$\boxed{1}$ (a) Problem $f(X) = Q_k \cdots Q_1 X$ data $X : A$ To emphasize that computation will be in the computer.

$\tilde{f}(A) = \widetilde{Q_k \cdots Q_1 (A)}$. Let $\tilde{f}(A) = f(\tilde{A}) = Q_k \cdots Q_1 (A+\delta A)$. Now we have to show $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{machine})$.

By Thm 3.1, it is equivalent to $\|Q_k \cdots Q_1 \cdot \delta A\| = \|A\| \, O(\varepsilon_{machine})$

Proof by Induction

① $k=1$

for each product and sum, $(1+\varepsilon_k)$ appears. But, it is all $O(\varepsilon_m)$, clearly

By Thm 14.1, this exercise doesn't depend on $\|\cdot\|$.

Consider $Q \cdot \delta A$. The ij-element of it is $q_i^* \delta a_j = \sum_{k=1}^{m} q_{i,k} \cdot \delta a_{k,j} = m\,O(\varepsilon_m) \underbrace{\left(\sum q_{i,k}\, \delta a_{k,j}\right)}_{\text{ideal computation}}$

As Q is unitary, we can think the columns of it as a orthonormal vector set i.e, $|q_{i,k}| \le 1$

$\Rightarrow q_i^* \delta a_j \le m\, O(\varepsilon_m) \sum |q_{i,k} \cdot \delta a_{k,j}| \le m\, O(\varepsilon_m) \sum_{k=1}^{m} |\delta a_{k,j}| \le m\, O(\varepsilon_m) \underbrace{\sum |\delta a_{k,l}|}$

also norm of δa clearly (proof is similar with L_1)

$\le m\, O(\varepsilon_{machine}) \|\delta A\|$ $(\because c \cdot O(\varepsilon_m) = O(\varepsilon_m))$

Let $\|\cdot\|_{F_1}$

So, for given A, we can choose any δA with $\|\delta A\| < \|A\|$.

$\therefore$ By norm equivalence Thm 14.1,

$\|Q\delta A\| \le C_1 \|Q\delta A\|_{F_1} = C_1 \sum_{i=1}^{m} \sum_{j=1}^{m} \left| \left(\sum_{k=1}^{m} q_{i,k}\, \delta a_{k,j} \right) \right| \le C_1 \cdot m^2 (m\, O(\varepsilon_{machine}) \|\delta A\|)$

$< \underbrace{m^3}_{\text{doesn't matter}} O(\varepsilon_{machine}) \|A\|$,,

② $k = l+1$ when $k = l$ satisfies the statement

$\widetilde{Q_l \cdots Q_1 A} = Q_l \cdots Q_1 \tilde{A}$ backward stable for some $\frac{\|\delta A'\|}{\|A\|} = O(\varepsilon_m)$

let $\tilde{B}$; let $A + \delta A'$

And, $\widetilde{Q\tilde{B}} = Q(\tilde{B} + \delta\tilde{B})$ also backward stable for some $\frac{\|\delta \tilde{B}\|}{\|\tilde{B}\|} = O(\varepsilon_m)$

$\therefore \widetilde{Q Q_l \cdots Q_1 A} = Q(Q_l \cdots Q_1 (A + \delta A') + \delta \tilde{B})$

$\leftrightarrow \|\delta\tilde{B}\| = \|\tilde{B}\|\, O(\varepsilon_m)$
$= \|Q_k \cdots Q_1 (A + \delta A')\|\, O(\varepsilon_m)$
$\le \|A\| O(\varepsilon_m) + \|\delta A'\| O(\varepsilon_m)$
$= \|A\|\, O(\varepsilon_m)$
by choose $\delta A'$ properly again
(ex) $\|\delta A'\| = \|A\|$, $\|\delta A'\| < 1$, etc)
$\|\delta A'\| = \|A\|\, O(\varepsilon_m)$,

Choose δA be $Q_1^* \cdots Q_l^* \delta\tilde{B} + \delta A'$ $\rightsquigarrow \|\delta A\| \le \|\delta\tilde{B}\| + \|\delta A'\| = \|A\|\, O(\varepsilon_m)$

$\rightsquigarrow Q Q_l \cdots Q_1 (A + \delta A)$

$\therefore \widetilde{Q \cdots Q_1 A} = Q Q_l \cdots Q_1 (A + \delta A)$ & $\|\delta A\| = \|A\|\, O(\varepsilon_m)$ ■

(b) $X = x\ y^*$ Then, $x^* y \cdot A$ = rank 1, but the computed result will be not ■