$\boxed{1}$ $\Rightarrow$) All diagonals of $U$ are not 0.

$A = LU \rightsquigarrow A_{1:k,1:k} = L_{1:k,1:k} U_{1:k,1:k}$, $A_{1:k,1:k}$ nonsingular for $\forall 1 \le k \le m$.

$\Leftarrow$) Induction on $m$.

① $m=1$. trivial

② $m=k$.

For $m=k+1$, Let $A' = A_{1:k,1:k}$. $\rightsquigarrow A_{1:k,1:k} = L'U'$

$$A = \left[\begin{array}{c|c} A' & b \\ \hline c^* & d \end{array}\right] = \left[\begin{array}{c|c} L' & 0 \\ \hline x^* & 1 \end{array}\right] \left[\begin{array}{c|c} U' & y \\ \hline 0 & z \end{array}\right]$$

$A' = L'U'$ $b = L'y$. $c^* = x^* U'$ $d = x^* y + z$

Set $x^* = c^*(U')^{-1}, y = (L')^{-1} b$, $z = d - x^* y$.

$\to$ $L$ is clearly nonsingular.
Then, $U = (L^{-1})A$
so $U$ is also nonsingular
Done. ■

* uniqueness.

Spse $A = L_1U_1 = L_2U_2$. As $L_i$ & $U_i$ are lower&upper triangular matrix with nonzero diagonals, they are invertible.

$\Rightarrow L_2^{-1}L_1 = U_2U_1^{-1}$

Note that inverse of upper/lower triangular matrix is also upper/lower triangular
products "
which can be proved by just simple calculation.

So, $L_2^{-1}L_1$ & $U_2U_1^{-1}$ are diagonals. $\to L_2^{-1}L_1 = U_2U_1^{-1} = I$ ■

$\boxed{2}$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,p+1} & & & \\ \vdots & \ddots & \vdots & \ddots & & \\ a_{p+1,1} & \cdots & \cdots & \cdots & \ddots & \\ & \ddots & \ddots & \vdots & \ddots & a_{m-p,m} \\ & & \ddots & \vdots & \ddots & \vdots \\ & & & a_{m,m-p} & \cdots & a_{mm} \end{bmatrix}$$

$L$ is lower triangular with $l_{ij} = 0$ for $i > j+p$.
$U$ is upper " $u_{ij} = 0$ for $j - i > p$.

$\boxed{3}$ (a)
$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} IA_{11} + 0A_{21} & IA_{12} + 0A_{22} \\ -A_{21} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$ ■

(b) After $n$ steps of Gaussian Elimination, we can write

$$L' = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I \end{bmatrix}, U' = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$ where $A = L'U'$. $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

Note that $U'$ is $A$ after $n$ steps of Gaussian Elimination.

$\Rightarrow L_{11}U_{11} = A_{11}$. By Exercise 20.1, $A_{11}$ invertible.

$L_{11}U_{12} = A_{12}$ Note that $n \times n$ $L_{11}$ is lower triangular which all diagonals are 1. ) So invertible.

$L_{21}U_{11} = A_{21}$ $\quad U_{11}$ upper "

$L_{21}U_{12} + U_{22} = A_{22}$.

Our Goal: $U_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$

$U_{22} = A_{22} - L_{21}\cdot U_{12}$. And $\left.\begin{array}{l} L_{21}U_{11}=A_{21} \rightsquigarrow L_{21}=A_{21}\cdot U_{11}^{-1} \\ L_{11}U_{12}=A_{12} \rightsquigarrow U_{12}=A_{12}\cdot L_{11}^{-1} \\ L_{11}U_{11}=A_{11} \rightsquigarrow A_{11}^{-1}=U_{11}^{-1}L_{11}^{-1} \end{array}\right) \Rightarrow U_{22}=A_{22}-A_{21}\underbrace{U_{11}^{-1}\cdot L_{11}^{-1}}_{A_{11}^{-1}}\cdot A_{12}$ ∎

$\boxed{4}$

**Algorithm 20.1. Gaussian Elimination without Pivoting**

$U = A,\ L = I$
for $k = 1$ to $m-1$
  for $j = k+1$ to $m$
    $\ell_{jk} = u_{jk}/u_{kk}$
    $u_{j,k:m} = u_{j,k:m} - \ell_{jk}u_{k,k:m}$

$\Rightarrow$ $U=A, L=I$
for $k=1$ to $m-1$
  $\ell_{k+1:m,k} = u_{k+1:m,k}/u_{kk}$
  $u_{k+1:m,k:m} = u_{k+1:m,k:m} - \ell_{k+1:m,k}\cdot u_{k,k:m}$

Note that this is the outer product.

$k^{th}$

pivot

$k^{th}$

$$a_{kk} \quad a_{k,k+1} \cdots a_{k,m}$$

$$\boxed{\begin{array}{l} a_{k+1,k} \\ \vdots \\ a_{m,k} \end{array}} - \begin{bmatrix} a_{k+1,k}/a_{kk} \\ \vdots \\ a_{m,k}/a_{kk} \end{bmatrix} [a_{kk} \ \cdots \ a_{k,m}]$$

eliminating the $k^{th}$ column below the pivot.

$u_{k+1:m,\,k:m}$ $((m-k)\times(m-k+1))$   $\ell_{k+1:m,k}$ $((m-k)\times 1)$   $u_{k,k:m}$ $(1\times(m-k+1))$

So, each rows of this outer product corresponds to the elementary row operation: subtracting the mutiple of $k^{th}$ row by $\frac{a_{i,k}}{\text{pivot}}$, which eliminates the entries of $k^{th}$ column below the pivot $a_{kk}$.
$\Rightarrow$ Gauss elimination

$\boxed{5}$ (a) $L = \begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1m}/a_{11} \\ & 1 & & a_{2m}/a_{22} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$ upper $\Delta$. And, $U = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & 0 \\ & & \ddots & \\ & & & * \end{bmatrix}$ lower $\Delta$.

(b) $\ell_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i<j) \\ \dfrac{\cancel{d_j} a_{ij}}{\cancel{d_j} a_{jj}} & (i>j) \end{cases}$ $\Rightarrow$ $L$ doesn't change.

$AD = LUD = LU'$
$\rightsquigarrow$ $U$ becomes $UD$.

$Ax=b \rightsquigarrow L(U'x)=b \rightsquigarrow LU\underbrace{Dx}_{=x'}=b$ so, solutions for this equation rescaled by $D^{-1}$

(c) $A = LU'$ Now, we are trying to apply (a) to $U' \rightsquigarrow U'=UD$

Note that $D'$ is diagonal: $(D)_{ij} = \begin{cases} 0 \ (i>j) & \text{since it is Gaussian eliminated from an upper } \Delta) \\ u_{ij}\ (i=j) & \text{"} \\ 0 \ (i<j) & \text{since it is Gaussian eliminated} \end{cases}$

$\therefore A = LUD$