$\boxed{1}$ (i) → (ii)

As z is an eigenvalue of $A+\delta A$, $\exists\, x \in \mathbb{C}^m$, $\|x\|_2=1$ s.t $(A+\delta A)x = zx \rightsquigarrow (A-zI)x = -(\delta A)x$.

So, $\|(A-zI)x\|_2 = \|(\delta A)x\|_2 \le \sup_{\|u\|_2=1} \|(\delta A)u\|_2 = \|\delta A\|_2 \le \varepsilon$ //

(ii) → (iii)

Claim : $\sigma_m(zI-A) = \inf_{\|u\|_2=1} \|(zI-A)u\|_2$

pf) By Thm 4.1, we can find the SVD of $zI-A$: $U \overbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix}}^{\Sigma} V^*$ where $\sigma_1 \ge \cdots \ge \sigma_m \ge 0$.

For $^\forall x \in \mathbb{C}^m$ with $\|x\|_2=1$, $\|\Sigma x\|_2 = \sqrt{\sigma_1^2 x_1^2 + \cdots + \sigma_m^2 x_m^2} \ge \sigma_m \|x\|_2 = \sigma_m$.

And, as U & V unitary & Thm 3.1, always $\exists\, u \in \mathbb{C}^m$ with $V^*u = x$ for arbitrary x so that

$\|(zI-A)u\|_2 = \|U\Sigma V^* u\|_2 = \|\Sigma V^* u\|_2 = \|\Sigma x\|_2 \ge \sigma_m$.

Thus, $\inf_{\|u\|_2=1} \|(zI-A)u\|_2 \ge \sigma_m$. Since this lower bound is obtainable when $x = e_m$ (so $u = Ve_m$, done //

Since $\exists\, u \in \mathbb{C}^m$ with $\|u\|_2=1$ s.t $\|(A-zI)u\|_2 \le \varepsilon$,

$\sigma_m(zI-A) = \inf_{\|u\|_2=1} \|(zI-A)u\|_2 \le \|(A-zI)u\|_2 \le \varepsilon$ //

(iii) → (iv)

As the singular values of $(zI-A)^{-1}$ are the inverses of that of $zI-A$,

$\exists$(the singular value of $zI-A$) $\le \varepsilon \longleftrightarrow$ (iv) clearly as $\|(zI-A)^{-1}\|_2 =$ (the largest singular val of $(zI-A)^{-1}$)

Done. ($\because \sigma_m(zI-A) \le \varepsilon$) //

(iv) → (ii)

$\exists\, v \in \mathbb{C}^m$ with $\|v\|_2 = 1$ s.t $\|(zI-A)^{-1}v\|_2 \le \varepsilon^{-1}$.

(If $^\forall x \in \mathbb{C}^m$ with $\|x\|_2=1$, $\|(zI-A)^{-1}x\|_2 < \varepsilon^{-1}$, contradiction since $\sup_{\|x\|_2=1} \| \,''\, \|_2 < \varepsilon^{-1}$.

Note that this inequality is strict since if $\sup_{\|x\|_2=1} \| \,''\, \|_2 = \varepsilon^{-1}$, by cptness of the unit sphere, $\exists\, y \in \mathbb{C}^m$ with $\|y\|_2=1$ s.t $\|(zI-A)^{-1}y\|_2 = \sup = \varepsilon^{-1}$, contradiction again.)

Let $u = (zI-A)^{-1}v$. By mutiplying $(zI-A)$, $(zI-A)u = v$. $\rightsquigarrow (zI - A - \frac{vu^*}{u^*u})u = 0$.

$\therefore$ z is an eigenvalue of $A + \frac{vu^*}{u^*u}$ whose eigenvector is u.

Let $\delta A = \frac{vu^*}{u^*u}$. $\|\delta A\|_2 \le \frac{\|v\|_2 \|u^*\|_2}{|u^*u|} = \frac{\|u^*\|_2}{|u^*u|} \underset{\text{Cauchy-Schwartz}}{\le} \frac{\|u^*\|_2}{|u^*u|}$

$\boxed{3}$ (a) Let $\hat{\lambda}$ be any eigenvalues of $A+\delta A$. If $\hat{\lambda}$ is also an eigenvalue of A, it's done.
Spse not. $\|\delta A\|_2 \le \varepsilon = \|\delta A\|_2$.

By Exercise 26.1, $\|(\hat{\lambda} I - A)^{-1}\|_2 \ge \dfrac{1}{\|\delta A\|_2}$. And, $\|(\hat{\lambda} I - A)^{-1}\|_2 = \|V^{-1}(\hat{\lambda} I - \Lambda)^{-1} V\|_2$
$\le K(V) \|(\hat{\lambda} I - \Lambda)^{-1}\|_2$

So, $1 \le \|(\Lambda - \hat{\lambda} I)^{-1}\|_2 \, K(V) \, \|\delta A\|_2$

Now, from this, it's clear that $\|(\Lambda - \hat{\lambda} I)^{-1}\|_2 = \max\limits_{1\le i\le m} \dfrac{1}{|\lambda_i - \hat{\lambda}|}$ since $\Lambda - \hat{\lambda} I$ diagonal.
(λ_i's are eigenvalues of A)

As m is finite, $\exists j \in \{1,2,\cdots,m\}$ s.t. $\max\limits_{1\le i\le m} \dfrac{1}{|\lambda_i - \hat{\lambda}|} = \dfrac{1}{|\lambda - \hat{\lambda}|}$ Note that $\lambda - \hat{\lambda} \ne 0$.

$\therefore |\hat{\lambda} - \lambda| \le K(V) \|\delta A\|_2$ ∎

Another way

$$\det(\hat{\lambda} I - A - \delta A) = 0 = \det(V^{-1}(\hat{\lambda} I - A - \delta A) V) = \det(\hat{\lambda} I - \Lambda - V^{-1}\delta A V)$$
$$= \det((\hat{\lambda} I - \Lambda)) \det(I - (\hat{\lambda} I - \Lambda)^{-1} V^{-1} \delta A V)$$

$\rightsquigarrow \det(I + (-\hat{\lambda} I + \Lambda)^{-1} V^{-1} \delta A V) = 0$.

-1 is an eigenvalue of $(-\hat{\lambda} I + \Lambda)^{-1} V^{-1} \delta A V$.

$$\Rightarrow 1 = |-1| \le \|(-\hat{\lambda} I + \Lambda)^{-1} V^{-1} \delta A V\|_2 \le \|(\Lambda - \hat{\lambda} I)^{-1}\|_2 \underbrace{\|V^{-1}\|_2 \|V\|_2}_{K(V)} \|\delta A\|_2$$

(b) By Thm 24.7, V is unitary, so $\|V\|_2 = \|V^{-1}\|_2 = 1$ (For $\forall x \in \mathbb{C}^m$ with $\|x\|_2 = 1$, $\|Vx\|_2 = \sqrt{x^* \cancel{V^* V} x}$
$= \|x\|_2 = 1$).

$\therefore$ By Bauer–Fike thm, for each eigenvalue $\hat{\lambda}_j$ of $A + \delta A$,
$\exists$ an eigenvalue λ_j of A s.t. $|\hat{\lambda}_j - \lambda_j| \le K(V) \|\delta A\|_2 = \|\delta A\|_2$ ∎