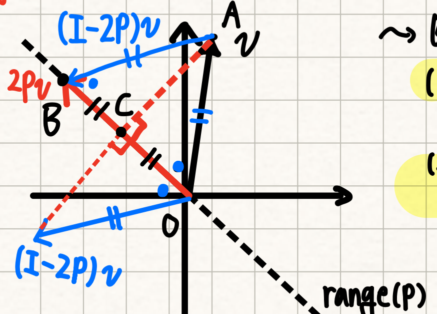
$\boxed{1}$ $P=P^*$. $(I-2P)^*(I-2P)=(I-2P^*)(I-2P)=I-2P-2P^*+4P^*P = I-\cancel{4P}+\cancel{4P^2}=I$ as P is projector

Analogous result for $(I-2P)(I-2P)^*=I$ ″ $\therefore (I-2P)$ unitary

* geometric interpretation



⇝ By the figure, we can guess that $I-2P$ is reflection about range(P).

(1) $\|(I-2P)v\|_2=\|v\|_2$

LHS $= v^*(I-2P^*)(I-2P)v = v^*v=\|v\|_2$ as $(I-2P)$ unitary

(2) $\dfrac{v\cdot Pv}{\|v\|_2\|Pv\|_2} = \dfrac{(I-2P)v\cdot P(I-2P)v}{\|(I-2P)v\|_2\cdot\|P(I-2P)v\|_2}$ (same angle between range(P))

RHS $= \dfrac{v^*(I-2P^*)(-P)v}{\|v\|_2\|-Pv\|_2} = \dfrac{v^*(-P+\underbrace{2P^*P}_{P^2=P})v}{\|v\|_2\|Pv\|_2} = \dfrac{v^*(Pv)}{\|v\|_2\|Pv\|_2}$ = LHS ■

As $\overline{AC}$ is orthogonal to the middle point of $\overline{OB}$,
ΔBAO is isoceles triangle.

$\boxed{2}$ ① E is projector

Let $\forall x\in\mathbb{C}^{m\times1}$. $E^2x=E\left(\frac{x+Fx}{2}\right)=\frac12Ex+\frac12EFx=\frac{x+Fx}{4}+\frac{Fx+F^2x}{4}=\frac12(x+Fx)$

$F^2x=F(Fx)=x$ as F^2 is flipping twice.

② Orthogonal

$E=\frac12 I_m+\frac12F$. And $F=\begin{bmatrix} & & 1\\ & \iddots & \\ 1 & & \end{bmatrix}$ $\left(\because \overbrace{\begin{bmatrix} & & 1\\ & \iddots & \\ 1 & & \end{bmatrix}}^{m}\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}=\begin{bmatrix}x_m\\ \vdots\\ x_1\end{bmatrix}\right)$. So, $I_m^*=I$, $F^*=F$, $E^*=E$.

⇝ E is orthogonal projector by Thm 6.1.

$$E=\begin{bmatrix}1/2 & & & & & & 1/2\\ & 1/2 & & & & 1/2 & \\ & & 1/2 & 0 & 1/2 & & \\ & & 0 & 1 & 0 & & \\ & & 1/2 & 0 & 1/2 & & \\ & 1/2 & & & & 1/2 & \\ 1/2 & & & & & & 1/2\end{bmatrix}$$

$\boxed{3}$ By SVD, let $A=U\Sigma V^*$. Denote $\Sigma=\begin{bmatrix}\sigma_1 & & \\ & \ddots & \sigma_n\\ \hline & 0 & \end{bmatrix}\begin{matrix}\}n\\ \}m-n\end{matrix}$ where $\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_n\ge0$

⇝ $\Sigma^*\Sigma=\left[\begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{matrix}\ \vdots\ 0\right]\begin{bmatrix}\sigma_1 \ddots \sigma_n\\ \hline 0\end{bmatrix}=\begin{bmatrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2\end{bmatrix}$

Then $A^*A=V\Sigma^*\Sigma V^*=V\Sigma'V^*$ ⇝ it is svd of A^*A.

So, (# of nonzero singular values of A^*A) = (# of nonzero singular values of A) as $\sigma_i^2=0\leftrightarrow\sigma_i=0$

By Thm 5.1, $\text{rank}(A^*A)=\text{rank}(A)$

A^*A is non-singular (full rank) $\leftrightarrow$ A full rank ■

$\boxed{4}$ (a) Let $q_1=\begin{bmatrix}1/\sqrt2\\0\\1/\sqrt2\end{bmatrix}$, $q_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$. Then it is clear that it's orthonormal

Let $\hat Q=[q_1\ q_2]$. Then, by our lecture, we know that $P=\hat Q\hat Q^*$ is the projection we want.

$P=\begin{bmatrix}1/\sqrt2 & 0\\ 0 & 1\\ 1/\sqrt2 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt2 & 0 & 1/\sqrt2\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1/2 & 0 & 1/2\\ 0 & 1 & 0\\ 1/2 & 0 & 1/2\end{bmatrix}$

(1) orthogonal : $P=P^*$ clearly

(2) for any $x\in\mathbb{C}^{3\times1}$, $Px\in\text{range}(A)$: $P\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1/2x_1+1/2x_3\\ x_2\\ 1/2x_1+1/2x_3\end{bmatrix}=\frac12(x_1+x_3)a_1+x_2a_2$

$\therefore P(1,2,3)^*=2\begin{bmatrix}1\\0\\1\end{bmatrix}+2\begin{bmatrix}0\\1\\0\end{bmatrix}=(2,2,2)$

(b) Let $a_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $a_2=\begin{bmatrix}2\\1\\0\end{bmatrix}$. Then, it is clear that it is linearly indep. Let $A=[a_1\ a_2]=B$.

Then, by our lecture, $P=B(B^*B)^{-1}B^*$ is the projector that we want ($\because$ a_1 & a_2 isn't orthonormal)

$B^*B=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 5\end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1/2 & 0 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & -1/3 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right] \quad (B^*B)^{-1}$$

*check $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/6 & -1/3 & 5/6 \\ 1/3 & 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

$P\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\therefore (2,0,2)$

$\boxed{5}$ ① $\|P\|_2 \geq 1$

$$\|P\|_2 = \sup_{\|x\|=1} \|Px\|_2 = \sup_{\|x\|=1} \|P^2x\|_2 \leq \sup_{\|x\|=1} \|P\|_2 \|Px\|_2 = \|P\|_2 \sup_{\|x\|=1} \|Px\|_2 = \|P\|_2^2.$$

As $\|P\|_2 \geq 0$, $\|P\|_2^2 \geq \|P\|_2 \leftrightarrow \|P\|_2 \geq 1$,,

We know $\|Ax\| \leq \|A\| \|x\|$:

pf) If $x=0$, it's trivial

If $x \neq 0$, spse not. Then $\|Ax\| > \|A\| \|x\|$

$\sim \dfrac{\|Ax\|}{\|x\|} > \|A\| \sim \left\|A\dfrac{x}{\|x\|}\right\| > \|A\|$

As $\dfrac{x}{\|x\|}$ nonzero unit vector, contradicts to definition,,

② $\|P\|_2 = 1 \leftrightarrow P = P^*$

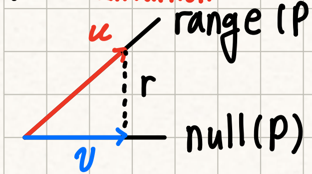
pf) ($\leftarrow$) Using reduced SVD, it is clear that

$P = \hat{Q} I_r \hat{Q}^*$ (r: rank(P), $\hat{Q}$: $m\times r$, unitary)

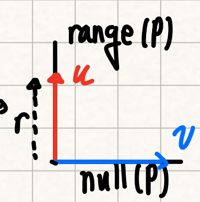
by $P^2 = P$, $P = P^*$

Then, $\|P\|_2 = \|I_r\|_2$ ($\because$ Thm 3.1: unitary matrix preserves norm)
$= 1$,,

($\rightarrow$) - Intuition



If P orthogonal $\sim$



$u = \dfrac{v^*u}{v^*v} v + r$. $Pu = Pr \sim \|Pu\|_2 = \|u\|_2 = \|Pr\|_2$

$\|u\|_2^2 = \left(\dfrac{v^*u}{v^*v}\right)^2 \|v\|_2^2 + \|r\|_2^2 = \|Pr\|_2^2 \leq \|P\|_2^2 \|r\|_2^2 = \|r\|_2^2 \sim$ only possible that $u \cdot v = 0$

$\therefore$ range$(P) \perp$ null(P)