38.3 The conjugate gradient is applied to a symmetric positive definite matrix A with the result $||e_0||_A = 1, ||e_{10}||_A = 2 \times 2^{-10}$. Based solely on this data,

(a) What bound can you give on $\kappa(A)$?

(b) What bound can you give on $||e_{20}||_A$?

Solution. (a) By theorem 38.5,

$$\frac{2 \times 2^{-10}}{2} = \frac{||e_{10}||_A}{||e_0||_A} \leqslant \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^{10}$$

i.e.

$$\frac{1}{2} \leqslant \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$$

After computation, we get $\kappa(A) \geqslant 9$.

(b) By theorem 38.5,

$$\frac{||e_{20}||_A}{||e_0||_A} \leqslant \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^{20}$$

We can observe when κ goes infinity, above inequality simply becomes $||e_{20}||_A \leqslant ||e_0||_A$. We know basic fact $||e_{20}||_A \leqslant ||e_{10}||_A$ by theorem. Therefore we can say

$$||e_{20}||_A \leqslant 2 \times 2^{-10}$$