1 * Recall : How to calculate SVD? (Real matrices)

i. Find eigenvalues of $A^TA \rightsquigarrow \sigma_1=\sqrt{\lambda_1}, \sigma_2=\sqrt{\lambda_2}, \cdots$ (nonzero eigenvalues)

ii. Find the unit eigenvectors of $A^TA \rightsquigarrow v_1, v_2, \cdots$

iii. Then, find $u_i = \frac{1}{\sigma_i} A v_i$

iv. Extend each $\{u_i\}, \{v_i\}$ to the basis sets of $\mathbb{R}^m, \mathbb{R}^n$

$A = U\Sigma V^*$
$A^*A = V\Sigma^* U^* U \Sigma V^*$
$= V(\Sigma\Sigma^*)V^*$
$\rightsquigarrow (A^*A)V = V(\Sigma\Sigma^*)$
$(AA^*)U = U(\Sigma^*\Sigma)$

So, U & V are eigenvectors of AA^* & A^*A.

(a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

$A^TA = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$ $\det(A^TA - \lambda I) = (9-\lambda)(4-\lambda)$ $\sigma_1 = 3, \sigma_2 = 2.$

$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9x \\ 9y \end{bmatrix} \rightsquigarrow v_1 = (1,0)$ $\quad u_1 = \frac{1}{3}Au_1 = (1,0)$

$v_2 = (0,1)$ $\quad u_2 = \frac{1}{2}Au_2 = (0,-1)$

$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, V^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b), (c), (d), (e) similar calculations.

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\det(A^TA - \lambda I_2) = \lambda^2 - 4\lambda \rightsquigarrow \sigma_1 = 2$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \end{bmatrix} \rightsquigarrow u_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $\quad v_1 = \frac{1}{2}Au_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$\underset{\text{extend}}{\rightsquigarrow} u_2 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ $\quad \underset{\text{extend}}{\rightsquigarrow} v_2 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

$\therefore U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ $\quad V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

2 Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \rightsquigarrow B = \begin{bmatrix} a_{m1} & \cdots & a_{11} \\ \vdots & & \vdots \\ a_{mn} & \cdots & a_{1n} \end{bmatrix}$ = mirror image of A^T about the column axis.

Let am $\quad$ Let a_1

Spse $A^T = U\Sigma V^* = \underbrace{[u_1 | u_2 | \cdots | u_m]}_{n\times m} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_k & \\ & & & 0 \ddots 0 \end{bmatrix}}_{m\times n} \underbrace{[v_1 | \cdots | v_n]}_{n\times n}$ where $k = \text{rank}(A)$

$B = A^T \underbrace{\begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix}}_{\text{Let } I' \, (m\times m)} = A^T I'$

* It is clear that A & A^T has the same singular value

$A = (V^*)^T \Sigma U^T$.

And I' is unitary : $I'I'^T = \begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix}\begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix} = I_m$

$\therefore B = U\Sigma V^* I' = U\Sigma (V')^*$ where $V' = (I')^* \cdot V$, unitary.

$\therefore$ It is the SVD of B, and has the same Σ with A^T. True

(by the uniqueness of SVD)

3 See the bottom of the file

4 Claim: A & B are unitarily equivalent $\rightarrow$ have the same singular value.

pf) Let $B = U\Sigma V^* \rightsquigarrow A = QU\Sigma V^* Q^* = U'\Sigma V' \rightsquigarrow$ it is the SVD of A. (by uniqueness of SVD)

But converse is not true.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Spse $\exists$ unitary $Q \in \mathbb{C}^{2\times 2}$ s.t $A = QBQ^*$

$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = Q\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}Q^* \leftrightarrow Q\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = Q\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

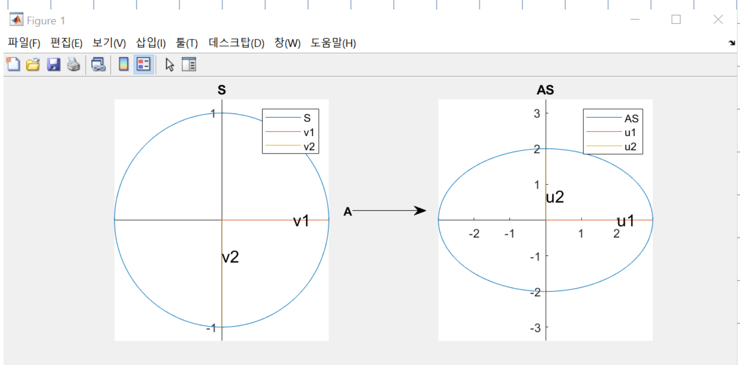
(contradiction)

5 Just slightly modify the proof of Thm 4.1 : $\mathbb{C}^n \to \mathbb{R}^n$.

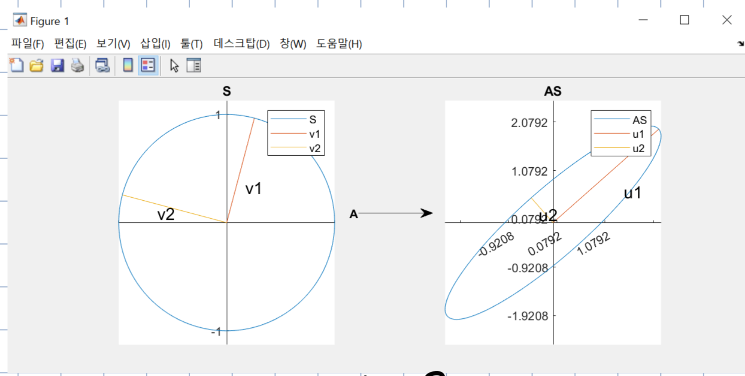
Then you can see that $v_1 \in \mathbb{R}^n$ ($\because v_1 \in \{x \in \mathbb{R}^n \mid \|x\|=1\}$) and $u_1 \in \mathbb{R}^m$ as $\underbrace{A}_{\in\mathbb{R}^{m\times n}}\overbrace{v_1}^{\in\mathbb{R}^n} = u_1$

And, by inductive argument, you can see that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ ∎

※ The result of MATLAB problem.



$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$