MA1522 Linear Algebra for computing: Chapter 2: Matrix Algebra

August 22, 2023

Section 2.1 Matrix Operations

If A is an m by n matrix—that is, a matrix with m rows and n columns—then the scalar entry in the i-th row and j-th column of A is denoted by a_{ij} and is called the (i,j)-entry of A.

$$A = \begin{array}{c c} \begin{array}{ccccc} & & j\text{th col.} & & \end{array} & \\ \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{l} \\ m \text{ row} \\ i\text{th row} \\ \\ \end{array} \\ n \text{ columns} & \end{array}$$

col is a vector in $\mathbb{R}^m$

Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.

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Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.

1. The columns are denoted by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and the matrix A is written as

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{pmatrix}.$$

2. The number a_{ij} is the i-th entry (from the top) of the j-th column vector $\mathbf{a}_j$.

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$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} \leftarrow \text{the } i\text{-th entry}$$

- The **diagonal entries** in an m by n matrix $A = (a_{ij})$ are $a_{11}, a_{22}, a_{33}, \ldots,$ and they form the **main diagonal** of A.
- A **diagonal matrix** is an n by n matrix whose non-diagonal entries are zero.

  An example is the n by n identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

$$A = \begin{pmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & a_{33} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{pmatrix}$$

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n×n matrix

$$I_n A = A I_n = A$$

An m by n matrix whose entries are all zero is a **zero matrix** and is written as $\mathbf{0}_{mn}$.

$$\mathbf{0}_{mn} = \underbrace{\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}}_{n \text{ col.}}. \qquad m \text{ row}$$

$$\begin{pmatrix} 3 & 5 \\ 6 & 9 \end{pmatrix} \neq \begin{pmatrix} 3 & 7 \\ 6 & 9 \end{pmatrix}$$

- Two matrices are equal if

  (i) they have the same size (i.e., the same number of rows and the same number of columns) and

  (ii) if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal.

- If A and B are m by n matrices, then the sum $A+B$ is the m by n matrix where entry in $A+B$ is the sum of the corresponding entries in A and B.

Example.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 10 & 20 \\ -2 & -4 & -8 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+10 & 3+20 \\ 4-2 & 5-4 & 6-8 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 12 & 23 \\ 2 & 1 & -2 \end{pmatrix}.$$

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col1 $\quad$ col1

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

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- The columns of the sum $A+B$ are the sums of the corresponding columns in A and B.
- The sum $A+B$ is defined only when A and B are the same size. We cannot add A and B if the two matrices have different sizes.

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Example 1

$$A+B = \begin{pmatrix} 4+1 & 0+1 & 5+1 \\ -1+3 & 3+5 & 2+7 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{pmatrix} \text{ and } C = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

Find (i) $A+B$ and (ii) $A+C$.

Solution. (i)

$$A+B = \begin{pmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{pmatrix}.$$

(ii) The sum $A+C$ is <u>not defined</u> because A and C have different sizes.

Example 1

Let
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Solution. (i)
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(ii) The sum $A+C$ is <u>not defined</u> because A and C have different sizes.

Cannot add

If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose every entry is obtained by r times the corresponding entry in A.

Example. Let $A = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix}$ and $r = 3$. Then

$$3A = \begin{pmatrix} 3\times 4 & 3\times 0 & 3\times 5 \\ 3\times(-1) & 3\times 3 & 3\times 2 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 15 \\ -3 & 9 & 6 \end{pmatrix}.$$

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Theorem

Let A, B, and C be m by n matrices, and let r and s be scalars.

1. $A+B=B+A$.
2. $(A+B)+C=A+(B+C)$.
3. $A+\mathbf{0}_{mn}=A$.
4. $r(A+B)=rA+rB$.
5. $(r+s)A=rA+sA$.
6. $(rs)A=r(sA)$.

distributive laws

$LHS = r\left(\begin{pmatrix} a_{11} & \cdots \\ \cdots & a_{mn}\end{pmatrix} + \begin{pmatrix} b_{11} & \cdots \\ \cdots & b_{mn}\end{pmatrix}\right) = \cdots$

same

$RHS = \cdots = \cdots$

Idea of Proof

Each quantity in Theorem 1 is verified by showing that

(1) the matrix on the left side has the same size as the matrix on the right and

(2) that corresponding matrix entries are equal.

Try supplying a proof.

Matrix Multiplication

$$\underset{\underset{\mathbb{R}^n}{\cap}}{\underline{u}} \longmapsto A\underline{u} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \underbrace{\underline{a}_1 u_1 + \underline{a}_2 u_2 + \cdots + \underline{a}_n u_n}_{\underset{\mathbb{R}^m}{\cap}}$$

Definition

Let A be an m by n matrix.

We define a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(\mathbf{u}) = A\mathbf{u} \;\text{ for }\; \mathbf{u} \in \mathbb{R}^n.$$

Then T is called a *matrix transformation*.

For vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c, we have

$$\begin{aligned} T(\underbrace{\mathbf{u}+\mathbf{v}}_{}) &= A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} \\ &= T(\mathbf{u})+T(\mathbf{v}), \end{aligned}$$

$\in \mathbb{R}^n$

$\in \mathbb{R}^m$

$$\begin{aligned} T(c\mathbf{u}) &= A(c\mathbf{u}) = cA\mathbf{u} \\ &= cT(\mathbf{u}). \end{aligned}$$

For vectors **u**, **v** in $\mathbb{R}^n$ and scalar c, we have

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} \\ &= T(\mathbf{u})+T(\mathbf{v}), \\ T(c\mathbf{u}) &= A(c\mathbf{u}) \overset{\checkmark}{=} cA\mathbf{u} \\ &= cT(\mathbf{u}). \end{aligned}$$

T is a linear operator

Remark

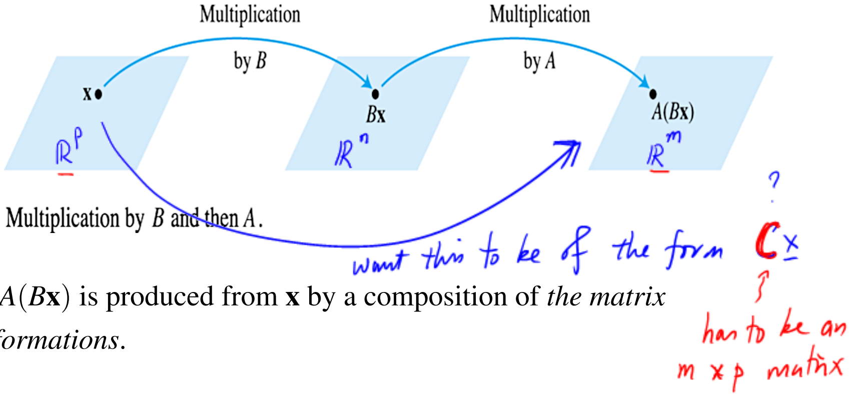
At the end of this course, we will define and study an important concept called *linear transformations*.

Matrix transformations are important examples of $\mathbb{R}$-linear transformations.

$\underline{u} \longmapsto T(\underline{u}) = A\underline{u}$

When a matrix B multiplies a vector $\mathbf{x}$ in $\mathbb{R}^n$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.

If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$.



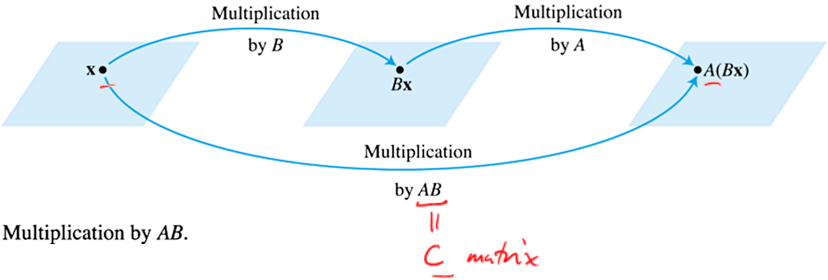
Multiplication by B and then A.

Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a composition of *the matrix transformations*.

Goal

We would like to represent the composite matrix transformations as multiplication by a single matrix C so that

$$C\mathbf{x} = A(B\mathbf{x}) \quad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n.$$



Multiplication by AB.

We will denote the matrix C by AB.

Suppose A is an m by n matrix, B is an n by p matrix, and $\mathbf{x}$ is column vector in $\mathbb{R}^p$. We write

$$B = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \\ | & | & \cdots & | \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_p\mathbf{b}_p.$$

defn of $B\underline{x}$

This is a n×1 col vector

By the linearity of multiplication by A,

$$
\begin{aligned}
A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_p\mathbf{b}_p) \\
&= x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_p A\mathbf{b}_p \\
&= \begin{pmatrix} | & | & \cdots & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \\ | & | & \cdots & | \end{pmatrix} \mathbf{x}.
\end{aligned}
$$

m×1 col vector

matrix multi to vector

col.

this is the matrix C

Thus multiplication by the matrix

$$AB = C = \begin{pmatrix} | & | & \cdots & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \\ | & | & \cdots & | \end{pmatrix}$$

transforms $\mathbf{x}$ into $A(B\mathbf{x})$.

We recall that we denote C by AB.

Thus multiplication by the matrix

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We recall that we denote C by AB.

Definition

If A is an m by n matrix, and if B is an n by p matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$, then the product AB is the m by p matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$.

That is,

$$\underline{AB} = A \underbrace{\begin{pmatrix} | & | & \cdots & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & \cdots & | \end{pmatrix}} = \begin{pmatrix} | & | & \cdots & | \\ \underline{A\mathbf{b}_1} & \underline{A\mathbf{b}_2} & \cdots & \underline{A\mathbf{b}_p} \\ | & | & \cdots & | \end{pmatrix}$$

Multiplication of matrices corresponds to composition of linear transformations.

Definition

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Multiplication of matrices corresponds to composition of linear transformations.

Example 2

Compute AB, where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}.$$

$$AB = \left(A\begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad A\begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad A\begin{pmatrix} 6 \\ 3 \end{pmatrix} \right)$$

Solution

$A\underline{u} = u_1\underline{a}_1 + \cdots + u_n\underline{a}_n$

Write $B = (\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3)$.

$$A\mathbf{b}_1 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix},$$

$$A\mathbf{b}_2 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix},$$

$AB = \begin{pmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{pmatrix}$

$$A\mathbf{b}_3 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ -9 \end{pmatrix}.$$

Then

$$AB = A(\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3) = \begin{pmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{pmatrix}. \qquad \square$$

Remarks

1. Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

2. Row-column rule for computing AB:
   If a product AB is defined, then the entry in row i and column j of AB is the dot product of row i of A and column j of B.
   If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an m by n matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

Remarks

$A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}| & |\\ a_1 & a_2\\ | & |\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = xa_1 + ya_2 \ (\in \mathbb{R}^3)$

$xa_1 + ya_2$

2 deg of freedom

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$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

$\begin{pmatrix}1 & 1\\ 2 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix}$

Theorem

Let A be an m by n matrix, and let B and C have sizes for which the indicated sums and products are defined.

(a) $A(BC) = (AB)C$ *(associative law of multiplication).*

(b) $A(B+C) = AB+AC$ *(left distributive law).*

(c) $(B+C)A = BA+CA$ *(right distributive law).*

(d) $r(AB) = (rA) = A(rB)$ *for any scalar r.*

(e) $I_mA = A = AI_n$ *(identity for matrix multiplication).*

Proof

We will only prove Part (a).

Parts (b) to (e) are easier and we will leave them to the students.

We will provide two proofs of (a).

Let A be an m by n matrix.

Let B be an n by p matrix.

Let C be an p by q matrix.

First Proof of (a)

We compare the matrices $(AB)C$ and $A(BC)$.

Let $X = AB$ which is an m by p matrix.
The (i,j)-th entry of X is

$$x_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

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First Proof of (a)

Now $(AB)C = XC$ is an m by q matrix.

The (i,k)-th entry of $XC = (AB)C$ is

$$
\begin{aligned}
& x_{i1}c_{1k} + x_{i2}c_{2k} + \ldots + x_{ip}c_{pk} \\
&= a_{i1}b_{11}c_{1k} + a_{i2}b_{21}c_{1k} + \ldots + a_{in}b_{n1}c_{1k} \\
&\quad + a_{i2}b_{22}c_{2k} + a_{i2}b_{22}c_{2k} + \ldots + a_{in}b_{n2}c_{2k} \\
&\quad + \ldots + a_{ir}b_{rs}c_{sk} + \ldots \\
&\quad + a_{i1}b_{1p}c_{pk} + a_{i2}b_{2p}c_{pk} + \ldots + a_{in}b_{np}c_{pk}.
\end{aligned}
$$

In short the (i,k)-th entry of $(AB)C$ is the sum of terms $a_{ir}b_{rs}c_{sk}$ where $r = 1, \ldots, n$ and $s = 1, \ldots, p$.

First Proof of (a)

Let $Y = BC$ which is an n by q matrix.

The (i,j)-th entry of Y is

$$y_{ij} = b_{i1}c_{1j} + b_{i2}c_{2j} + \ldots + b_{ip}c_{pj}.$$

First Proof of (a)

Now $A(BC) = AY$ is an m by q matrix.
The (i,k)-th entry of $AY = A(BC)$ is

$$\begin{aligned} & a_{i1}y_{1k} + a_{i2}y_{2k} + \ldots + a_{in}y_{nk} \\ &= a_{i1}b_{11}c_{1k} + a_{i1}b_{12}c_{2k} + \ldots + a_{i1}b_{1p}c_{pk} \\ &\quad + a_{i2}b_{21}c_{1k} + a_{i2}b_{22}c_{2k} + \ldots + a_{i2}b_{2p}c_{pk} \\ &\quad + \ldots + a_{ir}b_{rs}c_{sk} + \ldots \\ &\quad + a_{in}b_{n1}c_{1k} + a_{in}b_{n2}c_{2k} + \ldots + a_{in}b_{np}c_{pk}. \end{aligned}$$

The (i,k)-th entry of $A(BC)$ is the sum of terms $a_{ir}b_{rs}c_{sk}$ where $r = 1, \ldots, n$ and $s = 1, \ldots, p$.

First Proof of (a)

We have computed the (i,j)-th entries of $(AB)C = XC$ and $A(BC) = AY$ and they are the same.

Hence the two matices $(AB)C$ and $A(BC)$ are equal, i.e.

$$(AB)C = A(BC).$$

This completes the first proof of Part (a).

Second proof of Part (a)

The second proof is essentially the same as the first proof but more conceptual.
It follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known that the composition of functions is associative.

Let $C = (\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \cdots \ \ \mathbf{c}_p)$.

By the definition of matrix multiplication,

$$BC = (B\mathbf{c}_1 \ \ B\mathbf{c}_2 \ \ \cdots \ \ B\mathbf{c}_p)$$

$$A(BC) = (A(B\mathbf{c}_1) \ \ A(B\mathbf{c}_2) \ \ \cdots \ \ A(B\mathbf{c}_p)).$$

By the definition of AB, we have $A(B\mathbf{x}) = (AB)\mathbf{x}$ (Why?). Thus

$$A(BC) = ((AB)\mathbf{c}_1 \ \ (AB)\mathbf{c}_2 \ \ \cdots \ \ (AB)\mathbf{c}_p) = (AB)C.$$

This proves (a). $\square$

Warnings

Usually

$$AB \neq BA.$$

First AB and BA may not be of the same size.

For example, if A is a 3 by 5 matrix and B is a 5 by 3 matrix, then AB is a 3 by 3 matrix and BA is a 5 by 5 matrix.

If A and B are square n by n matrices, then it is still possible that

$$AB \neq BA.$$

One explanation is the columns of AB are linear combinations of the columns of A, whereas the columns of BA are constructed from the columns of B.

The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A.

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For specific matrices A and B, we may have $AB = BA$

Then we say that A and B *commute* with one another.

Warnings

1. The cancellation laws do not hold for matrix multiplication,
   i.e. if $AB = AC$, then it is **not always true** that $B = C$.
   Likewise if $BA = CA$, then it is **not always true** that $B = C$.
2. If $AB = \mathbf{0}$ is the zero matrix, you **cannot conclude** that either $A = \mathbf{0}$ or $B = \mathbf{0}$.

Could you give examples of the above two warnings?

Powers of a matrix

Suppose A is a square n by n matrix.

- If k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \overbrace{A\,A\,\cdots\,A}^{k \text{ copies}}.$$

- If $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.
- If $k = 0$, then A^0 is the identity n by n matrix and $A^0\mathbf{x} = \mathbf{x}$.

Transpose of a matrix

Given an m by n matrix A, the *transpose* of A is the n by m matrix, denoted by $A^\top$, whose columns are formed from the corresponding rows of A.

Example.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

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Given an m by n matrix A, the *transpose* of A is the n by m matrix, denoted by $A^\top$, whose columns are formed from the corresponding rows of A.

Example.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Theorem

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

1. $(A^\top)^\top = A$.
2. $(A+B)^\top = A^\top + B^\top$.
3. *For any scalar r,* $(rA)^\top = rA^\top$.
4. $(AB)^\top = B^\top A^\top$.

We warn that in (d), the transpose of a product of matrices equals the product of their transposes in the *reverse order*.

Exercise. Supply a proof to the theorem.

2.2 The Inverse of a matrix

Definition

An n by n matrix A is said to be *invertible* if there is an n by n matrix C such that

$$AC = I \text{ and } CA = I$$

where $I = I_n$ is the n by n identity matrix.
In this case, C is called an *inverse matrix* of A.

Remarks

(1) Not every n by n matrix has an inverse matrix.
For example the 2 by 2 zero matrix does not have an inverse matrix, i.e. it is not invertible (Why?).

(2) If A has an inverse matrix C, then it is unique.

Indeed if both B and C are inverse matrices of A, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

This unique inverse matrix C is denoted by A^{-1}, so that

$$AA^{-1} = I \text{ and } A^{-1}A = I.$$

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Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. *If* $ad - bc \neq 0$, *then* A *is invertible and*
$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$
2. *If* $ad - bc = 0$, *then* A *is not invertible.*

The quantity $ad - bc$ is called the *determinant* of A, and we write

$$\det A = ad - bc.$$

This theorem says that a 2 by 2 matrix A is invertible if and only if $\det A \neq 0$.

Theorem

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This theorem says that a 2 by 2 matrix A is invertible if and only if $\det A \neq 0$.

Proof

We denote $C = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $\delta = ad - bc$.
Calculation shows (check this!)

$$AC = \delta I_2 \ \text{ and } \ CA = \delta I_2.$$

(i) If $\delta \neq 0$, then

$$A(\frac{1}{\delta}C) = I_2 \ \text{ and } \ (\frac{1}{\delta}C)A = I_2.$$

Hence $\dfrac{1}{\delta}C$ is the inverse matrix A^{-1}.

Proof

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Hence $\dfrac{1}{\delta}C$ is the inverse matrix A^{-1}.

(ii) Suppose $\delta = 0$. Then

$$A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \delta I_2 = \mathbf{0}_{22}$$

is the zero 2 by 2 matrix.

Suppose A has an inverse matrix A^{-1}. Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = I_2 \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}\mathbf{0}_{22} = \mathbf{0}_{22}.$$

We get

$$a = b = c = d = 0$$

so $A = \mathbf{0}_{22}$ is the zero matrix.

Suppose A has an inverse matrix A^{-1}. Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = I_2 \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}\mathbf{0}_{22} = \mathbf{0}_{22}.$$

We get

$$a = b = c = d = 0$$

so $A = \mathbf{0}_{22}$ is the zero matrix.

This is a contradiction because

$$I_2 = A^{-1}A = A^{-1}\mathbf{0}_{22} = \mathbf{0}_{22}.$$

This shows that A^{-1} does not exist. $\square$

Theorem

If A is an invertible n by n matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. Take any $\mathbf{b}$ in $\mathbb{R}^n$.

We set $\mathbf{x} = A^{-1}\mathbf{b}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b}.$$

Hence $\mathbf{x} = A^{-1}\mathbf{b}$ is solution to $A\mathbf{x} = \mathbf{b}$.

A solution exists.

Theorem

If A is an invertible n by n matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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A solution exists.

Suppose $\mathbf{y}$ is another solution, i.e. $A\mathbf{y} = \mathbf{b}$.

Then

$$\mathbf{x} = A^{-1}\mathbf{b} = A^{-1}(A\mathbf{y}) = (A^{-1}A)\mathbf{y} = I\mathbf{y} = \mathbf{y}.$$

This shows that the solution is unique. $\square$

Theorem

1. *If A is an invertible matrix, then A^{-1} is invertible and*

$$(A^{-1})^{-1} = A.$$

2. *If A and B are n by n invertible matrices, then so is AB, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

3. *If A is an invertible matrix, then so is $A^\top$, and*

$$(A^\top)^{-1} = (A^{-1})^\top$$

Proof

(a) We have $AA^{-1} = I$ and $A^{-1}A = I$ so A is the inverse matrix of A^{-1}. In particular A^{-1} is invertible.

(b)

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = BB^{-1} = I.$$

Hence $B^{-1}A^{-1}$ is the inverse matrix of AB, and AB is invertible.

Proof

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Hence $B^{-1}A^{-1}$ is the inverse matrix of AB, and AB is invertible.

(c) We have

$$AA^{-1} = I \text{ and } A^{-1}A = I.$$

Taking transposes on both sides give

$$(A^{-1})^\top A^\top = I^\top = I \text{ and}$$
$$A^\top (A^{-1})^\top = I^\top = I.$$

Hence $(A^{-1})^\top$ is the inverse matrix of $A^\top$, and $A^\top$ is invertible. $\square$

We could generalize Theorem 6(b) is as follows:

Theorem

If $A_1, A_2, \ldots, A_k$ are invertible n by n matrices, then $A_1 A_2 \cdots A_k$ is invertible and

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

Elementary matrices

Fact. An invertible matrix A is row equivalent to an identity matrix I.

Later we will see how use this fact to compute A^{-1}.

Before that, we have to introduce elementary matrices.

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

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Fact. An invertible matrix A is row equivalent to an identity matrix I.

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Example 1

Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Compute E_1A, E_2A, and E_3A, and describe how they are related to the elementary row operations on A.

Solution

We compute

$$E_1A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{pmatrix}.$$

Hence E_1A is obtained by adding of -4 times Row 1 of A to Row 3.

We compute

$$E_2A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}.$$

Hence E_2A is obtained by interchanging Row 1 and Row 2 of A.

We compute

$$E_3A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ 5g & 5h & 5i \end{pmatrix}.$$

Hence E_3A is obtained by multiplying by 5 to Row 2 of A. $\square$

Remark

1. Left-multiplication by E_1, E_2, E_3 in Example 1 to a 3 by n matrix B has the same effect on B.
2. How do we remember these matrices E_1, E_2, E_3?
   For example, we multiply the matrix E_1 to the 3 by 3 identity matrix I_3.
   $$E_1 I_3 = E_1.$$
   We can see the effect of E_1 on I_3.

Example 1 illustrates the following general fact about elementary matrices.

1. Performing an elementary row operation on an m by n matrix A to get a matrix B is the same as multiplying a corresponding elementary matrix E to the left of A, i.e. $B = EA$.
2. Each elementary matrix E is invertible.
   The inverse of E is the elementary matrix of the same type that transforms E back into I_m.

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   The inverse of E is the elementary matrix of the same type that transforms E back into I_m.

Theorem (7)

An n by n matrix A is invertible if and only if A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Proof

Suppose that A is invertible.

By an previous theorem, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$.

Thus A has a pivot position in every row.

Since A is square, the n pivot positions must be on the diagonal.

Hence the reduced echelon form of A is I_n. That is, $A \sim I$.

Conversely suppose $A \sim I_n$.

Each step of the row reduction of A to I_n corresponds to left multiplication by an elementary matrix, there exist elementary matrices $E_1, E_2, \ldots, E_p$ such that

$$A \ \sim \ E_1 A \ \sim \ E_2(E_1 A) \ \sim \cdots \sim \ E_p(E_{p-1} \cdots E_1 A) = I_n.$$

Let $B = E_p E_{p-1} \cdots E_1$. Then we have

$$BA = I_n.$$

Now $B = E_p E_{p-1} \cdots E_1$ is the product of invertible matrices E_i so B is invertible, i.e. B^{-1} exists.

We check

$$\begin{aligned}AB = I_nAB &= (B^{-1}B)AB\\ &= B^{-1}(BA)B = B^{-1}I_nB\\ &= B^{-1}B\\ &= I_n.\end{aligned}$$

This shows that B is the inverse matrix of A and, A is invertible.

This completes the proof of the theorem. $\square$

Remark

The above proof shows that

$$A^{-1} = B = E_p E_{p-1} \cdots E_1$$

were $E_1, \ldots, E_p$ are the elementary row matrices which row reduced A to I_n.

We use this to devise a method to compute A^{-1}.

Remark

The above proof shows that

$$A^{-1} = B = E_p E_{p-1} \cdots E_1$$

were $E_1, \ldots, E_p$ are the elementary row matrices which row reduced A to I_n.

We use this to devise a method to compute A^{-1}.

Algorithm for finding A^{-1}

Let A be an n by n matrix.

1. Form the augmented matrix $(A \;\; I_n)$. This is an n by $2n$ matrix.
2. If A is row equivalent to I_n, then $(A \;\; I_n)$ is row equivalent to $(I_n \;\; A^{-1})$.
3. If $(A \;\; I_n)$ is **not** row equivalent to $(I_n \;\; B)$ then A is **not** row equivalent to I_n, and A **does not** have an inverse.

Algorithm for finding A^{-1}

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3. If $(A \;\; I_n)$ is **not** row equivalent to $(I_n \;\; B)$ then A is **not** row equivalent to I_n, and A **does not** have an inverse.

Example 2

Find the inverse of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix},$$

if it exists.

Solution

We perform row operations to the augmented matrix

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right) \sim \cdots \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right).$$

The above shows that $A \sim I_3$ (Why?)

By Theorem (7), A is invertible and

$$A^{-1} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}.$$

In order to play it safe, we check

$$AA^{-1} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix} \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not necessary to check that $A^{-1}A = I_3$ since A is invertible (Why?). $\square$

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be the columns of the matrix A^{-1}.
Multiplying the matrices A and $(I_n \;\; A^{-1})$ gives

$$\begin{aligned} A\;\;(I_n \;\; A^{-1}) &= A(I_n \;\; \mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \cdots \;\; \mathbf{x}_n) \\ &= (A \;\; A\mathbf{x}_1 \;\; A\mathbf{x}_2 \;\; \cdots \;\; A\mathbf{x}_n). \end{aligned}$$

On the other hand

$$\begin{aligned} A\;\;(I_n \;\; A^{-1}) &= (AI_n \;\;\; AA^{-1}) \\ &= (A \;\; I_n) \\ &= (A \;\; \mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \cdots \;\; \mathbf{e}_n). \end{aligned}$$

Comparing the above two equations gives n equations

$$A\mathbf{x}_1 = \mathbf{e}_1,\ \ A\mathbf{x}_2 = \mathbf{e}_2,\ \ \cdots,\ \ A\mathbf{x}_n = \mathbf{e}_n. \tag{2}$$

Hence the columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of A^{-1} are precisely the solutions of the equations in Equation (2).

Another way of saying this is that row reduction of $(A\ \ I_n)$ to $(I_n\ \ A^{-1})$ gives the solution of Equation (2).

Let A be an m by n matrix.

In the next few slides, we will make a list of statements about the matrix A.

Each statement is not always true.

It is true if and only if certain conditions in blue are met.

In order to have some fun, we will make this into game.

We will figure out the correct conditions in blue.

2.3 Characterizations of Invertible Matrices

Let A be an m by n matrix.

In the next few slides, we will make a list of statements about the matrix A.

Each statement is not always true.

It is true if and only if certain conditions in blue are met.

In order to have some fun, we will make this into game.

We will figure out the correct conditions in blue.

Hint to the game

The condition in blue is either one or both of these two statements below:

- There is a pivot in every row of A.
- There is a pivot in every column of A.

We have to choose the right statement(s).

The equation $A\mathbf{x} = \mathbf{0}_{m1}$ has only the trivial solution.

There is a pivot in every column of A.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

There is a pivot in every row of A.

The equation $A\mathbf{x} = \mathbf{0}_{m1}$ has only the trivial solution.

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The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

There is a pivot in every row of A.

The columns of A span $\mathbb{R}^m$.

There is a pivot in every row of A.

The columns of A are linearly independent.

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There is a pivot in every column of A.

The matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

There is a pivot in every column of A.

The matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

There is a pivot in every row of A.

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There is a pivot in every row of A.

Notice that when $m = n$,

There is a pivot in every column of A

if and only if

there is a pivot in every row of A.

Notice that when $m = n$,

There is a pivot in every column of A

if and only if

there is a pivot in every row of A.

The n by n matrix A is an invertible matrix.

The matrix A has a pivot in every row and column.

The n by n matrix A is row equivalent to the n by n identity matrix I_n.

The matrix A has a pivot in every row and column.

The n by n matrix $A^\top$ is invertible.

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The n by n matrix $A^\top$ is invertible.

The matrix A has a pivot in every row and column.

There is an n by n matrix C such that $CA = I$.

The matrix A has a pivot in every row and column.

There is an n by n matrix D such that $AD = I$.

The matrix A has a pivot in every row and column.

The last two statements show that

- There is an n by n matrix C such that $CA = I$ if and only if there is an n by n matrix D such that $AD = I$.

There is an n by n matrix C such that $CA = I$.

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The last two statements show that

- There is an n by n matrix C such that $CA = I$ if and only if there is an n by n matrix D such that $AD = I$.

The Invertible Matrix Theorem

Theorem

Let A be an n by n matrix. The following statements are equivalent:

(a) *The matrix A is an invertible matrix.*

(b) *The matrix A is row equivalent to the n by n identity matrix I_n.*

(c) *A has n pivot positions.*

(d) *The equation $A\mathbf{x} = \mathbf{0}_{n1}$ has only the trivial solution.*

(e) *The columns of A form a linearly independent set.*

(f) *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.*

(g) *The equation* $A\mathbf{x} = \mathbf{b}$ *has at least one solution for each* $\mathbf{b}$ *in* $\mathbb{R}^n$.

(h) *The columns of* A *span* $\mathbb{R}^n$.

(i) *The linear transformation* $\mathbf{x} \mapsto A\mathbf{x}$ *maps* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

(j) *There is an* n *by* n *matrix* C *such that* $CA = I_n$.

(k) *There is an* n *by* n *matrix* D *such that* $AD = I_n$.

(l) $A^\top$ *is an invertible matrix.*

Proof

First we have shown that (j) and (k) are equivalent in the slide before the statement of the Invertible Matrix Theorem.

The textbook contains a careful proof of the relationship between each statement for the interested reader.

For example, the proof will establish the "circle" of implications.

$$(a) \Rightarrow (j) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).$$

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If any one of these five statements is true, then so are the others, as each one implies the next.

Next, (a) $\Rightarrow$ (k) because we simply set $D = A^{-1}$.

We have (k) $\Rightarrow$ (g) and (g) $\Rightarrow$ (a).

Hence (a), (g) and (k) are equivalent.

Since (a) is equivalent to the statements in the circle, so are (g) and (k).

Further, (g), (h), and (i) are equivalent for any matrix.

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Now (d), (e), and (f) are all equivalent for any matrix A.

Since (d) is equivalent to the statements in the circle, so are (e) and (f).

Finally (a) $\Rightarrow$ (l) and (l) $\Rightarrow$ (a). (Why?)

This completes the proof of the theorem. $\square$

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Since (d) is equivalent to the statements in the circle, so are (e) and (f).

Finally (a) $\Rightarrow$ (l) and (l) $\Rightarrow$ (a). (Why?)

This completes the proof of the theorem. $\square$

Remark

Whenever A is invertible,

$$A^{-1} \text{ is invertible and } (A^{-1})^{-1} = A.$$

The Invertible Matrix Theorem divides the set of all n by n matrices into two disjoint classes:

1. the invertible (nonsingular) matrices, and
2. the non-invertible (singular) matrices.

Each statement in the theorem describes a property of every invertible matrix.

The Invertible Matrix Theorem divides the set of all n by n matrices into two disjoint classes:

1. the invertible (nonsingular) matrices, and
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The negation of a statement in the Invertible Matrix Theorem describes a property of every n by n singular matrix.

For instance, an n by n singular matrix is not row equivalent to I_n, does not have n pivot position, and has *linearly dependent* columns.

Example 1

Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}.$$

Solution.

$$A \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}.$$

The matrix A has three pivot positions.

By Part (c) of the Invertible Matrix Theorem, A is invertible. $\square$

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Remark

The Invertible Matrix Theorem **only applies to square matrices**.

For example, if the columns of a 4 by 3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $A\mathbf{x} = \mathbf{b}$.

2.4 Matrix Factorization

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.

Matrix multiplication involves a synthesis of data (combining the effects of two or more linear transformations into a single matrix).

On the other hand matrix factorization is an analysis of data.

The LU factorization

We will describe the LU factorization in the next few slides.

This is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$A\mathbf{x} = \mathbf{b}_1, \;\; A\mathbf{x} = \mathbf{b}_2, \;\; \ldots, \;\; A\mathbf{x} = \mathbf{b}_p.$$

When A is invertible, one could compute A^{-1} and then compute $A^{-1}\mathbf{b}_1$, $A^{-1}\mathbf{b}_2$, and so on.

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When A is invertible, one could compute A^{-1} and then compute $A^{-1}\mathbf{b}_1$, $A^{-1}\mathbf{b}_2$, and so on.

In actual computation, it is more efficient to solve the first equation $A\mathbf{x} = \mathbf{b}_1$ in the above sequence by row reduction and obtain the LU factorization of A at the same time.

Assume that A is an m by n matrix that can be row reduced to echelon form, without row interchanges.

Then A can be written in the form $A = LU$, were L is an m by m lower triangular matrix with 1's on the diagonal and U is an m by n echelon form of A.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \begin{pmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Such a factorization is called an LU *factorization* of A.

The matrix L is invertible and is called a *unit lower triangular matrix*.

What is good about LU factorization?

When $A = LU$, the equation $A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$.
Writing $\mathbf{y}$ for $U\mathbf{x}$, we can find $\mathbf{x}$ by solving two equations

$$L\mathbf{y} = \mathbf{b}$$
$$U\mathbf{x} = \mathbf{y}.$$

Each equation is easy to solve because L and U are triangular.

Example 1

It can be verified that

$$A = \begin{pmatrix} -3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix} \begin{pmatrix} -3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = LU.$$

Use this factorization of A to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} -9 \\ 5 \\ 7 \\ 11 \end{pmatrix}$.

Solution

The solution of $L\mathbf{y} = \mathbf{b}$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in Column 5.

$$(L \mid \mathbf{b}) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) = (I \mid \mathbf{y})$$

Then, for $U\mathbf{x} = \mathbf{y}$, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

$$(U \mid \mathbf{y}) = \left(\begin{array}{cccc|c} -3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) = (I \mid \mathbf{x}).$$

Hence $\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -6 \\ -1 \end{pmatrix}$.

To find $\mathbf{x}$ requires 28 arithmetic operations, excluding the cost of finding L and U.

In contrast, row reduction of $(A \mid \mathbf{b})$ to $(I \mid \mathbf{x})$ takes 62 operations.

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.

In this case, there exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that

$$E_p E_{p-1} \cdots E_1 A = U \tag{1}$$

Then

$$A = (E_p E_{p-1} \cdots E_1)^{-1} U = LU \tag{2}$$

where

$$L = (E_p E_{p-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_p^{-1}. \tag{3}$$

Each E_i is a unit lower triangular matrice.

Then E_i^{-1} a unit lower triangular matrice (Why?).

Products of unit lower triangular matrices is a unit lower triangular (Why?).

Thus L is unit lower triangular.

Note that row operations in Equation (1), which reduce A to U, also reduce the L in Equation (3) to I, because

$$E_pE_{p-1}\cdots E_1L = (E_pE_{p-1}\cdots E_1)(E_pE_{p-1}\cdots E_1)^{-1} = I.$$

This observation is the key to constructing L.

Algorithm for an LU Factorization

1. Reduce the n by m matrix A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists.

Example 2 below will show how to implement Step 2.

By construction, L will satisfy

$$(E_p E_{p-1} \cdots E_1)L = I$$

using the same $E_p, \ldots, E_1$ as in Equation (1).

Thus L will be invertible, by the Invertible Matrix Theorem, with

$$E_p E_{p-1} \cdots E_1 = L^{-1}.$$

By Equation (2)

$$A = LU.$$

Therefore Step 2 will produce an acceptable L.

Example 2

Find an LU factorization of

$$A = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}.$$

Solution

Since A has four rows, L should be a 4 by 4 matrix.

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix}.$$

We need to compute the entries of L denoted by $*$.

We will row reduce A to a upper triangular matrix U.

$$A = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$$

$$\sim A_1 = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix}$$

$$\sim A_2 = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{pmatrix} \sim U = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

We look at the row operations that create zeros in the first column of A, i.e. the reduction from A to A_1.

These row operators will also create zeros in the first column of L.

Hence the first column of L in blue is the first column of A divided by the top pivot entry:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ -3 & * & * & 1 \end{pmatrix}$$

Next we look at the row operations that create zeros in the second column of A_1, i.e. the reduction from A_1 to A_2.

$$A_1 = \begin{pmatrix} * & * & * & * & * \\ 0 & 3 & * & * & * \\ 0 & -9 & * & * & * \\ 0 & 12 & * & * & * \end{pmatrix} \sim A_2 = \begin{pmatrix} * & * & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}.$$

These row operators will also create zeros in the second column of L.
Hence the second column of L in blue is the second column of A_1 in blue divided by the pivot entry 3:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & * & 1 \end{pmatrix}.$$

Next we look at the row operations that create zeros in the second column of A_1, i.e. the reduction from A_1 to A_2.

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These row operators will also create zeros in the second column of L. Hence the second column of L in blue is the second column of A_1 in blue divided by the pivot entry 3:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & * & 1 \end{pmatrix}.$$

Next we look at the row operations that create zeros in the 4-th column of A_2, i.e. the reduction from A_2 to U.

$$A_1 = \begin{pmatrix} * & * & * & * & * \\ 0 & 3 & 1 & * & * \\ 0 & 0 & 0 & 2 & * \\ 0 & 0 & 0 & 4 & * \end{pmatrix} \sim A_2 = \begin{pmatrix} * & * & * & * & * \\ 0 & 3 & 1 & * & * \\ 0 & 0 & 0 & 2 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

These row operators will also create zeros in the third column of L.

Hence the second column of L in blue is the 4-th column of A_2 in blue divided by the pivot entry 2:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}.$$

Next we look at the row operations that create zeros in the 4-th column of A_2, i.e. the reduction from A_2 to U.

$$A_1 = \begin{pmatrix} * & * & * & * & * \\ 0 & 3 & 1 & * & * \\ 0 & 0 & 0 & {\color{blue}2} & * \\ 0 & 0 & 0 & {\color{blue}4} & * \end{pmatrix} \sim A_2 = \begin{pmatrix} * & * & * & * & * \\ 0 & 3 & 1 & * & * \\ 0 & 0 & 0 & 2 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

These row operators will also create zeros in the third column of L.
Hence the second column of L in blue is the 4-th column of A_2 in blue divided by the pivot entry 2:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & {\color{blue}1} & 0 \\ -3 & 4 & {\color{blue}2} & 1 \end{pmatrix}.$$

Finally an easy calculation verifies that this L and U satisfy $LU = A$. $\square$

Warning

In the algorithm for LU factorization, the first step is to reduce the n by m matrix A to an echelon form U by a sequence of row replacement operations. This step is applicable to **most but not all** n by m matrices.

Hence a small portion of matrices **does not have** LU factorization.

In Lie Theory, we have a beautiful formula called the *Brubat decomposition* to explain this discrepancy. Fortunately (or unfortunately) for us, this decomposition is beyond the scope of this module.