$\boxed{1}$ $A=\begin{bmatrix}2&1&1&0\\4&3&3&1\\8&7&9&5\\6&7&9&8\end{bmatrix}$

(a) $A=LU=\begin{bmatrix}1&&&\\2&1&&\\4&3&1&\\3&4&1&1\end{bmatrix}\begin{bmatrix}2&1&1&0\\&1&1&1\\&&2&2\\&&&2\end{bmatrix}$

$\det(L)=1\cdot\begin{vmatrix}1&&\\3&1&\\4&1&1\end{vmatrix}=1\cdot1\cdot\begin{vmatrix}1&\\1&1\end{vmatrix}=1\cdot1\cdot1\cdot1=1$ by choosing 1^{st} row

$\det(U)=2\cdot1\cdot2\cdot2=8$

$\therefore \det(A)=\det(LU)=\det(L)\det(U)=8$

(b) $PA=LU$

$\begin{bmatrix}&&1&\\&&&1\\&1&&\\1&&&\end{bmatrix}A=\begin{bmatrix}1&&&\\3/4&1&&\\1/2&-2/7&1&\\1/4&-3/7&1/3&1\end{bmatrix}\begin{bmatrix}8&7&9&5\\&7/4&9/4&17/4\\&&-6/7&-2/7\\&&&2/3\end{bmatrix}$

$\det(L)=1$

$\det(U)=8\cdot\frac{\cancel{7}}{\cancel{4}}\cdot\frac{\cancel{-6}^{\,2}}{\cancel{7}}\cdot\frac{\cancel{2}}{\cancel{3}_{\,1}}=-8$

$\det(P)\underset{\text{choose }1^{st}\text{ row}}{=}1\cdot\begin{vmatrix}&&1\\&1&\\1&&\end{vmatrix}\underset{\text{choose }1^{st}\text{ row}}{=}1\cdot1\cdot\begin{vmatrix}&1\\1&\end{vmatrix}=1\cdot1\cdot(-1)\cdot1=-1$

$\therefore \det(PA)=\det(LU)=-8$

$\det(PA)=\det(P)\cdot\det(A)=-\det(A)$

$\det(A)=8$

(c) Note that the elementary operation matrix of changing rows is invertible (just changing again conversely), $\det(P)\neq0$.

Actually, $\det(P)=\pm1$ as changing rows of matrix changes only the sign of determinant, not the absolute value of it.

So, $\det(A)=\dfrac{\det(L)\cdot\det(U)}{\det(P)}$ (from $PA=LU$, and the basic fact that $\det(AB)=\det(A)\det(B)$)

And, the det of lower/upper triangular matrix is just the multiplication of its diagonal entries.

Thus, if we know the LU factorization of matrix, we can compute the det of matrix much simply.

$\boxed{2}$ Same with 20.2. (We changes rows only in the band)

$\boxed{3}$ (a) Since A nonsingular, it must have some nonzero entries in row 1. Let j be the column that has the maximum among the first low. Let Q_1 swaps 1^{st} & j^{th} columns.

Then, 1^{st} diagonal entry of AQ_1 is nonzero, so 1^{st} step of Gaussian Elimination can proceed.

And, we can see that remaining $(m-1)\times(m-1)$ submatrix of L_1AQ_1.

$\because$ First, L_1AQ_1 is nonsingular clearly. And it is form of $\begin{bmatrix}a_{11}&w\\0&B\end{bmatrix}\rightsquigarrow \det=\underbrace{a_{11}}_{\neq0}\cdot\det(B)$

By repeating this procedure for each rows, we can get $\underbrace{L_{m-1}\cdots L_1}_{L^{-1}}A\underbrace{Q_1\cdots Q_{m-1}}_{Q}=U$.

(b) Since A is singular, there can exist zero pivot, which breaks the algorithm by division by 0.

For example, let $A=\begin{bmatrix}0&0\\1&1\end{bmatrix}$. Since 2 columns are same, $AQ=A$ for any elementary operation matrix Q for changing columns.

Spse $\exists L$ & U s.t. $AQ=LU$

$\because \begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\l_{21}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}\\0&u_{22}\end{bmatrix}=\begin{bmatrix}u_{11}&u_{12}\\u_{11}\cdot l_{21}&l_{21}\cdot u_{12}+u_{22}\end{bmatrix}\rightsquigarrow u_{11}=0$ but $u_{11}\cdot l_{21}$ must be 1 (contradiction) ∎

$\boxed{4}$ (a) ① Compute $A=LU \rightsquigarrow \frac{2}{3}m^3$ flops.

② $AA^{-1}=LUA^{-1}=I$.

Let $UA^{-1}=X \rightsquigarrow$ Solve $Lx_k=e_k$ for each $k=1,\cdots,m$, which x_k denotes k^{th} column of X.
(by forward-subtitution)

$$\Rightarrow m\cdot\sum_{j=1}^{m}2j = m^3 \text{ flops}$$

③ Again, solve $UA^{-1}=X$ with back-substitution $\Rightarrow m^3$ flops ∎

(b) For solving $Lx_k=e_k$,

$$\begin{bmatrix} 1 & & & \\ * & 1 & & \\ \vdots & & \ddots & \\ * & * & \cdots & 1 \end{bmatrix}\begin{bmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{kk} \\ \vdots \\ x_{mk} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$x_{1k}=0 \sim x_{k-1,k}=0$ without calculation

So, we only need to compute $\sum_{k=1}^{m}\sum_{j=k}^{m}2(m-j) = \sum_{k=1}^{m}(m^2-2mk+k^2+m-k)$

$= \frac{1}{3}m^3$ flops ∎

(c) (i) $\underbrace{L(Ux)=b}_{n\cdot m^2} \rightarrow \underbrace{Ux=y}_{n\cdot m^2}$ $\quad \therefore \frac{2}{3}m^3+2nm^2$

(ii) $\underbrace{m^3}_{\text{for } A^{-1}}+\underbrace{2nm^2}_{\text{matmul}}$ ∎

$\boxed{6}$

Solution. $|a_{11}| = \max\{|a_{1k}| : 1 \leqslant k \leqslant m\}$, so no row interchange takes place for the first step. **(+2 points)** Let

$$L_1A = \begin{bmatrix} 1 & & & \\ -l_{21} & 1 & & \\ \vdots & & \ddots & \\ -l_{m1} & & & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{1,2:m} \\ \mathbf{0} & A_2 \end{bmatrix},$$

where

$$A_2 = \begin{bmatrix} a_{22}-l_{21}a_{12} & a_{23}-l_{21}a_{13} & \cdots & a_{2m}-l_{21}a_{1m} \\ a_{32}-l_{31}a_{12} & a_{33}-l_{31}a_{13} & \cdots & a_{3m}-l_{31}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-l_{m1}a_{12} & a_{m3}-l_{m1}a_{13} & \cdots & a_{mm}-l_{m1}a_{1m} \end{bmatrix}.$$

For each $2 \leqslant k \leqslant m$,

$$\begin{aligned} \sum_{j\neq k, j\geqslant 2}|a_{jk}-l_{j1}a_{1k}| &\leqslant \sum_{j\neq k, j\geqslant 2}|a_{jk}| + |a_{1k}|\sum_{j\neq k, j\geqslant 2}|l_{j1}| \\ &< |a_{kk}|-|a_{1k}|+\frac{|a_{1k}|}{|a_{11}|}\sum_{j\neq k, j\geqslant 2}|a_{j1}| \\ &< |a_{kk}|-|a_{1k}|+\frac{|a_{1k}|}{|a_{11}|}(|a_{11}|-|a_{k1}|) \\ &= |a_{kk}|-|l_{k1}a_{1k}| \\ &< |a_{kk}-l_{k1}a_{1k}|. \end{aligned}$$

Therefore, $A_2 \in \mathbb{C}^{(m-1)\times(m-1)}$ is a also strictly column diagonally dominant matrix **(+5 points)**. So no row interchange takes palce for the second step, too. Then, by induction, we can conclude that no row interchanges take place during Gaussian elimination with partial pivoting. **(+3 points)**

By using $|a_{kk}| > \sum_{j\neq k} |a_{jk}|$ from the problem, first I'll prove the following claim.

Claim : The maximality of diagonal entries (precisely, $|a_{kk}| > |a_{ik}|$ for $\forall m \geq i \geq k$) holds after the elimination of each columns. (without pivoting) if - is strictly column diagonally dominant.

pf) Proof by induction on l^{th} column. $(1 \leq l \leq m)$

① $l=1$.

After the elimination for 1^{st} column (let U_1 be the matrix after Gaussian eliminating without pivoting of 1^{st} column), $u_{11} = a_{11}$. where $u_{ij} = (U_1)_{ij}$

$$\text{And, for } k\geq 2, \sum_{\substack{j\neq k\\(j\geq 2)}} |u_{jk}| = \sum_{\substack{j\neq k\\(j\geq 2)}} \left(|a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}|\right) \leq \sum_{\substack{j\neq k\\(j\geq 2)}} |a_{jk}| + \sum_{\substack{j\neq k\\(j\geq 2)}} |\frac{a_{j1}}{a_{11}} a_{1k}| < |a_{kk}| - \left|\frac{a_{k1}}{a_{11}} a_{1k}\right| - |a_{1k}| + \left(\sum_{j\neq 1} |a_{j1}|\right)\cdot \frac{|a_{1k}|}{|a_{11}|}$$

$$< |a_{kk}| - \left|\frac{a_{k1}}{a_{11}} a_{1k}\right| - \cancel{|a_{1k}|} + \cancel{|a_{1k}|}$$

$$< |a_{kk} - \frac{a_{k1}}{a_{11}} a_{1k}| = |u_{kk}|.$$

So, for $\forall j \geq k$, $|u_{jk}| < \sum_{\substack{j\neq k\\(j\geq 2)}} |u_{jk}| < |u_{kk}|$. for $\forall\, 2 \leq k \leq m$.

Note that $|u_{11}| > 0 = |u_{j1}|$ for $\forall\, 2 \leq j \leq m$,,

② Spse it is true for $l = i$ $(1 \leq i \leq m-1)$

For $l = i+1$, let U_{i+1} be the matrix after Gaussian eliminating without pivoting of $1^{st}, 2^{nd}, \cdots, (i+1)^{th}$ columns successively.

For convenience, let's denote U_i by A. Let $u_{xy} = (U_{i+1})_{xy}$, $a_{xy} = (U_i)_{xy}$.

Note that $u_{kk} > 0 = u_{jk}$ $(1 \leq j \leq k-1)$ for all $k = 1, 2, 3, \cdots, i+1$.

It's because we are considering the situation after the $(i+1)^{th}$ elimination.

$$\text{For } k > i+1, \sum_{\substack{j\neq k\\(j\geq i+1)}} |u_{jk}| = \sum_{\substack{j\neq k\\(j\geq i+1)}} \left(|a_{jk} - \frac{a_{j,i+1}}{a_{i+1,i+1}} a_{i+1,k}|\right) \leq \sum_{\substack{j\neq k\\(j\geq i+1)}} |a_{jk}| + \sum_{\substack{j\neq k\\(j> i+1)}} |\frac{a_{j,i+1}}{a_{i+1,i+1}} a_{i+1,k}| < |a_{kk}| - \left|\frac{a_{j,i+1}}{a_{i+1,i+1}} a_{i+1,k}\right| - |a_{i+1,k}| + \left(\sum_{j\neq i+1} |a_{j,i+1}|\right)\cdot \frac{|a_{i+1,k}|}{|a_{i+1,i+1}|}$$

$$< |a_{kk}| - \left|\frac{a_{j,i+1}}{a_{i+1,i+1}} a_{i+1,k}\right|$$

$$< |a_{kk} - \frac{a_{j,i+1}}{a_{i+1,i+1}} a_{i+1,k}| = |u_{kk}|$$

So, for $\forall j \geq k$, $|u_{jk}| < \sum_{\substack{j\neq k\\(j\geq i+1)}} |u_{jk}| < |u_{kk}|$ for $\forall\, i+1 \leq k \leq m$ ∎

Now, let's prove the problem by Induction on 2^{nd} m.

① $m=1$.

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

It's enough to show that $|a_{11}|$ is the maximum among A.

$k=1$. $|a_{11}| > |a_{21}| + |a_{31}| + \cdots + |a_{m1}| \geq |a_{j1}|$ for $\forall j \in \{2, 3, \cdots, m\}$. by hypothesis.

So, $|a_{11}|$ is chosen as a pivot. No row changes occurs,,

② Spse it's true for $m = n-1$ $(m \geq n \geq 2)$

For $m = n$,

$$A = \left[\begin{array}{c:c} A_{1:m,1:n-1} & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix} \end{array}\right].$$

By induction hypothesis, Gaussian Elimination without pivoting can be proceeded for $A_{1:m,1:n-1}$.

And, by previous claim, we can say that $a_{nn} = \max_{n\leq j\leq m} \{a_{jn}\}$.

So, the last column can be Gaussian eliminated without partial pivoting ∎

(Another Solution)

Induction on the dimension of m.

① $m=1$ trivial.

② Spse it's true for $m=\ell\ (\ell\geq 1)$

For $m=\ell+1$,

let $A=\begin{bmatrix} a_{11} & v \\ u & B \end{bmatrix}$ where B is $(m-1)\times(m-1)$

If $a_{11}=0$, $a_{j1}=0\ (j=2,3,\cdots,m)$ trivially from the inequality, which is meaningless for the problem.
So, we may assume $a_{11}\neq 0$.

Since A is strictly column diagonally dominant, $|a_{11}|>\sum_{j\neq 1}|a_{j1}| \rightsquigarrow a_{11}$ is chosen as a pivot in the 1^{st} column

After the 1^{st} Gaussian Elimination with partial pivoting

$\rightsquigarrow \begin{bmatrix} a_{11} & v \\ 0 & B-\frac{u}{a_{11}}v \end{bmatrix}$. Note that no row interchanges occur.

Now it suffices to show $B-\frac{u}{a_{11}}v$ is still strictly column diagonally dominant.
Then, we can say that no row interchanges occur in the Gaussian Elimination of $B-\frac{u}{a_{11}}v$ by induction hypothesis. Then, the statement is proved since $2^{nd}\sim$ final steps of Gaussian Elimination correspond to eliminations of $B-\frac{u}{a_{11}}v$, i.e. $L_A=\begin{bmatrix} 1 & 0 \\ \frac{u}{a_{11}} & L_B \end{bmatrix}$ where $A=L_AU_A$, $B-\frac{u}{a_{11}}v=L_BU_B$

For $\forall k=1,\cdots,m-1$,

$$\sum_{j\neq k}\left|\left(B-\frac{u}{a_{11}}v\right)_{jk}\right|\leq\sum_{j\neq k}|B_{jk}|+\sum_{j\neq k}\left|\frac{1}{a_{11}}u_jv_k\right|$$

Be careful of indices, it implies $\sum_{j=1,j\neq k}^{m-1}$

As A is strictly column diagonally dominant, $|v_k|+\sum_{j=1,j\neq k}^{m-1}|B_{jk}|<|B_{kk}|$

$\sum_{j=1}^{m-1}|u_j|<|a_{11}| \rightsquigarrow \sum_{j=1,j\neq k}^{m-1}|u_j|<|a_{11}|-|u_k|$.

$$\therefore \sum_{j\neq k}\left|\left(B-\frac{u}{a_{11}}v\right)_{jk}\right|\leq|B_{kk}|-|v_k|+\sum_{j\neq k}\left|\frac{1}{a_{11}}u_jv_k\right| = |B_{kk}|-|v_k|+\frac{1}{|a_{11}|}|v_k|\sum_{j\neq k}|u_j|$$
$$<|B_{kk}|-|v_k|+\frac{|v_k|}{|a_{11}|}(|a_{11}|-|u_k|).$$
$$=|B_{kk}|-\left|\frac{1}{a_{11}}u_kv_k\right|$$
$$\leq\left|B_{kk}-\frac{1}{a_{11}}u_kv_k\right|=\left|\left(B-\frac{u}{a_{11}}v\right)_{kk}\right| \blacksquare$$

$\boxed{7}$ L_k' has the same structure as L_k.

Consider $L_k' = P_{k+1} L_k P_{k+1}^{-1}$ at $k+1^{th}$ step.

Note that P_{k+1} swaps $k+1^{th}$ & i^{th} $(k+1 \le i \le m)$ rows $\rightarrow$ $P_{k+1} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 0 \cdots 0 & 1 & 0 \cdots 0 \\ & & & \vdots & 1 & & \\ & & & & & \ddots & \\ & & & \cdots 0 & 1 & 0 \cdots 0 & 0 \\ & & & & & & 1 \\ & & & & & & & \ddots & 1 \end{bmatrix} = P_{k+1}^{-1}$

$L_k = I - l_k e_k^*$.

Then $L_k' = \underbrace{P_{k+1} \cdot P_{k+1}^{-1}}_{I.} - P_{k+1} \underbrace{l_k \cdot e_k^*}_{\begin{bmatrix} 0 & \begin{matrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{matrix} & 0 \end{bmatrix}} \cdot P_{k+1}^{-1}$. $= I - \begin{array}{c} k^{th} \\ \begin{bmatrix} 0 & \begin{matrix} 0 \\ \vdots \\ 0 \\ l_{i,k} \\ \vdots \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{matrix} & 0 \end{bmatrix} \end{array} P_{k+1}^{-1}$

$= I - \begin{array}{c} k^{th} \\ \begin{bmatrix} 0 & \begin{matrix} 0 \\ \vdots \\ 0 \\ l_{i,k} \\ \vdots \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{matrix} & 0 \end{bmatrix} \end{array} = I - l_k' e_k^*$ as P_{k+1}^{-1} swaps $k+1^{th}$ & i^{th} columns

Then, other $L_{k-1}', L_{k-2}', \cdots, L_1'$ can be proved inductively ∎