$\boxed{1}$ $A=\begin{bmatrix}1&2\\0&2\end{bmatrix}$ $AA^T=\begin{bmatrix}1&2\\0&2\end{bmatrix}\begin{bmatrix}1&0\\2&2\end{bmatrix}=\begin{bmatrix}5&4\\4&4\end{bmatrix}$ $\det(AA^T-\lambda I)=\lambda^2-9\lambda+4=0$. $\therefore \lambda=\frac{9\pm\sqrt{65}}{2}$

$\begin{bmatrix}5&4\\4&4\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}5u_1+4u_2\\4u_1+4u_2\end{bmatrix}=\frac{9+\sqrt{65}}{2}\begin{bmatrix}u_1\\u_2\end{bmatrix}$ ⇝ eigenvalue : $\underbrace{\frac{9+\sqrt{65}}{2}}_{\text{Let } \lambda_1}$, eigenvector : $\underbrace{\left(\frac{\sqrt{65}+1}{8},1\right)}_{}$ Let u_1 & normalize it.

$\quad\quad '' \quad = \frac{9-\sqrt{65}}{2}\begin{bmatrix}u_1\\u_2\end{bmatrix}$ ⇝ eigenvalue: $\underbrace{\frac{9-\sqrt{65}}{2}}_{\text{Let } \lambda_2}$, eigenvector: $\underbrace{\left(\frac{1-\sqrt{65}}{8},1\right)}_{}$ Let u_2 & normalize it.

Let $\sigma_1=\sqrt{\lambda_1}$, $\sigma_2=\sqrt{\lambda_2}$.

Define $v_i=\frac{A^*u_i}{\sigma_i}$. Then, $A=U\Sigma V^*$ $\left(U=[u_1\ u_2],\ \Sigma=\begin{bmatrix}\sigma_1&0\\0&\sigma_2\end{bmatrix},\ V=[v_1\ v_2]\right)$

$$=[u_1\ \ u_2]\begin{bmatrix}\sqrt{\frac{1}{2}(9+\sqrt{65})}&0\\0&\sqrt{\frac{1}{2}(9-\sqrt{65})}\end{bmatrix}\begin{bmatrix}v_1^*\\v_2^*\end{bmatrix}$$

$\therefore$ $\sigma_{max}(A)=\frac{\sqrt{2}}{2}(9+\sqrt{65})=2.92080962\ldots$

$\sigma_{min}(A)=\frac{\sqrt{2}}{2}(9-\sqrt{65})=0.684741649\ldots$

$\boxed{2}$ Let $\forall\varepsilon>0$ be given. We want to show that : $\exists$full rank $B\in\mathbb{C}^{m\times n}$ s.t $\|A-B\|_2<2\varepsilon$.

Let $A=U\Sigma V^*$. Construct $B=U\left(\Sigma+\begin{array}{c}\overset{n}{}\\ \begin{array}{c}n\\m-n\end{array}\begin{bmatrix}\varepsilon&&\\&\varepsilon&\\&&\ddots\ \varepsilon\\&0&\end{bmatrix}\end{array}\right)V^*$ (WLOG, let $m\ge n$)

Then, it is SVD of B. As all singular values of B are nonzero ($\because$ each singular values of $A\ge 0$ and $\varepsilon>0$) B is full rank by Thm 5.1.

$$\|A-B\|_2=\left\|U\begin{bmatrix}\varepsilon&&\\&\varepsilon&\\&&\ddots\ \varepsilon\\&0&\end{bmatrix}V^*\right\|_2\underset{\text{Thm 3.1}}{=}\left\|\begin{bmatrix}\varepsilon&&\\&\varepsilon&\\&&\ddots\ \varepsilon\\&0&\end{bmatrix}\right\|_2=\varepsilon<2\varepsilon\ \blacksquare$$

$\boxed{3}$ (a) $A^TA=\begin{bmatrix}-2&-10\\11&5\end{bmatrix}\begin{bmatrix}-2&11\\-10&5\end{bmatrix}=\begin{bmatrix}104&-72\\-72&146\end{bmatrix}$ $\det(\lambda I-A^TA)=\lambda^2-250\lambda+104\cdot146-72^2$
$=\lambda^2-250\lambda+10000$

$\Rightarrow$ eigenvalue : 200 : $\begin{bmatrix}104&-72\\-72&146\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=200\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}104x-72y\\-72x+146y\end{bmatrix}$

$96x=-72y$ ⇝ $4x=-3y$.

Then, eigenvector is $\left(x,-\frac{4}{3}x\right)\xrightarrow{\text{normalize}}\left(-\frac{3}{5},\frac{4}{5}\right)$ or $\left(\frac{3}{5},-\frac{4}{5}\right)$ choose it v_1

50 : $\begin{bmatrix}104&-72\\-72&146\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=50\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}104x-72y\\-72x+146y\end{bmatrix}$

$54x=72y$ ⇝ $3x=4y$

Then, eigenvector is $\left(x,\ \frac{3}{4}x\right)\xrightarrow{\text{normalize}}\left(-\frac{4}{5},-\frac{3}{5}\right)$ or $\left(\frac{4}{5},\ \frac{3}{5}\right)$ choose it v_2

Let $\sigma_1=\sqrt{200}=10\sqrt{2}$, $\sigma_2=5\sqrt{2}$

Define $u_1=\frac{1}{\sigma_1}Av_1=\frac{1}{10\sqrt{2}}\begin{bmatrix}\frac{6}{5}+\frac{44}{5}\\6+4\end{bmatrix}=\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}$, $u_2=\frac{1}{\sigma_2}Av_2=\frac{1}{5\sqrt{2}}\begin{bmatrix}\frac{8}{5}-\frac{33}{5}\\\frac{40}{5}-\frac{15}{5}\end{bmatrix}=\begin{bmatrix}-\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}$

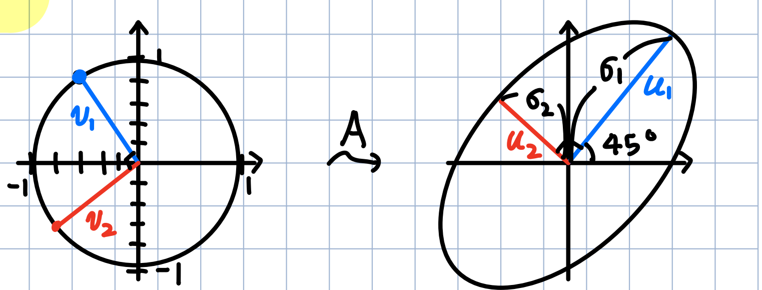
$\therefore A=[u_1\ \ u_2]\begin{bmatrix}\sigma_1&0\\0&\sigma_2\end{bmatrix}\begin{bmatrix}v_1^T\\v_2^T\end{bmatrix}=U\Sigma V^T$

$$=\begin{bmatrix}\sqrt{2}/2&-\sqrt{2}/2\\\sqrt{2}/2&\sqrt{2}/2\end{bmatrix}\begin{bmatrix}10\sqrt{2}&0\\0&5\sqrt{2}\end{bmatrix}\begin{bmatrix}-3/5&4/5\\-4/5&-3/5\end{bmatrix}$$

(b) right singular vectors $v_1=(-3/5,4/5)$
$v_2=(-4/5,-3/5)$

left singular vectors $u_1=(\sqrt{2}/2,\sqrt{2}/2)$
$u_2=(-\sqrt{2}/2,\sqrt{2}/2)$

singular values : $\sigma_1=10\sqrt{2}$
$\sigma_2=5\sqrt{2}$



(c) By Thm 5.3, $\|A\|_2 = \sigma_1 = 10\sqrt{2}$, $\|A\|_F = \sqrt{\sigma_1^2+\sigma_2^2} = \sqrt{250} = 5\sqrt{10}$

Claim: $\|A\|_1 =$ (maximum column sum), $\|A\|_\infty =$ (maximum column sum).

pf) Consider $C = \{x \mid \sum_{i=1}^{n} |x_i| = \|x\|_1 = 1\}$.

For $\forall x \in C$, $\|Ax\|_1 = \|\sum_{i=1}^{n} x_i a_i\|_1 \le \sum_{i=1}^{n} |x_i| \|a_i\|_1 \le \max_{1\le i\le n} \|a_i\|_1$. If max is $\|a_j\|_1$, let $x = e_j$. Then we can attain this bd.

$\therefore \|A\|_1 = \sup_{\|x\|_1=1} \|Ax\|_1 = \max_{1\le i\le n} \|a_i\|$

Consider $D = \{x \mid \|x\|_\infty = 1 \leftrightarrow |x_i| \le 1, i=1,\cdots,n\}$

For $\forall x \in C$, $\|Ax\|_\infty = \left\| \begin{matrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{matrix} \right\|_\infty = \max_{1\le i\le m} |\sum_{j=1}^{n} a_{ij}x_j| \le \max_{1\le i\le m} (\sum_{j=1}^{n} |a_{ij}x_j|)$

$\le \max_{1\le i\le n} (\sum_{j=1}^{n} |a_{ij}|) \max_{1\le j\le n} |x_j| = \max_{1\le i\le n} (\sum_{j=1}^{n} |a_{ij}|)$

Choose $x = (1,\cdots,1)$, then we can attain this bound.
↳ $\|x\|_\infty = 1$

$\therefore \|A\|_\infty = \sup_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{1\le i\le m} (\sum_{j=1}^{n} |a_{ij}|)$

$\therefore \|A\|_1 = 16$, $\|A\|_\infty = 15$

(d) $U^{-1} = U^* = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ $\Sigma^{-1} = \begin{bmatrix} 1/10\sqrt{2} & \\ & 1/5\sqrt{2} \end{bmatrix}$, $(V^*)^{-1} = V = \begin{bmatrix} -3/5 & -4/5 \\ 4/5 & -3/5 \end{bmatrix}$.

Then $A^{-1} = V\Sigma^{-1}U^* = \begin{bmatrix} -3/5 & -4/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} 1/10\sqrt{2} & \\ & 1/5\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -3/5 & -4/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} 1/20 & 1/20 \\ -2/20 & 2/20 \end{bmatrix}$

$= \begin{bmatrix} 1/20 & -11/100 \\ 1/10 & -1/50 \end{bmatrix}$

We can see $AA^{-1} = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 1/20 & -11/100 \\ 1/10 & -1/50 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(e) $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+2 & -11 \\ 10 & \lambda-5 \end{bmatrix}\right)$

$= \lambda^2 - 3\lambda - 10 + 110 = \lambda^2 - 3\lambda + 100 = 0$ $\therefore \lambda = \frac{3 \pm \sqrt{391}\, i}{2}$

$\therefore \lambda_1 = \frac{3+\sqrt{391}\, i}{2}$

$\lambda_2 = \frac{3-\sqrt{391}\, i}{2}$

(f) $\det A = -10 + 110 = 100 = \frac{3+\sqrt{391}\, i}{2} \cdot \frac{3-\sqrt{391}\, i}{2} = \frac{9+391}{4} = 100$. And $|\det A| = 100 = 10\sqrt{2} \cdot 5\sqrt{2} = \sigma_1 \cdot \sigma_2$

(g) recall that the area of ellipse is πab where a: length of semi-major axis
b: length of semi-minor axis

$\therefore 100\pi$

4 $\begin{bmatrix}0 & A^*\\ A & 0\end{bmatrix} = \begin{bmatrix}0 & I_m\\ I_m & 0\end{bmatrix}\begin{bmatrix}A & 0\\ 0 & A^*\end{bmatrix} = \begin{bmatrix}0 & I_m\\ I_m & 0\end{bmatrix}\begin{bmatrix}U & \\ & V\end{bmatrix}\begin{bmatrix}\Sigma & \\ & \Sigma^*\end{bmatrix}\begin{bmatrix}V^* & \\ & U^*\end{bmatrix}$ ⤳ we can say that it is svd of X as $\begin{bmatrix}0 & I_m\\ I_m & 0\end{bmatrix}\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix} = \begin{bmatrix}0 & V\\ U & 0\end{bmatrix}$ unitary.

let X

$$= \begin{bmatrix} & V\\ U & \end{bmatrix}\begin{bmatrix}\Sigma & \\ & \Sigma^*\end{bmatrix}\begin{bmatrix}0 & I_m\\ I_m & 0\end{bmatrix}\begin{bmatrix} & U^*\\ V^* & \end{bmatrix}$$

Let $P = \begin{bmatrix} & V\\ U & \end{bmatrix} \rightsquigarrow P^{-1} = \underbrace{\begin{bmatrix} & U^*\\ V^* & \end{bmatrix}}_{=P^*} \rightsquigarrow X = P\begin{bmatrix}0 & \Sigma\\ \Sigma^* & 0\end{bmatrix}P^{-1}$

As $X = X^*$, by Thm 5.5, $\Sigma = \Sigma^*$ as $\begin{bmatrix}\Sigma & \\ & \Sigma^*\end{bmatrix}$ is singular value matrix of X.

$X = P\begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix}P^{-1}$

Now, let's show $\begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix}$ is similar to some diagonal matrix.

$\det\left(\lambda I_{2m} - \begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix}\right) = \det\left(\begin{bmatrix}\lambda I_m & -\Sigma\\ -\Sigma & \lambda I_m\end{bmatrix}\right) = \det(\lambda^2 I_m - \Sigma^2)$

so, eigenvalues of $\begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix}$ is $\pm\sigma_1, \cdots, \pm\sigma_m$ if $\Sigma = \begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_m\end{bmatrix}$

Let $\begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix}[q_1 \cdots q_{2m}] = [q_1 \cdots q_{2m}]\begin{bmatrix}\Sigma & 0\\ 0 & -\Sigma\end{bmatrix} \rightsquigarrow 1 \le i \le m:$ $\begin{bmatrix}\sigma_1 q_{i,m+1} + \cdots + \sigma_m q_{i,2m}\\ \vdots\\ \sigma_m q_{i,2m}\\ \sigma_1 q_{i,1} + \cdots + \sigma_m q_{i,m}\\ \vdots\\ \sigma_m q_{i,m}\end{bmatrix} = \sigma_i q_i$

Then, $q_i = e_i + e_{m+i}$ satisfies the equation.

same way, $q_i = e_{i-m} - e_i$ $(m+1 \le i \le 2m)$

$\therefore \begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix}\begin{bmatrix}I_m & I_m\\ I_m & -I_m\end{bmatrix} = \begin{bmatrix}I_m & I_m\\ I_m & -I_m\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & -\Sigma\end{bmatrix}$

multiply : $2I_{2m}$ (so, it is invertible)

$$\therefore X = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0 & V\\ U & 0\end{bmatrix}\begin{bmatrix}I_m & I_m\\ I_m & -I_m\end{bmatrix}}_{\text{let } Q}\underbrace{\begin{bmatrix}\Sigma & 0\\ 0 & -\Sigma\end{bmatrix}}_{\text{let } \Lambda}\underbrace{\begin{bmatrix}I_m & I_m\\ I_m & -I_m\end{bmatrix}\begin{bmatrix}0 & U^*\\ V^* & 0\end{bmatrix}\frac{1}{\sqrt{2}}}_{\text{we can see that it is } Q^{-1} = Q^*}$$

$$= Q\Lambda Q^* \blacksquare$$

check : $\frac{1}{\sqrt{2}}\begin{bmatrix}0 & V\\ U & 0\end{bmatrix}\begin{bmatrix}I_m & I_m\\ I_m & -I_m\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}V & -V\\ U & U\end{bmatrix}$

$\begin{bmatrix}I_m & I_m\\ I_m & -I_m\end{bmatrix}\begin{bmatrix}0 & U^*\\ V^* & 0\end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}V^* & U^*\\ -V^* & U^*\end{bmatrix}$

multiply : $\frac{1}{2}\begin{bmatrix}2I_m & 0\\ 0 & 2I_m\end{bmatrix}$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}V & -V\\ U & U\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & -\Sigma\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}V^* & U^*\\ -V^* & U^*\end{bmatrix} = \frac{1}{2}\begin{bmatrix}V & -V\\ U & U\end{bmatrix}\begin{bmatrix}\Sigma V^* & \Sigma U^*\\ \Sigma V^* & -\Sigma U^*\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}0 & 2V\Sigma U^*\\ 2U\Sigma V^* & 0\end{bmatrix} = \begin{bmatrix}0 & A^*\\ A & 0\end{bmatrix}$$