$\boxed{1}$ ($\rightarrow$) Let $r(x) = \frac{x^*Ax}{x^*x} = z$ for some $x \in \mathbb{C}^m$

Extend $\{\frac{x}{\|x\|_2}\}$ to the orthonormal basis for $\mathbb{C}^m$, $\{\frac{x}{\|x\|_2}, q_2, \cdots, q_m\}$. (Note that $\|x\|_2 = \sqrt{x^*x}$

Set $Q = [x/\|x\|_2 \mid q_2 \mid \cdots \mid q_m]$. Then, $(Q^*AQ)_{11} = \underbrace{\left(\frac{x}{\|x\|_2}\right)^*}_{\text{1st row of } Q^*}\underbrace{(AQ)_1}_{\text{1st column of } Q} = \left(\frac{x}{\|x\|_2}\right)^* A \frac{x}{\|x\|_2} = \frac{x^*Ax}{x^*x} = z.$

($\leftarrow$)

Let $z = (Q^*AQ)_{ii}$ for some $1 \le i \le m$.

Choose q_i (ith column of Q). As we saw, then $z = q_i^*Aq_i = \frac{q_i^*Aq_i}{q_i^*q_i} = r(q_i)$ ∎

$\boxed{2}$ convex hull of A : the smallest convex set (intersection of all convex sets containing A) of A.

(a) • $W(A)$ is a convex set. i.e. if $\alpha, \beta \in W(A)$ then $p\alpha + (1-p)\beta \in W(A)$ $\forall p \in [0,1]$

pf) $\exists x, y \in \mathbb{C}^m$ s.t $\alpha = x^*Ax$ $\beta = y^*Ay$ where $\|x\|_2 = \|y\|_2 = 1$.

Assume x & y linearly indep.

Let $B = \frac{-\beta}{\alpha-\beta} I + \frac{1}{\alpha-\beta} A$, $X = \frac{1}{2}(B^*+B)$ $Y = \frac{1}{2i}(B - B^*)$ $\leadsto B = X + iY$

$x^*Bx = 1$ $x^*Xx = 1$ $x^*Yx = 0$

$y^*By = 0$ $y^*Xy = 1$ $y^*Yy = 0$

WLOG, let u^*Yv be purely imaginery (otherwise, replace v by $\exp(i\theta)v$)

Let $z(t) = \frac{tu + (1-t)v}{\| \quad '' \quad \|_2}$, $t \in [0,1]$. Then $z(t)^*Bz(t) = z(t)^*Xz(t) = \frac{t^2 + 2t(t-1)\mathrm{Re}(v^*Xu)}{\|tu + (1-t)v\|_2}$ $\leadsto$ conti, $[0,1] \to [0,1]$

$\therefore \exists t \in [0,1]$ s.t $z(t)^*Bz(t) = p \in [0,1]$ $\leadsto$ $z(t)^*Az(t) = p\alpha + (1-p)\beta$,,

And, as all the eigenvalues are in $W(A)$, the convex hull of eigenvalues of $A \subseteq W(A)$ ∎

(b) A normal $\leadsto$ unitarily diagonalizable $A = Q^*\Lambda Q$

Fix $\alpha = u^*Au \in W(A)$.

Let $u = Qr$. $\alpha = \sum_i r_i^2 \lambda_i$ where $\sum_i r_i^2 = 1$ $\therefore$ $\forall \alpha \in \mathrm{conv}(\{\lambda_1, \cdots, \lambda_m\})$

By (a), $W(A)$ is equal to the convex hull of eigenvalues of A ∎

Remark

If A is hermitian, $W(A)$ is the closed interval in $\mathbb{R}$

pf) $A = Q^*\Lambda Q$, $\Lambda \in \mathbb{R}^{m \times m}$.

Again $W(A)$ is the convex hull of eigenvalues of A: for $\forall x \neq 0 \in \mathbb{R}$, $r(x) = \frac{x^TAx}{x^Tx} = (Q\bar{x})^*\Lambda(Q\bar{x})$ $\quad \bar{x} = x/\|x\|$

Let $Q\tilde{x} = [q_1 \cdots q_m]^T$ & $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$

Then $r(x) = \sum_i q_i^2 \lambda_i$ & $\sum_i q_i^2 = 1$ as Q unitary & $\|\bar{x}\| = 1$

x is arbitrary $\Rightarrow$ q_i can be any complex # satisfies $\sum_i q_i^2 = 1$

Then, $W(A) = [\min \lambda_i, \max \lambda_i]$ ∎

$\boxed{3}$ For non-hermitian, $\nabla r(q_j) \neq 0$. generally.

$$\frac{\partial r(x)}{\partial x_j} = \frac{1}{x^Tx} \cdot \frac{\partial}{\partial x_j}(x^TAx) - \frac{x^TAx}{(x^Tx)^2} \cdot \frac{\partial}{\partial x_j}(x^Tx) = \frac{1}{x^Tx} \cdot (Ax)_j + \frac{1}{x^Tx} \cdot (x^TA)_j - \frac{x^TAx}{(x^Tx)^2} \cdot 2x_j$$

$$= \frac{1}{x^Tx}\left((A^T+A)x - 2r(x) \cdot x\right)_j$$

$\leadsto \nabla r(x) = \frac{1}{x^Tx}\left((A^T+A)x - 2r(x) \cdot x\right)$

So, by Taylor expansion, generally $r(x) = r(q_J) + \overset{\neq 0 \text{ generally}}{\nabla r(q_J)} \|x - q_J\| + O(\|x - q_J\|^2)$

$= r(q_J) + O(\|x - q_J\|)$ first-order (linear accurate)

Then, the order of convergence rate of eigenvalues will be reduced by half ∎