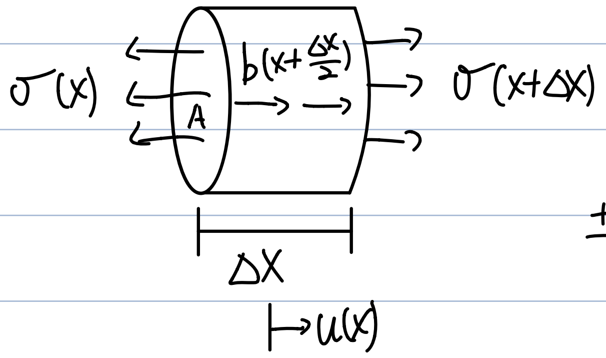
P1

ⓐ The assumption is that the defomation is 1D/Linear

ⓑ



$A\sigma' + b = 0, \quad F = A\sigma$

$\overset{+}{\rightarrow} \sum F = -F(x) + [b(x+\frac{\Delta x}{2})]\Delta x + F(x+\Delta x) = 0$

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} + b(x+\frac{\Delta x}{2}) = 0$$

$$\lim_{\Delta x \to 0}\Big[\underbrace{\frac{F(x+\Delta x) - F(x)}{\Delta x}} + b(x+\frac{\Delta x}{2})\Big] = 0$$

$$\frac{d}{dx}F(x) + b(x) = 0$$

$$\frac{d}{dx}A\sigma + b(x) = 0$$

So, $\boxed{A\sigma' + b = 0}$ for Δx

ⓒ $\varepsilon = u'$

$$\varepsilon(x) = \frac{\text{elong}}{\text{original}} = \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0}\Big[\frac{u(x+\Delta x) - u(x)}{\Delta x}\Big] = \frac{du}{dx} = u'$$

So, $\boxed{\varepsilon = u'}$ for Δx

P2 The Governing Equation : $AE\,u''(x) + b(x) = 0$

(a) Using GE : $AE\,u''(x) + \frac{x^3}{100} = 0$

$$\int AE\,u''(x) + \frac{x^3}{100}\,dx = AE\,u'(x) + \frac{x^4}{400} + C_1 = 0$$

$$\int AE\,u'(x) + \frac{x^4}{400} + C_1\,dx = AE\,u(x) + \frac{x^5}{2000} + C_1 x + C_2 = 0$$

When $X=0$, $u(0)=0$,

$$AE(0) + 0 + C_1(0) + C_2 = 0 \,. \qquad \boxed{C_2 = 0}$$

When $X=L$, $P = A\sigma = AEu' = 2000$, $u'(L) = \frac{2000}{AE}$

$$2000 + \frac{60^4}{400} + C_1 = 0 \,, \qquad \boxed{C_1 = -34400}$$

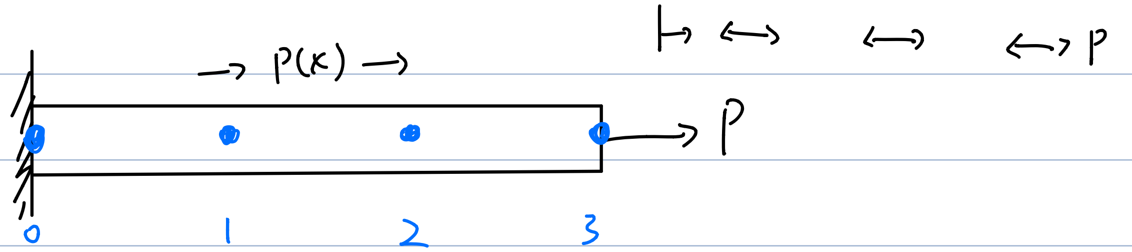
So, $\boxed{u(x) = \frac{1}{30000}\left(34400x - \frac{x^5}{2000}\right)}$ $\qquad u'(x) = \left(34400 - \frac{x^4}{400}\right)\frac{1}{AE}$

(b)

Weak Form : $\int_0^L AE\,u'\bar{u}'\,dx - \int_0^L b\,\bar{u}\,dx - A\sigma\bar{u}\Big|_0^L = 0$

For this problem: $\boxed{\int_0^L AE\,u'\bar{u}'\,dx - \int_0^L \left(\frac{x^3}{100}\right)\bar{u}\,dx - P\,\bar{u}\Big|_0^L = 0}$

(c)

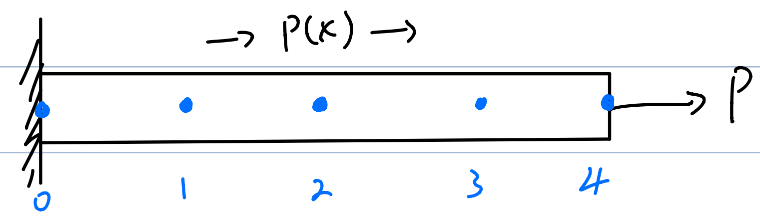


Three elements:

$$\begin{bmatrix} P_0 \\ 0 \\ 0 \\ 2000 \end{bmatrix} + \begin{bmatrix} 80 \\ 2400 \\ 14400 \\ 15520 \end{bmatrix} = \frac{AE}{L/3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

End force   Truss   k global

$$\boxed{u_1 = 0.0229 \qquad u_2 = 0.0442 \qquad u_3 = 0.0558}$$



$$\begin{bmatrix} P_0 \\ 0 \\ 0 \\ 0 \\ 2000 \end{bmatrix} + \begin{bmatrix} 25.3125 \\ 759.375 \\ 4556.25 \\ 14428.125 \\ 12630.9375 \end{bmatrix} = \frac{AE}{L/4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\boxed{u_1 = 0.0172 \qquad u_2 = 0.0340 \qquad u_3 = 0.0485 \qquad u_4 = 0.0558.}$$

(d) Plots on Matlab

For displacement, the FEA work fine, for stress, the FEA results are not close to the exact solutions. The max σ is at left end of the bar for FEM solution

(e) Computational: $u = \frac{1}{2}\int^{60} \cancel{EA} \frac{1}{(EA)^{2}} (3400 - \frac{x^4}{400})^2 dx = \boxed{854.18 \text{ lb} \cdot \text{in}}$

$$E = \int_0^{60} \left[\frac{1}{30000}\left(34400x - \frac{x^5}{2000}\right)\right] \frac{x^3}{100}\, dx + 2000(55.84) = \boxed{1708.36 \text{ lb}\cdot\text{in}}$$

$$\boxed{\Pi_p = E - U = 854.18 \text{ lb}\cdot\text{in}}$$

Three Element :

$$U = \frac{1}{2} EA \left[\int_0^{20} \underbrace{\overset{\frac{u_1}{20}}{(0.001145)^2}}_{393.31} dx + \int_0^{20} \underbrace{\overset{\frac{u_2-u_1}{20}}{(0.001064)^2}}_{339.63} dx + \int_0^{20} \underbrace{\overset{\frac{u_3-u_2}{20}}{(0.000584)^2}}_{102.32} dx \right] = \boxed{835.26}$$

$$E = \int_0^{20} \left(\frac{x^3}{100}\right)(0.02288)\, dx + \int_0^{20} \frac{(x+20)^3}{100} (0.04416)\, dx + \int_0^{20} \frac{(x+40)^3}{100} (0.05588)\, dx + 2000(0.05588)(20)$$

$$\boxed{= 1670.52 \text{ lb}\cdot\text{in}}$$

$$\boxed{\Pi_p = 1670.52 - 835.26 = 835.26 \text{ lb}\cdot\text{in}}$$

Four Elements

$$U = \frac{1}{2} EA \left[\int_0^{15} (0.001146)^2 dx + \int_0^{15} (0.00112)^2 dx + \int_0^{15} (0.000969)^2 dx + \int_0^{15} (0.000488)^2 dx \right.$$

$$\boxed{= 842.6 \text{ lb}\cdot\text{in}}$$

$$E = \int_0^{15} \frac{x^3}{100} (0.01719)\, dx + \int_0^{15} \frac{(x+15)^3}{100} (0.03399)\, dx + \int_0^{15} \frac{(x+30)^3}{100} (0.04852)\, dx$$

$$+ \int_0^{15} \frac{(x+45)^3}{100} (0.05584)\, dx + 2000(0.05584)(15) = \boxed{1685.16}$$

$$\Pi_p = 1685.16 - 842.6 = \boxed{842.6 \text{ lb}\cdot\text{in}}$$

P3

Transfer to WF first by times $\bar{u}(x)$ and $\int_0^L$

$$\int_0^L \frac{d^2u(x)}{dx^2}\bar{u}(x)dx + \int_0^L u(x)\bar{u}(x) = 0$$

$$\int_0^L \frac{du(x)}{dx}\frac{du(x)}{dx}\bar{u}(x)\,dx + \int_0^L u(x)\bar{u}(x) = 0$$

Integ by parts

$$-\int_0^L \frac{du(x)}{dx}\frac{\bar{u}(x)}{dx}dx + \frac{du(0)}{dx}\bar{u}(0) - \frac{du(L)}{dx}\bar{u}(L) + \int_0^L u(x)\bar{u}(x)dx = 0$$

$$\bar{u}(0) = 0, \quad \bar{u}(L) = 0$$

So,

$$-\int_0^L \frac{du(x)}{dx}\frac{\bar{u}(x)}{dx}dx + \int_0^L u(x)\bar{u}(x)dX = 0$$

$$u(x) = [N][u] \quad , \bar{u}(x) = [N][\sigma u]$$

So, $$\left[\int_0^L \left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right] - [N]^T[N]dx\right][u] = 0 = [K][u]$$

$$\left[\frac{dN}{dx}\right] = [-2, 2]$$

$$[K] = \int_0^{0.5} \begin{bmatrix} -2 \\ 2 \end{bmatrix}[-2 \;\; 2] - \begin{bmatrix} 1-2x \\ 2x \end{bmatrix}[1-2x \;\; 2x]\,dx$$

$$[K] = \begin{bmatrix} 1.833 & -2.0833 \\ -2.0833 & 1.833 \end{bmatrix}$$

$\Downarrow$

$$\begin{bmatrix} F_0 \\ 0 \\ 0 \\ 0 \\ F_4 \end{bmatrix} = \begin{bmatrix} 1.833 & -2.0833 & 0 & 0 & 0 \\ -2.0833 & 3.667 & -2.0833 & 0 & 0 \\ 0 & -2.0833 & 3.667 & -2.0833 & 0 \\ 0 & 0 & -2.0833 & 3.667 & -2.0833 \\ 0 & 0 & 0 & -2.0833 & 1.833 \end{bmatrix} \begin{bmatrix} 0.025 \\ u_1 \\ u_2 \\ u_3 \\ 0 \end{bmatrix}$$

$$\boxed{u_1 = 0.0271 \qquad u_2 = 0.0228 \qquad u_3 = 0.0129}$$

P4.

Matlab

I think 2 elements & 8 elements FEM are good at the beginning but inaccurate at the end. 20 elements is better than both of them.

P5

$\sigma = E\varepsilon \Rightarrow \sigma = Eu' \Longrightarrow \hat{\sigma} = E\hat{u}' \Rightarrow \hat{\sigma} = EKN'$

$\varepsilon = u'$, $\hat{u} = K \cdot N(x)$

$$E = \int_0^L (EKN' - Eu')^2 dx$$

$$\frac{dE}{dK} = \int_0^L 2(EKN' - Eu')(EN')dx = 0$$

$\Downarrow$

$$\cancel{2E}\int_0^L N'(EKN' - Eu')\, dx = 0$$

$\Downarrow$

$$\Downarrow$$

$$\int_0^L (N')^2 E K \, dx - \int_0^L N' E u' \, dx = 0$$

$$\int_0^L (N')^2 E K \, dx = \int_0^L N' E u' \, dx$$

$$\boxed{K = \frac{\int_0^L N' u' \, dx}{\int_0^L (N')^2 \, dx}}$$