$\boxed{1}$ Let's denote $a_{ij}^{(k)}$ be the ij-entry after the k^{th} step of Gaussian Elimination(partial pivoting)

Let $A^{(i)}$ be A after i^{th} steps of Gaussian Elimination(partial pivoting), $A^{(0)} = A$

For example,

$$\underbrace{\begin{bmatrix} a_{11} & & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & & a_{mm} \end{bmatrix}}_{A} \xrightarrow{1\text{ step}} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & \cdots & a_{mm}^{(1)} \end{bmatrix}}_{A^{(1)}} \xrightarrow{1\text{ step}} \cdots$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1k} & a_{1\,k+1} & \cdots & a_{1m} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & \cdots & a_{2k}^{(1)} & a_{2\,k+1}^{(1)} & \cdots & a_{2m}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} & \cdots & a_{3k}^{(2)} & a_{3,k+1}^{(2)} & \cdots & a_{3m}^{(2)} \\ 0 & 0 & 0 & a_{44}^{(3)} & \cdots & a_{4k}^{(3)} & a_{4,k+1}^{(3)} & \cdots & a_{4m}^{(3)} \\ & & & & \ddots & & & & \\ & & & & & a_{kk}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{km}^{(k-1)} \\ & & & & & \vdots & \ddots & & \vdots \\ & & & & & a_{mk}^{(k-1)} & a_{m,k+1}^{(k-1)} & \cdots & a_{mm}^{(k-1)} \end{bmatrix}$$

After $(k-1)$ steps. $A^{(k-1)}$

$$\xrightarrow[1\text{step}]{} \sim \xrightarrow[1\text{step}]{} U = A^{(m-1)}$$

after $(m-2)$ steps.

Note that to obtain U, we need total $(m-1)$ steps. & partial pivoting is included.

Consider the entries after 1^{st} step.

$$(A^{(1)})_{ij} = \begin{cases} a_{ij} & \text{for some appropriate } i \,\&\, j \\ 0 & \text{"} \\ a_{ij}^{(1)} & \text{for some "} \end{cases}$$

And, claim that $|a_{ij}^{(1)}| < 2 \max_{i,j}\{|a_{i,j}|\}$, which implies $|(A^{(1)})_{ij}| < 2 \max_{i,j}\{|a_{i,j}|\}$ for all $i \,\&\, j$.

$$a_{ij}^{(1)} = a_{ij} - l_{ik} \cdot \underbrace{\overset{\text{appropriate sign}}{\pm} \max_{j \le i \le m}\{|a_{ij}|\}}_{\because \text{ partial pivoting. Let } M}$$

As we apply partial pivoting too, $|l_{ik}| \le 1$.

$$\therefore |a_{ij}^{(1)}| = |a_{ij} - l_{ik} M| \le |a_{ij}| + |l_{ik} M| = |a_{ij}| + |l_{ik}||M| \le |a_{ij}| + |M| \le 2 \max_{i,j}\{|a_{i,j}|\}$$

Now, Consider the entries after p^{th} step. $(2 \le p \le m-1)$

$$(A^{(p)})_{ij} = \begin{cases} (A^{(p-1)})_{ij} & \text{for some appropriate } i \,\&\, j. \\ 0 & \\ a_{ij}^{(p)} & \end{cases}$$

Similarly, claim that $|a_{ij}^{(p)}| \le 2 \max_{i,j}\{(A^{(p-1)})_{ij}\}$

$$a_{ij}^{(p)} = a_{ij}^{(p-1)} - l_{ik} \underbrace{M_j^{(p-1)}}_{\text{it means pivot of } j^{th} \text{column}}$$

Note that $a_{ij}^{(p-1)}$ & pivot $M_j^{(p-1)}$ are the entries of $A^{(p-1)}$

So, similarly, $|a_{ij}^{(p)}| \le 2 \max_{i,j} |(A^{(p-1)})_{ij}|$ $\therefore (A^{(p)})_{ij} \le 2 \max_{i,j} |(A^{(p-1)})_{ij}|$

That means, after a step of Gaussian Elimination, the changed entries cannot be larger than $2 \cdot$ maximum among the entries before elimination.

$\therefore$ Inductively, $|u_{ij}| \le 2 \max_{i,j} |(A^{(m-2)})_{ij}| \le 2^2 \max_{i,j} |(A^{(m-3)})_{ij}| \le \cdots \le 2^{m-2} \max_{i,j} |A_{ij}^{(1)}| \le 2^{m-1} \max_{i,j} |a_{ij}|$ for $\forall i,j$

$\Rightarrow \max_{i,j} |u_{ij}| \le 2^{m-1} \max_{i,j} |a_{ij}|$.

We can assume $A \neq 0$. (If $A = 0$, the statement is trivially true, so it's meaningless).

So $\max_{i,j} |a_{ij}| > 0$. $\therefore \rho = \dfrac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|} \le 2^{m-1}$ ∎

(Another Solution)

Solution. Gaussian elimination with partial pivoting consists of the following procedures:

1. Permute the rows of A according to P.

2. Apply Gaussian elimination to matrix PA without pivoting.

The procedure [1] preserves the maximal norm of entries $\left(\text{i.e. } \max_{i,j}|a_{ij}| = \max_{i,j}\left|a_{ij}^{(0)}\right|\right)$.
For the procedure [2], consider P_2 and Gaussian elimination matrix L_1 on P_1A

$$P_1A = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \cdots & a_{1m}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & \cdots & a_{2m}^{(0)} \\ a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & \cdots & a_{3m}^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{(0)} & a_{m2}^{(0)} & a_{m3}^{(0)} & \cdots & a_{mm}^{(0)} \end{bmatrix} \to P_2L_1P_1A = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \cdots & a_{1m}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2m}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3m}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & a_{m3}^{(1)} & \cdots & a_{mm}^{(1)} \end{bmatrix},$$

where $a_{ij}^{(1)} = a_{\bar{i}j}^{(0)} - \frac{a_{\bar{i}1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}$ (By P_2, $\bar{i} \to i$). By partial pivoting, we have $\left|\frac{a_{\bar{i}1}^{(0)}}{a_{11}^{(0)}}\right| \leqslant 1$, so

$$\left|a_{ij}^{(1)}\right| = \left|a_{\bar{i}j}^{(0)} - \frac{a_{\bar{i}1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}\right| \leqslant \left|a_{\bar{i}j}^{(0)}\right| + \left|\frac{a_{\bar{i}1}^{(0)}}{a_{11}^{(0)}}\right| \left|a_{1j}^{(0)}\right| \leqslant \left|a_{\bar{i}j}^{(0)}\right| + \left|a_{1j}^{(0)}\right| \leqslant 2\max_{i,j}\left|a_{ij}^{(0)}\right| = 2\max_{i,j}|a_{ij}|.$$

Similarly, for k-th step of Gaussian elimination,

$$\left|a_{ij}^{(k)}\right| \leqslant \left|a_{\bar{i}j}^{(k-1)}\right| + \left|a_{kj}^{(k-1)}\right| \leqslant 2\max_{i,j}\left|a_{ij}^{(k-1)}\right| \leqslant 2^2\max_{i,j}\left|a_{ij}^{(k-2)}\right| \leqslant \cdots \leqslant 2^k\max_{i,j}\left|a_{ij}^{(0)}\right| = 2^k\max_{i,j}|a_{ij}| \tag{1}$$

holds.
After $m-1$ steps of [1] and [2], we can get an upper triangular matrix U, in detail,

$$U = L_{m-1}P_{m-1}\cdots L_1P_1A = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & a_{14}^{(0)} & \cdots & a_{1m}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & \cdots & a_{2m}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} & \cdots & a_{3m}^{(2)} \\ 0 & 0 & 0 & a_{44}^{(3)} & \cdots & a_{4m}^{(3)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{mm}^{(m-1)} \end{bmatrix}.$$

Thus, using (1) and the fact that U consists of $a_{ij}^{(k)}$, $k \leqslant m-1$,

$$|u_{ij}| \leqslant 2^{m-1}\max_{i,j}|a_{ij}|.$$

Therefore, the growth factor ρ satisfies

$$\rho = \frac{\max_{i,j}|u_{ij}|}{\max_{i,j}|a_{ij}|} \leqslant 2^{m-1}.$$

3 Since the lecture's discussion is focusing on not the distribution of randomness, but just random-ness, there will be not significant difference between 2 results.

For example, I reproduced:

Figure 22.1

```matlab
function [] = Experiment(trial)
    x = zeros(1, trial); y = zeros(1, trial);
    m = 2;
    increase = 2;
    for i = 1:trial
        % mxm 'uniformly' random matrix (Exercise 12.3)
        A = rand(m, m)/sqrt(m);
        [~, U, ~] = lu(A);
        % growth factor
        p = max(abs(U), [], 'all') / max(abs(A), [], 'all');
        x(1, i) = m; y(1, i) = p;
        if (mod(i, 8) == 0)
            m = m + increase;
        end
        if (mod(i, 32) == 0)
            increase = increase + 2;
        end
    end
    hold on;
    % m^(1/2)
    plot(x, sqrt(x), '--');
    scatter(x, y, zeros(1, trial) + 10, 'filled');
    title('Figure 22.1'); legend('m^1/2', '');
    xlabel('dimension m'); ylabel('growth factor p');
end
```

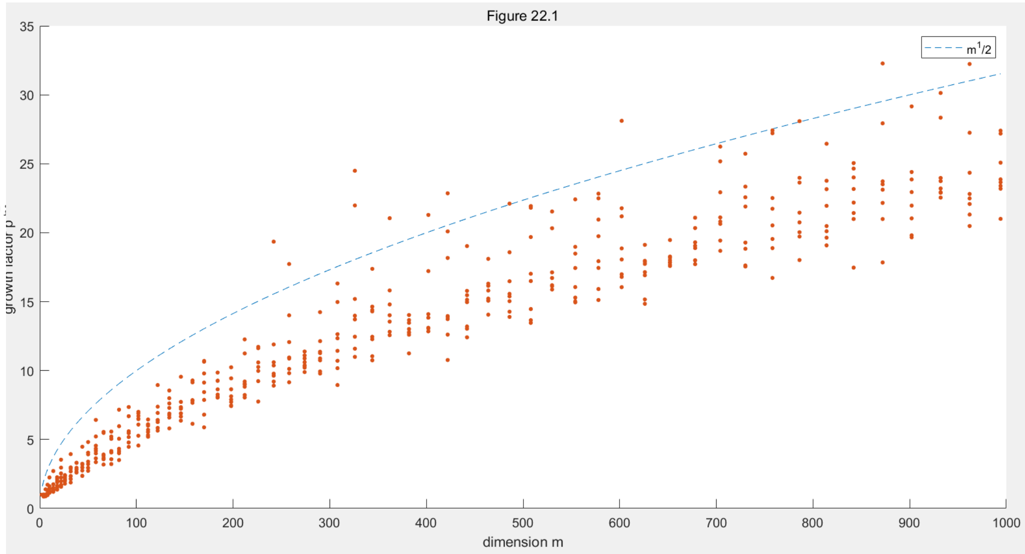
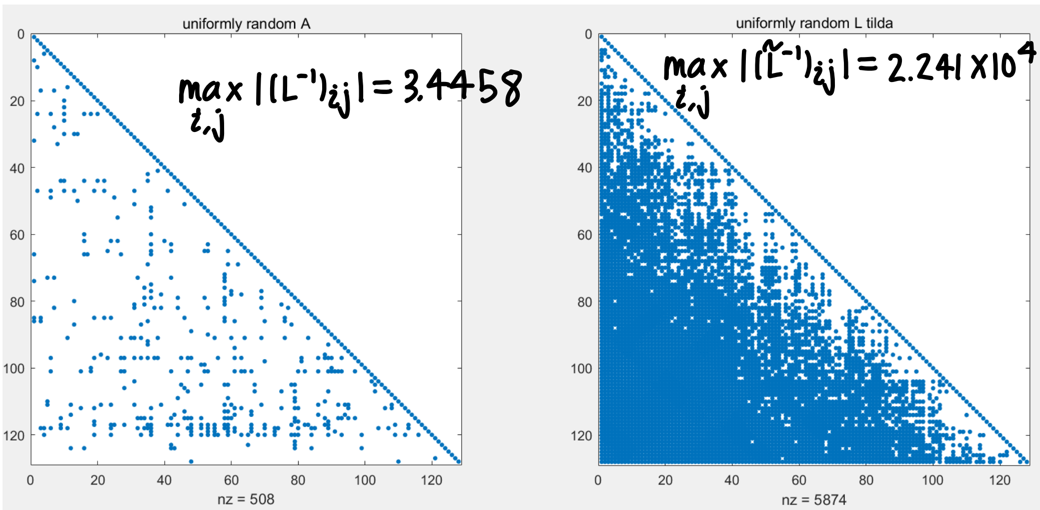


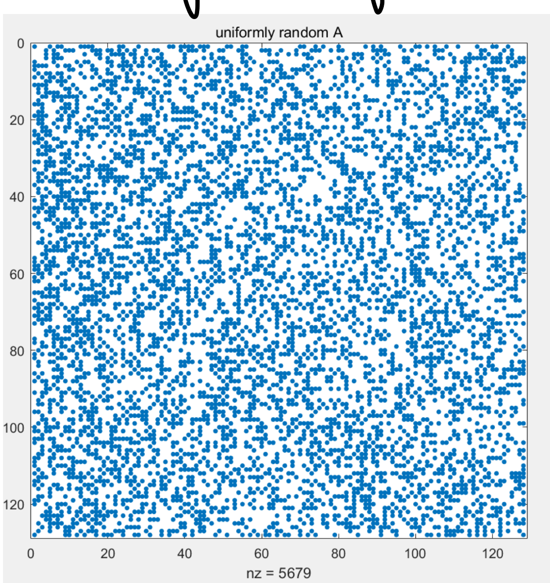
Figure 22.3

```matlab
>> A = 2 * rand(128) - 1;
>> [L, U, P] = lu(A);
>> [Q, R] = qr(A);
>> L_inv_no_sign_randomize = inv(L);
>> uniform_random_sign = 2*(rand(128,128)<0.5)-1;
>> L_inv_sign_randomize = inv(uniform_random_sign .* L);
>> subplot(1, 2, 1);
>> spy(abs(L_inv_no_sign_randomize)>=1);
>> title('uniformly random A');
>> subplot(1, 2, 2);
>> spy(abs(L_inv_sign_randomize)>=1);
>> title('uniformly random L_tilda');
>> max(abs(L_inv_no_sign_randomize), [], 'all')

ans =

    3.4458

>> max(abs(L_inv_sign_randomize), [], 'all')

ans =

   2.2410e+04
```




```matlab
>> spy(abs(Q)>1/sqrt(128))
>> title('uniformly random A');
```

left subfigure of Figure 22.4.



No significant different pattern is found.
It's because the reason of unstability of LU factorization was skewness in a very special fashion, which is exponentially rare.
As a uniform distribution is one in which all values are equally likely, it will be sufficiently random to satisfy with the lecture's argument.
(Also, uniform distribution is somewhat scattered than normal distribution).

(a). given that $PA = LU$ — (1)

and $A = QR$. — (2)

from (1) & (2) we have

$P(QR) = LU$

∴ When a matrix mostly does is to multiply a vector x.

Multiplying $Lx = Q$. by L^{-1}.

gives $(L^{-1}L)x = L^{-1}(Q)$.

This is $\boxed{x = L^{-1}Q}$.

The product is $L^{-1}L$ is like multiplying by a number and then dividing by that number.

(b). If A is Random in sense of having Normal distributed independent entries. Then column spaces are randomly oriented; particularly last column of Q is random vector.

for eg:- gauss Jordan elimination on triangular $[L\ I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{bmatrix}$

$[I\ L^{-1}] \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 11 & -5 & 1 \end{bmatrix}$

(c) Now; combining results obtained from part (a) & part (b) we conclude about final row of L^{-1} in gauss jordan elimination.

Since All pivots were 1!. So; we didn't need to divide rows by pivots to get I.

The inverse matrix L^{-1} looks like L itself; except odd numbered diagonals have minus signs.

∴ Hence; showed