$\boxed{1}$ $A^{(0)} = Q^{(\,)} R^{(\,)}$

$A^{(0)T} A^{(0)} = I = R^{(1)T} R^{(1)} \rightsquigarrow R^{(1)} = I$

So, $Q^{(1)} = A^{(0)}$

$A^{(1)} = Q^{(1)}$

$Q^{(2)} = A^{(1)}$

$A^{(2)} = Q^{(2)}$

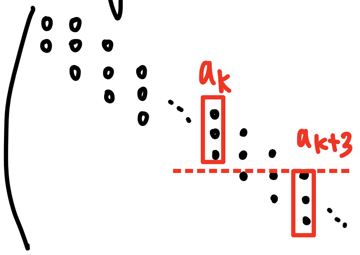
$\vdots$

Algorithm doesn't proceed.

But notice that Thm 28.4 doesn't break since A doesn't satisfy the 1^{st} assumption : $\lambda = \pm 1$.

$\boxed{2}$ (a) From the structure of QR, $r_{ij} = q_i^* a_j$ & $q_j \in \langle a_1, \cdots, a_j \rangle$. $\rightsquigarrow$ $(*)$ so, q_j is a linear combination of them

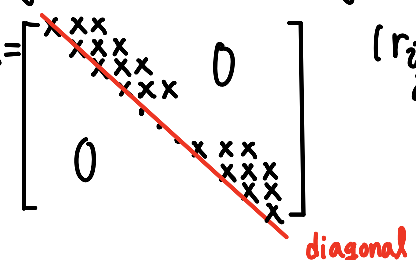
So, in general, we can assert that $r_{ij} = 0$ when $i+2<j$ for arbitrary tridiagonal A. ($\because$ The structure of tridiagonal A)



Denote a_i be i^{th} column of A.
As we can see, for $^\forall$entries, $a_i^* a_j = 0$ if $i+2<j$.

So, $a_j \perp \langle a_1, \cdots, a_i \rangle$ for $^\forall j > i+2$ ($\rightsquigarrow q_i^* a_j = 0$ for $^\forall j > i+2$)

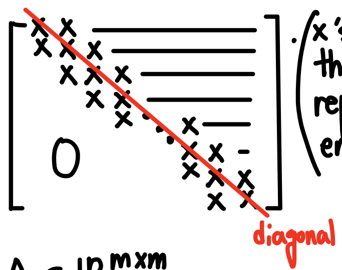
$\therefore R =$



($r_{ij} = 0$ for $i \neq j, i \neq j-1, i \neq j-2$) always.

And, also by $(*)$, Q is an upper Hessenberg

$\therefore Q =$



(x's are representing the form of A, and the line represents generally $\neq 0$ entries.

(b) Note that our discussion in the lecture assumed $A^T = A \in \mathbb{R}^{m \times m}$.
Now, consider RQ. We have to show it is tridiagonal.
Notice that $RQ = Q^T Q R Q = Q^T A Q$. As A is symmetric, so does the similar matrix RQ.
Claim: RQ is upper-Hessenberg (Then, it is automatically tridiagonal by symmetry)
pf) Let's consider $(RQ)_{i,j} = \sum_{k=1}^{m} r_{i,k} q_{k,j}$ with arbitrary i&j in $1,2,\cdots,m$ where $i > j+1$. (If it is 0, done)
By (a), $r_{i,k} = 0$ for $^\forall k > i+2$. & $i > k$ $\rightsquigarrow$ $(RQ)_{i,j} = (r_{i,i})\, q_{i,j} + (r_{i,i+1})\, q_{i+1,j} + (r_{i,i+2})\, q_{i+2,j}$
And, by (a) Q is upper-Hessenberg, $q_{l,k} = 0$ for $^\forall l > k+1$. So $q_{i,j} = q_{i+1,j} = q_{i+2,j} = 0$ as $i > j+1$.
$\therefore (RQ)_{i,j} = 0$ for $^\forall i > j+1$

(c) Consider Householder reflect. At each i^{th} column, we don't have to consider $a_{i+2,i} \sim a_{m,i}$
So, we can consider only one below the diagonal, $a_{i+1,i}$. Then, Householder reflector can be reduced to 2×2 in our algorithm:

for $k=1$ to m

$\quad x = A_{k:k+1,k}$ $\rightsquigarrow 2 \times 1$

$\quad v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$

$\quad v_k = v_k / \|v_k\|_2$

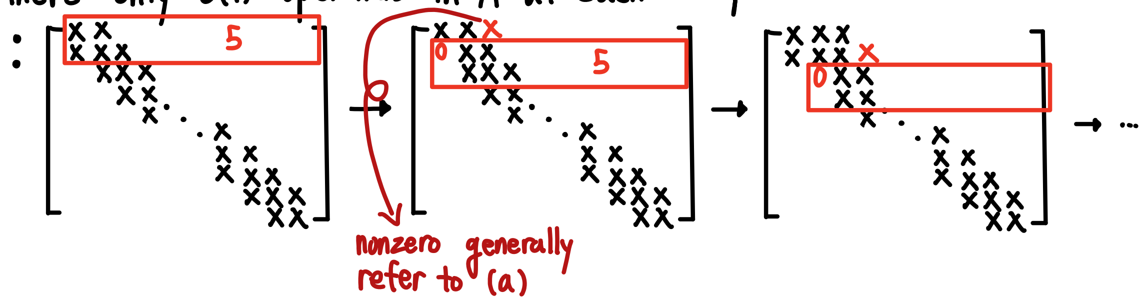
$\quad A_{k:k+1,\,k:k+1} = A_{k:k+1,\,k:k+1} - 2v_k(v_k^* A_{k:k+1,\,k:k+1})$ $\rightsquigarrow$ 2×2 matrices operation

$O(1)$ flops clearly. (as x, e_1 are 2×1 vectors!)

It was the dominant part in the original algorithm ($O(m^2)$) but it becomes $O(1)$!

So, $O(m^3)$ (original) $\longrightarrow O(m)$: reduced dramatically.

This argument can be applied to Givens rotation, too. In QR Factorization with Givens rotation, it is needed to calculate sin & cos of $a_{i,i}$ & $a_{i+1,i} \sim a_{m,i}$ and multiply matrixically to A at each i^{th} step.
But, as A is tridiagonal, we can calculate only sin & cos for $a_{i,i}$ & $a_{i+1,i}$. (Notice that it takes O(1) flops)
And, in terms of matrix multiplication, there only O(1) operands in A at each step



So, total O(m) flops are needed, not O(m³) ∎