Homework 2

CESG 504 – Finite Element Methods

Due: Friday, April 23, 2021

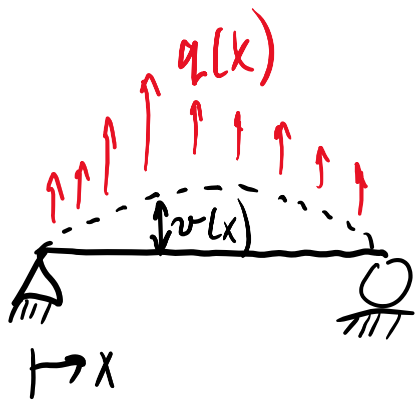
Part 1 – 1D/1D Capstone Problems

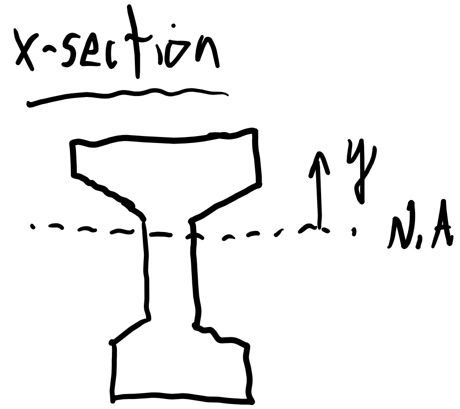
This part will cover two different kinds of problems from beginning (the fundamental governing equations) to end (solution of weak form with FEM). The problems both have 1D/1D governing differential equations. The intention for this part is for you to try to connect the approach used here to the approach we used for the 1D elasticity formulation of the stiffness method.

1. ***Euler-Bernoulli (EB) Beam Theory:*** This problem will go through the derivation of the EB beam-theory stiffness matrix equations using FEM. The EB beam theory is an example of what I like to call a classical (or ad hoc) mechanics formulation. These classical methods have also been called "reasonable approximation theories". In the classical methods, instead of solving a beam as what it really is, i.e., a 3D solid governed by the laws of 3D mechanics (which we will cover later), a simpler approximate 1D/1D governing equation can be derived using some very clever approximations/observations. The main simplification is the kinematic assumption called "Plane-Sections Remain Plane + 90°" (PSRP+90), this allows us to go from a 3D problem down to a 1D/1D problem (i.e., a line-element or "stick"). Mechanics has a long history of clever ways to get around needing to solve full 3D problems (which are difficult to solve) for special types of problems. The main examples are 1D elasticity (which we already covered), Timoshenko beam theory, and plate theory all of which use similar observations to reduce the problem complexity. Many ad hoc methods actually predate the realization that 3D solid mechanics equations work for everything.

(a) ***Do not turn in this part***. Make sure you are comfortable with the fundamental EB governing equations below. Note that equilibrium, kinematics, and the constitutive law are all present (as in any solid mechanics problem, though statically-determinate problems only need equilibrium). The sign convention diagrams shown on next page.

$\varepsilon(y) = -v''y$	(i) Kinematics (derived from PSRP + 90)
$\sigma(y) = E\varepsilon(y)$	(ii) Constitutive Law
$M = -\int_A \sigma(y)\, y\, dA = EI\, v''$	(*) ad hoc introduction of net x-section moment
$I = \int_A y^2\, dA$	(**) useful quantity from (*)
$M' = V$	(iiia) Moment Equilibrium
$V' = q$	(iiib) Vertical Force Equilibrium
$EIv'''' = q$	(*SF*) Put it all together - EB Strong Form





Note that the ‘y’ coordinate is only temporarily needed to derive the cross section quantity I. In full 3D formulation which we will do later, the x,y,z coordinates would all be needed, and a ‘cross-section’ does not really exist.

(b) The final EB governing equation in (*SF*) does have additional assumptions that were applied. What are those assumptions? ***Hint:*** If you put the governing equations together, you will get something a bit more general from what is seen in (*SF*). Note that the final EB equation is actually an equation of vertical equilibrium, written in terms of displacement.

(c) Apply the method of weighted residuals to show that the weak form is given by the equation below. You will need to apply integration by parts twice to get the particular version shown below (you will have to rearrange things a bit to get what is shown below). The version below is ideal as it minimizes the highest derivative order. Note that Min PE could be used to get the exact same result. Note also that the equation WF is in units of work, so it can again be called the virtual work expression.

$$M\bar{v}'|_0^L - V\bar{v}|_0^L + \int_0^L q\bar{v}\,dx \;=\; \int_0^L EI\, v''\bar{v}''\,dx$$

Where $M = EIv''$, and $V = EIv'''$.

(d) Apply the Galerkin method, i.e., let $v = [N]\{d\}$ and $\bar{v} = [N]\{\bar{d}\}$ to produce the stiffness equations below:

$$\{\bar{d}\}^T \left([K]\{d\} - \{P_{FEF}\} - \{P\}\right) = 0 \quad for\ all \quad \{\bar{d}\}$$

Which Implies that

$$\{P_{FEF}\} + \{P\} = [K]\{d\}$$

where

$$\{P\} = M[N']^T|_0^L - V[N]^T|_0^L$$

$$\{P_{FEF}\} = \int_0^L q[N]^T dx$$

$$[K] = \int_0^L EI\ [B]^T[B]\ dx$$

where

$$[B] = [N'']$$

Discussion 1: Note that these terms all look very similar to the 1D formulation, but the [B] matrix is now the 2nd derivative instead of the first.

Discussion 2: Note that the size of $\{d\}$ (which is also the number of columns of $[N]$) can be left general in this formulation.

Discussion 3: The components of the $\{d\}$ vector are often called degrees of freedom (DOFs). In an FEM context, these are nodal displacements (and/or rotations), while in a Rayleigh-Ritz context, they are simply scaling factors for the shape functions and are called “modal amplitudes”. At this stage, we have not decided which of these routes we will go, and the equations above are general.

Discussion 4: The term degrees of freedom now becomes a bit ambiguous, as it can be used to describe the $\{d\}$ vector components, or the type of differential equation. To clarify that, one should formally describe this problem as having a 1D/1D governing differential equation (since $v(x)$ has one input x and one output v), however, in the following parts we are going to solve it as a 1D/2D stiffness formulation since each node will have two DOFs (a displacement and a rotation).

(e) ***FEM assembly: Do not turn in this part***, there is no work actually assigned in this question, but please read it. Okay, we will go with FEM instead of Rayleigh-Ritz. To do that, we will chop the beam up into multiple elements. The elements will usually be chosen to line up with point loads or supports, but in principle you can chop it up however you like. The FEM idea behind that is that no matter how you chop up an integral domain, it should still work as long as your piecewise integral covers the entire domain. Though we are not using the Direct Stiffness Method (DSM), the equivalent DSM idea is that we always enforce that all elements (element stiffness equations) and all nodes (global stiffness equations) are in equilibrium, and it does not matter how many elements you decide to evaluate, all parts will always be in equilibrium.

The first step is to evaluate all of the components in a piecewise fashion over each element as shown below (again, nothing to do here, just make sure the expressions below feel right).

$$\{P\} = M\left[N^{G'}\right]^T\Big|_0^L - V[N^G]^T|_0^L = \sum_e \left(M\left[N^{G'}\right]^T\Big|_{L^e} - V[N^G]^T|_{L^e} \right) = \boldsymbol{A}_{\boldsymbol{e=1}}^{\boldsymbol{N}}\, \{f^e\}$$

$$\{P_{FEF}\} = \int_0^L q[N^G]^T dx = \sum\nolimits_e \int_{L^e} q[N^G]^T dx = \boldsymbol{A}_{\boldsymbol{e=1}}^{\boldsymbol{N}} \int_{L^e} q[N^e]^T dx = \boldsymbol{A}_{\boldsymbol{e=1}}^{\boldsymbol{N}}\, \{P_{FEF}^e\}$$

$$[K] = \int_0^L EI\, [B^G]^T[B^G]\, dx = \sum\nolimits_e \int_{L^e} EI\, [B^G]^T[B^G]\, dx = \boldsymbol{A}_{\boldsymbol{e=1}}^{\boldsymbol{N}} \int_{L^e} EI^e\, [B^e]^T[B^e]\, dx = \boldsymbol{A}_{\boldsymbol{e=1}}^{\boldsymbol{N}}\, [K^e]$$

Discussion 1: The $\{P\}$ equation is not really going to be used. We have discussed a few times about how this will always just end up being the forces (and moments) applied at nodes.

Discussion 2: The summation sign is formally correct, but the new "assembly" operator $\boldsymbol{A}$ is introduced to denote the algorithmic stenciling approach that we use in practice to assemble the global integrals (of global shape functions) to element integrals (of local element shape functions) by matching nodes. If you have used the stiffness method before, you will know how to do assembly from experience, however, a good challenge (which I don't expect you to do unless you have time) is to see if you can formalize the assembly symbol. The assembly symbol is actually slightly different from a summation, as it maps the stiffnesses to their proper DOF.

Discussion 3: The first expression on each line is the physical quantity. The second is the global definition, the third is a piecewise version of the same integral, the fourth introduces the same but expresses all shape functions and [B] matrices in their element in local coordinates and nodes. This is why the new assembly operator symbol needs to be introduced to ensure the nodal mapping is maintained.

(f) The problem is now distilled down into assembly of element stiffness matrices and FEF vectors. That means we can now look at a single element in isolation and thereby create a stencil. The most common approach is the 4-DOF element (2-node * 2-dof) shown below. Prove the 16 components of the stiffness matrix by evaluating the integrals (please use Mathematica or software of your choice).

$v(x) = v_i N_1(x) + \theta_i N_2(x) + v_j N_3(x) + \theta_j N_4(x)$
$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$
$N_2(x) = L\left[\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right]$
$N_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$
$N_4(x) = L\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right]$

$$[K^e] = \int_{L^e} EI^e\,[B^e]^T[B^e]\,dx = EI\begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix}$$

Discussion 1: When someone says “beam element” this is usually what they mean. Though they might also mean Timoshenko beam elements which are slightly more accurate. It also so happens that these shape functions are also exact when the shear is constant (i.e., when q=0, so they are the homogeneous solution of the Strong Form), and are therefore also called constant shear elements.

Discussion 2: These shape functions are designed to have a special property. Note that the following is true of the shape functions:

$$\begin{array}{llll} N_1(0) = 1, & N_1'(0) = 0, & N_1(L) = 0, & N_1'(L) = 0 \\ N_2(0) = 0, & N_2'(0) = 1, & N_2(L) = 0, & N_2'(L) = 0 \\ N_3(0) = 0, & N_3'(0) = 0, & N_3(L) = 1, & N_3'(L) = 0 \\ N_4(0) = 0, & N_4'(0) = 0, & N_4(L) = 0, & N_4'(L) = 1 \end{array}$$

This means that the four coefficients of these shape functions will be identically equal to displacements and rotations at the left and right ends of the element. This is a result of the clever design of these shape functions. If the shape functions did not partition unity in this way, the DOFs would not necessarily be equal to nodal displacements and rotations. In fact, one could do a displacement only approach such as $v^e(x) = v_i N_1(x) + v_j N_2(x)$. This would end up being a really bad result as the rotations would not be independent, but FEM would still give you the best result with these two shape functions.

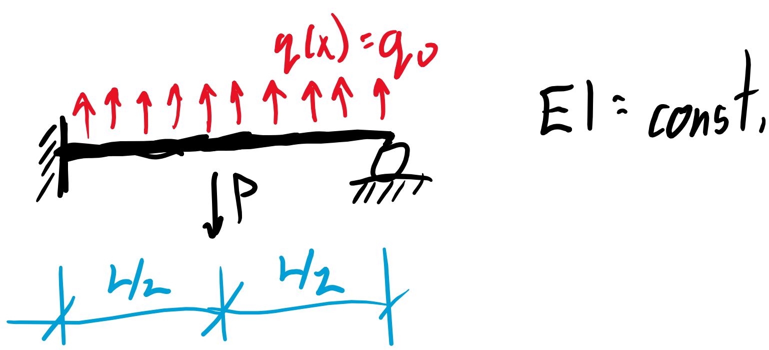
(g) For the same shape functions, and for $q(x) = q_0 = constant$, on the element, prove the following by evaluating the integrals:

$$\{P^e_{FEF}\} = \int_{L^e} q[N^e]^T dx = \begin{Bmatrix} q_0L/2 \\ q_0L^2/12 \\ q_0L/2 \\ -q_0L^2/12 \end{Bmatrix}$$

These are the equivalent fixed end forces (and moments) for a beam element. If you have done this using the direct stiffness method in the past, you will recall that there is an alternative definition of this that requires solving the particular solution of the strong form for clamped BCs.

Discussion: The beam equation results in second derivatives appearing. Thus, the shape functions must be at least quadratic. Fortunately, the onces used here are actually cubic,

(h) Well, we have gone through so much work to get the stiffness matrix equilibrium equations. We should actually use it. Set up the equilibrium equations for the problem below, and denote the 3 unknowns (please use 3 nodes). Don't solve it. You can go straight to the algorithmic assembly, no need to rederive everything.



(i) Describe how you would plot the moment and shear diagrams using shape functions. Note, if you have the reactions you could also plot these diagrams using the traditional "follow the forces" approach, but this question is instead about using the shape functions (which is how FEM software does it).

2. ***1D Steady-State Heat Conduction:*** This problem is a bit shorter than Q1, and it will be because the equation will have the exact same ODE form as 1D elasticity, and hence the stiffness equations are already available.

(a) ***Do not turn in this part***. In class we derived the following governing laws:

$q = -k\,T'$ (i) Fourier's law of conduction (a constitutive law)

$-q' + h = \dot{e}$ (ii) Conservation of energy (where h is a heat source/sink)

$\dot{e} = c\dot{T}$ (iii) Specific heat equation (another constitutive law)

$k\,T'' + h = c\dot{T}$ (*) Putting it all together - The general heat conduction equation

$k\,T'' + h = 0$ (*SF*) The steady-state heat conduction equation (not all problems reach steady state. The pre-steady state behavior is called "Transient")

(b) Apply the method of weighted residuals to (*SF*) using a weight function $\bar{T}$ to get the weak form. Also apply integration by parts once to minimize the order of derivatives. This should look almost identical to 1D elasticity.

(c) Apply the Galerkin method, i.e., let $T = [N]\{d\}$, and $\bar{T} = [N]\{\bar{d}\}$ to derive the equations for the global $\{P_{FEF}\}$ and $[K]$ in terms of integrals and the generic global $[N]$ and $[B]$ matrices. This should also look almost identical to 1D elasticity.

(d) What is the physical meaning of the $\{P\}$ vector that appears in the weak form? Fourier's law will give some guidance. Another bit of guidance is that you cannot provide both the temperature and the thermal flux (q) at one location, one of these will be known and one unknown (just like to cannot dictate both the force at a node and the displacement for 1D elasticity).

(e) The highest derivatives that appear are first order, just like 1D elasticity. So let's use the same simple linear shape functions. Derive the 2x2 $[K^e]$ matrix.

(f) Let's solve a very simple problem using just one element. The temperature on the inside of a wall was measured to be T_i. It was also observed that the steady-state thermal energy loss rate on the outside of the wall was q_o. What is the steady state temperature T_o on the outside? Don't overthink it, this comes from a very simple 2x2 element formulation. In this case, it will also be the exact answer as the linear shape functions are exact without any heat generation.

Part 2 – Some Exploration of the 3D & 2D Elasticity Governing Equations

3. ***Linear Elastic Constitutive Law:*** Two equivalent versions of the constitutive law for 3D linear elasticity are shown below. The first one shows the "material stiffness matrix" $[C_{3D}]$. The second one shows the "material flexibility/compliance matrix" $[F_{3D}]$. These two matrices are inverses of one another.

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{C_{3D}}]\{\boldsymbol{\varepsilon}\}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{F_{3D}}]\{\boldsymbol{\sigma}\}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

Discussion: To get the 2D constitutive law for plane stress & plane strain it is tempting to simply cross out the yz, xz, and zz components of the $[C_{3D}]$ and $[F_{3D}]$, but it is a little more subtle than that, we will show that in the next two parts.

(a) ***Plane Strain (Do not turn in this part):*** To derive the 2D constitutive law for Plane Strain, apply $\varepsilon_{zz} = 0, \gamma_{yz} = 0, \gamma_{xz} = 0$ to the right side of the $[C_{3D}]$ equation above which will directly produce the $[C_{2D}^{\varepsilon}]$ matrix below. Then the $[F_{2D}^{\varepsilon}]$ can be obtained by inverting $[C_{2D}^{\varepsilon}]$. This $[F_{2D}^{\varepsilon}]$ is not the same as you would get by just crossing out rows and columns from $[F_{3D}]$. The superscript ε denotes that this is for plane strain. Note that even though $\varepsilon_{zz} = 0$, you can use the full 3D constitutive law to see that σ_{zz} is not zero, and it can be postprocessed from the in-plane stresses. This z-direction normal stress is called confining stress. Lastly, though three parameters are used out of convenience, only two are actually needed for any isotropic material, hence any one can be replaced with the other two.

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{C}_{2D}^{\varepsilon}]\{\boldsymbol{\varepsilon}\}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{F}_{2D}^{\varepsilon}]\{\boldsymbol{\sigma}\}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

Additional Equations

$$\varepsilon_{zz} = 0, \gamma_{yz} = 0, \gamma_{xz} = 0$$

$$\tau_{yz} = 0, \tau_{xz} = 0$$

$$\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy})$$

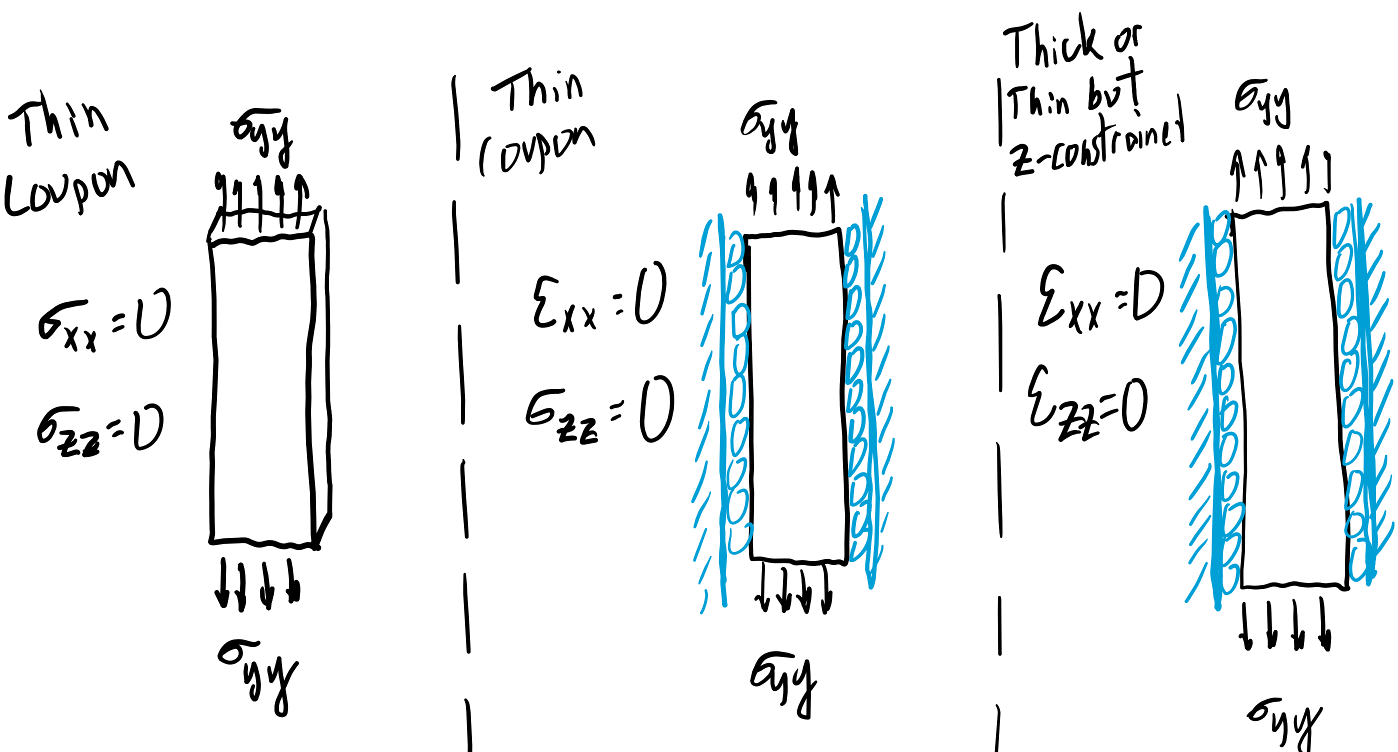
$$G = \frac{E}{2(1+\nu)}$$

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$$

(b) ***Plane Stress/Thin Structures:*** Derive the 2D constitutive law for Plane Stress. This means finding $[C_{2D}^{\sigma}]$ and $[F_{2D}^{\sigma}]$. Also use the 3D constitutive law to derive ε_{zz}, which will not be zero, even though σ_{zz} = 0. This z-direction longitudinal strain is out of plane relaxation due to the Poisson effect.

Hint: Do this by applying $\sigma_{zz} = 0, \tau_{yz} = 0, \tau_{xz} = 0$ to the right side of the $[F_{3D}]$ equation which will directly produce the $[F_{2D}^{\sigma}]$ matrix. Then invert that to get $[C_{2D}^{\sigma}]$. Note, this is the reversed process from what is needed for plane strain. Also note, this solution is all over the place, feel free to check your work against another resource.

(c) Prove the following effective stiffnesses for the following three scenarios using the appropriate 2D constitutive law. Note that sometimes it ends up being easier to use the material flexibility matrix, and other times the material stiffness matrix.



$\sigma_{yy} = E\varepsilon_{yy}$

$\sigma_{yy} = \frac{E}{1-\nu^2}\varepsilon_{yy} > E\varepsilon_{yy}$

$\sigma_{xx} \neq 0$

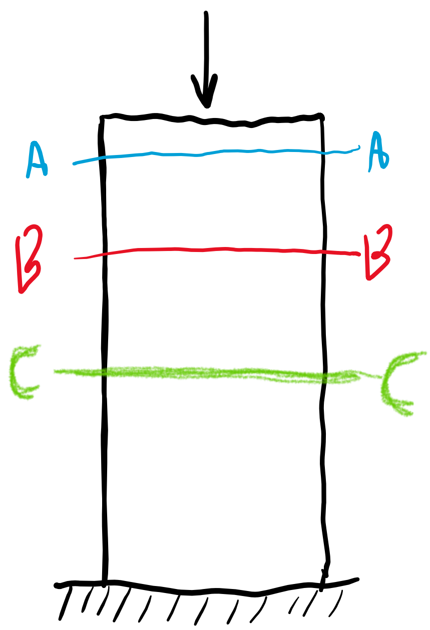
$\sigma_{yy} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\varepsilon_{yy}$

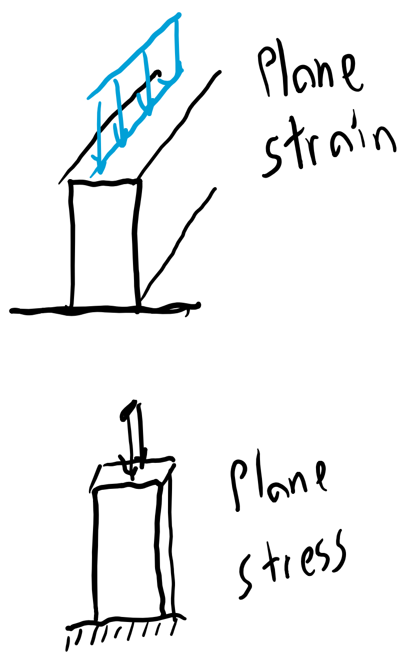
$\sigma_{xx} \neq 0$

$\sigma_{zz} \neq 0$

Discussion: This has some implication in how we should test for E. It means we need to make sure our coupons are not providing significant constraint in the xx or zz directions. Otherwise, when we plot stress vs strain, the slope we would be get would be one of the two latter options above which have some effects due to the Poisson ratio. It also means that when we use $\sigma = E\varepsilon$ in beams, it assumes the beam is not very wide. If it was wide, we should consider Poisson constraint. This is why one-way slabs (very wide beams) actually have a higher bending stiffness (they are like the middle case). Lastly, very deep beams are also constrained through-thickness and this means we can approach the right-most case from above. This is part of the reason why plane-sections don't remain plane for deep beams.

4. Sketch what you think the σ_{yy} would look like at the three different slices below. Note, the load is literally a point load at the center of the top of the column. Note, it won't affect the stress distribution (though it would affect the deflections), but this 2D problem could be either a thin plate, or a deep wall with a line load as shown on the right. This question is intended to pose questions about when/where 1D elasticity is valid.





5. Sketch what you think the deflected shape would look like. The bottom clamped boundary does not allow any deformation. This question is intended to indicate the complexity of even simple problems. The strong form of 2D elasticity for this problem is quite tricky to solve (not surprising given how complex the deflected shape is), but we will solve this problem later using 2D elasticity.