10.1 Determine the (a) eigenvalues and (c) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as algebraic proof.

Solution. (a) $\exists x \in \mathbb{C}^{m-k+1}$, $v = ||x||e_1 - x$. Householder reflector F is reflection across H in $\mathbb{C}^{m-k+1}$, where H is a hyperplane orthogonal to v ($H = \text{null}(v)$). We can write F as

$$F = I - 2\frac{vv^*}{v^*v}.$$

Then,

$$\begin{aligned} Fx &= x \qquad \forall x \in H = span\{q_1, q_2, \cdots, q_{m-k}\}, \\ Fv &= -v, \end{aligned}$$

where q_i's and $\frac{v}{||v||}$ are orthonormal basis of $\mathbb{C}^{m-k+1}$.
Therefore, eigenvalues are 1 and -1. Geometrically, F is reflection across H, so if $x \in H$, $Fx = F$. Also, $Fv = -v$ for $v \in H^{\perp}$.
(c) $F^*F = FF^* = I$. Therefore, singular value is 1.