LECTURE 3

Investing with Risk-Free Asset

EC3333 Financial Economics I

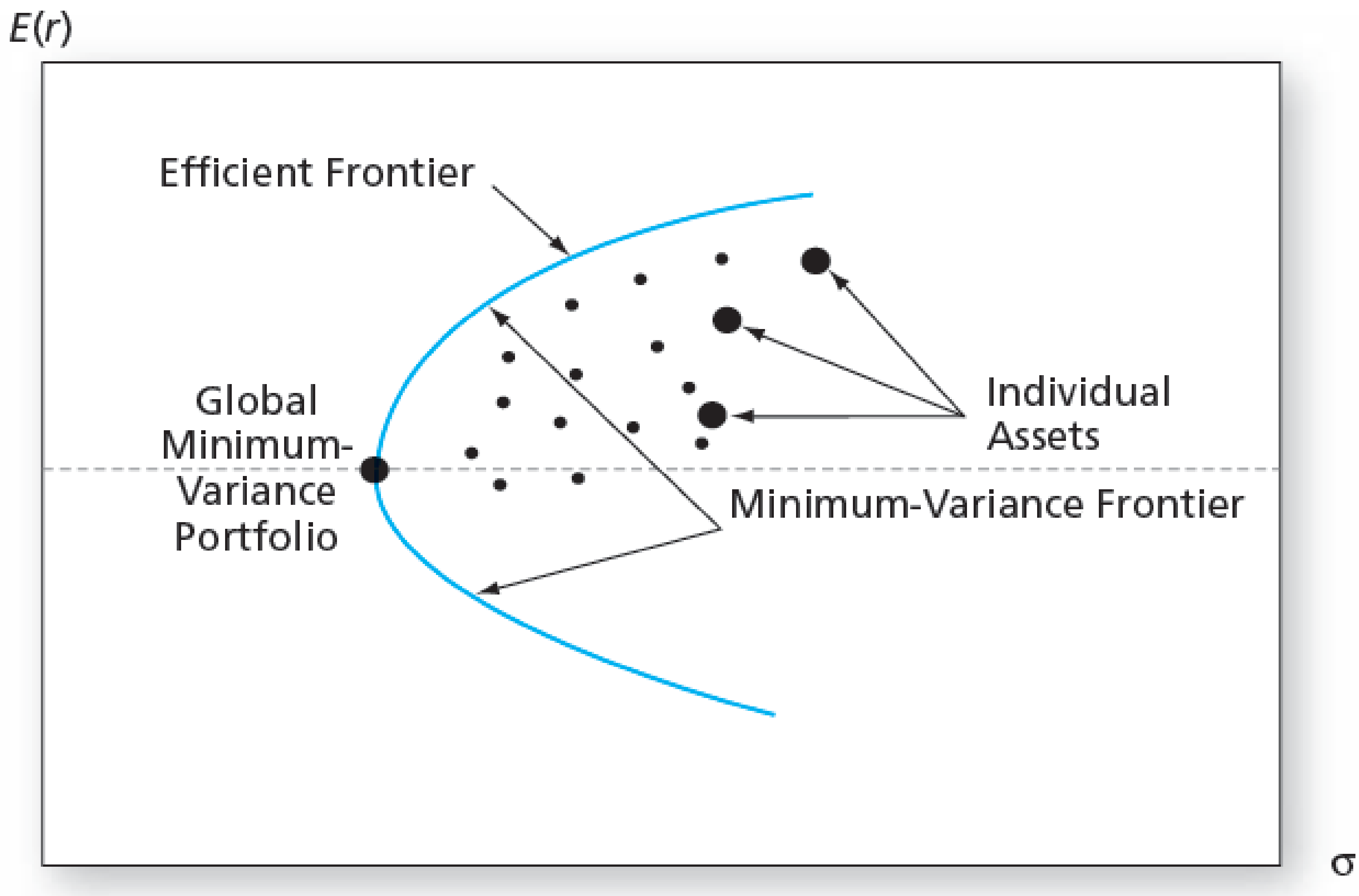
Learning Objectives

- Explain the effect of combining a risk-free asset with a portfolio of risky assets, and compute the expected return and volatility for that combination.
- Define the Sharpe ratio, and explain how it helps identify the portfolio with the highest possible expected return for any level of volatility, and how this information can be used to identify the tangency (efficient) portfolio.
- Show how risk aversion can be characterized by a utility function.
- Demonstrate the 2-step process of portfolio construction:
    1. determine of the tangency (efficient) portion of risky assets in complete portfolio, and
    2. allocate capital in the complete portfolio to risk-free versus tangency (efficient) portfolio of risky assets.

Part 1

- Identifying the optimal risky portfolio in the presence of a risk-free asset – i.e. the tangency (efficient) portfolio

Recall: Figure 7.10 The Efficient Frontier with Multiple Risky Assets
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Introducing a Risk-Free Asset

- Now suppose we introduce the opportunity to invest in a risk-free asset.
  - How does this affect the investors' portfolio choices?

Risk-Free Asset

- Only the government can issue default-free securities
  - A security is risk-free in real terms only if its price is indexed and maturity is equal to investor's holding period
- T-bills viewed as "the" risk-free asset
- Money market funds also considered risk-free in practice

The world around us: Index-linked Gilts

- Image source: https://dmo.gov.uk/data/gilt-market/index-linked-gilts/

United Kingdom
Debt Management
Office

Gilt Market

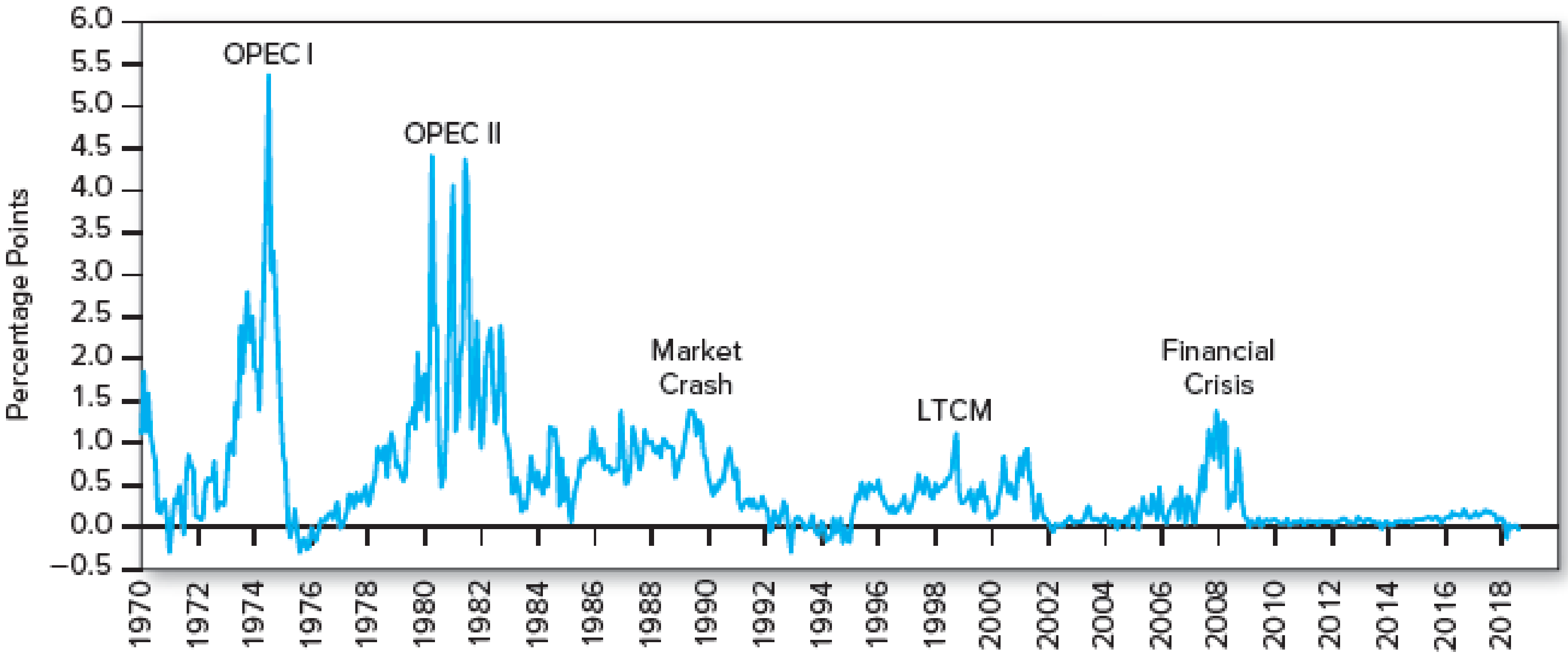
- **Gilt Market**
- Results of Gilt Operations
- Historical Prices and Yields
- Gilts in Issue
- Index-linked Gilts
- Government Holdings

Index-linked Gilts

The UK was one of the earliest developed economies to issue inflation-indexed bonds for institutional investors, with the first index-linked gilt issue being in 1981. A brief history of the main developments in the index-linked gilt market is available below.

Index-linked gilts differ from conventional gilts in that both the semi-annual coupon payments and the principal payment are adjusted in line with movements in the General Index of Retail Prices in the UK (also known as the RPI). RPI data since June 1980 and full details on all Index-linked gilts currently in issue are both available below.

Figure 2.2 Spread Between the federal funds and T-bill rates
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



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Risk and Return with a Risk-Free Asset

- Consider the following complete portfolio where:
  - x = portfolio weight on the risky portfolio, P
  - $(1 - x)$ = portfolio weight on the risk-free asset, F

- The expected return on the complete portfolio:

$$E(r_c) = (1 - x)r_f + xE(r_P)$$
$$E(r_c) = r_f + x[E(r_P) - r_f]$$

- The standard deviation of the complete portfolio:

$$\sigma_C = x\sigma_P$$

Risk and Return with a Risk-Free Asset

- Rearrange and substitute $x = \sigma_C/\sigma_P$

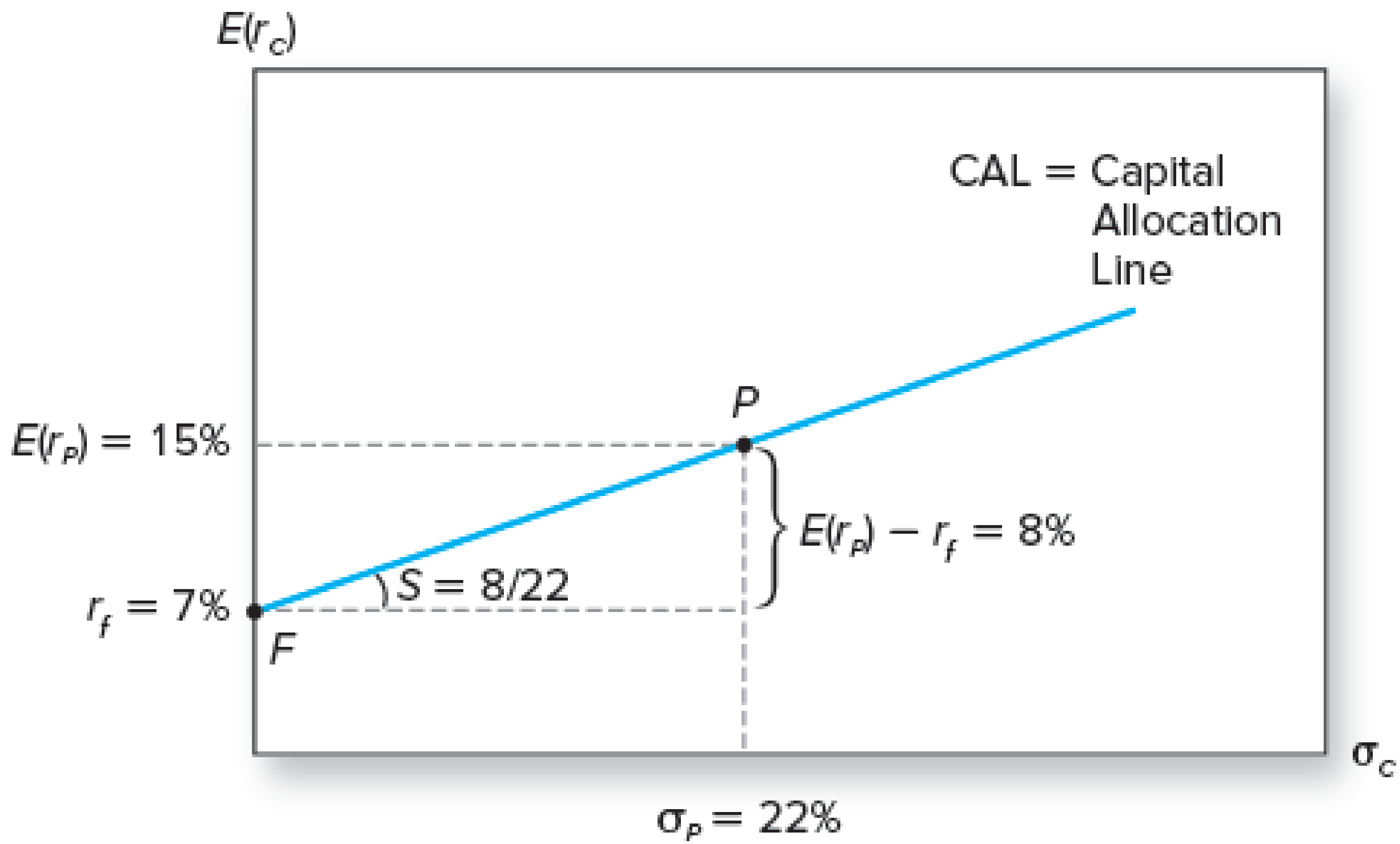
$$E(r_C) = r_f + \frac{[E(r_P) - r_f]}{\sigma_P}\sigma_C$$

- This equation is called the capital allocation line (CAL)

$$\text{Slope} = \frac{E(r_P) - r_f}{\sigma_P}$$

- The slope is the reward-to-volatility ratio: it is also called the **Sharpe ratio**

Figure 6.3 The Capital Allocation Line (CAL)
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



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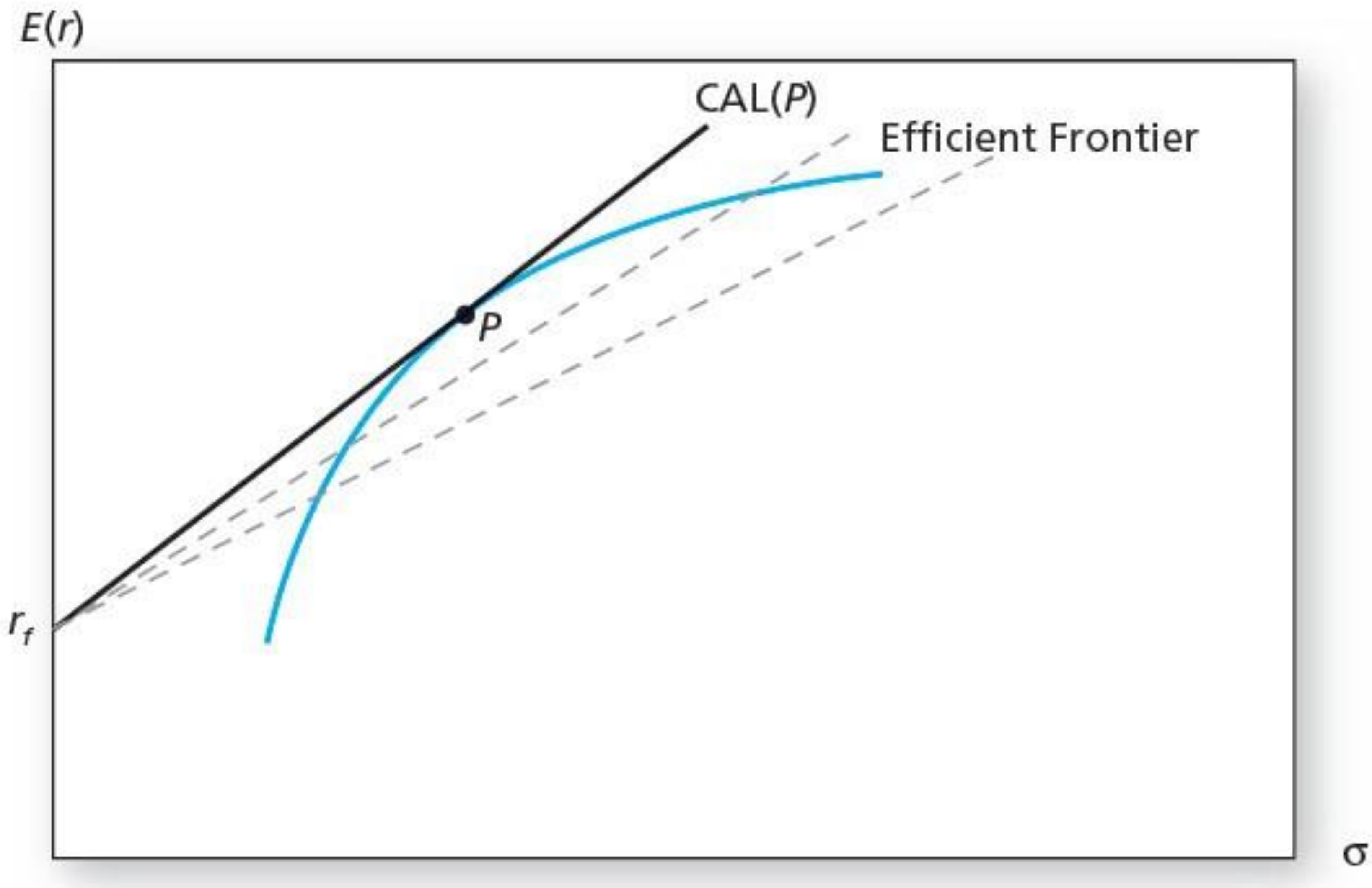
Optimal Risky Portfolio or Tangent Portfolio

- Investors will find the CAL with the highest reward-to-volatility ratio to find the optimal risky portfolio on the efficient frontier
- Maximize the slope of the CAL for any possible portfolio, *P*
- The objective function is the slope:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

- Recall that the slope is also called the Sharpe ratio

Figure 7.11 The Efficient Frontier of Risky Assets with the Optimal CAL (from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



Efficient Portfolio & Required Returns

- Consider an arbitrary portfolio of risky securities, P
- Consider whether we could raise its Sharpe ratio by adding investment i to the portfolio P
- Specifically, we will short the risk-free assets (i.e., borrow at the risk-free rate) and invest the proceeds in investment i

Efficient Portfolio & Required Returns

- Adding *i* to the portfolio *P* will improve our Sharpe ratio if

$$\frac{E[R_i]-r_f}{SD(R_i)} > \mathrm{Corr}(R_i, R_P) \times \frac{E[R_P]-r_f}{SD(R_P)}$$

$$\underbrace{E[R_i]-r_f}_{\text{Additional return from investment } i} > \underbrace{\underbrace{SD(R_i)\times \mathrm{Corr}(R_i, R_P)}_{\text{Incremental volatility from investment } i} \times \underbrace{\frac{E[R_P]-r_f}{SD(R_P)}}_{\text{Return per unit volatility available from portfolio } P}}_{\text{Additional return from taking the same risk investing in } P}$$

Efficient Portfolio & Required Returns

- Define Beta of Portfolio *i* with Portfolio P

$$\beta_i^P = \frac{SD(R_i) \times \text{Corr}(R_i, R_P)}{SD(R_P)}$$

$$\beta_i^P = \frac{SD(R_i)}{SD(R_P)} \times \frac{\text{Cov}(R_i, R_P)}{SD(R_i)SD(R_p)}$$

$$\beta_i^P = \frac{\text{Cov}(R_i, R_P)}{\text{Var}(R_P)}$$

β_i^P Measures the sensitivity of investment *i* to the fluctuations of the portfolio *P*.
For each 1% change in the portfolio's return, investment *i*'s return is expected to change by *beta* percent due to the risk *i* has in common with *P*.

Expected Returns & the Efficient Portfolio

- The required return is the expected return necessary to compensate for the risk investment *i* will contribute to the portfolio *P*.

$$r_i \equiv r_f + \beta_i^P \times (E[R_P] - r_f)$$

- If $E[R_i] > r_f + \beta_i^{\mathrm{P}} \times (E[R_P] - r_f)$

- Then, increasing the amount invested in *i* will increase the Sharpe ratio of the portfolio *P.*

- Thus, a portfolio is efficient if and only if the expected return of every available security equals its required return (defined with reference to the portfolio).

- Expected Return of a Security

$$E[R_i] = r_i \equiv r_f + \beta_i^{eff} \times (E[R_{eff}] - r_f)$$

- R_{eff} is the return of the efficient (tangent) portfolio

Part 2

- Allocating capital in the complete portfolio to risk-free versus tangency (efficient) portfolio of risky assets – by maximising utility given a risk preference

Risk Aversion and Utility Values

- We assume each investor can assign a welfare, or utility, score to competing portfolios

$$U = E(r) - \tfrac{1}{2} A\sigma^2$$

- Utility function
- U = Utility value
- E(r) = Expected return
- A = Index of the investor's risk aversion
- σ^2 = Variance of returns
- ½ = Scaling factor

Table 6.1 Available Risky Portfolio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Table 6.1

Available risky portfolios (risk-free rate = 5%)

Table 6.2 Utility scores of alternative portfolios for investors with varying degrees of risk aversion
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

Investor Risk Aversion (A)	Utility Score of Portfolio L $[E(r) = .07; \sigma = .05]$	Utility Score of Portfolio M $[E(r) = .09; \sigma = .10]$	Utility Score of Portfolio H $[E(r) = .13; \sigma = .20]$
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	$\mathbf{.13 - ½ \times 2 \times .2^2 = .09}$
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	$\mathbf{.09 - ½ \times 3.5 \times .1^2 = .0725}$	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	$\mathbf{.09 - ½ \times 5 \times .1^2 = .0650}$	$.13 - ½ \times 5 \times .2^2 = .03$

Investor Types

- Risk-averse investors consider risky portfolios only if they provide compensation for risk via a risk premium
  - $A > 0$
- Risk-neutral investors find the level of risk irrelevant and consider only the expected return of risk prospects
  - $A = 0$
- Risk lovers are willing to accept lower expected returns on prospects with higher amounts of risk
  - $A < 0$

Risk Aversion & Mean-Variance (M-V) Criterion

- Investors who are risk averse reject investment portfolios that are fair games or worse

- A fair game is a risky investment with a risk premium or expected excess return of zero

- Mean-Variance (M-V) Criterion: Portfolio A dominates portfolio B if:

$$E(r_A) \geq E(r_B)$$

$$\& \quad \sigma_A \leq \sigma_B$$

Figure 6.1 Trade-off between risk and return
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

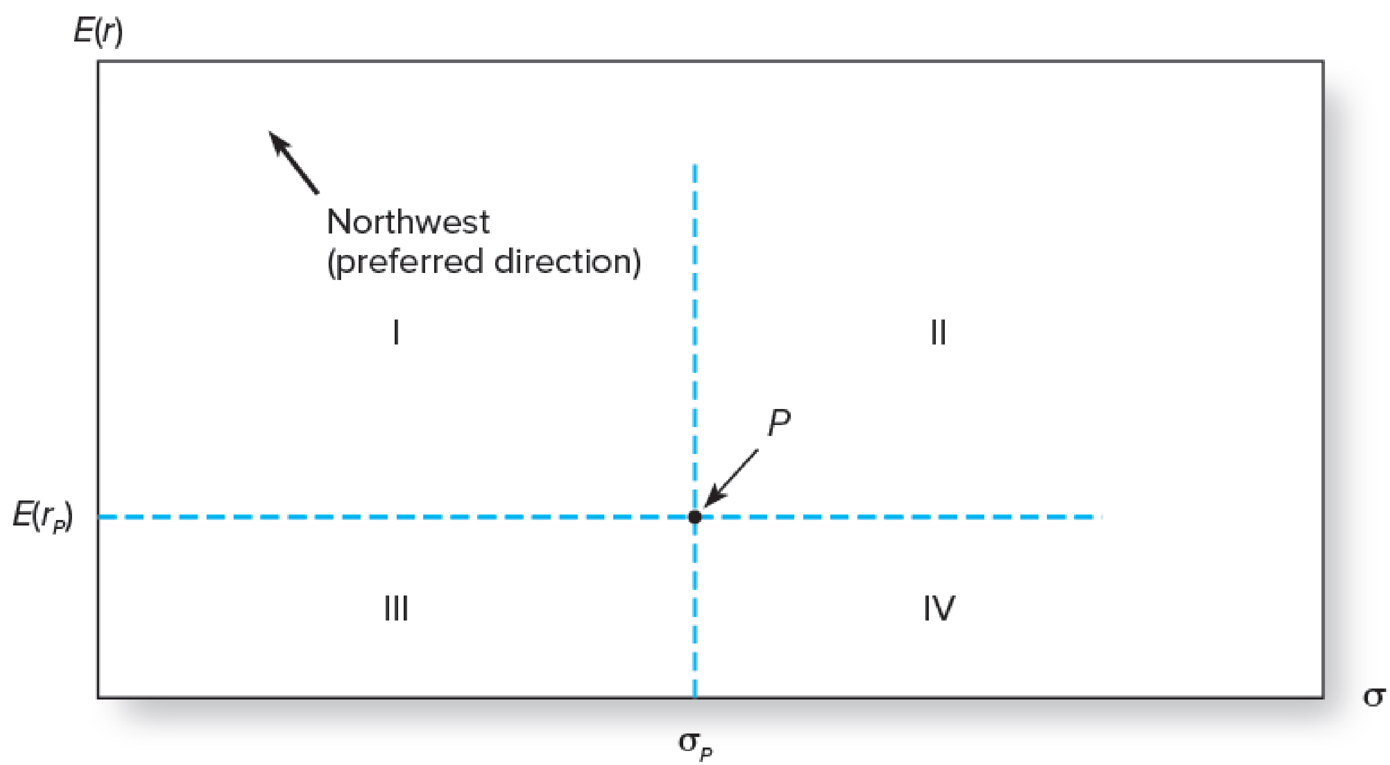


Figure 6.1 The trade-off between risk and return of a potential investment portfolio, *P*

Figure 6.2 The Indifference Curve
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

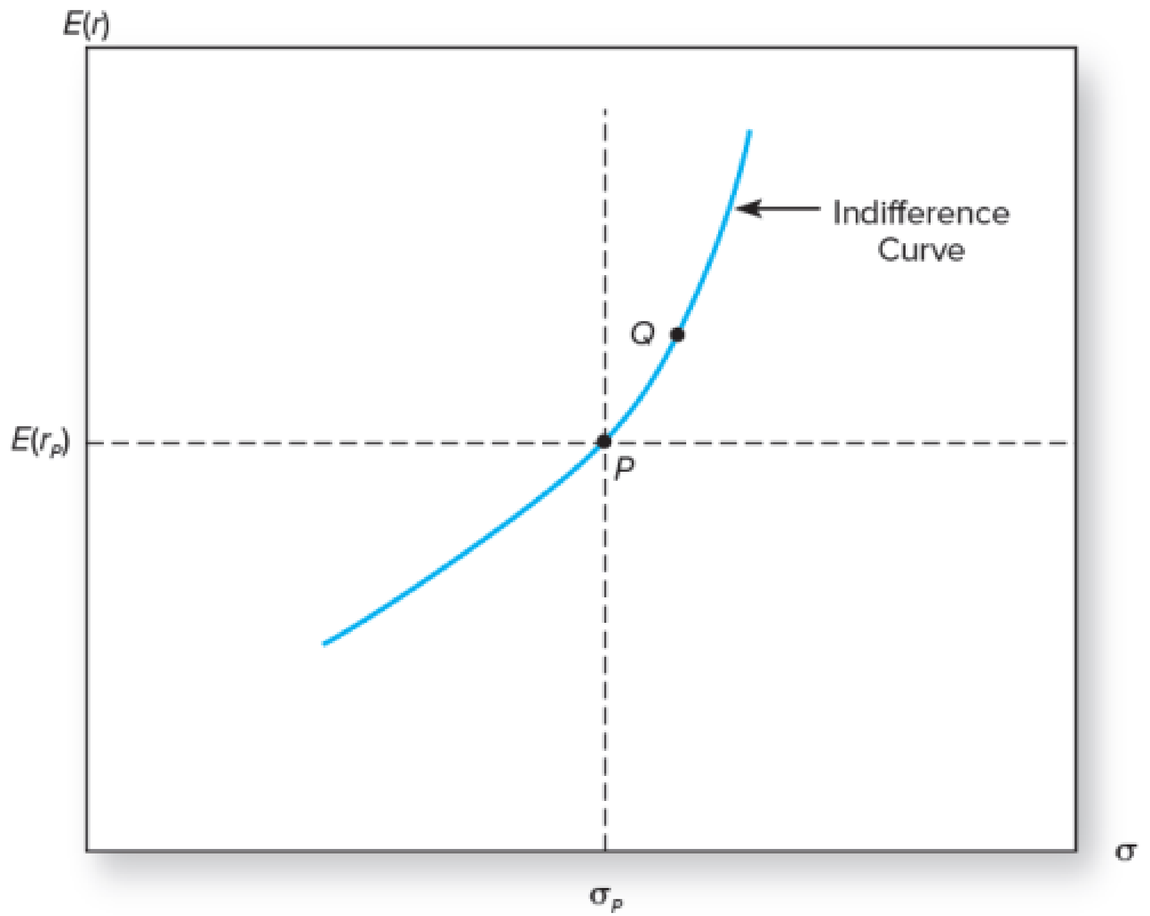


Figure 6.2 The indifference curve

Figure 6.6 Indifference Curves for *A* = 2 and *A* = 4, where U = E(r) – 0.5 Aσ^2
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

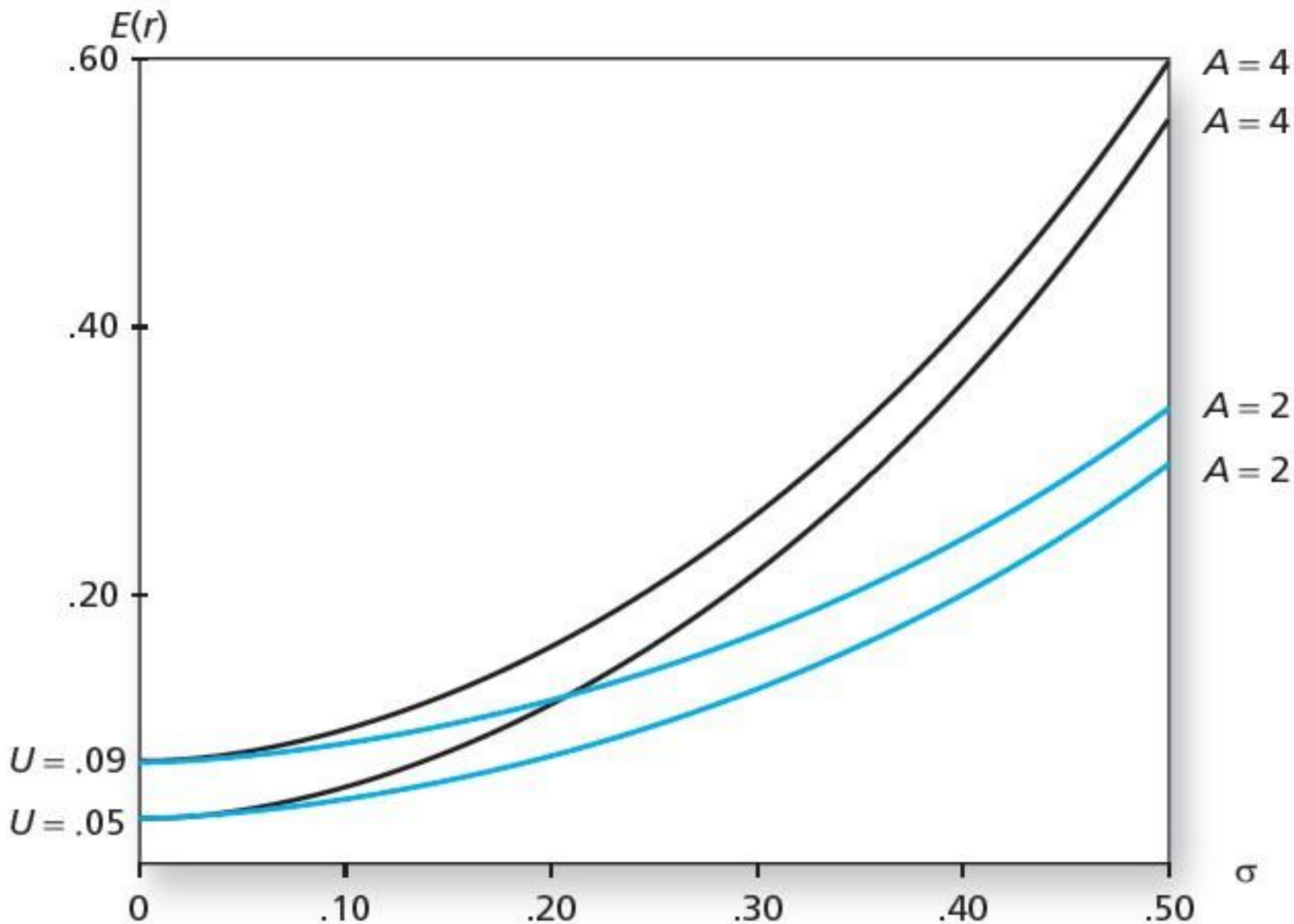


Figure 6.7 Portfolio C is the Optimal Complete Portfolio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)

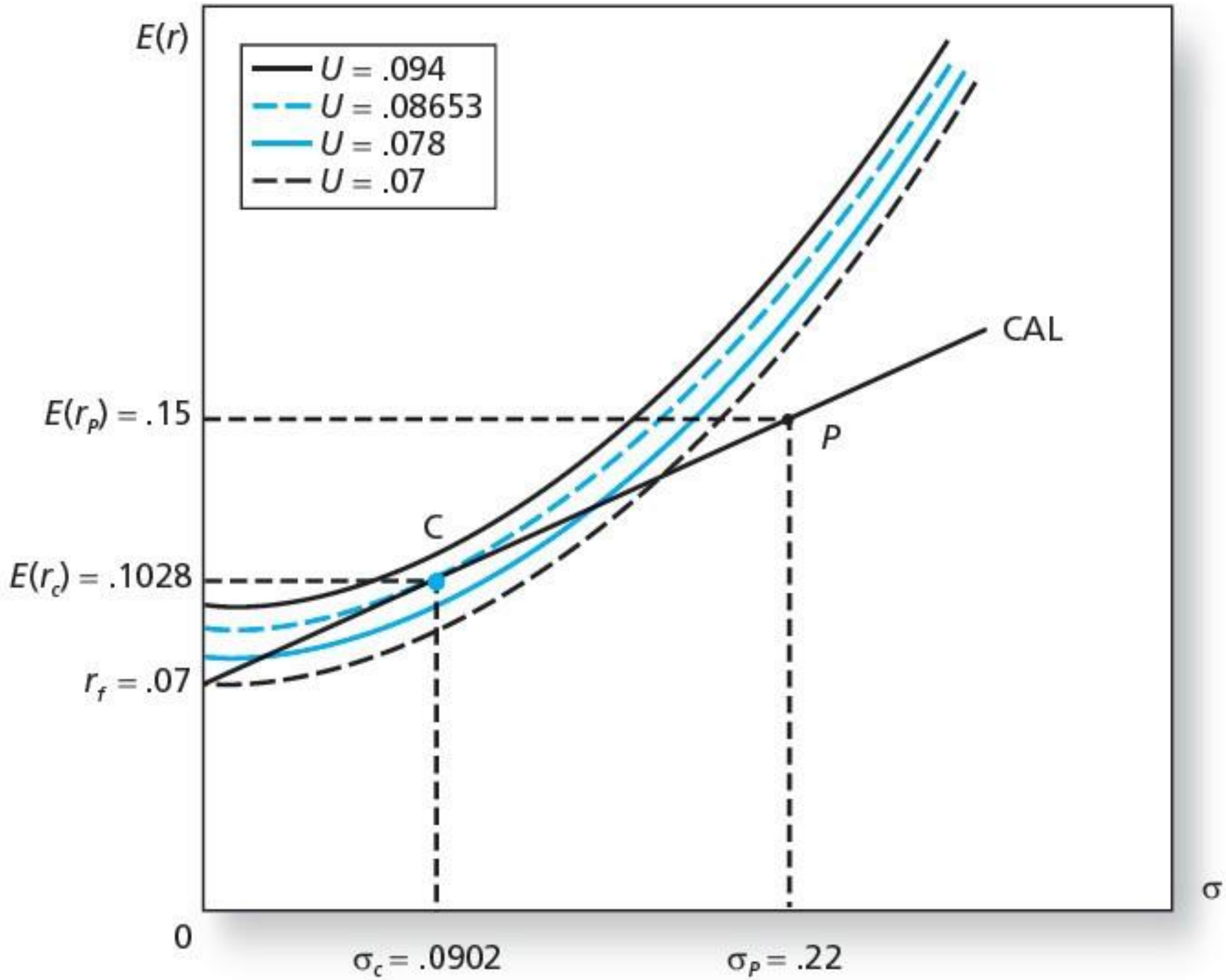
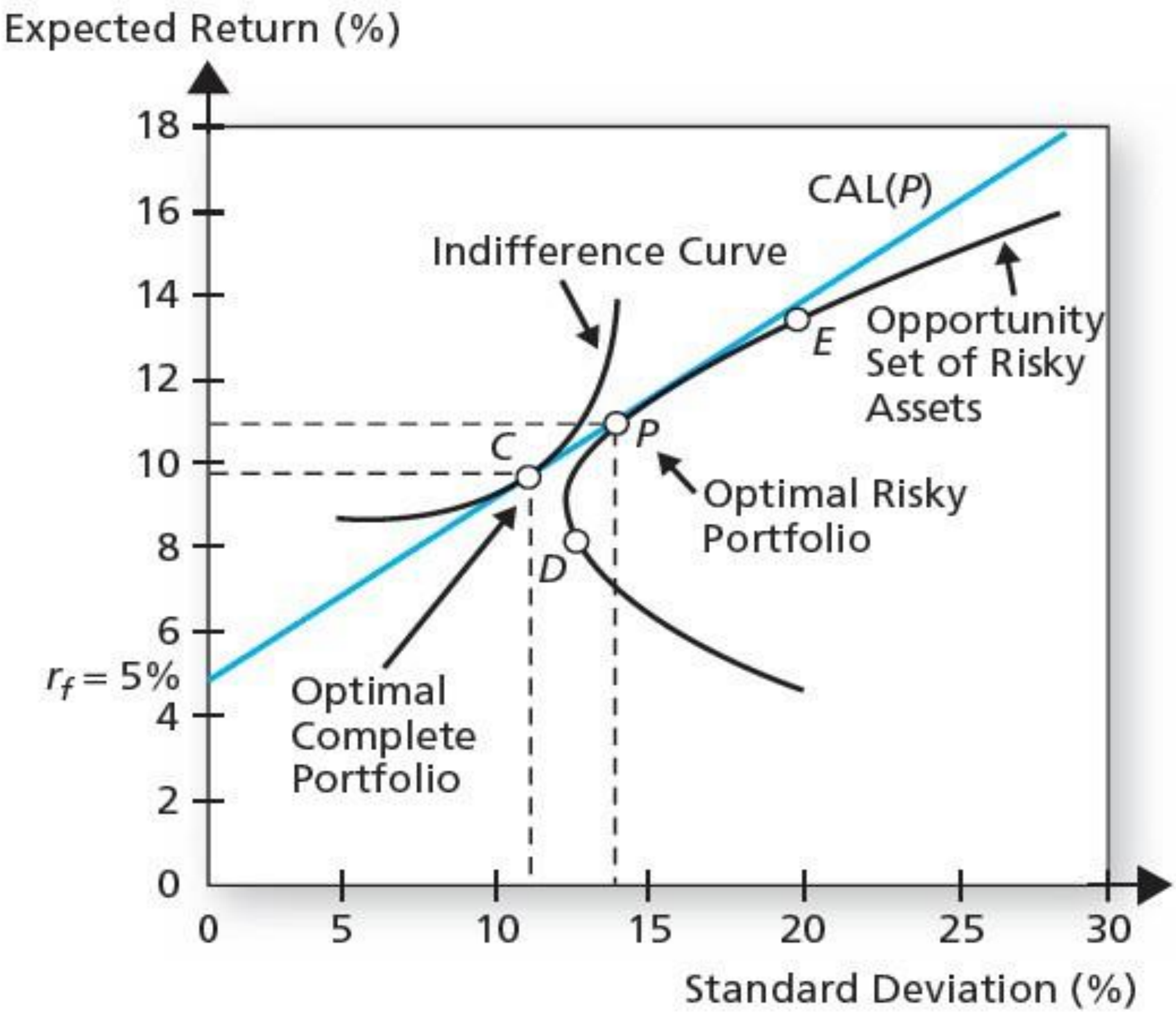


Figure 7.8 Determination of the Optimal Overall Portfolio
(from adopted text, Bodie, Kane and Marcus, Investments, McGraw Hill, 12e)



The Separation Property / Theorem

- Portfolio choice problem may be separated into two independent tasks
  - Determination of the optimal risky portfolio is purely technical
    - Everyone invests in *P*, regardless of their degree of risk aversion
  - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference
    - More risk averse investors put more in the risk-free asset