1 (a) backward stable

Let $f(x)=2x$, $\tilde{f}(x)=fl(x)\oplus fl(x)$ By def of $\tilde{f}$ in our textbook, $\tilde{f}(x)$ also contains rounding x.

By axiom, $\tilde{f}(x)=x(1+\varepsilon_1)\oplus x(1+\varepsilon_1) \overset{\text{by 13.7}}{=} 2x(1+\varepsilon_1)(1+\varepsilon_2)$ for some $|\varepsilon_1|,|\varepsilon_2|\le \varepsilon_{machine}$. Let $\tilde{x}=x(1+\varepsilon_1)(1+\varepsilon_2)$

Then $\dfrac{\|\tilde{x}-x\|}{\|x\|}=\dfrac{\|x\cdot(\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2)\|}{\|x\|}=|\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2|<|\varepsilon_1+\varepsilon_2+\varepsilon_1|\le 3\cdot\varepsilon_{machine}$.

as $|\varepsilon_2|\le\varepsilon_{machine}<1$

Let $C=4$, $\delta=1$.

$\dfrac{\|\tilde{x}-x\|}{\|x\|}<3\varepsilon_{machine}<C\cdot\varepsilon_{machine}$ whenever $|\varepsilon_{machine}|<\delta$.

$\Rightarrow \dfrac{\|\tilde{x}-x\|}{\|x\|}=O(\varepsilon_{machine})$ as $\varepsilon_{machine}\to 0$ and $f(\tilde{x})=2x(1+\varepsilon_1)(1+\varepsilon_2)=\tilde{f}(x)$ ∎

(b) backward stable

$\dfrac{\|\tilde{x}-x\|}{\|x\|}=\sqrt{1+\varepsilon}-1\le\sqrt{1+\varepsilon_{machine}}-1=O(\varepsilon_{machine})$ as $\dfrac{\sqrt{1+\varepsilon_{machine}}-1}{\varepsilon_{machine}}=\dfrac{1}{\sqrt{1+\varepsilon_{machine}}+1}$

$\to 0$ as $\varepsilon_m\to 0$

(c) stable, not backward stable

Let $f(x)=1$, $\tilde{f}(x)=fl(x)\oplus fl(x)$

By 13.5&7, $\tilde{f}(x)=(x(1+\varepsilon_1)\div x(1+\varepsilon_1))(1+\varepsilon_2)=(1+\varepsilon_2)$ for some $|\varepsilon_1|,|\varepsilon_2|\le\varepsilon_{machine}$

Claim: stable

pf) Let $\tilde{x}=x$, $\dfrac{\|\tilde{f}(x)-f(\tilde{x})\|}{\|f(\tilde{x})\|}=\dfrac{|1+\varepsilon_2-1|}{|1|}=|\varepsilon_2|\le\varepsilon_{machine}$.

Let $C=2$, $\delta=1$.

$\dfrac{\|\tilde{f}(x)-f(\tilde{x})\|}{\|f(\tilde{x})\|}\le\varepsilon_{machine}<C\cdot\varepsilon_{machine}$ whenever $|\varepsilon_{machine}|<\delta$.

Also, it is clear that $\dfrac{\|\tilde{x}-x\|}{\|x\|}=O(\varepsilon_{machine})$ as $\dfrac{\|\tilde{x}-x\|}{\|x\|}=0<\varepsilon_{machine}$. Done by def.

Claim: not backward stable

pf) Since we don't assume that our computer doesn't satisfy (13.5), $\exists\, fl(x)\in F$ s.t.

$\tilde{f}(x)=fl(x)\oplus fl(x)\ne \underbrace{fl(fl(x)\div fl(x))}_{fl(1)=1}$. Now, choose that $x\in X$. Then, we can say that $\varepsilon_2\ne 0$.

So, for any $\tilde{x}\in X$, $f(\tilde{x})=1\ne\tilde{f}(x)$, which shows our claim ∎

(d) backward stable

$\tilde{f}(x)=fl(x)\ominus fl(x)=(x(1+\varepsilon)-x(1+\varepsilon))(1+\varepsilon')=0=f(x)$ always

(e) Unstable

terms)

$$\left(\cdots\left(\left(\frac{1}{0!}(1+2\cdot0\cdot\epsilon_1)+\frac{1}{1!}(1+2\cdot1\cdot\epsilon_2)\right)(1+\epsilon_3)+\frac{1}{2!}(1+2\cdot2\cdot\epsilon_4)\right)(1+\epsilon_5)\cdots\right)$$

where $|\epsilon_i|\le\epsilon_{\text{machine}}$. The largest error is obtained if all $\epsilon_i=\epsilon_{\text{machine}}$, in which case the coefficient a_1 for all first powers of $\epsilon_{\text{machine}}$ is:

$$a_1=\sum_{k=0}^{n}\left(\frac{2k+1}{k!}+\sum_{j=0}^{k-1}\frac{1}{j!}\right)$$

But this coefficient grows with increasing number of terms n, a simple lower bound is:

$$a_1\ge\sum_{k=1}^{n}\frac{1}{0!}=n$$

Therefore, since n grows with decreasing $\epsilon_{\text{machine}}$, the error cannot be bounded as $O(\epsilon_{\text{machine}})$ and the algorithm is unstable.

(f) Stable, but not backward stable

Not backward stable as no $\tilde{x}$ can be chose

In this case, the algorithm computes

$$\left(\cdots\left(\left(\frac{1}{n!}(1+2n\epsilon_1)+\frac{1}{(n-1)!}(1+2(n-1)\epsilon_2)\right)(1+\epsilon_3)+\frac{1}{(n-2)!}(1+2(n-2)\epsilon_4)\right)(1+\epsilon_5)\cdots\right)$$

and the coefficient a_1 for all first powers of ϵ_i becomes

$$a_1=\sum_{k=0}^{n}\left(\frac{2(n-k)+1}{(n-k)!}+\sum_{j=0}^{k-1}\frac{1}{(n-j)!}\right)$$

This time, the coefficient can be bounded from above independently of n:

$$a_1=2\sum_{k=0}^{n}\frac{n-k}{(n-k)!}+\sum_{k=0}^{n}\frac{1}{(n-k)!}+\sum_{k=0}^{n}\frac{n-k}{(n-k)!}=3\sum_{k=0}^{n-1}\frac{1}{(n-k-1)!}+\sum_{k=0}^{n}\frac{1}{(n-k)!}\leq 4e$$

Therefore, the total relative error is bounded by $O(\epsilon_{\text{machine}})$, and the algorithm is stable.

$\boxed{2}$ $f(A)=U\Sigma V^*$ $\tilde{f}(A)=\tilde{U}\tilde{\Sigma}\tilde{V}^*$

(a) It means, the SVD matrices of A on the computer, $\tilde{U}$, $\tilde{\Sigma}$, $\tilde{V}$ are the exact ideal SVD matrices of small perturbed $A+\delta A$ where $\frac{\|\delta A\|}{\|A\|}=O(\varepsilon_{machine})$

"on the computer" & "ideal" mean that to compute SVD of A on the computer ($\tilde{f}(A)$), A must be rounded, which makes slightly different SVD matrices computed theoretically ($f(A)$).

(b) Because of rounding, computed $\tilde{U}$, $\tilde{\Sigma}$, $\tilde{V}$ are not unitary in general. So, if we try to compare them with ideal SVD matrices which are unitary, there must be the difference between them.

(c) It means, the relative error of SVD matrices of A on the computer, $\tilde{U}$, $\tilde{\Sigma}$, $\tilde{V}$ and the exact ideal SVD matrices of small perturbed $A+\delta A$ is the order of $O(\varepsilon_{machine})$

$\left(\frac{\|\tilde{U}\tilde{\Sigma}\tilde{V}^*-(A+\delta A)\|}{\|A+\delta A\|}=O(\varepsilon_{machine})\right)$ with $\frac{\|\delta A\|}{\|A\|}=O(\varepsilon_{machine})$ ■