7.4 Let $x^{(1)}, y^{(1)}, x^{(2)}$, and $y^{(2)}$ be nonzero vectors in $\mathbb{R}^3$ with the property that $x^{(1)}$ and $y^{(1)}$ are linearly independent and so are $x^{(2)}$ and $y^{(2)}$. Consider the two planes in $\mathbb{R}^3$,

$$P^{(1)} = \langle x^{(1)}, y^{(1)} \rangle, \qquad P^{(2)} = \langle x^{(2)}, y^{(2)} \rangle.$$

Suppose we wish to find a nonzero vector $v \in \mathbb{R}^3$ that lies in the intersection $P = P^{(1)} \bigcap P^{(2)}$. Devise a method for solving this problem by reducing it to the computation of QR factorization of three 3×2 matrices.

Solution. Let

$$A_1 = \begin{bmatrix} x^{(1)} & y^{(1)} \end{bmatrix} = Q_1 R_1, \quad A_2 = \begin{bmatrix} x^{(2)} & y^{(2)} \end{bmatrix} = Q_2 R_2,$$

where Q_1, Q_2 are 3×3 matrices with orthonormal columns and R_1, R_2 are 3×2 and upper triangular.

Next, denote

$$Q_1 = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \end{bmatrix}, \quad Q_1 = \begin{bmatrix} q_1^{(2)} & q_2^{(2)} & q_3^{(2)} \end{bmatrix}$$

Then, by definition, $q_3^{(1)} \in \text{range}(A_1)^{\perp}$, $q_3^{(2)} \in \text{range}(A_2)^{\perp}$. i.e. $q_3^{(1)} \perp x^{(1)}, y^{(1)}$, $q_3^{(2)} \perp x^{(2)}, y^{(2)}$. Therefore,

$$q_3^{(1)} \perp P^{(1)}, \quad q_3^{(2)} \perp P^{(2)} \qquad \cdots\cdots (*)$$

Define new matrix,

$$A_3 = \begin{bmatrix} q_3^{(1)} & q_3^{(2)} \end{bmatrix} = Q_3 R_3,$$

where Q_3 is 3×3 matrix with orthonormal columns and R_3 is 3×2 and upper triangular.

Similarly denote Q_3 as

$$Q_3 = \begin{bmatrix} q_1^{(3)} & q_2^{(3)} & q_3^{(3)} \end{bmatrix}$$

Then, by definition, $q_3^{(3)} \perp q_3^{(1)}, q_3^{(2)} \qquad \cdots\cdots (**)$.

From $(*), (**)$ P is parallel with $q_3^{(3)}$.

P contains zero trivially, so, we can choose v as a scalar(nonzero) multiple of $q_3^{(3)}$.