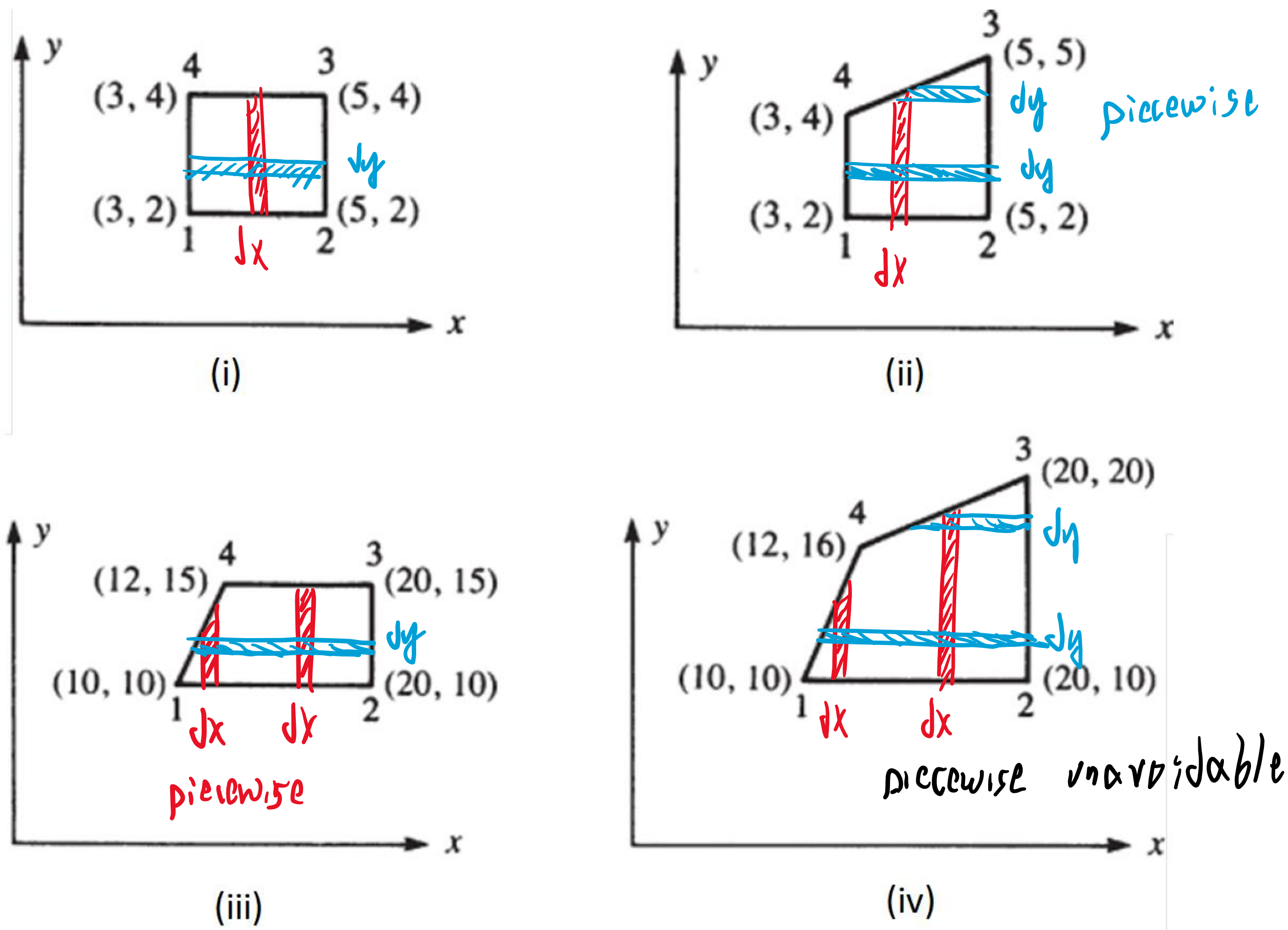
Homework 3 – Partial Solution

You are not required to turn in the two problems below this (the solution for these is also already posted), but you are responsible for this material on the midterm.

6. ***Isoparametric Elements***

   (a) ***Solution***



(b) ***Solution***

First, must define the geometry of each element in (ξ,η) space:

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \{x_1 \quad y_1 \quad x_2 \quad y_2 \quad x_3 \quad y_3 \quad x_4 \quad y_4\}^T$$

Using the shape functions:

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Doing this, one obtains:

$$\text{Elem a}: \begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = [N]\{3 \quad 2 \quad 5 \quad 2 \quad 5 \quad 4 \quad 3 \quad 4\}^T = \begin{Bmatrix} 4+\xi \\ 3+\eta \end{Bmatrix}$$
$$\text{Elem b}: \begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = [N]\{3 \quad 2 \quad 5 \quad 2 \quad 5 \quad 5 \quad 3 \quad 4\}^T = \begin{Bmatrix} 4+\xi \\ \frac{1}{4}(13+\xi+\xi\eta+5\eta) \end{Bmatrix}$$
$$\text{Elem c}: \begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = [N]\{10 \quad 10 \quad 20 \quad 10 \quad 20 \quad 15 \quad 12 \quad 15\}^T = \begin{Bmatrix} \frac{1}{2}(31+9\xi-\xi\eta+\eta) \\ \frac{1}{2}(25+5\eta) \end{Bmatrix}$$
$$\text{Elem d}: \begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = [N]\{10 \quad 10 \quad 20 \quad 10 \quad 20 \quad 20 \quad 12 \quad 16\}^T = \begin{Bmatrix} \frac{1}{2}(31+9\xi-\xi\eta+\eta \\ 14+\xi+\xi\eta+4\eta \end{Bmatrix}$$

Then, to calculate the Jacobian:

$$J = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix}$$

$$\text{Elem a}: \ [J_a] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Elem b}: \ [J_b] = \begin{bmatrix} 1 & \frac{1}{4}(1+\eta) \\ 0 & \frac{1}{4}(5+\xi) \end{bmatrix}$$
$$\text{Elem c}: \ [J_c] = \begin{bmatrix} \frac{1}{2}(9-\eta) & 0 \\ \frac{1}{2}(1-\xi) & \frac{5}{2} \end{bmatrix} \qquad \text{Elem d}: \ [J_d] = \begin{bmatrix} \frac{1}{2}(9-\eta) & 1+\eta \\ \frac{1}{2}(1-\xi) & 4+\xi \end{bmatrix}$$

Finally, the determinant of the Jacobian:

$$\text{Elem a}: \ |\mathbf{J}_a| = 1 \qquad \text{Elem b}: \ |\mathbf{J}_b| = \frac{1}{4}(5+\xi)$$
$$\text{Elem c}: \ |\mathbf{J}_c| = \frac{5}{4}(9-\eta) \qquad \text{Elem d}: \ |\mathbf{J}_d| = \frac{1}{2}(35+10\xi-5\eta)$$

7. ***Discrete Integration - Solution***

$$I = \int_{-1}^{1} f(\xi)\, d\xi = \sum_{k=1}^{3} W_k f(\xi_k) = W_1 f(\xi_1) + W_2 f(\xi_2) + W_3 f(\xi_3)$$

$$f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5$$

$$I = \int_{-1}^{1} \left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5\right) d\xi = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4$$

Because we require that the formulation is unbiased with respect to $\xi = 0$, we require $\xi_1 = -\xi_3$, $\xi_2 = 0$, and $W_1 = W_3$. We can define $\xi_1 = -\tilde{\xi}$, thus $\xi_3 = \tilde{\xi}$, and we can define $W_1 = W_3 = W^*$:

$$I_{gauss} = W_1 f(\xi_1) + W_2 f(\xi_2) + W_3 f(\xi_3) = W^* f\left(-\tilde{\xi}\right) + W_2 f(0) + W^* f\left(\tilde{\xi}\right)$$

$$f\left(-\tilde{\xi}\right) = a_0 - a_1\tilde{\xi} + a_2\tilde{\xi}^2 - a_3\tilde{\xi}^3 + a_4\tilde{\xi}^4 - a_5\tilde{\xi}^5$$

$$f\left(\tilde{\xi}\right) = a_0 + a_1\tilde{\xi} + a_2\tilde{\xi}^2 + a_3\tilde{\xi}^3 + a_4\tilde{\xi}^4 + a_5\tilde{\xi}^5$$

$$f(0) = a_0$$

$$I_{gauss} = 2W^*\left(a_0 + a_2\tilde{\xi}^2 + a_4\tilde{\xi}^4\right) + W_2 a_0 = (2W^* + W_2)\, a_0 + 2W^*\tilde{\xi}^2 a_2 + 2W^*\tilde{\xi}^4 a_4$$

Setting $I_{gauss} = I$, we get

$$(2W^* + W_2)\, a_0 + 2W^*\tilde{\xi}^2 a_2 + 2W^*\tilde{\xi}^4 a_4 = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4$$

$$\text{From } a_2 \text{ and } a_4:\quad 2W^*\tilde{\xi}^2 = \frac{2}{3} \quad\to\quad 2W^*\tilde{\xi}^4 = \frac{2}{5} \quad\to\quad \left(2W^*\tilde{\xi}^2\right)\tilde{\xi}^2 = \frac{2}{3}\tilde{\xi}^2 = \frac{2}{5} \quad\to\quad \tilde{\xi} = \pm\sqrt{\frac{3}{5}}$$

$$2W^*\tilde{\xi}^2 = \frac{2}{3} \quad\to\quad 2W^*\left(\frac{3}{5}\right) = \frac{2}{3} \quad\to\quad W^* = W_1 = W_3 = \frac{5}{9}$$

$$\text{From } a_0:\quad 2W^* + W_2 = 2 \quad\to\quad W_2 = 2 - 2W^* = 2 - \frac{10}{9} \quad\to\quad W_2 = \frac{8}{9}$$

Use two and three point quadrature to solve the following integrals and comment on your results.

$$\int_{-1}^{1} \cos\frac{s}{2} ds \qquad \int_{-1}^{1} s^2 ds \qquad \int_{-1}^{1} s^4 ds$$

For two point quadrature: $x_1 = -\sqrt{1/3},\ x_2 = -x1 = \sqrt{1/3},\ W_1 = W_2 = 1.0$
For three point quadrature: $x_1 = -\sqrt{3/5},\ x_2 = 0,\ x_3 = -x1 = \sqrt{3/5},\ W_1 = W_3 = 5/9,\ W_2 = 8/9$

$$\int_{-1}^{1} \cos\frac{s}{2} ds = 1.9177$$

$$\text{Two point quadrature :}\quad 1.0\cos(-\sqrt{1/3}/2) + 1.0\cos(\sqrt{1/3}/2) = 1.9172$$

$$\text{Three point quadrature :}\quad (5/9) * \cos(-\sqrt{3/5}/2) + (8/9) * \cos(0) + (5/9)\cos(\sqrt{3/5}/2) = 1.9177$$

$$\int_{-1}^{1} s^2 ds = 0.6667$$

$$\text{Two point quadrature :}\quad 1.0(-\sqrt{1/3})^2 + 1.0(\sqrt{1/3})^2 = 0.6667$$

$$\text{Three point quadrature :}\quad (5/9) * (-\sqrt{3/5})^2 + (8/9) * (0)^2 + (5/9)(\sqrt{3/5})^2 = 0.6667$$

$$\int_{-1}^{1} s^4 ds = 0.40$$

$$\text{Two point quadrature :}\quad 1.0(-\sqrt{1/3})^4 + 1.0(\sqrt{1/3})^4 = 0.2222$$

$$\text{Three point quadrature :}\quad (5/9) * (-\sqrt{3/5})^4 + (8/9) * (0)^4 + (5/9)(\sqrt{3/5})^4 = 0.4$$

Of note, both two and three point quadrature were able to solve the second term exactly, because two point quadrature is exact for polynomials up to 3rd order and three point quadrature is exact for polynomials up to 5th order. This is also why two point quadrature is incorrect for the s^4 equation but three point quadrature works. While the cosine function is not a polynomial, it expands into an infinite series of polynomials, thus additional levels of quadrature will improve the solution with higher order error terms.