$\boxed{1}$ * Recall: $Q^*Q = I \rightsquigarrow R = Q^*A$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) $a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, r_{11} = \sqrt{2}$

$a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow q_2 = \dfrac{a_2 - \overbrace{(q_1^* a_2)}^{= r_{12}} q_1}{r_{22}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\rightsquigarrow \boxed{A = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\rightsquigarrow \boxed{A = QR = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

(b) $b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, r_{11} = \sqrt{2}$

$b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - (q_1^* b_2) q_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\rightsquigarrow q_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, r_{12} = \sqrt{2}, r_{22} = \sqrt{3}$

$$\rightsquigarrow \boxed{B = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}}$$

$$\rightsquigarrow \boxed{B = QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}}$$

$\boxed{2}$ It will be a checker board matrix:

$$\hat{R} = \begin{bmatrix} r_{11} & 0 & r_{13} & 0 & & \\ & r_{22} & 0 & r_{24} & & \\ & & r_{33} & 0 & & \\ & & & \ddots & & \\ & & & & & r_{nn} \end{bmatrix}$$

$r_{ij} = 0$ if different parity of i & j

ex) $r_{12} = 0$, $r_{36} = 0$, r_{24} doesn't have to be 0.

pf) To observe the speciality of $\hat{R}$ clearly, spse A has a full rank.

It suffices to show for $\forall j = 1, \cdots, n$, $r_{ij} = 0$ if $i + j = 2k+1$

Note that we cannot assert about others

: $I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \rightsquigarrow \hat{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Claim: $q_j \in \langle a_1, a_3, \cdots, a_{j-2}, a_j \rangle$ or $\langle a_2, a_4, \cdots, a_{j-2}, a_j \rangle$

(j is odd) (j is even)

Strong Induction for j:

① $j = 1$

$q_1 = \dfrac{a_1}{r_{11}}$ true.

② Assume that $j \le k$ satisfies the statement. WLOG, let k be odd.

Now, for $j = k+1$, (even)

$v_{k+1} = a_{k+1} - \sum_{l=1}^{k} r_{l,k+1} q_l$. where $r_{l,k+1} = q_l^* a_{k+1}$.

For odd l: $q_l = \dfrac{1}{r_{ll}} (a_l - r_{1l} \underbrace{q_1}_{\in \langle a_1 \rangle} - r_{3l} \underbrace{q_3}_{\in \langle a_1, a_3 \rangle} - \cdots - r_{l-2,l} \underbrace{q_{l-2}}_{\in \langle a_1, \cdots, a_{l-2} \rangle})$, so it is clear that $q_l \in \langle a_1, a_3, \cdots, a_{l-2}, a_l \rangle$

$\rightsquigarrow r_{l,k+1} = q_l^* a_{k+1} = 0$ (odd l) as a_{k+1} orthogonal to odd columns of A.

$\therefore v_{k+1} =$ (linear combination of a_{k+1} and $q_2, q_4, \cdots, q_{k-1}$ where each q_{even} is in $\langle a_2, a_4, \cdots, a_{k-1} \rangle$

$\therefore q_{k+1} = \dfrac{v_{k+1}}{r_{k+1,k+1}} = \dfrac{v_{k+1}}{\underbrace{\|v_{k+1}\|}} \in \langle a_2, a_4, \cdots, a_{k+1} \rangle$

nonzero. If it is 0, it means $a_{k+1} =$ (linear combination of a_{even})

which contradicts to the full rank.

This claim implies $r_{ij} = 0$ if $i + j = 2k+1$.

$\boxed{3}$ By Thm 7.1, $\exists$ unitary Q & upper triangular R

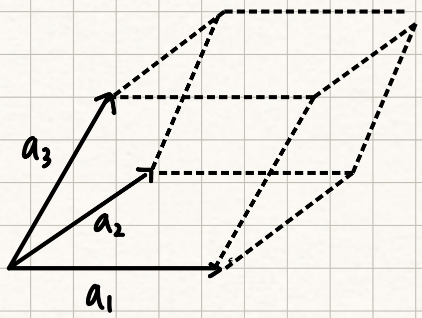
As Q is unitary, $|\det Q| = 1$. As upper triangular, $\det R = \prod_{i=1}^{m} |r_{ii}|$

So, $|\det A| = |\det Q||\det R| = |\det R| = \prod_{i=1}^{m} |r_{ii}|$

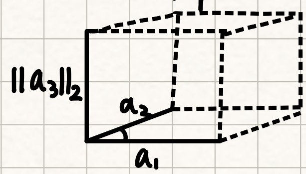
And, we know $r_{ii} = \| v_i \|_2 = \| a_i - \sum_{j=1}^{i-1} (q_j^* a_i) q_j \|_2 \le \| a_i \|_2$ $\quad$ $\therefore |\det A| \le \prod_{i=1}^{m} \| a_i \|_2$

* Geometric interpretation.

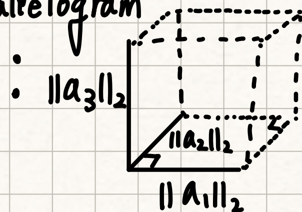
For visualizing, consider $m=3$. Let $A = [a_1 \ a_2 \ a_3]$. Then, we can draw the parallelepiped:



Then volume of this is smaller than a hexahedron which height is $\|a_3\|_2$ and with parallelogram underside



and again its volume is smaller than rectangular parallelogram

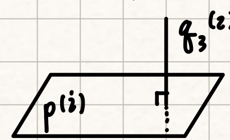


So, (volume of parallelepied) $= |\det A| \le \prod_{i=1}^{3} \|a_i\|_2$

$\boxed{4}$ First, apply QR Factorization to $P^{(1)}$ & $P^{(2)}$

: $P^{(1)} = [q_1^{(1)} \ q_2^{(1)}] R^{(1)}$ $P^{(2)} = [q_1^{(2)} \ q_2^{(2)}] R^{(2)}$. Then, extend each $(q_1^{(i)}, q_2^{(i)})$ to orthonormal basis for $\mathbb{R}^3$

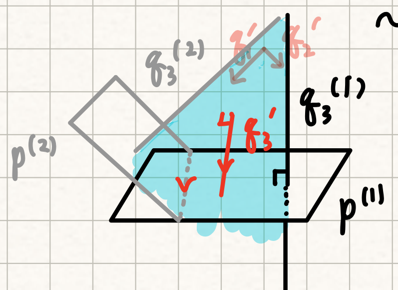
$\rightsquigarrow (q_1^{(i)}, q_2^{(i)}, q_3^{(i)})$. Then $q_3^{(i)}$ is a normal vector of $P^{(i)}$



Again, let $A = [q_3^{(1)}, q_3^{(2)}]$. Apply QR Factorization to it.

: $A = [q_1' \ , q_2'] R'$. Then, extend each (q_1', q_2') to orthonormal basis for $\mathbb{R}^3$

$\rightsquigarrow (q_1', q_2', q_3')$. Now q_3' is parallel to the intersection of 2 planes.



From q_3', now we can find v easily by substituting $q_3' t$ $(t \in \mathbb{R})$ to both planes $q_3^{(1)} \cdot (x,y,z) = 0$ & $q_3^{(2)} \cdot (x,y,z) = 0$ ∎

$\boxed{5}$ (a) $r_{ii} \neq 0 \leftrightarrow a_i - \sum_{j=1}^{i-1} (q_j^* a_i) q_j \neq 0$

We know that $\langle q_1, \cdots, q_j \rangle = \langle a_1, \cdots, a_j \rangle$, so $a_i - \sum_{j=1}^{i-1} (q_j^* a_i) q_j \neq 0 \leftrightarrow a_i \notin \langle a_1, \cdots, a_{i-1} \rangle$

$\therefore r_{ii} \neq 0$ for $i = 1, \cdots, n \longleftrightarrow A$ has a full rank

(b) Exactly k.

Claim: $\hat{R}$ has k nonzero diagonal entries $\leftrightarrow$ rank k.

By rank$(AB) \le$ rank(B), rank$(\hat{Q}^* \hat{Q} \hat{R}) =$ rank$(\hat{R}) \le$ rank$(\hat{Q}\hat{R}) =$ rank(A)

Again, rank$(\hat{Q}\hat{R}) =$ rank$(A) \le$ rank$(\hat{R})$ $\quad$ rank$(A) =$ rank$(\hat{R})$