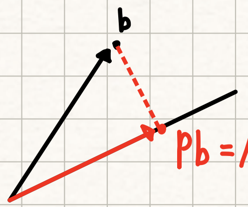
$\boxed{1}$ Let $P$ be a orthogonal projector onto range$(A)$.

Then, $\|P\|_2 = 1$ by SVD : $P = U\Sigma U^* \rightsquigarrow P^2 = U\Sigma^2 U^* = U\Sigma U^* \rightsquigarrow$ all singular values are $1$ or $0$.

$\therefore \|P\|_2 = \sigma_1 = 1.$




$Pb = Ax$ $\quad x = A^+ b.$

$x = A^+ b$ so $b = \underbrace{(A^+)^{-1}}_{\text{nonsingular.}} x$

$$\|A^+\|_2 = \sup_{\substack{b\in\mathbb{C}^m\\(b\neq 0)}} \frac{\|A^+ b\|_2}{\|b\|_2} \le \sup_{\substack{b\in\mathbb{C}^m\\(b\neq 0)}} \frac{\|A^+ b\|_2}{\|Pb\|_2} = \sup_{\substack{x\in\mathbb{C}^m\\(x\neq 0)}} \frac{\|x\|_2}{\|Ax\|_2}$$

$$Ax = \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{n} x = \begin{bmatrix} A_1 x \\ A_2 x \end{bmatrix} \rightsquigarrow \|Ax\|_2^2 = (Ax)^* Ax = \begin{bmatrix} x^* A_1^* & x^* A_2^* \end{bmatrix} \begin{bmatrix} A_1 x \\ A_2 x \end{bmatrix} = x^* A_1^* A_1 x + x^* A_2^* A_2 x = \|A_1 x\|_2^2 + \|A_2 x\|_2^2$$

($m$)

$\therefore \|Ax\|_2 \ge \|A_1 x\|_2$

$$\sup_{\substack{x\in\mathbb{C}^m\\(x\neq 0)}} \frac{\|x\|_2}{\|Ax\|_2} \le \sup_{\substack{x\in\mathbb{C}^m\\(x\neq 0)}} \frac{\|x\|_2}{\|A_1 x\|_2} = \sup_{\substack{x'\in\mathbb{C}^m\\(x'\neq 0)}} \frac{\|A_1^{-1} x'\|_2}{\|x'\|_2} = \|A_1^{-1}\|_2$$

$A_1$ is nonsingular so $\frac{\|x\|}{\|A_1 x\|_2}$ for each $x$ corresponds to $\frac{\|A_1^{-1} x'\|_2}{\|x'\|_2}$, which $x'$ is uniquely determined by $x$ i.e. $x' = A_1^{-1} x$.

$\therefore \|A^+\|_2 \le \|A_1^{-1}\|_2$ ∎