$\boxed{1}$ (a) Fix $\forall \lambda \in \mathbb{C}$. Consider $A-\lambda I$. Notice that $B=(A-\lambda I)_{2:m,2:m}$ (eliminating 1^{st} row & last column) is full rank as it is an upper triangular with nonzero diagonals which correspond to A's subdiagonal.

$$\begin{bmatrix} \times & \times & & \\ \times & \times & \times & \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

$: B_{ii} = (A-\lambda I)_{i+1,i} = A_{i+1,i} \quad (1 \le i \le m-1)$

and $B_{ij} = (A-\lambda I)_{i+1,j} = A_{i+1,j} = 0$ for $\forall i > j$ by def of tridiagonal

$\hookrightarrow i+1 > j$

So, if rank $(A-\lambda I) \le m-2$, there must be a linearly dependent pair of rows in B, which contradicts to full rank of B.

$\Rightarrow \text{rank}(A-\lambda I) \ge m-1$

Now, as A hermitian, by Thm 24.7, it diagonalizable by $A = X^* \Lambda X$. And by Thm 24.3, Λ & A have the exactly same eigenvalues which appear on Λ.

$$A - \lambda I = X^* \Lambda X - \lambda I = X^*(\Lambda - \lambda I) X.$$

As $A-\lambda I$ & $\Lambda - \lambda I$ similar, have the same rank i.e, rank $(\Lambda - \lambda I) \ge m-1$.

Denote $\lambda_i = (\Lambda)_{ii}$ $(1 \le i \le m)$. And each λ_i is the eigenvalue of A. Spse the eigenvalues of A are not distinct, i.e. $\lambda_i = \lambda_j$ for some i & j. Fix $\lambda = \lambda_i = \lambda_j$

Since $\Lambda - \lambda I$ is diagonal, rank $(\Lambda - \lambda I) = $ (# of nonzero diagonals). But, as $(\Lambda - \lambda I)_{ii} = (\,'' \,)_{jj} = 0$, rank $(\Lambda - \lambda I) = $ rank $(A - \lambda I) \le m-2$ (contradiction) $\Rightarrow$ the eigenvalues of A are distinct. ∎

(b) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = A$ $\quad$ nonzero

$\hookrightarrow$ Hessenberg as $a_{i,j} = 0$ for all $i > j+1$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda+1 & 1 \\ 0 & -1 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda^2-1+1) - (\lambda-1)$$
$$= (\lambda^2-1)(\lambda-1) = (\lambda+1)(\lambda-1)^2$$

$\hookrightarrow \lambda = 1$ with algebraic mult $2 \Rightarrow$ not distinct. ∎

$\boxed{3}$ (a) (ii)

By left multiplication of some Householder reflectors, $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$

For some appropriate Householder reflectors, we can construct $Q_2 \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}^* = Q_2 \begin{bmatrix} x & 0 & 0 \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ x & x & x \\ 0 & x & x \end{bmatrix}$

So, $\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix} Q_2^* = \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$.

Again, left multiplication of some Householder reflectors Q_3 makes $Q_3 \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & x & x \end{bmatrix} = \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$

$\therefore Q_3 Q_1 A Q_2^*$

But, notice that it can't be obtained only by (i) "generally".

Consider $A = \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix}$ for arbitrary a, b, c where at least one of a, b, c is nonzero. Fix any unit vector q. $(q \neq 0)$

To make the right upper entry to 0, we need $Q = I_3 - 2qq^*$. let $\begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$ $\rightarrow$ so do V_{12}, V_{22}, V_{32} & V_{13}, V_{23}, V_{33}.

(it can't be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cdot & \\ 0 & & \cdot \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ & & \cdot \end{bmatrix}$ for $a \neq 0$)

$\rightsquigarrow QA = A - \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} A \rightsquigarrow \begin{matrix} \frac{1}{2}V_{11} = q_1^2 \\ \frac{1}{2}V_{21} = q_2 q_1 \\ \frac{1}{2}V_{31} = q_3 q_1 \end{matrix}$ one of them must be nonzero as $q_1^2 + q_2^2 + q_3^2 = 1$.

Then, we can see that left multiplication of these Q is the row operation ($*$const, add/sub between rows)

$\Rightarrow$ applying Q again, it is easy to show that is again the form of $\begin{bmatrix} a & a' & a' \\ b' & b' & b' \\ c' & c' & c' \end{bmatrix}$.

So, the form $\begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$ of A must be $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

But, as Q invertible, rank $(QA) = \text{rank}(A) = 1$ can't be 0 forever

Thus, again, at least one of a', b', c' is nonzero.

$\therefore$ It implies inductively we can't get rid of this nonzero rows, so A can't be $\begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$ by (i).

It is much straightforward if Q is Givens rotation. As it is rotation, Q is the form of

$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}}_{\text{let } Q_1}$ or $\underbrace{\begin{bmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{bmatrix}}_{Q_2}$ or $\underbrace{\begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_3}$. with $a^2+b^2=1$. For $A = \begin{bmatrix} x & x & x \\ y & y & y \\ z & z & z \end{bmatrix}$, $Q_1 A = \begin{bmatrix} x & x & x \\ ay-bz & ay-bz & ay-bz \\ by+az & by+az & by+az \end{bmatrix}$

it can't be $a=b=0$.

$Q_2 A = \cdots$, $Q_3 A = \cdots$

(it is simple calculation & shows the same results)

So, again, left multiplication of each Q_i is the row operation ($*$const, add/sub between rows)

(b) (ii)

From (a), $\begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & x & x \end{bmatrix}$ can be obtained (so, left $\to$ right $\to$ left $\to$ right)

Note that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_1 & \\ & I - 2\cdot\frac{vv^*}{v^*v} \end{bmatrix}$ for $v = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ $\rightsquigarrow$ $\frac{vv^*}{v^*v} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

so Householder reflector,,

For same reason with (a), it can't be obtained by just left multiplications for arbitrary 3x3 matrix

(c) (iii)

$\det\left(\begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}\right) = 0$ clearly.

Spse a nonsingular 3x3 matrix A is given and for appropriate Q_j's,

$\underbrace{Q_k \cdots Q_1}_{Q} A \underbrace{Q_1' \cdots Q_l'}_{Q'} = \begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$. Then $\det(QAQ') = 0$

$\det(QAQ') = \det(Q)\det(Q')\det(A)$.

Since $|\det(Q)| = 1$ for unitary matrix Q, $\det(A)$ must be 0 which contradicts to nonsingular.

So, it can't be obtained by any seq of multiplications by Q_j ∎

Solution.

(a) Note that the first and last rows are orthogonal. Since the left multiplication of Q_j preserves orthogonality, (a) cannot be obtained from general matrix by a sequence of left-multiplications by matrices Q_j However, by allowing right multiplication also, (a) can be obtained from general matrix (by singular value decomposition, we can get a stricter structure).

(b) Suppose that structure (b) is obtained from the matrix whose entries are all 1 by just left multiplication. Since all columns of (b) should be the same, (b) is the zero matrix. This is a contradiction because the left multiplication of a unitary matrix preserves the rank. Therefore, (b) cannot be obtained from general matrices by just left multiplication. On the other hand, since (b) can be obtained by changing the order of rows of the structure (a), (b) can be obtained from general matrix by allowing left and right multiplications.

(c) Since multiplication of unitary matrix preserves the rank of matrix, (c) cannot be obtained from general matrix even if we allow both left and right multiplications of unitary matrices.