$\boxed{1}$ $K_n = [b \mid Ab \mid \cdots \mid A^{n-1}b]$

$x \in K_n \rightsquigarrow$ linear comb of them

i.e., $x = \sum_{j=0}^{n-1} a_j \cdot A^j b = b\left(\sum_{j=0}^{n-1} a_j \cdot A^j\right) = p(A)b$ ∎

$\boxed{2}$ (a) Note that $\tilde{H}_n = \begin{bmatrix} h_{11} & \cdots & & h_{1n} \\ h_{21} & h_{22} & \ddots & \vdots \\ & \ddots & h_{n,n-1} & h_{nn} \\ & & & h_{n+1,n} \end{bmatrix}$. As $h_{n+1,n} = 0$,

$$A[q_1|\cdots|q_n] = [q_1|\cdots|q_n|q_{n+1}] \begin{bmatrix} h_{11} & \cdots & & h_{1n} \\ h_{21} & h_{22} & \ddots & \vdots \\ & \ddots & h_{n,n-1} & h_{nn} \\ & & & 0 \end{bmatrix} = [q_1|\cdots|q_n] \begin{bmatrix} h_{11} & \cdots & & h_{1n} \\ h_{21} & h_{22} & \ddots & \vdots \\ & \ddots & h_{n,n-1} & h_{nn} \end{bmatrix}$$

$= Q_n H_n$ clearly. (We denoted H_n as the Hessen-berg matrix by removing the last row of $\hat{H}_n$)

So, $AQ_n = Q_n H_n$.

(b) It suffices to show for $\forall x \in AK_n$, $x \in K_n$ too.

Fix $\forall x \in AK_n = A\langle q_1, \cdots, q_n\rangle = \langle Aq_1, \cdots, Aq_n\rangle$ where q_i is the i^{th} column of Q_n.

From (a), $[Aq_1|\cdots|Aq_n] = Q_n H_n. = [q_1|\cdots|q_n] H_n$

And, it means each Aq_j can be represented as a linear combination of $\{q_i \mid i=1,\cdots,n\}$.

Then, since x can be also represented as a linear combination of $\{Aq_j \mid j=1,\cdots,n\}$, also linear combination of $\{q_i \mid i=1,\cdots,n\}$ i.e., $x \in K_n$ ∎

(c) $K_n = \langle b, Ab, \cdots, A^{n-1}b\rangle = \langle q_1, \cdots, q_n\rangle$. $\rightsquigarrow$ Let $A^{n-1}b = \sum_{i=1}^{n} a_i q_i$ (a_i constant)

$K_{n+1} = \langle b, Ab, \cdots, A^{n-1}b, A^n b\rangle \rightsquigarrow K_n \subseteq K_{n+1}$ clearly.

It suffices to show $A^n b \in K_n$.

$A^n b = A \cdot A^{n-1} b = A \sum_{i=1}^{n} a_i q_i = \sum_{i=1}^{n} a_i A q_i \in AK_n \subseteq K_n$ (from (b)). $\therefore K_n = K_{n+1}$

Now, consider K_{n+1} & $K_{n+2} = \langle b, Ab, \cdots, A^{n-1}b, A^n b, A^{n+1}b\rangle$. where $K_{n+1} = \langle q_1, \cdots, q_n\rangle$. ($\because = K_n$)

Again, it suffices to show $A^{n+1}b \in K_{n+1}$. $\hookrightarrow$ let $A^n b = \sum_{i=1}^{n} a_i q_i$.

$\overset{\shortparallel}{\sum_{i=1}^{n} a_i A q_i} \subseteq \langle q_1, \cdots, q_n\rangle$ from (b). $\therefore K_{n+1} = K_{n+2}$.

$= K_{n+1} = K_n$.

So, for $\forall k \in \mathbb{N}$, with induction, we can show that $\underbrace{K_n = K_{n+1} = \cdots = K_{n+k}}_{\text{induction}} = K_{n+k+1}$

$\therefore K_n = K_{n+1} = K_{n+2} = \cdots$

(d) By Thm 24.3, similar matrices have the same eigenvalues & from (a), $H_n \sim A$ as $A = Q_n H_n Q_n^*$

Also, it can be proved without thm easily: λ is the eigenvalue of H_n

$\leftrightarrow \det(\lambda I - H_n) = 0$

Then $\det(Q_n)\det(\lambda I - H_n)\det(Q_n^*)$

$= \det(Q_n(\lambda I - H_n)Q_n^*) = \det(\lambda I - Q_n H_n Q_n^*)$

$= \det(\lambda I - A) = 0$

$\leftrightarrow \lambda$ is the eigenvalue of A ∎

(e) A nonsingular, $Ax = b \leftrightarrow x = A^{-1}b$.

So, it suffices to show $A^{-1}b \in K_n = \langle b, Ab, \cdots, A^{n-1}b\rangle$

$= \langle q_1, \cdots, q_n\rangle$.

As A is nonsingular ($\det(A) \neq 0$), so does H_n ($\det(H_n) \neq 0$) from $AQ_n = Q_n H_n$ ($\because \det(Q_n) \neq 0$)

So, $AQ_n H_n^{-1} = Q_n$, i.e. $[Aq_1|\cdots|Aq_n] H_n^{-1} = [q_1|\cdots|q_n]$.

$\rightsquigarrow$ Each q_j can be represented as a linear combination of $\{Aq_i \mid i=1,\cdots,n\}$.

$\rightsquigarrow$ $K_n \subseteq AK_n$ clearly, $K_n = AK_n$ by (c).

So, $b \in AK_n$ i.e, $b = \sum_{j=1}^{n} b_j A q_j$ $\rightsquigarrow$ $x = A^{-1}b = \sum_{j=1}^{n} b_j q_j \in K_n$ ∎