$\boxed{1}$ (a) True.

By Thm 24.1, $p_A(z) = \det(\lambda I - A) = 0$.

$p_{A-\mu I}(\lambda-\mu) = \det((\lambda-\cancel{\mu})I - A + \cancel{\mu} I) = \det(\lambda I - A) = 0$.

By Thm 24.1, $\lambda-\mu$ is an eigenvalue of $A-\mu I$ ∎

(b) False.

counter ex) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow p_A(\lambda) = \det(\lambda I_2 - A) = (\lambda-1)^2$, so $\lambda=1$ is an eigenvalue of A

but $\lambda=-1$ is not by Thm 24.1 ∎

(c) True.

The coefficients of char poly of real A are also real as $\det(\lambda I - A) = p_A(\lambda)$; computed by arithmetic operation of real numbers.

By Fundamental Theorem of Algebra, if λ is a root of complex polynomials, so is the conjugate pair $\bar{\lambda}$.

So, $\bar{\lambda}$ is also an eigenvalue of A by Thm 24.1 ∎

(d) True.

$p_{A^{-1}}(\lambda^{-1} = \frac{1}{\lambda}) = \det(\frac{I}{\lambda} - A^{-1})$, $p_A(\lambda) = \det(\lambda I - A) = 0$.

Note that $\det(A) \neq 0$. $\rightsquigarrow \det(A^{-1}) = \frac{1}{\det(A)} \neq 0$.

$\det(A)\det(\frac{I}{\lambda} - A^{-1})$

$= \det(\frac{1}{\lambda}A - I) = (-\lambda)^m \det(\lambda I - A) = 0$. $\therefore \det(\frac{I}{\lambda} - A^{-1}) = p_{A^{-1}}(\lambda^{-1} = \frac{1}{\lambda}) = 0$ ∎

(e) False.

counter ex) $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \rightsquigarrow p_A(z) = \det(zI_2 - A) = \begin{vmatrix} z-1 & -1 \\ 1 & z+1 \end{vmatrix} = z^2 - 1 + 1 = z^2$

So, all the eigenvalues of A are 0 ($\because z^2 = 0$ iff $z=0$), but A is not a zero matrix ∎

(f) True

By Thm 24.7, $\exists$ unitarily diagonalization $A = Q\Lambda Q^*$ where diagonals are eigenvalue of A.

$A = Q\Lambda Q^* = Q\,\mathrm{sign}(\Lambda)\,|\Lambda|\,Q^*$ is the singular value decomposition of A.

So, if λ is an eigenvalue of A, $|\lambda|$ is the singular value of A ∎

(g) False.

counter ex) $A = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$, $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & i \\ -i & \lambda+1 \end{vmatrix} = \lambda^2 - 1 - i^2 = \lambda^2$

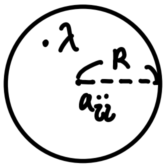
So, all the eigenvalues of A are 0

As A is hermitian : $A^* = \begin{bmatrix} \bar{1} & \bar{i} \\ \overline{-i} & \overline{-1} \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} = A$, diagonalizable by Thm 24.7

But it is not a diagonal ∎

$\boxed{2}$ (a) Let any eigenvalue λ of A. Let x be a corresponding eigenvector with largest entry 1.
(If $Ax' = \lambda x'$, x can be obtained by $x = \dfrac{x'}{\max\{|x'|\}} \cdot \underbrace{\pm 1}_{\text{appropriate sign}}$)
Let $x_i = 1$ for some $1 \le i \le m$.
$\lambda = \lambda x_i$, and $\lambda x_i = \sum_{j=1}^{m} a_{ij} x_j$ from $A \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \lambda x = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_m \end{bmatrix}$

$\therefore |\lambda - a_{ii}| = \left| \sum_{j=1}^{m} (a_{ij} x_j) - a_{ii} \right| = \left| \sum_{j \ne i} (a_{ij} x_j) + \cancel{a_{ii} x_i} - \cancel{a_{ii}} \right| \le \sum_{j \ne i} |a_{ij}| |x_j| \le \sum_{j \ne i} |a_{ij}|$ ■ by $\max_{1 \le j \le m} |x_j| = 1$.



$R = \sum_{j \ne i} |a_{ij}|$

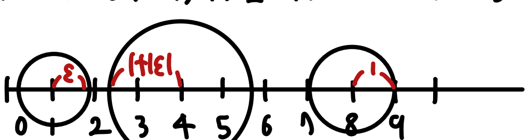
λ is in the m circular disk (may be on the boundary) with center a_{ii} and radius $\sum_{j \ne i} |a_{ij}|$

(b) Let $D = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mm} \end{bmatrix}$, $f(t) = (1-t)D + tA$.
Claim: The eigenvalue of A w.r.t t ($\lambda(t)$) is conti, so if any eigenvalue moves from one of the unions to the other, then it must be outside all the discs for some t (contradiction)

pf) True for $t = 0$.
$\text{diag}(f(t)) = A \rightsquigarrow$ center of each circles are the same, but radii are t times that of A.

So, the union of n disks of $f(t)$ is still disjoint from the union of remaining $n-k$ for $^{\forall} t \in [0,1]$.
Let the distance of the 2 unions for A is d_A ($d_A > 0$)
This distance for $f(t)$ is decreasing i.e, $d_{f(t)} \le d_A$ for $^{\forall} t \in [0,1]$
$\lambda(t)$ is conti $\rightsquigarrow$ for $^{\forall}\lambda(t)$ of $f(t)$ in the union of n disks, its distance $d(t)$ from the union of remaining $n-k$ is also conti.
Obviously $d(0) \ge d_A$. Spse $\lambda(1)$ lies in the union of remaining $n-k$ disks
Then $d(1) = 0 \rightsquigarrow {}^{\exists} t_0 \in (0,1)$ s.t $0 < d(t_0) < d_A$
But this $\lambda(t_0)$ lies outside the discs ■

(c) $|\lambda_1 - 8| \le 1$, $|\lambda_2 - 4| \le 1 + |\varepsilon|$, $|\lambda_3 - 1| < |\varepsilon|$



Note that these 3 disks are disjoint each other since $|\varepsilon| \le 1$.
So, the estimation cannot proceed any more by the theorem
If ε is real, we can estimate each λ by the interval of $\mathbb{R}$:
$\therefore 7 \le \lambda_1 \le 9$, $2 < 3 - |\varepsilon| \le \lambda_2 \le 5 + |\varepsilon| < 6$, $0 < 1 - |\varepsilon| \le \lambda_3 \le 1 + |\varepsilon| < 2$ ■

(d) Let $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \varepsilon \end{bmatrix} \rightsquigarrow DAD^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \varepsilon \end{bmatrix} \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/\varepsilon \end{bmatrix}$

$= \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon^2 & \varepsilon \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/\varepsilon \end{bmatrix} = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & \varepsilon^2 & 1 \end{bmatrix}$

Now, $|\lambda_1 - 8| \le 1$, $|\lambda_2 - 4| \le 2$, $|\lambda_3 - 1| \le \varepsilon^2$
$\rightsquigarrow$ As $\lambda_3 \le 1 + \varepsilon^2 < 2 \le \lambda_2$, λ_3 is the smallest eigenvalue of A ($\because$ Thm 24.3: DAD^{-1} & A are similar, so λ_1, λ_2, and λ_3 are eigenvalues of both DAD^{-1} & A).
$\therefore$ Done by $|\lambda_3 - 1| \le \varepsilon^2$ ■

(Another Solution)

Solution.

(a) Let λ be an eigenvalue of A and $x \neq 0$ be the corresponding eigenvector. Let ith entry of x, x_i, has the largest absolute value. Since $(\lambda I - A)x = 0$,

$$(\lambda - a_{ii})x_i + \sum_{j \neq i} a_{ij}x_j = 0$$

$$\Rightarrow |\lambda - a_{ii}| = \left|\sum_{j \neq i} a_{ij}\frac{x_j}{|x_i|}\right| \leqslant \sum_{j \neq i} |a_{ij}|,$$

by triangular inequality. Therefore, the first part of Gerschgorin's theorem holds.

(b) Let A_ϵ be the matrix such that $(A_\epsilon)_{ii} = (A)_{ii}$ and $(A_\epsilon)_{ij} = (A)_{ij} \times \epsilon$ for $i \neq j$. That is, $A = A_1$. Suppose there is an empty connected domain, which is a union of the disks of the Gerschgorin's theorem, where no eigenvalue of A_1 is contained. As ϵ goes to zero, the radius of the disks converges to 0, meanwhile the eigenvalues of A_ϵ continuously converge to all a_{ii}. This is a contradiction because no eigenvalue of A_ϵ can touch the empty connected domain. This proves the second part.

(c) By Gerschgorin's theorem,

$$|\lambda_1 - 8| < 1, \quad |\lambda_2 - 4| < 1 + \epsilon, \quad |\lambda_3 - 1| < \epsilon.$$

(d) Consider the transformation $A' = XAX^{-1}$ where $X = diag(1, 1, \epsilon)$. Then

$$A' = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & \epsilon^2 & 1 \end{bmatrix}.$$

By Gerschgorin's theorem, the smallest eigenvalue of A' is contained in the disk with radius ϵ^2 centered at 1. Since A' and A has the same eigenvalues, $|\lambda_3 - 1| < \epsilon^2$.