P5.1

2D Fourier transform (continuous def.)

$$S(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)\, e^{-j2\pi f_x x}\, e^{-j2\pi f_y y}\, dx\, dy$$

Vector notation:

$$S(\underline{f}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\underline{x})\, e^{-j2\pi \underline{f}^T \underline{x}}\, d\underline{x}$$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{f} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

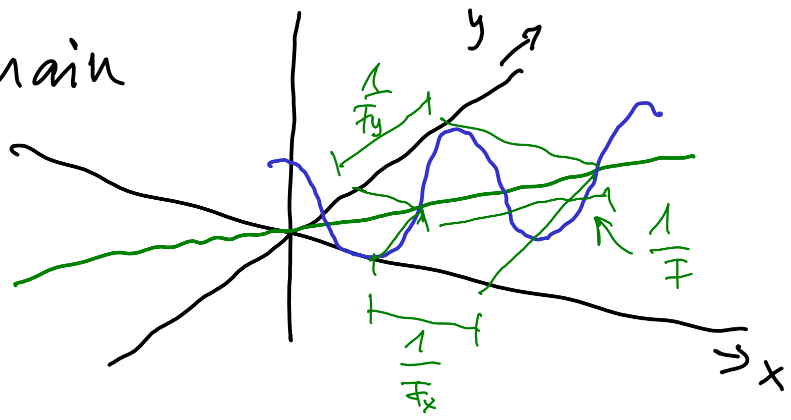
Example:

$$s(x,y) = \cos\left(2\pi\left[F_x x + F_y y\right]\right)$$

$$\circ\!\!-\!\!\bullet$$

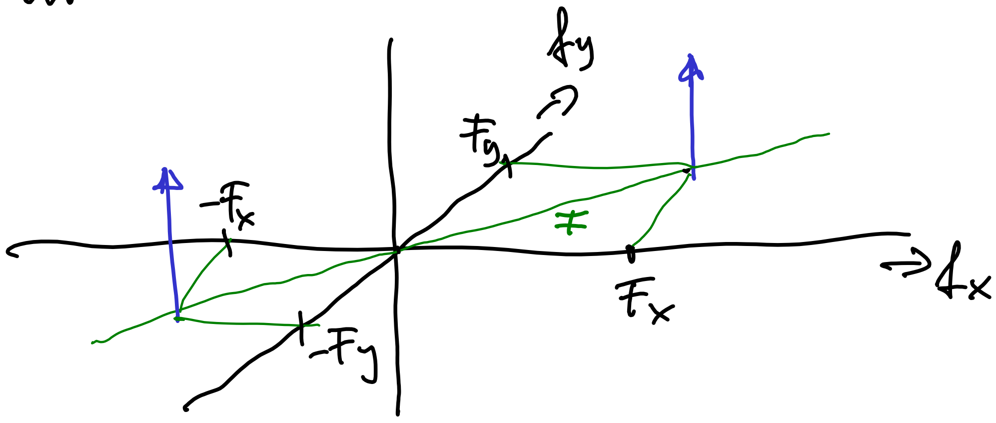
$$S(f_x, f_y) = \frac{1}{2}\left[\delta(f_x + F_x, f_y + F_y) + \delta(f_x - F_x, f_y - F_y)\right]$$

Spat. domain

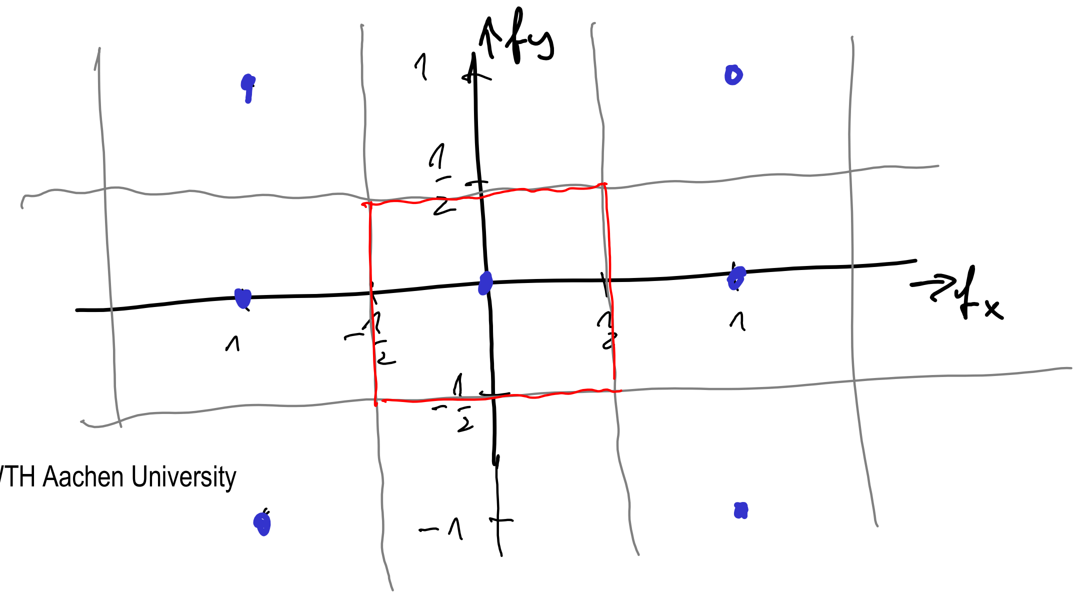


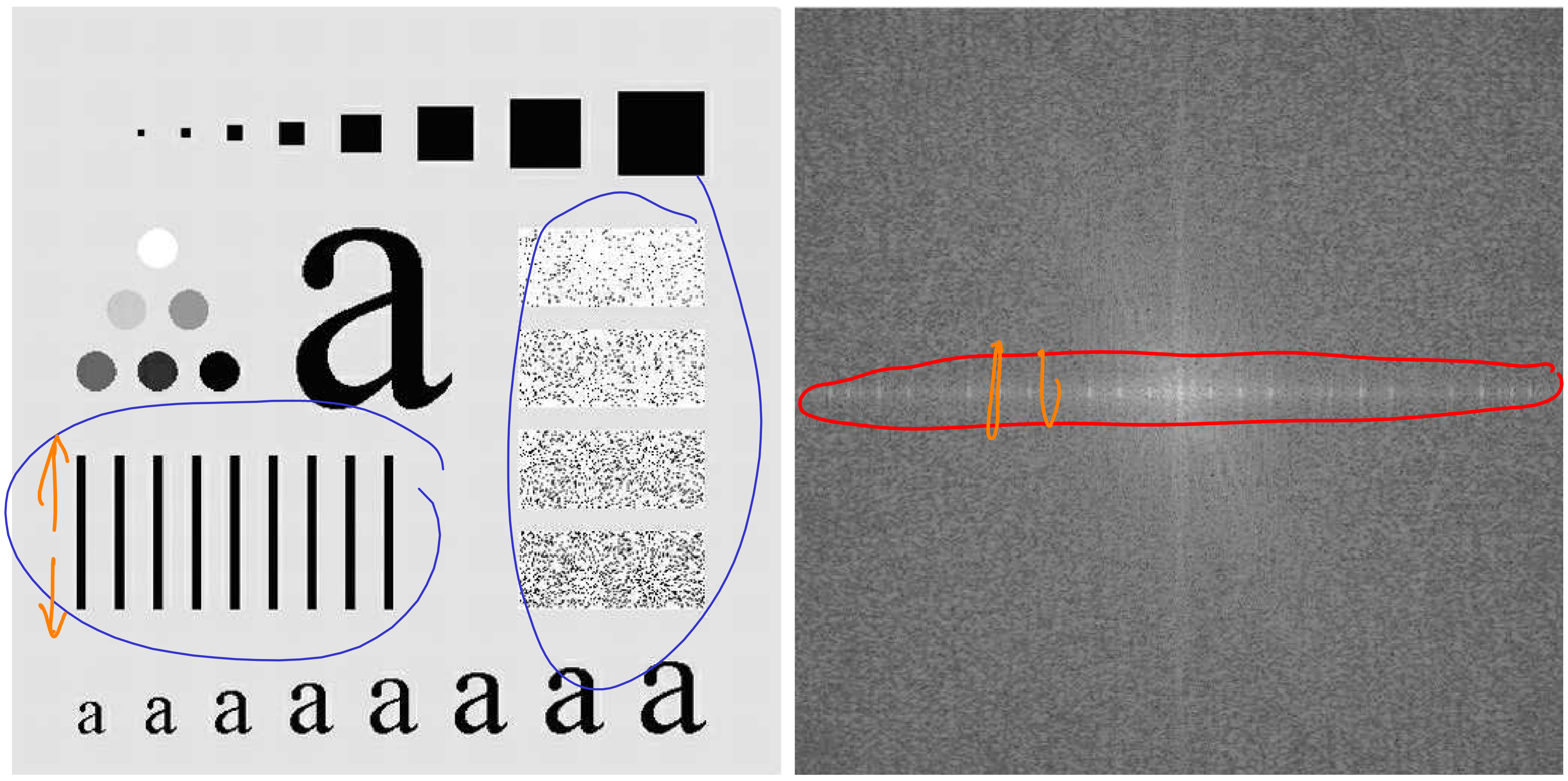
$$F = \sqrt{F_x^2 + F_y^2}$$

Freq. domain



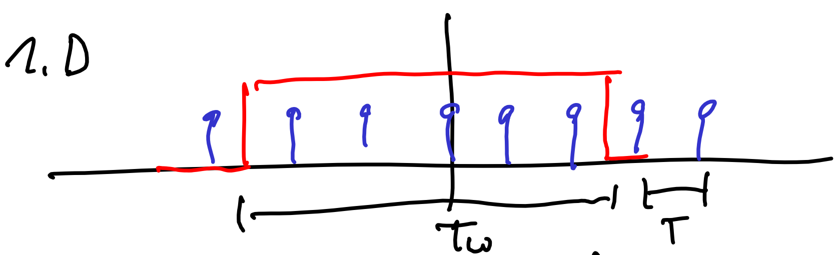
Images: Discrete sample arrays → 2D periodic spectra





where do these dots come from?

Observation from 1D: Impulse train in time domain transforms into impulse train in freq. domain!



$$\mathrm{rect}\left(\frac{t}{T_w}\right) \cdot \sum \delta(t - nT)$$

$$\circ\!\!-\!\!\bullet$$

$$T_w \, \mathrm{si}(\pi T_w f) * \frac{1}{T} \sum \delta\left(f - \frac{n}{T}\right)$$

$\Rightarrow$ Periodic stripes in image produce the bright dots on the f_x axis.

P5.2)

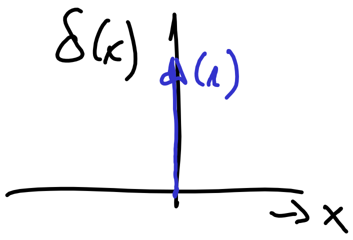
a) Separable function: $s(x,y) = \underline{f(x)\cdot g(y)}$

b) $$S(f_x,f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{s(x,y)}_{f(x)\cdot g(y)}\, e^{-j2\pi f_x x}\, e^{-j2\pi f_y y}\, dx\, dy$$

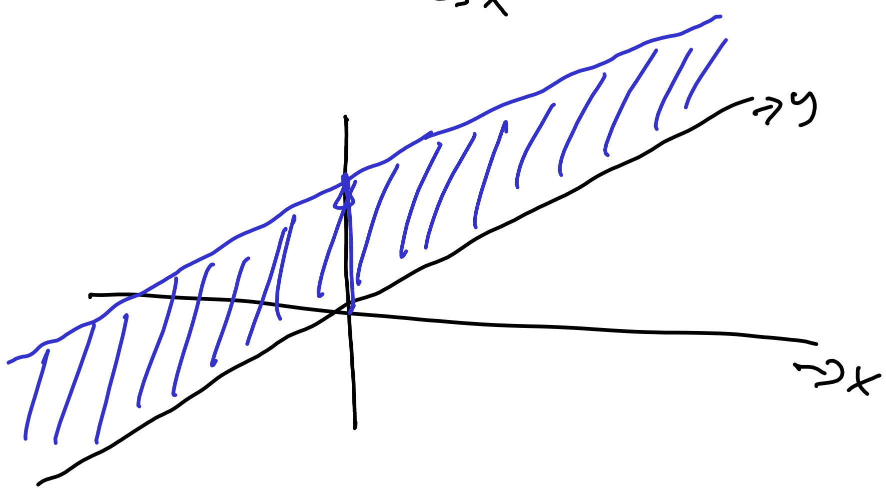
$$= \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi f_x x}\, dx \cdot \int_{-\infty}^{\infty} g(y)\, e^{-j2\pi f_y y}\, dy$$

$$= F(f_x)\cdot G(f_y)$$

c) 1D



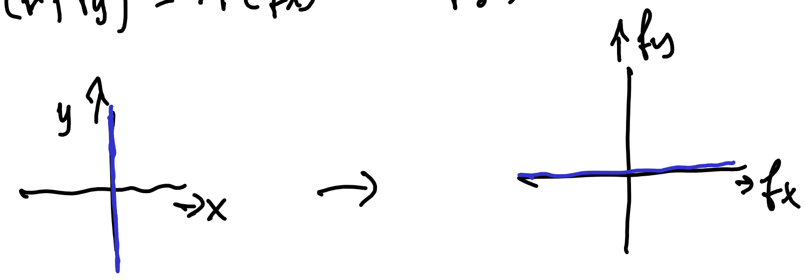
2D



$\delta(x)\cdot 1(y)$

$\Rightarrow \delta$-line

$$s(x,y) = \delta(x)\cdot 1(y)$$
$$\circ\!\!-\!\!\bullet$$
$$S(f_x,f_y) = 1(f_x)\cdot \delta(f_y)$$



d)

$$\delta(x) \;\overset{1D}{\circ\!\!-\!\!\bullet}\; 1$$

$$\delta(x)\,1(y) \;\overset{2D}{\circ\!\!-\!\!\bullet}\; 1(f_x)\cdot\delta(f_y)$$

$$\underbrace{\delta(x)\,1(y)\,1(z)}_{\delta\text{-plane}} \;\overset{3D}{\circ\!\!-\!\!\bullet}\; \underbrace{1(f_x)\cdot\delta(f_y)\cdot\delta(f_z)}_{\delta\text{-line in 3D freq. domain}}$$

also

$$\underbrace{1(x)\,\delta(y)\delta(z)}_{\substack{\delta\text{ line}\\ \text{in space}}} \overset{3D}{\circ\!\!-\!\!\bullet} \underbrace{\delta(f_x)\,1(f_y)\,1(f_z)}_{\substack{\delta\text{ plane}\\ \text{in freq.}}}$$

$$\underbrace{\delta(x)\,\delta(y)\,\delta(z)}_{\text{point}} \overset{3D}{\circ\!\!-\!\!\bullet} \underbrace{1(f_x)\cdot 1(f_y)\cdot 1(f_z)}_{\text{volume}}$$

e)

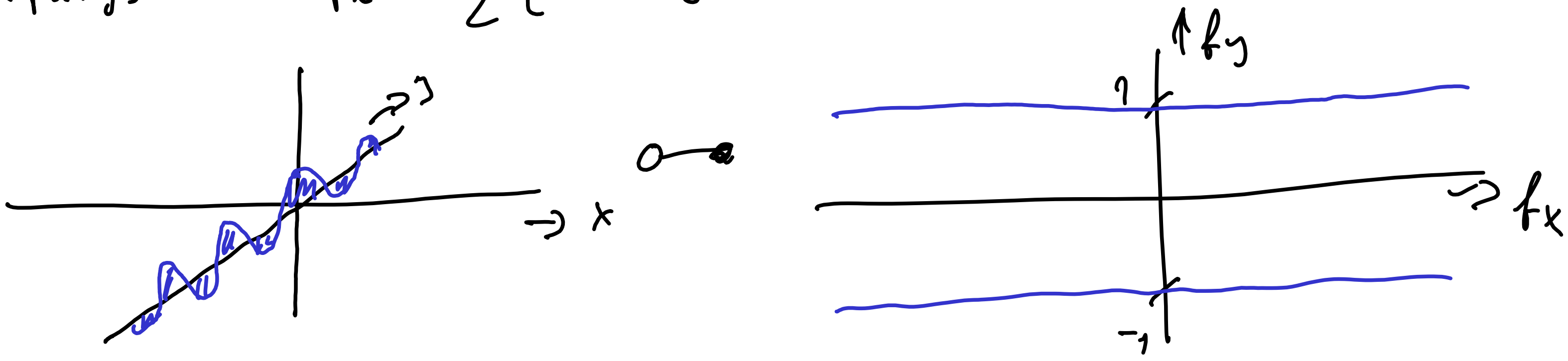
$$s(x,y) = \delta(x)\cdot\sum_{n=-\infty}^{\infty}\delta(y-n)$$

$$S(f_x,f_y) = 1(f_x)\cdot\sum_{n=-\infty}^{\infty}\delta(f_y-n)$$

f)

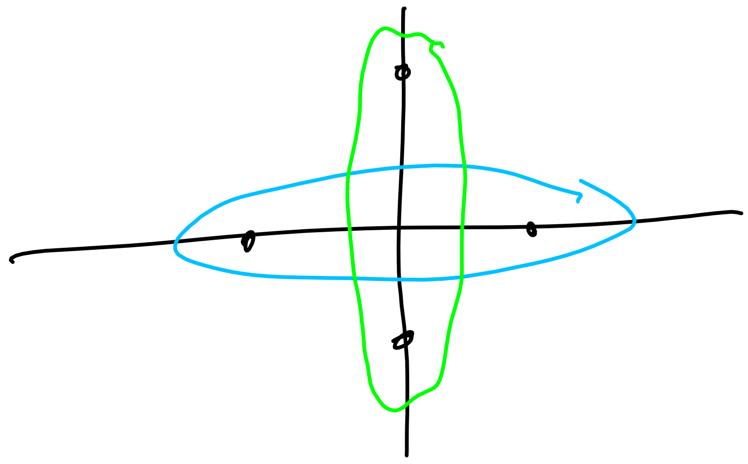
$$s(x,y) = \delta(x)\cdot\cos(2\pi 1 y)$$

$$S(f_x,f_y) = 1(f_x)\cdot\frac{1}{2}\left[\delta(f_y-1)+\delta(f_y+1)\right]$$

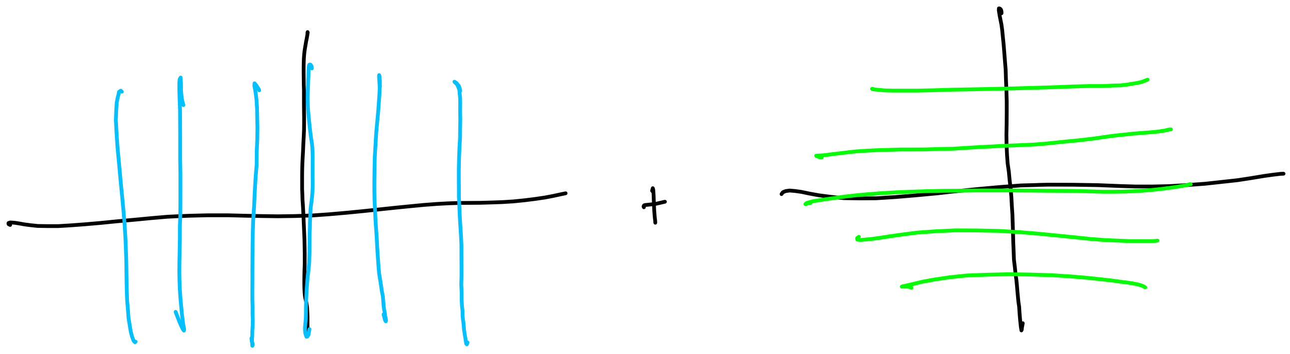


P5.3)

a)

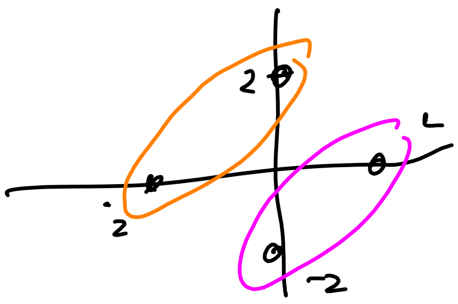


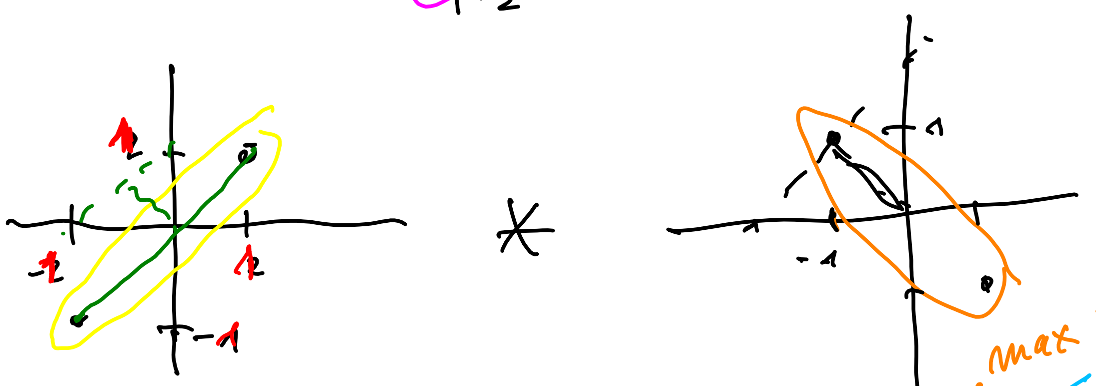
Option 1:



(difficult to draw)

Option 2:



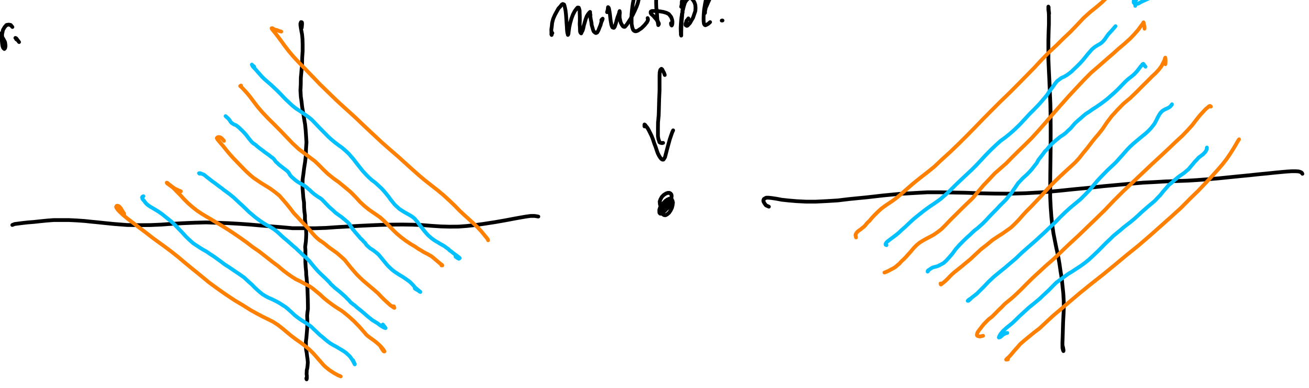


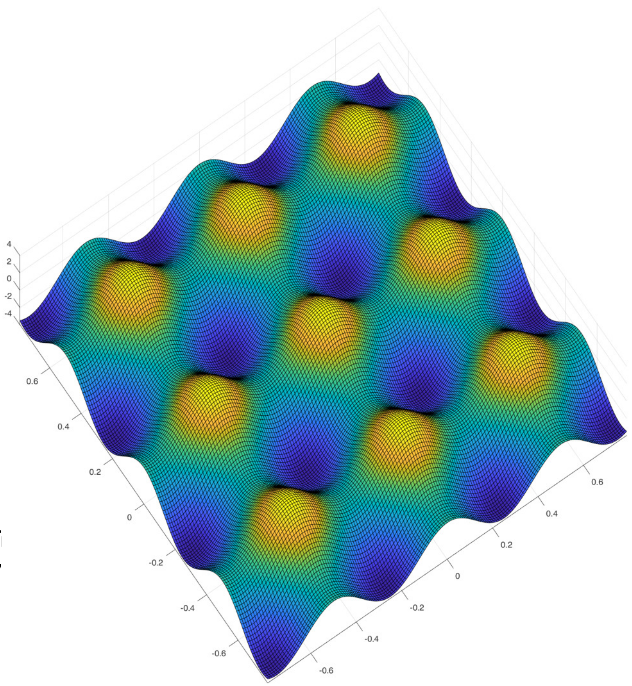
max value

zero crossing

Spectr.

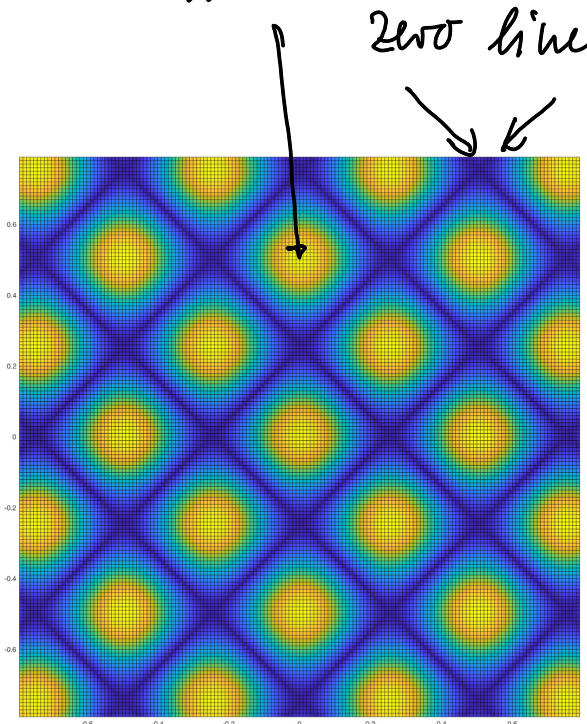
multipl.





max values

zero lines



$|S(f_x, f_y)|$

b) $$S(f_x, f_y)\big|_{f_y=0} = 2\cos(2\pi\, 2m\, f_x) + \underbrace{2\cos(2\pi\, 2m \underbrace{f_y}_{=0})}_{=2}$$