Worked Example

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1 Simulation-optimisation worked example

This document contains an example illustrating the working of the simulation-optimisation model described in §3 of [1]. The data used to carry out the worked example may be accessed in Excel file format[1]. The example opens in §1.1 with a brief description of the hypothetical test instance. The approach followed to generate sales quantities by sampling from an empirical distribution, in conjunction with the inverse-transform method, is presented in §1.2. In §1.3, the method of determining order sizes during individual product replenishments is explained, while §1.4 contains a description of the formation of consolidated replenishment orders, taking into account container filling or MOV requirements. The methods for computing service levels and statistical confidence levels are discussed next, in §1.5, and this is followed in §1.6 by a comparison of the proposed heuristic optimisation approaches adopted to adjust safety stock durations.

1.1 Worked example test instance description

In order to present the worked example, a hypothetical test instance of the joint inventory replenishment problem was created in which a supplier offers a basket of four products. The information pertaining to these products may be found in Table 1.

The following model inputs were decided upon:

- A lead time of $\lambda = 42$ days for each product,
- a target service level of $\bar{\zeta} = 98\%$ for the supplier's entire basket,
- a desired confidence level of 98.5%,
- an inventory review period of 7 days with the first review taking place 3 days after the simulation begins,
- an MRI of 7 days,
- a safety stock duration threshold factor of $\delta = 0.5$ for all $p \in \mathcal{P}$,
- a planning horizon of $T = 100$ days, and
- a volume capacity of $V_{\max} = 25\,\mathrm{m}^3$.

[1] Available at: `https://github.com/InventoryManager101/Supporting-Material`.

Table 1: Product-specific information pertaining to each product $p \in \mathcal{P}$ in the basket of a hypothetical supplier considered in the worked example. Here s_p denotes the safety stock duration (in days), I_p denotes the inventory on hand (in product units), L_p denotes the inventory level (in product units), E_p denotes the MOQ (in product units), e_p denotes the ILS (in product units), c_p denotes the unit cost (in Rands), and v_p denotes the unit volume (in cubic metres) for product $p \in \mathcal{P}$.

p	s_p	I_p	L_p	E_p	e_p	c_p	v_p
1	24.7919	142	184	6	6	7.10	0.0079
2	21.6688	195	235	4	4	7.15	0.0193
3	24.2501	260	308	4	4	6.98	0.0201
4	23.6284	359	391	4	4	6.52	0.0204

1.2 Sales simulation example

Sales quantities were simulated based on the empirical distribution of past sales quantities and invoking the inverse-transform method. Important takeaways from the simulation of sales are:

1. The simulated sales quantities are non-negative integers (that is, $S_p(t) \geq 0$).

2. The historical sales quantities that occur most frequently in the historical sales data set of a product $p \in \mathcal{P}$ are also the most frequent among the simulated sales quantities.

3. Each historical sales quantity histogram bin contains at least five observations.

4. Sales quantities in the outlier bin of a product $p \in \mathcal{P}$ occur at very low simulated frequencies. Furthermore, the larger a sales quantity in the outlier bin, the smaller its simulated frequency of occurrence, owing to sampling from an exponential distribution in this bin.

Table 2 contains a summary of the maximum historical sales quantity for each inventory product $p \in \mathcal{P}$, and the resulting histogram bin boundaries as well as the simulated frequencies of sales associated with these bins before performing bin width adjustment. For example, the maximum sales quantity of Product 1 is 39, which results in the twelve histogram bins: $[0,1)$, $[1,4)$, $[4,7)$, $[7,10)$, $[10,13)$, $[13,16)$, $[16,19)$, $[19,22)$, $[22,25)$, $[25,28)$, $[28,39)$ and $[39,\infty)$. Since the historical sales quantities are discrete values, the bin [0,1) represents the occurrence of zero sales of Product 1.

From the second part of Table 2, it is evident that the requirement in the third point above was not met. In order to address this deficiency, the simulation model combines those bins containing fewer than five observations with their adjacent bins that contain more than five observations, yielding the results

Table 2: Properties of the histogram bins of the empirical sales quantity distribution for each product $p \in \mathcal{P}$ before bin width adjustment.

Product	Maximum
1	39
2	86
3	87
4	67

Product	Bin boundaries
1	[0 | 1 | 4 | 7 | 10 | 13 | 16 | 19 | 22 | 25 | 28 | 39 | ∞]
2	[0 | 1 | 9 | 17 | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 86 | ∞]
3	[0 | 1 | 9 | 17 | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 87 | ∞]
4	[0 | 1 | 7 | 13 | 19 | 25 | 31 | 37 | 43 | 49 | 55 | 67 | ∞]

Product	Number of bin observations
1	[463 | 263 | 214 | 88 | 43 | 11 | 8 | 3 | 1 | 0 | 0 | 1]
2	[398 | 524 | 152 | 14 | 3 | 1 | 1 | 0 | 0 | 1 | 0 | 1]
3	[377 | 424 | 249 | 38 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 1]
4	[358 | 318 | 304 | 92 | 13 | 7 | 0 | 1 | 0 | 0 | 0 | 2]

summarised in Table 3. The cumulative probabilities associated with each bin are also provided in the third part of the table.

Based on the information provided in Table 3, a sales quantity for a product $p \in \mathcal{P}$ may be generated by invoking the inverse-transform method, as described in §3.2 of the referenced paper[1]. Two sales quantity examples are presented for Product 1 in Table 4. In the first example, a random number is generated between 0 and 1 according to a uniform distribution, as indicated in the first row of Table 4. Referring to Table 3, it is clear that $0.1812 < 0.4228$, which corresponds to the first bin in the normalised cumulative distribution of Product 1, as also confirmed in the second row of Table 4. A random sales quantity is uniformly generated within the boundaries of the selected bin for all non-outlier bins. Bin 1, however, represents the range $[0, 1)$, resulting in a simulated sales quantity of 0.

In order to compare the simulated sales quantities with the historical sales quantities of each product $p \in \mathcal{P}$, a total of 1 095 sales quantities were generated for each product, matching the number of historical sales quantities for each product. Table 5 contains the maximum sales quantities and frequencies of occurrence simulated for the histogram bin boundaries specified in Table 3.

The similarities between the numerical values in Table 3 and those in Table 5 clearly demonstrate the characteristics in Take-aways 1, 2, and 4 above.

Table 3: Properties of, and cumulative probabilities derived from, the histogram bins of the empirical sales quantity distribution for each product $p \in \mathcal{P}$ after bin width adjustment.

Product	Maximum	Bin boundaries
1	39	$[0 \mid 1 \mid 4 \mid 7 \mid 10 \mid 13 \mid 16 \mid \infty]$
2	86	$[0 \mid 1 \mid 9 \mid 17 \mid \infty]$
3	87	$[0 \mid 1 \mid 9 \mid 17 \mid \infty]$
4	67	$[0 \mid 1 \mid 7 \mid 13 \mid 19 \mid 25 \mid \infty]$

Product	Number of bin observations
1	[463 | 263 | 214 | 88 | 43 | 11 | 13]
2	[398 | 524 | 152 | 21]
3	[377 | 424 | 249 | 45]
4	[358 | 318 | 304 | 92 | 13 | 10]

Product	Normalised cumulative probability
1	[0.4228 | 0.6630 | 0.8584 | 0.9388 | 0.9781 0.9881 | 1]
2	[0.3635 | 0.8420 | 0.9808 | 1]
3	[0.3443 | 0.7315 | 0.9589 | 1]
4	[0.3269 | 0.6174 | 0.8950 | 0.9790 | 0.9909 | 1]

Table 4: The inverse-transform method applied to the empirical sales quantity distribution of Product 1 aimed at generating two stochastic sales quantities for Product 1.

Examples		Product 1
1	Random value	0.1812
	Selected bin	1
	Generated value	0
2	Random value	0.7358
	Selected bin	3
	Generated value	6

Table 5: Maximum sales quantities and numbers of bin observations for the generated sales quantities of each product $p \in \mathcal{P}$.

Product	Maximum	Number of bin observations
1	39	[469 | 234 | 217 | 100 | 46 | 16 | 13]
2	91	[393 | 483 | 177 | 42]
3	71	[384 | 395 | 273 | 43]
4	74	[358 | 287 | 299 | 120 | 22 | 9]

1.3 Order size example

The following conditions are important when simulating replenishment order sizes for each product in the supplier's basket:

5. The order size agrees with the expression in (9) of the referenced paper [1].
6. The order size is computed as

$$R_p(t) = \begin{cases} E_p & \text{if } R_p(t) \leq E_p, \\ E_p + ne_p & \text{if } R_p(t) > E_p, \end{cases}$$

as described in §3.3 of the referenced paper [1].

The model output for the order size generation process of Product 1 during time period $t = 25$ is summarised in Table 6. Product 1 has an MOQ value of $E_p = 6$ units and an ILS value of $e_p = 6$ units, as may be seen in Table 1.

Table 6: Parameters related to the simulated individual replenishment order size $R_1(25)$ for Product 1 on day 25.

Day	Inventory level	ROP	Order quantity
25	219	223	30

From Table 6, it is evident that during time period $t = 25$, it holds that $219 = L_1(25) \leq r_1(25) = 223$, leading to the placement of a replenishment order for Product 1 of 30 units when the inventory for Product 1 is considered completely independently of the inventories of the other products. This replenishment order quantity exceeds the MOQ and is also a multiple of the ILS value greater than the MOQ.

Given a lead time of $\lambda = 42$ days, an MRI of $\mu = 7$ days, and the forecast sales quantities $S_1^*(68)$ to $S_1^*(74)$ captured in the vector $[7 \mid 0 \mid 4 \mid 0 \mid 0 \mid 11 \mid 0]$, the required replenishment quantity according to the expression in (9) of the referenced paper [1] is

$$\begin{aligned} R_1(25) &= r_1(25) - L_1(25) + \sum_{\tau=25+42+1}^{25+42+1+7} S_p^*(\tau) \\ &= 223 - 219 + (7 + 0 + 4 + 0 + 0 + 11 + 0) \\ &= 26. \end{aligned}$$

The value $R_1(25) = 26$ is, however, not a multiple of the ILS above 6. As a result, the order quantity is adjusted to 30, as shown in Table 6, which complies with points 5 and 6 above.

1.4 Consolidated inventory replenishment example

The consolidation of replenishment orders must adhere to the following requirements:

7. A consolidated inventory replenishment order should be initiated if, for any product $p \in \mathcal{P}$, it holds that $L_p(t) \leq r_p(t)$.
8. Separate product inventory replenishment orders should then be brought forward until the container is filled to capacity or the MOV criterion is attained.
9. Separate product inventory replenishment orders should be considered for consolidation purposes in chronological order.
10. Separate product inventory replenishment orders scheduled for the same day should be considered for consolidation purposes according to decreasing order of volume for container filling or increasing order cost for MOV.

Since the distributor specified the initial inventory review period to occur three days after the start of the simulation, the review conducted on Day 4 marks the first inventory level assessment. Table 7 contains the results of this review, offering information on the value of the difference $L_p(4) - r_p(4)$ for each product $p \in \mathcal{P}$.

Table 7: The difference $L_p(4) - r_p(4)$ for each product p in the supplier's basket $\mathcal{P}$, computed to determine whether or not a consolidated inventory replenishment order should be placed.

Product	$L_p(4) - r_p(4)$
1	51
2	-29.6688
3	76.7488
4	-24

Since $178 = L_2(4) \leq r_2(4) = 207.6688$ and $322 = L_4(4) \leq r_4(4) = 346$, it is necessary to place a consolidated inventory replenishment order. Table 8 contains a summary of the timings, order quantities and order volumes for the first five inventory replenishment orders when the inventories of these products are considered completely independently of one another. These orders resulted from conducting individual simulation runs for each product independently and ranking the orders chronologically by day and then by order volume.

In accordance with the data provided in Table 8, the total volume in the container, after having brought forward the first three orders in chronological order and subject to the constraint on container filling, is $11.0770 + 6.2050 + 6.6823 = 23.9643\,\mathrm{m}^3$. When including the fourth order in Table 8, the total volume would become $11.0770 + 6.2050 + 6.6823 + 1.0723 = 25.0366\,\mathrm{m}^3$. This volume exceeds the $25\,\mathrm{m}^3$ container constraint specified. Therefore, the fourth order in Table 8 is not included in the consolidated inventory replenishment order. Order 3 is

Table 8: The timings, order quantities and order volumes of the first five individual product inventory replenishment orders when the inventories of these products are considered completely independently of one another.

Order #	Order day	Product	Quantity	Order volume
1	4	4	92	$11.0770\,\mathrm{m}^3$
2	4	2	52	$6.2050\,\mathrm{m}^3$
3	11	2	56	$6.6823\,\mathrm{m}^3$
4	11	1	60	$1.0723\,\mathrm{m}^3$
5	18	1	48	$0.8578\,\mathrm{m}^3$

Table 9: The consolidated inventory replenishment order placed by the simulation model during time period $t = 4$ subject to a constraint on container filling of at least $21\mathrm{m}^3$.

Product	Quantity
4	92
2	52
2	56
1	48

chosen over Order 4 for inclusion in the consolidated replenishment order since it has a larger order volume, although both orders would have occurred on the same day. The inclusion of Order 3 makes for a more efficient container utilisation. Moreover, Order 5 is also included in the consolidated replenishment order, because it does not violate Constraint (11) of the referenced paper [1] — that is, $11.0770 + 6.2050 + 6.6823 + 0.8578 = 24.8221\,\mathrm{m}^3 < 25\,\mathrm{m}^3$. As this volume exceeds the lower bound on the volume constraint set by the distributor (that is, $24.8221\,\mathrm{m}^3 \geq 21\,\mathrm{m}^3$) and since there is no order with an order volume of $25 - 24.8221 = 0.1779\,\mathrm{m}^3$ or less, the consolidated order is deemed feasible, and it constitutes the final consolidated replenishment order for day $t = 4$ under the condition on container filling, as shown in Table 9. According to the simulation model, this joint replenishment order is scheduled to arrive on day $t = 46$, which is $\lambda = 42$ days after the joint order replenishment date. The subsequent inventory review is scheduled for $t = 11$, occurring seven days after the previous review, as specified by the review period.

In the case of the MOV of \$ 2 633 specified by the supplier, a similar procedure to that carried out in order to achieve container filling is followed. The first seven orders in the order list generated, after having conducted individual order replenishment simulations, are presented in Table 10.

From Table 10, the total cost of the first seven orders, when brought forward chronologically, is $411.62 + 253.31 + 284.97 + 439.45 + 379.71 + 569.94 + 347.60 =$ \$ 2 686.61. This total value exceeds the MOV threshold of \$ 2 633.35, as specified by the supplier, and so satisfies Constraint (13) of the referenced paper [1]. Table 11 contains a summary of the final consolidated inventory replenishment

Table 10: The timings, order quantities, and order values of the first seven individual product inventory replenishment orders when the inventories of these products are considered completely independently of one another.

Order #	Order day	Product	Quantity	Order value
1	4	2	52	$ 411.62
2	18	2	32	$ 253.31
3	18	2	36	$ 284.97
4	25	4	56	$ 439.45
5	25	1	48	$ 379.71
6	32	2	72	$ 569.94
7	32	3	44	$ 347.60

Table 11: The consolidated inventory replenishment order placed by the simulation model during time period $t = 4$ subject to a constraint on the MOV of at least $ 2 633.

Product	Quantity
2	52
2	32
2	36
4	56
1	48
2	72
3	44

order placed by the simulation model during time period $t = 4$. Moreover, it is evident from the order list that orders are placed chronologically, with the smallest order values given priority to ensure that the smallest value above the MOV is obtained, thus satisfying requirements 9 and 10 above.

1.5 Service level and confidence level example

When estimating the service level and corresponding level of statistical confidence, it is important that:

11. The service level estimate associated with the original safety stock durations of each product $p \in \mathcal{P}$ should be at least as large as the desired service level as a result of the imposition of container filling or MOV constraints.
12. The number of simulation runs required in a simulation replication in order to achieve the overall target service level $\bar{\zeta}$ with a user-desired half-width should align with (18) of the referenced paper [1].

The simulation-optimisation procedure for the worked example was conducted for a desired service level of 98% and a desired confidence level of 98.5%

by applying the safety stock duration heuristic. After nineteen simulation replications and approximately eighteen minutes of simulation time, the desired output of a 98% service level was achieved with a half-width of 0.1213.

For the initial safety stock durations shown in Table 12, an overall service level estimate of $\zeta^* = 98.6\%$ was achieved after the first simulation replication. This value was expected to be larger than the overall target service level of $\bar{\zeta} = 98\%$ due to the effects of consolidated replenishment orders, thereby confirming requirement 11 above. A total of 1 986 simulation runs were required during simulation replication 19. This determination was made by initially executing $\Omega = 50$ simulation runs of the model, which yielded the following outputs:

- A mean overall service level $\zeta^* = 98.4\%$,
- a service level variance $\mathrm{Var}(\zeta^*) = 1.9154$, and
- a confidence interval half-width $h(50) = 1.4091$ for the service level estimate.

These values were substituted into the equation (18) of the referenced paper [1] to calculate the number of simulation runs needed to achieve the target service level with a user-desired half-width $h^* = 0.1$. The value $\Omega^* = \left\lceil 50\,(1.4091/0.1)^2 \right\rceil = 1\,986$ was thus obtained. This means that 1 986 simulation runs are required, as performed by the simulation during replication 19, thereby complying with requirement 12 above.

The safety stock duration s_p for each product $p \in \mathcal{P}$ required to obtain an overall target service level estimate of $\zeta^* = 98\%$, given the safety stock duration heuristic approach followed, may be found in the *Recommended safety stock duration* column of Table 12.

Table 12: A comparison of the safety stock duration s_p (in days) of each product $p \in \mathcal{P}$ associated with an overall service level estimate of $\zeta^* = 98.6\%$ before having applied the simulation model and an overall service level estimate of $\zeta^* = 98\%$ after having applied the simulation model, respectively.

Product	Initial	Recommended
1	24.7919	15.7919
2	21.6688	15.6688
3	24.2501	15.2501
4	23.6284	15.6284

The impact of the container filling or MOV constraint is evident in Table 12. Consolidating inventory replenishment orders to fill containers sufficiently or to meet the MOV leads to an increase in the inventory held, consequently resulting in higher service levels. In order to attain the initial desired service level and reduce the amount of inventory held, shorter safety stock durations are required, as indicated by the recommended values for the products supplied by a specific supplier. This reduction in safety stock durations helps minimise the amount of inventory carried, thereby reducing costs.

The exact effects of consolidated inventory replenishment on inventory costs were also determined for the worked example test problem. The value of stock on hand inventory was calculated for each day, given the initial safety stock durations. The values were accumulated over the planning horizon (of $T = 100$ days) and the average inventory on hand value was calculated. The same values were computed for the case of the recommended safety stock durations. The total inventory costs over the planning horizon and the average inventory cost per day may be found for the initial safety stock durations and for the recommended safety stock durations in Table 13.

Table 13: A comparison of the total inventory cost over the entire planning horizon and the average inventory cost per day for the initial safety stock durations and for the recommended safety stock durations.

	Total cost	Average cost
Initial	\$ 39 679.43	\$ 396.79
Recommended	\$ 28 030.68	\$ 280.31

For the worked example, it was found that the simulation-optimisation model saves an average of \$ 116.49 per day and a total of \$ 11 648.75 over the entire planning horizon of $T = 100$ days for a single supplier by mitigating the effects of container filling and MOV.

1.6 Comparison of heuristic optimisation approaches

In addition to the safety stock duration heuristic described above, the simulation-optimisation model was also executed for the other heuristic optimisation approaches (excluding the random selection approach, as it comprises the other four heuristics) outlined in §3.7 of the referenced paper [1]. The requirement that must be adhered to when adjusting safety stock durations is:

13. The safety stock duration associated with any product $p \in \mathcal{P}$ must not fall below the specified threshold level $\delta_p s_p$.

Table 14 contains a summary of the results obtained for each of the heuristic optimisation approaches, confirming the fulfilment of requirement 13 above. The various heuristic optimisation approaches were also compared in terms of the total inventory cost over the planning horizon and the average inventory cost per day incurred. A summary of these results may be found in Table 15. Based on the data presented in Table 15, it is evident that the four heuristic approaches yielded relatively similar results in terms of reducing inventory costs. All four heuristic approaches led to improvements in inventory costs relative to the situation before applying the simulation-optimisation model. They effectively mitigate the impacts of container filling or MOV while consistently achieving the overall target service level at the desired level of statistical confidence.

Table 14: The safety stock durations (in days) returned by the simulation-optimisation model when equipped with the various heuristic optimisation approaches of §3.7 of the referenced paper.

Product p	Unit cost s_p	Safety stock duration s_p
1	24.7919	15.7919
2	12.6688	15.6688
3	24.2501	15.2501
4	23.6284	15.6284

Product p	Sensitivity s_p	Sensitivity/unit cost s_p
1	24.7919	24.7919
2	15.6688	21.6688
3	15.2501	12.2501
4	23.6284	17.6284

Table 15: The total inventory cost and the mean inventory cost per day returned by the simulation-optimisation model when equipped with the various heuristic optimisation approaches of §3.7 of the referenced paper [1].

	Unit cost s_p	Safety stock duration s_p
Total cost	\$ 28 817.31	\$ 28 030.68
Average cost	\$ 288.17	\$ 280.31

	Sensitivity s_p	Sensitivity/unit cost s_p
Total cost	\$ 29 103.61	\$ 27 975.72
Average cost	\$ 291.04	\$ 279.76

References

[1] Anonymous, 2024. *A practical method for joint replenishment under supplier constraints*, Submitted to the International Journal of Production Economics.