = 0.10

From the standard deviations and the correlation coefficient we generate the covariance matrix [note that $Cov(r_S, r_B) = \rho \times \sigma_S \times \sigma_B$]:

	Bond	Stock
Bonds	0.0225	0.0045
Stocks	0.0045	0.0900

Finding the portfolio with minimum portfolio variance is equivalent to solving the following minimization problem, where $\sigma_{SB} = \rho_{BS}\sigma_B\sigma_S$.

$$\min_{x_S, x_B} \sigma_p^2 = x_S^2\sigma_S^2 + x_B^2\sigma_B^2 + 2x_S x_B \sigma_{SB}$$
$$\text{s.t. } x_S + x_B = 1$$

Using substitution method, with $x_B = 1 - x_S$

We get a minimization problem with only one variable.

$$\min_{x_S} \sigma_p^2 = x_S^2\sigma_S^2 + (1 - x_S)^2\sigma_B^2 + 2x_S(1 - x_S)\sigma_{SB}$$

First order condition of the minimization problem is:

$$0 = \frac{d}{dx_S}\sigma_p^2 = \frac{d}{dx_S}\left(x_S^2\sigma_S^2 + (1 - x_S)^2\sigma_B^2 + 2x_S(1 - x_S)\sigma_{SB}\right)$$
$$= 2x_S\sigma_S^2 - 2(1 - x_S)\sigma_B^2 + 2\sigma_{SB}(1 - 2x_S)$$
$$\Rightarrow x_S^{min} = \frac{\sigma_B^2 - \sigma_{SB}}{\sigma_S^2 + \sigma_B^2 - 2\sigma_{SB}}, \qquad x_B^{min} = 1 - x_S^{min}$$

The minimum-variance portfolio is computed as follows:

$x_S^{min} = \frac{\sigma_B^2 - \text{Cov}(r_S, r_B)}{\sigma_S^2 + \sigma_B^2 - 2\text{Cov}(r_S, r_B)} = \frac{0.0225 - 0.0045}{0.0900 + 0.0225 - (2 \times 0.0045)} = \mathbf{0.1739}$

$x_B^{min} = 1 - 0.1739 = \mathbf{0.8261}$

The minimum variance portfolio mean and standard deviation are:

$\boldsymbol{E(r_{\text{Min}})} = (0.1739 \times 0.20) + (0.8261 \times 0.12) = 0.1339 = \mathbf{13.39\%}$

$\boldsymbol{\sigma}_{\textbf{Min}} = [x_S^2\sigma_S^2 + x_B^2\sigma_B^2 + 2x_Sx_B\text{Cov}(r_S, r_B)]^{1/2}$

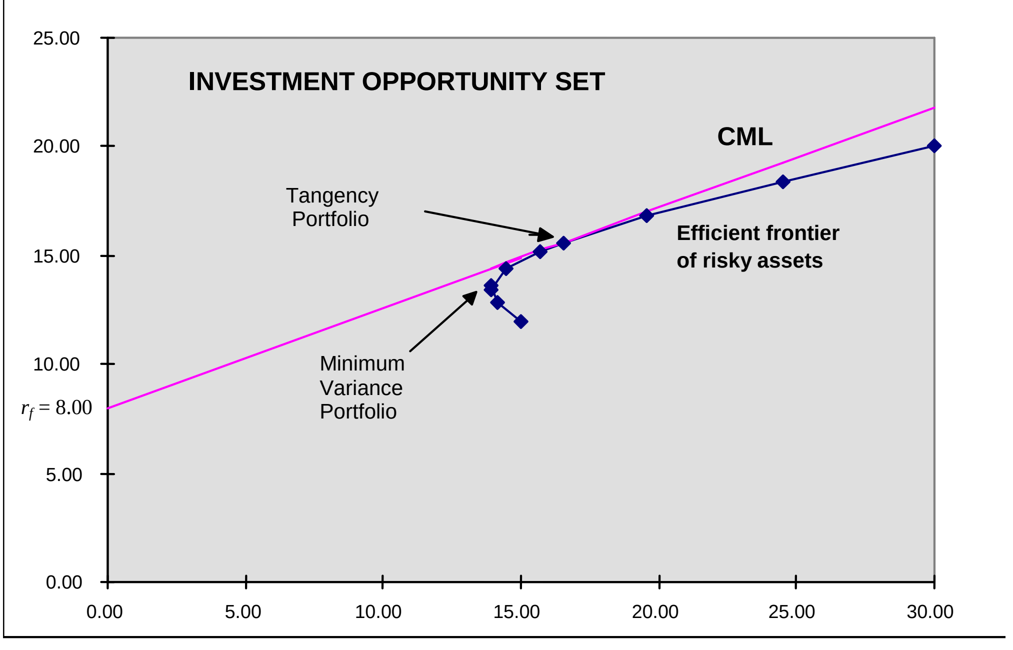
$= [(0.1739^2 \times 0.0900) + (0.8261^2 \times 0.0225) + (2 \times 0.1739 \times 0.8261 \times 0.0045)]^{1/2}$

= 13.92%

b. Tabulate and draw the investment opportunity set of the two risky funds. Use investment proportions for the stock fund of 0% to 100% in increments of 20%.

Proportion in Stock Fund %	Proportion in Bond Fund %	Expected Return %	Standard Deviation %	
0.00	100.0	12.00	15.00	
17.39	82.61	13.39	13.92	Min variance
20.00	80.00	13.60	13.94	
40.00	60.00	15.20	15.70	
60.00	40.00	16.80	19.53	
80.00	20.00	18.40	24.48	
100.00	0.00	20.00	30.00	

c. Draw a tangent from the risk-free rate to the opportunity set. What does your graph show for the expected return and standard deviation of the optimal portfolio?



The **graphical solution** indicates that the optimal portfolio is the tangency portfolio with expected return approximately 15.6% and standard deviation approximately 16.5%.

d. Solve numerically for the proportions of each asset and for the expected return and standard deviation of the optimal risky portfolio.

Finding the tangency portfolio is equivalent to solving the following optimization problem:

$$\max_{x_S, x_B} \mathrm{SR}_p = \frac{E(r_p) - r_f}{\sigma_p} \text{ subject to}$$

$$E(r_p) = x_S E(r_S) + x_B E(r_B)$$

$$\sigma_p^2 = x_S^2 \sigma_S^2 + x_B^2 \sigma_B^2 + 2 x_S x_B \sigma_{SB}$$

$$1 = x_S + x_B$$

Numerical solution

Using the Excel solver function by maximizing the Sharpe ratio, subject to the sum of the weights to1 (see Excel file posted), the numerical solution for the optimal risky portfolio and its characteristics are:

	E(r)	SD	rho	cov
S	0.2	0.3	0.1	0.004500
B	0.12	0.15		
T-bill	0.08	0		

w(S)	**w(B)**	sum w	**E(rp)**	**SD**	**Sharpe Ratio (SR)**
0.4516	0.5484	1	0.156129	0.1654	0.4603
45%	**55%**		**16%**	**17%**	

Analytical solution

Using substitution method, with $x_B = 1 - x_S$, and from the first order condition of the maximization problem, the weights of the tangent portfolio are given below. (The proof of the following asset weights in the tangent portfolio is not required for the exam.)

$$x_S^{\tan} = \frac{\left(E(r_S) - r_f\right)\sigma_B^2 - \left(E(r_B) - r_f\right)\sigma_{SB}}{\left(E(r_S) - r_f\right)\sigma_B^2 + \left(E(r_B) - r_f\right)\sigma_S^2 - \left(E(r_S) - r_f + E(r_B) - r_f\right)\sigma_{SB}}$$

$$x_B^{\tan} = 1 - x_S^{\tan}$$

The proportion of the optimal risky portfolio invested in the stock fund is given by:

$$x_S^{\tan} = \frac{[(0.20-0.08)\times 0.0225]-[(0.12-0.08)\times 0.0045]}{[(0.20-0.08)\times 0.0225]+[(0.12-0.08)\times 0.0900]-[(0.20-0.08+0.12-0.08)\times 0.0045]} = \mathbf{0.4516}$$

$$x_B^{\tan} = 1 - 0.4516 = \mathbf{0.5484}$$

The mean and standard deviation of the optimal risky portfolio are:

$\boldsymbol{E(r_P)}$ = (0.4516 × 0.20) + (0.5484 × 0.12) = 0.1561

= 15.61%

$\boldsymbol{\sigma_P}$ = $[(0.4516^2 \times 0.0900) + (0.5484^2 \times 0.0225) + (2 \times 0.4516 \times 0.5484 \times 0.0045)]^{1/2}$

= 16.54%

e. What is the Sharpe ratio of the best feasible CAL?

The Sharpe ratio of the optimal CAL is:

$$\frac{E(r_p) - r_f}{\sigma_p} = \frac{0.1561 - 0.08}{0.1654} = \mathbf{0.4601}$$

f. You require that your portfolio yield an expected return of 14%, and that it be efficient, on the best feasible CAL.

i. What is the standard deviation of your portfolio?

If you require that your portfolio yield an expected return of 14%, then you can find the corresponding standard deviation from the optimal CAL. The equation for this CAL is:

$$E(r_C) = r_f + \frac{E(r_p) - r_f}{\sigma_P}\sigma_C = 0.08 + 0.4601\sigma_C$$

If $E(r_C)$ is equal to 14%, then the **standard deviation** of the portfolio is **13.04%.**

ii. What is the proportion invested in the T-bill money market fund and each of the two risky funds?

To find the proportion invested in the T-bill fund, remember that the mean of the complete portfolio (i.e., 14%) is an average of the T-bill rate and the optimal combination of stocks and bonds (P). Let y be the proportion invested in the portfolio P.

The mean of any portfolio along the optimal CAL is:

$$E(r_C) = (1 - y) \times r_f + y \times E(r_P) = r_f + y \times [E(r_P) - r_f]$$
$$= 0.08 + y \times (0.1561 - 0.08)$$

Setting $E(r_C)$ = 14% we find: $y = 0.7884$ and $(1 - y) = 0.2119$ (the proportion invested in the T-bill fund).

To find the proportions invested in each of the funds, multiply 0.7884 times the respective proportions of stocks and bonds in the optimal risky portfolio:

Proportion of stocks in complete portfolio = 0.7884 × 0.4516 **= 0.3560**

Proportion of bonds in complete portfolio = 0.7884 × 0.5484 **= 0.4323**

g. If you were to use only the two risky funds, and still require an expected return of 14%, what would be the investment proportions of your portfolio? Compare its standard deviation to that of the optimized portfolio in Part f. What do you conclude?

Using only the stock and bond funds to achieve a portfolio expected return of 14%, we must find the appropriate proportion in the stock fund (x_S) and the appropriate proportion in the bond fund ($x_B = 1 - x_S$) as follows:

$0.14 = 0.20 \times x_S + 0.12 \times (1 - x_S) = 0.12 + 0.08 \times x_S \Rightarrow x_S = 0.25$

Thus, the proportions are 25% invested in the stock fund and 75% in the bond fund. The **standard deviation** of this portfolio will be:

$\boldsymbol{\sigma_P} = [(0.25^2 \times 0.0900) + (0.75^2 \times 0.0225) + (2 \times 0.25 \times 0.75 \times 0.0045)]^{1/2}$ **= 14.13%**

This is considerably **greater than the standard deviation of 13.04% achieved using T-bills and the optimal portfolio.**