QUANTUM BUILDING BLOCKS

Recap

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Single-Qubit Quantum Systems

The Quantum Mechanics of Photon Polarization

There is a simple experiment with light polarization illustrates some of the **nonintuitive** behavior of quantum systems, behavior that is **exploited** to good effect in quantum algorithms and protocols.

The Quantum Mechanics of Photon Polarization

Shine a beam of light on a projection screen. When polaroid **A** is placed between the light source and the screen, the intensity of the light reaching the screen is **reduced**.

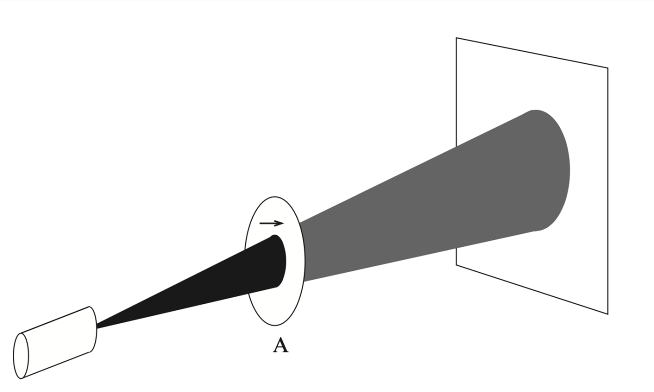


Figure 1: Single polaroid weaken unpolarized light by 50 percent.

The Quantum Mechanics of Photon Polarization

Next, if we place polaroid **C** and if polaroid C is rotated so that its polarization is orthogonal (vertical) to the polarization of **A**, no light reaches the screen.

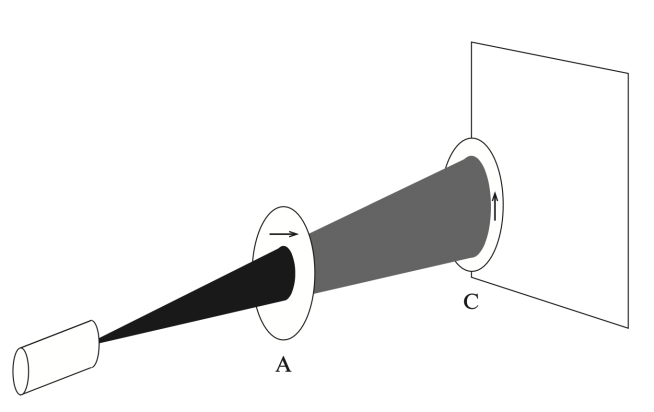


Figure 2: Two orthogonal polaroids block all photons.

The Quantum Mechanics of Photon Polarization

Finally, place polaroid **B** between polaroids **A** and **C**. Surprisingly, at most polarization angles of **B**, light shines on the screen. The intensity of this light will be maximal if the polarization of **B** is at 45° to both **A** and **C**.

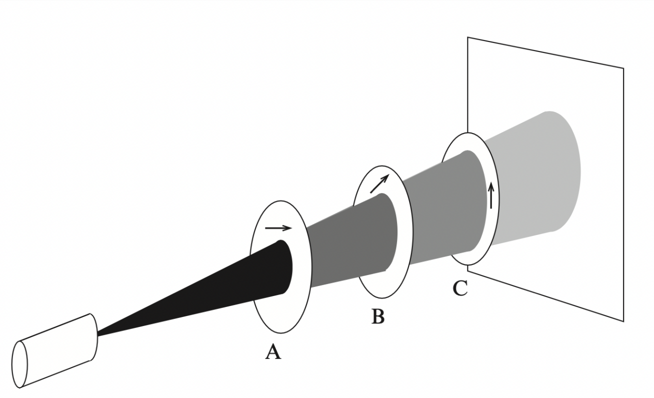


Figure 3: Inserting a third polaroid allows photons to pass.

The Quantum Mechanics of Photon Polarization

Clearly the polaroids cannot be acting as **simple sieves**; otherwise, inserting polaroid B could not increase the number of photons reaching the screen.

Some notation before an explanation

Quantum mechanics models a photon's polarization state by a unit vector, a vector of length 1, pointing in the appropriate direction.

We write $|\uparrow\rangle$ and $|\rightarrow\rangle$ for the unit vectors that represent vertical and horizontal polarization respectively.

In quantum mechanics, the standard notation for a vector representing a quantum state is $|v\rangle$, just as $\vec{v}$ or $\mathbf{v}$ are notations used for vectors in other settings.

An arbitrary polarization can be expressed as a linear combination

$$|v\rangle = a\,|\uparrow\rangle + b\,|\rightarrow\rangle$$

of the two basis vectors $|\uparrow\rangle$ and $|\rightarrow\rangle$.

Polaroid interacts with a photon

When a photon with polarization $|v\rangle = a\,|\uparrow\rangle + b\,|\rightarrow\rangle$ meets a polaroid with preferred axis $|\uparrow\rangle$, the photon will get through with probability $|a|^2$ and will be absorbed with probability $|b|^2$.

Any photon that passes through the polaroid will now be polarized in the direction of the polaroid's preferred axis.

A Quantum Explanation

In the experiment, any photons that pass through polaroid **A**, will leave as a $|{\rightarrow}\rangle$.

So it has no chance of passing through polaroid **C**, which was vertical.

To understand what will happen, when we insert **B**, let's rewrite

$$|{\rightarrow}\rangle = \frac{1}{\sqrt{2}}\,|{\nearrow}\rangle - \frac{1}{\sqrt{2}}\,|{\nwarrow}\rangle\,.$$

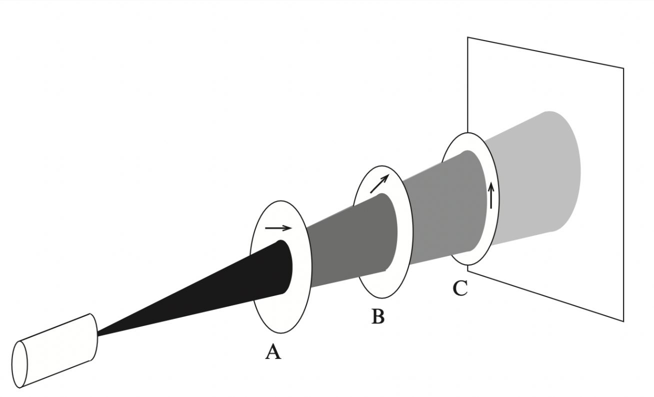
So, any photon in $|{\rightarrow}\rangle$ state will have a $\frac{1}{\sqrt{2}}$ amplitude in $|{\nearrow}\rangle$ direction. Thus, it has a probability of $\frac{1}{2}$ passing through **B**.

The same happens with **C** as

$$|{\nearrow}\rangle = \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\rightarrow}\rangle\,.$$

The Quantum Mechanics of Photon Polarization

We see how the model explains that $\frac{1}{8}$ of light pass through **A, B, C**.



Single Quantum Bits

The space of possible polarization states of a photon is an example of a quantum bit, or **qubit**. A qubit has a continuum of possible values: any state represented by a unit vector $a\left|\uparrow\right\rangle + b\left|\rightarrow\right\rangle$ is a legitimate qubit value.

Any quantum mechanical system that can be modeled by a two-dimensional complex vector space can be viewed as a qubit.

It includes photon polarization, electron spin, and the ground state together with an excited state of an atom.

It is as yet unclear which two-state systems will be most suitable for physical realizations of quantum computers; it is likely that a **variety of physical representation** of qubits will be used.

$|0\rangle$ and $|1\rangle$

We just need to specify any two orthonormal (perpendicular) states and call them $|0\rangle$ and $|1\rangle$. It can be any states. For example in the photon polarization example we could equally set $\{|0\rangle, |1\rangle\} = \{|\rightarrow\rangle, |\uparrow\rangle\}$ or $\{|0\rangle, |1\rangle\} = \{|\nearrow\rangle, |\nwarrow\rangle\}$.

We will use the convention that $|0\rangle = |\uparrow\rangle, |1\rangle = |\rightarrow\rangle$, which means $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\nwarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

We will call $\{|0\rangle, |1\rangle\}$ as standard basis.

And write any qubit in the state $a|0\rangle + b|1\rangle$ as $\begin{pmatrix} a \\ b \end{pmatrix}$.

Single-Qubit Measurement

Quantum theory postulates that any device that measures a two-state quantum system must have two preferred states whose representative vectors, $\{|u\rangle, |u^{\perp}\rangle\}$, form an **orthonormal** basis for the associated vector space.

The probability of measuring $|u\rangle$ or $|u^{\perp}\rangle$ is equal to the square of the magnitude of $|u\rangle$ and $|u^{\perp}\rangle$, respectively.

And measured qubit has to gain amplitude of 1 in the measured state, so the measurement outcome is always one of the two basis vectors.

This behavior of measurement is an **axiom** of quantum mechanics.

For this reason, whenever we say *"measure a qubit,"* we must specify with respect to which basis the measurement takes place.

Pop-science misconception: $|v\rangle$ is not prob. mix of $|0\rangle, |1\rangle$

Rather, $|v\rangle$ is a definite state, which, when measured in certain bases, gives deterministic results, but for some other bases gives random results.

For example a state

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle + |\uparrow\rangle\right)$$

is deterministic in Hadamard basis $\{|\nearrow\rangle, |\nwarrow\rangle\}$, however random in $\{|\rightarrow\rangle, |\uparrow\rangle\}$.

It is OK to think that $a|0\rangle + b|1\rangle$ is at the same time $|0\rangle, |1\rangle$, however we have to be careful to distinguish between

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \text{ and } |i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

that behave differently in many cases, yet have the same proportion of $|0\rangle$ and $|1\rangle$.

A global phase vs relative phase

Let's consider a qubit state $|1\rangle$, $-|1\rangle$, and $i|1\rangle$. Can we differentiate them?

Our measuring devices can not. This is what we call a global phase. We can not tell a difference between states $|\phi\rangle$ and $c \cdot |\phi\rangle$, when $|c| = 1$ in other symbols, ($c = e^{i\theta}$ for some θ).

We only care the difference between the given states, their relative phases.

While multiplication with a unit constant does not change a quantum state vector, relative phases in a superposition do represent distinct quantum states: eventhough $|v_1\rangle \sim e^{i\theta}|v_1\rangle$,

$$\frac{1}{\sqrt{2}}\left(|v_1\rangle + |v_2\rangle\right) \nsim \frac{1}{\sqrt{2}}\left(e^{i\theta}|v_1\rangle + |v_2\rangle\right).$$

Bloch Sphere

It turns out it is also convenient to represent a qubit as a point in a 3 dimensional space, where each diameter is a orthornormal basis of a qubit. Note that the angles are doubled in this representation.

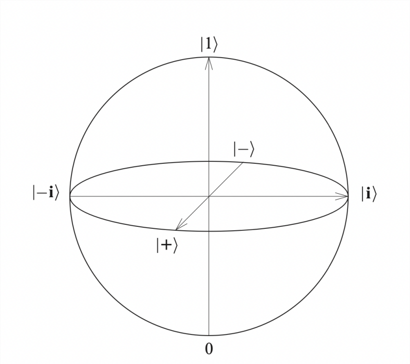


Figure 4: Location of certain single-qubit states on the surface of the Bloch sphere.