$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}\,,\qquad \begin{aligned} m &= 10\,kg \\ k &= 90\,N/m \\ y(0) &= 0.2\,m \\ y'(0) &= 0\,m/s \end{aligned}$$

① 2nd order ODE

$$10y'' + 100y' + 90y = 0\,,\quad y(0) = 0.2\,,\quad y'(0) = 0$$

Characteristic eqn: $10\lambda^2 + 100\lambda + 90 = 0$

$$10(\lambda+1)(\lambda+9) = 0 \implies \boxed{\begin{cases} \lambda_1 = -1 \\ \lambda_2 = -9 \end{cases}}$$

General Solution: $y = C_1e^{-t} + C_2e^{-9t}$

$$y' = -C_1e^{-t} - 9C_2e^{-9t}$$

$$\begin{aligned} y(0) &= C_1e^0 + C_2e^0 = 0.2 \\ y'(0) &= -C_1e^0 - 9C_2e^0 = 0 \end{aligned} \implies \begin{aligned} C_1 + C_2 &= 0.2 \\ -C_1 - 9C_2 &= 0 \end{aligned} \implies \boxed{\begin{cases} C_1 = 0.225 \\ C_2 = -0.025 \end{cases}}$$

$$\boxed{\begin{aligned} y &= 0.225e^{-t} - 0.025e^{-9t} \\ y' &= -0.225e^{-t} + 0.225e^{-9t} \end{aligned}}$$

① System of linear ODEs.

$$10y'' + 100y' + 90y = 0 \quad . \quad y(0) = 0.2 \ , \ y'(0) = 0$$

Let $y_1 = y$

$y_2 = y' = y_1'$ ①

New ODE $\Rightarrow$ $10y_2' + 100y_2 + 90y_1 = 0$ ②

① $\longrightarrow$ $y_1' = y_2$

② $\longrightarrow$ $y_2' = -9y_1 - 10y_2$

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{\tilde{y}'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -9 & -10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\tilde{y}} \Rightarrow \tilde{y}' = A\tilde{y}$$

$$A = VDV^{-1}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -9 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & -1 \\ 1 & 9 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} -\frac{9}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

$$\tilde{y} = e^{At}\tilde{c} = Ve^{Dt}V^{-1}\tilde{c}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-9t} \end{bmatrix} \begin{bmatrix} -\frac{9}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} & -e^{-9t} \\ e^{-t} & 9e^{-9t} \end{bmatrix} \begin{bmatrix} -\frac{9}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{8}e^{-t} - \frac{1}{8}e^{-9t} & \frac{1}{8}e^{-t} - \frac{1}{8}e^{-9t} \\ -\frac{9}{8}e^{-t} + \frac{9}{8}e^{-9t} & -\frac{1}{8}e^{-t} + \frac{9}{8}e^{-9t} \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 0.225e^{-t} - 0.025e^{-9t} \\ -0.225e^{-t} + 0.225e^{-9t} \end{bmatrix}}$$

② 2nd Order ODE

$$10y'' + 60y' + 90y = 0 \quad . \quad y(0) = 0.2 \ , \ y'(0) = 0$$

char eqn : $10\lambda^2 + 60\lambda + 90 = 0$

$$\lambda^2 + 6\lambda + 9 = 0 \implies \boxed{\lambda = -3}$$

Solution : $\boxed{y_1 = e^{-3t}}$, $y_1' = -3e^{-3t}$, $y_1'' = 9e^{-3t}$

$$y_2 = uy_1 \ , \ y_2' = u'y_1 + uy_1' \ , \ y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Sub in to ODE

$$(u''y_1 + 2u'y_1' + uy_1'') + 6(u'y_1 + uy_1') + 9(uy_1) = 0$$

$$u''(y_1) + u'(2y_1' + 6y_1) + u\underbrace{(y_1'' + 6y_1' + 9y_1)}_{=0} = 0$$

Let $U = u'$

$$U'y_1 + U(2y_1' + 6y_1) = 0$$

$$\frac{dU}{dt}y_1 + U(2\frac{dy}{dt} + 6y_1) = 0$$

$$\frac{dU}{U} + (\frac{2}{y_1}\frac{dy}{dt} + 6)dt = 0$$

$$\ln|U| + 2\ln|y_1| + 6t + C = 0$$

$$\ln|U| = -2\ln|y_1| - 6t - C$$

$$U = \frac{1}{y_1^2}e^{-6t} \longleftarrow y_1 = e^{-3t}, y_1^2 = e^{-6t}$$

$$u = \int U\,dt = \int \frac{1}{e^{-6t}}e^{-6t}\,dt = t$$

$$\boxed{y_2 = ty_1 = te^{-3t}}$$

General Solution: $y = C_1 e^{-3t} + C_2 t e^{-3t}$

$y' = -3C_1 e^{-3t} + C_2 e^{-3t} - 3C_2 t e^{-3t}$

$y(0) = C_1 = 0.2$

$y'(0) = -3C_1 + C_2 = 0$

$\begin{cases} C_1 = 0.2 \\ C_2 = 0.6 \end{cases}$

$$\boxed{\begin{aligned} y &= 0.2e^{-3t} + 0.6te^{-3t} \\ y' &= \cancel{-0.6e^{-3t}} \cancel{+0.6e^{-3t}} - 1.8te^{-3t} \end{aligned}}$$

② System of ODEs

$10y'' + 60y' + 90y = 0 \quad . \quad y(0) = 0.2 \quad , \quad y'(0) = 0$

Let $y_1 = y$

$y_2 = y'$ ①

$10y_2' + 60y_2 + 90y_1 = 0$ ②

$$\left.\begin{matrix} ① \\ ② \end{matrix}\right\} \quad \begin{cases} y_1' = y_2 \\ y_2' = -9y_1 - 6y_2 \end{cases} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underset{\sim}{y}' = (A \quad \underset{\sim}{y})$$

Let $\underset{\sim}{y} = \underset{\sim}{x} e^{\lambda t}$, $\underset{\sim}{y}' = x\lambda e^{\lambda t}$

$A\underset{\sim}{x}e^{\lambda t} = \underset{\sim}{x}\lambda e^{\lambda t}$

$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$(A - \lambda I)\underset{\sim}{x} = 0$

$\det|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -9 & -6-\lambda \end{vmatrix} = 0$

$-\lambda(-6-\lambda) + 9 = 0 \Rightarrow \boxed{\lambda = -3}$

$A = HJH^{-1}$

$\downarrow$ matlab

$$H = \begin{bmatrix} 3 & 1 \\ -9 & 0 \end{bmatrix} \quad J = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \quad H^{-1} = \begin{bmatrix} 0 & -0.111 \\ 1 & 0.333 \end{bmatrix}$$

$$e^{At} = e^{HJH^{-1}t} = He^{Jt}H^{-1} = H \begin{bmatrix} e^{-3t} & te^{t} \\ 0 & e^{-3t} \end{bmatrix} H^{-1}$$

$$\underset{\sim}{y} = e^{At}c = H \begin{bmatrix} e^{-3t} & te^{t} \\ 0 & e^{-3t} \end{bmatrix} H^{-1} c$$

$$\underset{\sim}{y} = \begin{bmatrix} 3 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} e^{-3t} & te^{t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 & -0.111 \\ 1 & 0.333 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{y} = \begin{bmatrix} 3e^{-3t} & t3e^{t} + e^{-3t} \\ -9te^{-3t} & -9e^{t} \end{bmatrix} \begin{bmatrix} 0 & -0.111 \\ 1 & 0.333 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{y} = \begin{bmatrix} 3e^{t} + e^{-3t} & -\frac{1}{3}e^{-3t} + te^{t} + \frac{1}{3}e^{-3t} \\ -9te^{t} & e^{-3t} - 3e^{t} \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$\boxed{\underset{\sim}{y} = \begin{bmatrix} 0.6e^{t} + 0.2e^{-3t} \\ -1.8te^{t} \end{bmatrix}}$$

③ 2nd order ODE

$$10y'' + 10y' + 90y = 0 \quad . \quad y(0) = 0.2 \ , \ y'(0) = 0$$

Char eqn: $10\lambda^2 + 10\lambda + 90 = 0 \qquad \begin{cases} \lambda_1 = -0.5 + 2.96i \\ \lambda_2 = -0.5 - 2.96i \end{cases}$ $\boxed{\left(-0.5 \pm \frac{\sqrt{35}}{2} i\right)}$

General Solution:

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$
$$y' = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t}$$

$\Downarrow$

$$\begin{cases} y(0) = C_1 + C_2 = 0.2 \\ y'(0) = C_1\lambda_1 + C_2\lambda_2 = 0 \end{cases} \qquad \begin{cases} C_1 = 0.1 - 0.017i \\ C_2 = 0.1 + 0.017i \end{cases}$$

$$y = 0.1e^{(-0.5+2.96i)t} - 0.017ie^{(-0.5+2.96i)t} + 0.1e^{(-0.5-2.96i)t} + 0.017ie^{(-0.5-2.96i)t}$$
$$= e^{-0.5t}(-0.017i)(\cos 2.96t + i\sin 2.96t) + e^{-0.5t}(0.017i)(\cos 2.96t - i\sin 2.96t)$$
$$+ e^{-0.5t}(0.1)(\cos 2.96t + i\sin 2.96t) + e^{-0.5t}(0.1)(\cos 2.96t - i\sin 2.96t)$$

$$= e^{-0.5t}\left(\cancel{-0.017i\cos 2.96t} \cancel{+0.017i\cos 2.96t} \ 0.017\cancel{i}i\sin 2.96t + 0.017\cancel{i}i\sin 2.96t\right)$$
$$+ e^{-0.5t}\left(0.1\cos 2.96t \cancel{+0.1i\sin 2.96t} + 0.1\cos 2.96t \cancel{-0.1i\sin 2.96t}\right)$$

$$= \boxed{e^{-0.5t}\left(0.034\sin 2.96t + 0.2\cos 2.96t\right)}$$

$$\boxed{y' = -0.609e^{-0.5t}\sin(2.96t) + 0.00064e^{-0.5t}\cos(2.96t)}$$

③ System of ODEs

$$10y'' + 10y' + 90y = 0 \quad . \quad y(0) = 0.2 \quad , \quad y'(0) = 0$$

Let $y_1 = y$

$$y_2 = y' = y_1' \qquad ① \quad y_1' = y_2$$

$$10y_2' + 10y_2 + 90y_1 = 0 \quad ② \quad y_2' = -9y_1 - y_2$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underset{\sim}{y}' = (A \quad \underset{\sim}{y})$$

$$\underset{\sim}{y} = e^{At} C \quad , \quad A = VDV^{-1}$$

Same

$$[V, D] = eig[A] \Rightarrow V = \begin{bmatrix} -0.527-0.312i & -0.0527+0.312i \\ 0.949 & 0.949 \end{bmatrix}$$

$\frac{\sqrt{35}}{2}$

$$D = \begin{bmatrix} -0.5+2.958i & 0 \\ 0 & -0.5-2.958i \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 1.604i & 0.527+0.0891i \\ -1.604i & 0.527-0.0891i \end{bmatrix}$$

$$\underset{\sim}{y} = e^{At}C = Ve^{Dt}V^{-1}C$$

$$= \begin{bmatrix} -0.527-0.312i & -0.0527+0.312i \\ 0.949 & 0.949 \end{bmatrix} \begin{bmatrix} e^{(-0.5+2.958i)t} & 0 \\ 0 & e^{(-0.5-2.958i)t} \end{bmatrix}$$

$$\begin{bmatrix} 1.604i & 0.527+0.0891i \\ -1.604i & 0.527-0.0891i \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} e^{-0.5t}(0.034\sin 2.96t + 0.2\cos 2.96t) \\ -0.609e^{-0.5t}\sin(2.96t) + 0.00064e^{-0.5t}\cos(2.96t) \end{bmatrix}}$$