NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1522 Linear Algebra for Computing **Tutorial 2**

1. Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices respectively.

   (a) Suppose the homogeneous linear system $\mathbf{Bx} = \mathbf{0}$ has infinitely many solutions. How many solutions does the system $\mathbf{ABx} = \mathbf{0}$ have?

   (b) Suppose $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution. Can we tell how many solutions are there for $\mathbf{ABx} = \mathbf{0}$.

2. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Find a 4×3 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I}_3$.

   Hint: Write $\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix}$, where $\mathbf{x}_i$ is a 4×1 matrix, for $i = 1, 2, 3$.

   (b) Let $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find a 3×4 matrix $\mathbf{Y}$ such that $\mathbf{YB} = \mathbf{I}_3$.

3. (i) Reduce the following matrices $\mathbf{A}$ to its reduced row-echelon form $\mathbf{R}$.

   (ii) For each of the elementary row operation, write the corresponding elementary matrix.

   (iii) Write the matrices $\mathbf{A}$ in the form $\mathbf{E}_1\mathbf{E}_2 \dots \mathbf{E}_n\mathbf{R}$ where $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n$ are elementary matrices and $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

   (c) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

4. Determine if the following matrices are invertible. If the matrix is invertible, find its inverse.

   (a) $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$.

   (b) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

5. Write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

6. (a) Suppose $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{0}$. Show that $\mathbf{I} - \mathbf{A}$ is invertible, with inverse $\mathbf{I} + \mathbf{A}$.

   (b) Suppose $\mathbf{A}^3 = \mathbf{0}$. Is $\mathbf{I} - \mathbf{A}$ invertible?

   (c) A square matrix $\mathbf{A}$ is said to be *nilpotent* if there is a positive integer n such that $\mathbf{A}^n = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\mathbf{I} - \mathbf{A}$ is invertible.

Extra problems

1. Show that a linear system $\mathbf{Ax} = \mathbf{b}$ has either no solution, only one solution or infinitely many solutions. (Hint: Suppose $\mathbf{Ax} = \mathbf{b}$ has two different solutions $\mathbf{u}$ and $\mathbf{v}$. Use $\mathbf{u}$ and $\mathbf{v}$ to construct infinitely many solutions.)

2. Determine which of the following statements are true. Justify your answer.

   (a) If $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices of the same size, then $\mathbf{AB} = \mathbf{BA}$.

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size, $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB}$.

   (c) If $\mathbf{A}$ is a square matrix, then $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric.

   (d) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size, then $\mathbf{A} - \mathbf{B}$ is symmetric.

   (e) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size, then $\mathbf{AB}$ is symmetric.

   (f) If $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$.

   (g) If $\mathbf{A}$ is an n by m matrix such that $\mathbf{AA}^T = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$.

3. Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same order. Prove that if $\mathbf{A}$ is singular, then $\mathbf{AB}$ and $\mathbf{BA}$ are singular. (Prove the statement without using determinant.)

4. (**Polynomial Interpolation**)
   Given any n points in the xy-plane that has distinct x-coordinates, it is known that there is a unique polynomial of degree $n-1$ or less whose graph passes through those points. A degree $n-1$ polynomial has the following expression

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

   Suppose its graph passes through the points (x_1, y_1), (x_2, y_2),...,(x_n, y_n), it follows that the coordinates of the points must satisfy

$$\begin{array}{ccccccccc} a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + \cdots + & a_{n-1} x_1^{n-1} & = y_1 \\ a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + \cdots + & a_{n-1} x_2^{n-1} & = y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a_0 & + & a_1 x_n & + & a_2 x_n^2 & + \cdots + & a_{n-1} x_n^{n-1} & = y_n \end{array}$$

   This is a linear system in the unknowns $a_0, a_1, ..., a_{n-1}$. The augmented matrix for the system is

$$\left(\begin{array}{cccccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & & y_n \end{array}\right) \tag{V}$$

   which has a unique solution whenever $x_1, x_2, ..., x_n$ are distinct.

   (a) Find a cubic polynomial whose graph passes through the points

x	1	2	3	4
y	3	-2	-5	0

(b) **(MATLAB)** The coefficient matrix of the linear system (V) is called a *Vandermonde Matrix*. The function fliplr(vander(v)) returns the Vandermonde matrix such that its rows are powers of the vector v. For example,

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>> v=[1;2;3;4;5;6;7;8];

>> A=fliplr(vander(v))
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will generate the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 \\ 1 & 3 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 & 3^7 \\ 1 & 4 & 4^2 & 4^3 & 4^4 & 4^5 & 4^6 & 4^7 \\ 1 & 5 & 5^2 & 5^3 & 5^4 & 5^5 & 5^6 & 5^7 \\ 1 & 6 & 6^2 & 6^3 & 6^4 & 6^5 & 6^6 & 6^7 \\ 1 & 7 & 7^2 & 7^3 & 7^4 & 7^5 & 7^6 & 7^7 \\ 1 & 8 & 8^2 & 8^3 & 8^4 & 8^5 & 8^6 & 8^7 \end{pmatrix}$$

Use the Vandermonde matrix function to find a degree 7 polynomial that passes through

x	1	2	3	4	5	6	7	8
y	12	70	1244	10500	54268	205682	630540	1657024