Homework 1

CESG 504 - Finite Element Methods

Due: Friday, April 9, 2021

Problem 1 (15 points)

Derive the following - make sure to show free body diagrams and sketches of the problem as needed.

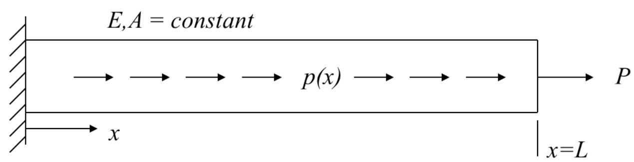
a. What are the main assumptions that are made in the relationship $\sigma = E\epsilon$?

b. Derive $A\sigma' + b = 0$ (An FBD of a differential element is required).

c. Derive $\epsilon = u'$.

Problem 2 (30 points)

Consider the idealized system shown below with $A = 3$ in^2, $E = 10,000$ ksi, $L = 60$ in, $p(x) = \frac{x^3}{100}$ lb/in, and $P = 2$ kip.



a. Determine the exact solution for the displacement.

b. Derive the weak/variational form of the boundary value problem using the method of weighted residuals.

c. Using the variational form of the boundary value problem and the finite element method with two-node elements, determine the nodal displacements in the structure for a three-element (u_{FE1}) and a four-element mesh (u_{FE2}).

d. Plot the following (you will need the element shape functions and nodal displacements from 1.c.):

   1. $u(x)$, $u_{FEi}(x)$, $0 \le x \le L$ (exact *and* approximate nodal displacement fields)
   2. $\sigma(x)$, $\sigma_{FEi}(x)$, $0 \le x \le L$ (exact *and* approximate element stress fields)

   where $\sigma(x) = E\frac{du(x)}{dx}$ is the exact stress field and $\sigma_{FEi}(x)$ is the corresponding approximation. Comment on each of your plots. How do the approximate and exact displacement fields compare? Where does the maximum stress occur in the FE solutions?

e. Compute the strain energy, $U(u(x)) = \frac{1}{2}\int_0^L EA\left(\frac{du(x)}{dx}\right)^2 dx$, and the total potential energy, Π_p, for all solutions. For a system subjected to an imposed load field, the strain energy fo the exact solution will exceed that of the approximate solution, thus the approximate solution provides a "stiffer" system. Show that the exact solution provides the lowest total potential energy.

Problem 3 (20 points)

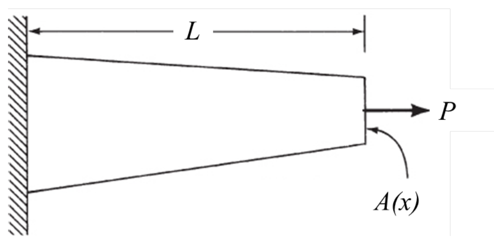
Consider the boundary value problem shown below. As noted in class, we can convert this problem to weak form and solve a problem that is equivalent to $[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$. Assuming that $u(x) = [\mathbf{N}]\{\mathbf{u}\}$ with piecewise linear shape functions, apply the method of weighted residuals and Galerkin's method to determine the stiffness matrix $[\mathbf{K}]$ for an element of length $L^e = 0.5$, then solve the problem using a mesh of four equal length elements and find the nodal displacements.

$$\frac{d^2u(x)}{dx^2} + u(x) = 0 \quad , \quad x \in [0,2]$$
$$u(2) = 0 \quad , \quad u(0) = 0.025$$

Without context, one can see that this simply a second order differential equation with prescribed boundary conditions. This could represent one of many things (a structural mechanics problem - such as a foundation pile, a heat transfer problem - with initial temperatures on either end, an electromagnetism problem - with whatever boundary conditions are used for those, etc.). Feel free to ask us why!

Problem 4 (20 Points)

For the tapered bar shown below, the area is defined: $A(x) = A_0\left(1 - \left(\frac{x}{L}\right)^2\right) + A_1\left(\frac{x}{L}\right)^2$



The corresponding exact solution to the displacement is:

$$u(x) = \frac{PL}{E\left(A_0\left(A_1 - A_0\right)\right)^{1/2}} \tan^{-1}\left(\frac{x}{L}\left(\frac{A_1 - A_0}{A_0}\right)^{1/2}\right)$$

Using the Matlab scripts from the website (or if you are feeling ambitious, by hand) compute a finite-element approximation of the displacement and stress and compare with your exact solution. Evaluate this problem for $L = 24$ in, $E = 10,000$ ksi, $A_0 = 4$ in^2, $A_1 = 2$ in^2, and $P = 50$ kips. For your finite element solution, the area of each element should be the area calculated at the center of the element. Start with one element and refine your mesh until you feel you have a reasonable solution. Plot the exact and approximate displacements and stresses and comment on your mesh refinement. **Note: the areas A_1 and A_2 are correct. You will get an imaginary number in the denominator of the first term but it will go away when multiplied by the inverse tangent term.**

Problem 5 (15 Points)

Prove that the virtual work expression, which is the same as the method of weighted residuals, minimizes the following squared error in stress when using the Galerkin method. Assume a 1-mode Ritz type shape function. (Hint: $\hat{\sigma} = E\hat{u}'$, $\sigma = Eu'$, Let $\hat{u} = k * N(x)$ (i.e. one-mode Ritz apprx.), and set $\frac{\partial E_r}{\partial K} = 0$.)

$$E_r := Error = \int_0^L (\hat{\sigma} - \sigma)^2 \, dx.$$