Homework 5

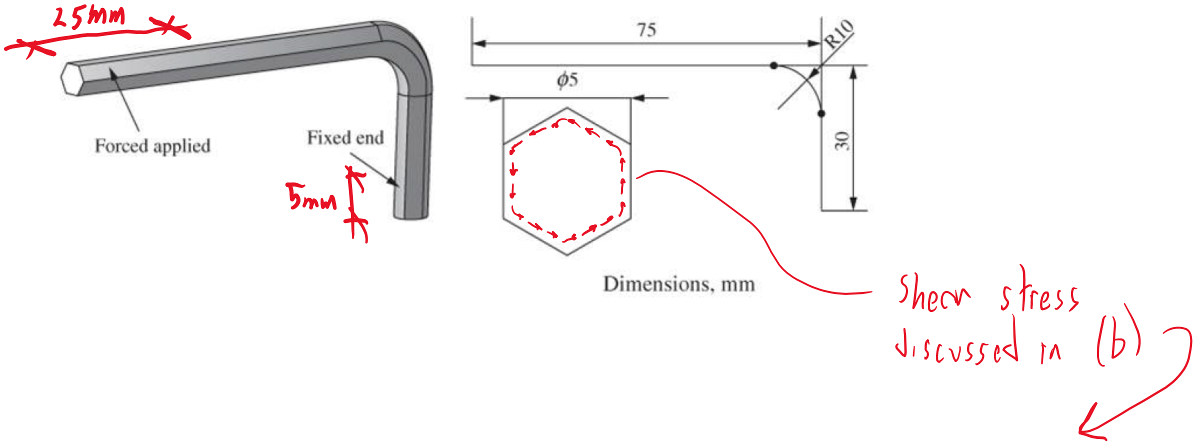
CESG 504 – Finite Element Methods

Due: Thursday, June 3, 2021

1. ***3D Stress Analysis Problem (See Tutorial 3 in Google Drive)***:

(a)

An Allen wrench will be used to loosen a bolt with a hex-head cross section. This wrench is 5-mm in size and is made of quenched and tempered carbon steel with modulus of elasticity 200 GPa, Poisson's ratio of 0.29, and a yield strength of 615 MPa. The wrench is to be used to loosen a rusty bolt. To simulate fixity, a surface 2.5-mm in depth from the bottom is held fixed. A total force of 125 N is applied uniformly over 25 mm at the end of the horizontal section of the wrench. Determine the maximum von Mises stress in the wrench, and determine the maximum displacement. Comment on the safety of the wrench based on whether it will yield or not.



Note: As a part of this question, please do a convergence study with increasing # elements.

(b) In the first part, the Von-Mises stress is requested. Also find the peak shear stress on the vertically-oriented leg (as shown in the 3D diagram) above the fixed support. This stress will be almost equal the torsion-induced shear stress 'flowing' around the outer perimeter as shown on the cross-section diagram.

(c) Try to estimate the same shear stress with a hand calculation.

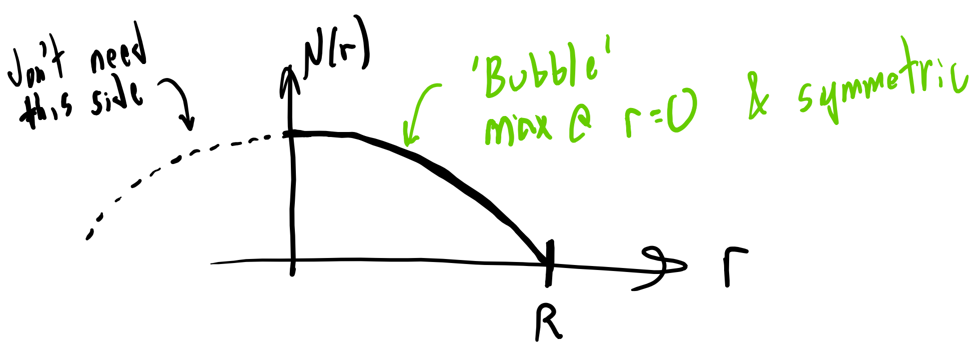
Tip Part 1: There are many ways to calculate shear stress. One possible estimate that you can use is discussed here: Approximate the cross-section with a circle (use engineering judgement to choose the radius). Because you can obtain the torsion (using engineering statics), you could estimate the value of $\phi(x, y) = \phi(r, \theta) = \phi(r)$ through the use of a shape function as shown below. In polar coordinates, the shape function is a lot easier to use and integrate. If this is a circular cross-section, the value of ϕ should not change with the theta coordinate. θ

$$T = \text{known} = 2\int_A \phi \, dA = 2\int_A \phi_0 N(x,y) \, dA$$

$$= 2\phi_0 \int_A N(x,y) \, dA = 2\phi_0 \int_0^r \int_0^{2\pi} N(r) \, r \, dr \, d\theta$$

To be solved

Tip Part 2: Using the membrane analogy, I feel like the shape function should look kind of like the sketch below. Come up with a function that looks like this, and use it.



Tip Part 3: Now that you know ϕ_0 and hence also $\phi(r)$, convert it back to $\phi(x, y)$. This will allow the final step of taking the necessary partial derivates to recover the stress. I recommend taking the derivative either at the left/right or top/bottom as the stress magnitude is constant around the perimeter.

(d) Comment on the result of the hand calculation vs. FEM. We talked about infinite stress at re-entrant corners in class, but what is happening here at the non-re-entrant corners?

Discussion 1: Try to take a look at what this shear stress is at the six corners, you will see that material is actually kind of wasted (you might need a fine mesh). Well, it is not really wasted, as it is needed within the contacting region with the bolt to get leverage, but outside of that region these could be made circular with almost the same performance (that is not usually done as it would be expensive to manufacture).

Discussion 2: This question highlights some classical mechanics analysis. In the pre-FEM days, all sorts of ad-hoc approaches were necessary to estimate solutions for complex problems (of course sometimes they are still used). Often, the ad-hoc approaches are just as intellectually challenging (or more) as 3D FEM itself, just a bit more tractable. However, what most ad-hoc approaches require is a deep understanding of the fundamentals, as you never know what you might need to solve a given problem. That part can be frustrating, but it is also rewarding, each problem is a new puzzle. Analytical approaches can also sometimes give more insight, even if they are very approximate.

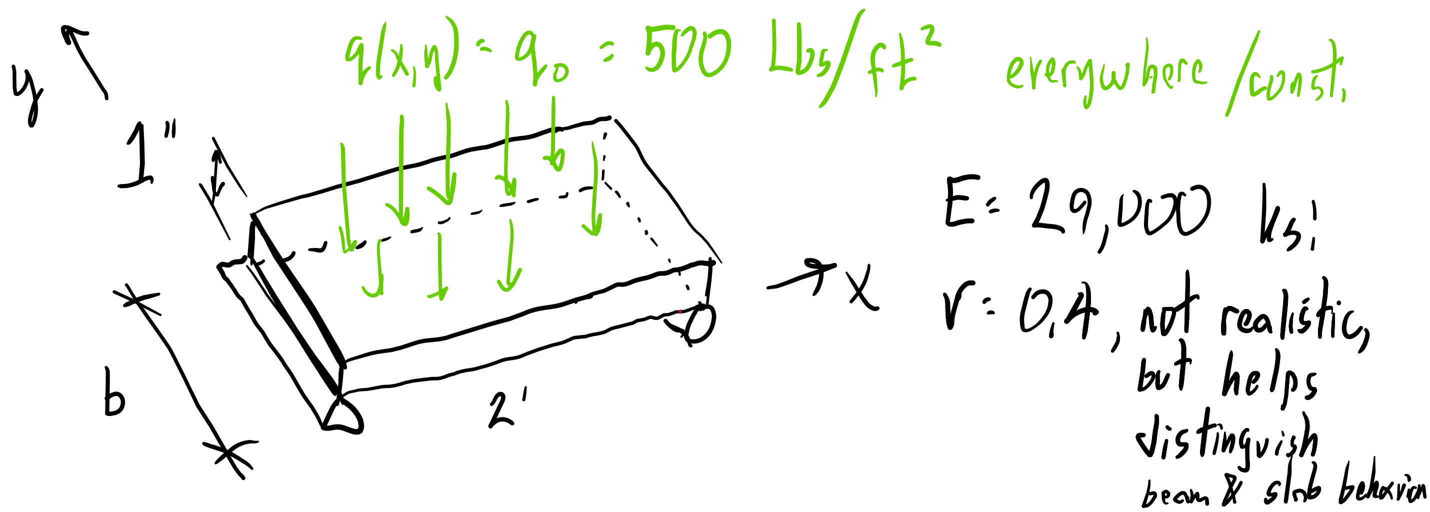
2. ***3D Stiffness Matrix:*** Evaluate component K_{11}^{e} of the 24 x 24 stiffness matrix for simple brick elements. This means that the elements are rectangular prisms. Formally, this means calculating the entire $[B^e]^T[C][B^e]$,then taking the 1,1 component and integrating it over the volume, however, there are shortcuts if you think through the bookkeeping carefully. The first shape function is shown below, you can obtain the rest in a similar manner (if you need them). Note that in the physical coordinates, node 1 is usually taken to be at the origin (and node 2 at (Lx, 0, 0), node 3 at (Lx, Ly, 0), node 4 at (0, Ly, 0), etc.), which is different from in the reference coordinates (which we are not using here) where the origin is in the interior of the element.

$$N_1(x,y,z) = \left(1-\frac{x}{Lx}\right)\left(1-\frac{y}{Ly}\right)\left(1-\frac{z}{Lz}\right) = \begin{cases} 1 \text{ @ } x=y=z=0 \text{, Node } 1 \\ 0 \text{ @ other 7 nodes} \end{cases}$$

Discussion 1: to make it easier the shape functions are written in the real physical coordinates, as opposed to the reference coordinates. One could also evaluate the integrals in the reference coordinates (using the Jacobian), however, there is no need to do so as the physical element is also rectangular. If the element was a skewed prism, then we would need to use the isoparametric approach.

Discussion 2: These shape functions borrow exactly from the approach we used for 2D rectangles. All we did was add the z-coordinate. We are still constructing these such that we have partitioning of unity.

3. ***Basic Plate Analysis:*** This problem focuses on the simply supported plate shown below.



(a) ***Analytical limiting solutions:*** Solve the deflection at the center by integrating the governing differential equations. Do this for the two extremes, (i) b = small, i.e., beam theory (use Euler-Bernoulli beam theory), and (ii) very wide, i.e., one-way slab theory. For option (ii), use the Kirchoff-Love Plate equation, but note that derivatives wrt y will vanish. Hence, you are solving the same problem twice, but with slightly difference coefficients. Additionally, you won't actually need to choose a b value, it will cancel out in both cases.

(b) ***From beams to one-way slabs.*** Use Abaqus to solve the same problem. Try to find the regimes of width b for which you would be comfortable using (i) beam theory, (ii) one-way slab theory, or (iii) neither.

Note: See Tutorial 4 on Abaqus plate modeling for some tips. The tutorial focuses on a two-way slab. The stresses are a lot more complex for two-way slabs than one-way types.

4. **Distorted Elements (Pathology)**: Read the first page, Table 1, and Section 5 of the included paper, “Effects of Element Distortions on the Performance of Isoparametric Elements" by Lee, and Bathe and answer the questions below. Of course feel free to read more, the part you are asked to read gives you a sense of the problem of element distortion, the rest of the paper adds a lot more detail. Elements may be highly distorted either due to poor meshing, or large deformations (in nonlinear analysis). Sometimes it is just unavoidable.

   Discussion 1: While the goal of this question is to improve your understanding of element distortion, a second goal is to give you a sense of what the literature looks like (this also happens to be a very good paper). Note though that no one paper is “representative” of the literature. This paper has academic, educational, and practical value. Not all papers seek to achieve impact in all those places, nor should they need to. Additionally, this paper is mostly numerical analysis. Many other papers will use closed-form analysis, or experiments, or mixtures of the three methods of solving mechanics problems (numerical, analytical, experimental).

   Discussion 2: Good research papers also lead to many new ideas. For example, this paper focuses on numerical analysis. It begs the question as to whether these observed behaviors can be more deeply described with closed-form analysis (though approximations are often needed). Indeed, many additional papers have come out that have explored this kind of question.

   (a) Calculate the Euler-Bernoulli prediction for the exact free end deflection as it will be a useful point of comparison for the values given in Table 1. Will this actually be the exact answer?

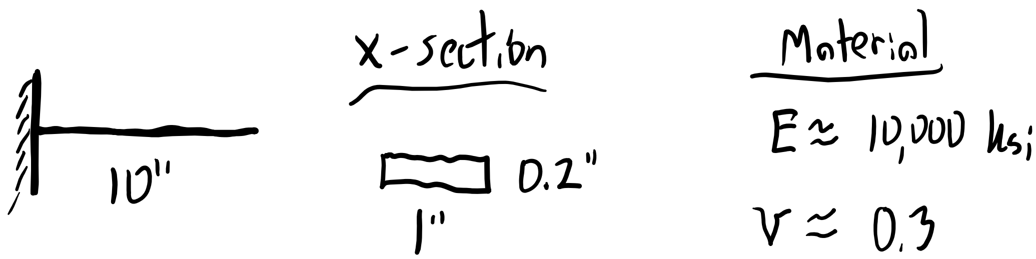
   (b) Sketch where the nodes are for 8-, 9-, and 16-node elements.

   (c) Define “Serendipity” and “Lagrangian” elements. Which ones from part (b) are which? We have primarily focused on Lagrangian elements in class, however, for linear shape functions they end up being the same thing as no interior nodes are needed.

   Discussion 3: The two classes of shape functions dominate common practice. However, there is really no limit to what you can use for shape functions, and everything including the kitchen sink has likely been tried at some point. If you decide not to care about partitioning of unity approach, you can get really crazy, but then again partitioning unity is very useful as it makes the coefficients equal to nodal displacements.

5. ***Structural Dynamics (Do not turn in, solution will be posted early):*** Obtain the requested natural frequencies and mode shapes of the structure below using each of the modeling approaches listed.

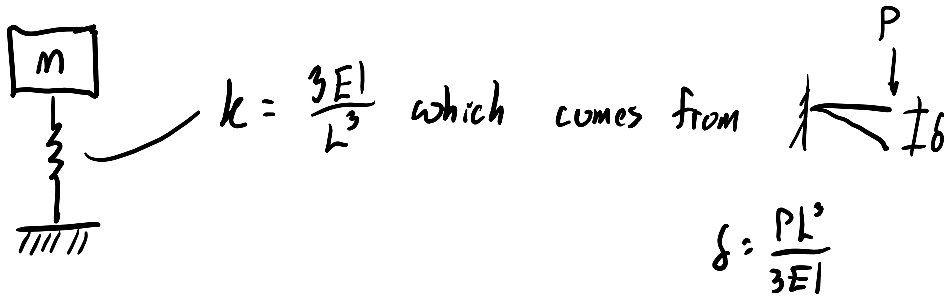
Note: this cantilever cross-section and material are modeled after common aluminum yard sticks. This is to give some intuition for what the frequency might be, or at least the order of magnitude. I could not make it a full yard in span as that would eat up the 1000-node limit that we have to work with.



(a) Using beam elements with Abaqus obtain the first three natural frequencies and mode shapes.

(b) Using 3D solid element with Abaqus obtain the first three natural frequencies and mode shapes and compare with the results from part (a). Also find the first natural frequency that the beam elements would be incapable of capturing.
Note: The results in this part will likely be worse than the results in part (a). This will be because you will find 1000 elements is not actually that much and will result in poor aspect ratios.

(c) This part looks at mass lumping using a basic 1DOF oscillator as shown below. Find (i) what the natural frequency would be with ½ the total mass at the free end, and (ii) what the free-end mass lumping should be to get the same first natural frequency as was obtained in part (a).



(d) Use a 1-mode Rayleigh-Ritz/Galerkin method with the shape function given below to estimate the first natural frequency. The given shape function seems reasonably appropriate for the first mode. This approach is very commonly used in vibrations to estimate natural frequencies. It is surprisingly accurate, but also requires experience as one needs to have an idea of the mode shapes.