P1

$$(x^2+1)y''+2xy'=0\ ,\ y(0)=0\ ,\ y'(0)=1$$

$$\begin{cases} y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \cdots \\ y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 \cdots \\ y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 \end{cases}$$

Plug into Eqn:

$$\begin{array}{lr} & 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 \cdots\cdots \\ + & 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 \cdots\cdots \\ + & 2a_1x + 4a_2x^2 + 6a_3x^3 + 8a_4x^4 + 10a_5x^5 \\ = & 0 \end{array}$$

$$\begin{array}{ll} 2a_2 = 0 & a_2 = 0 \\ 2a_1 + 6a_3 = 0 & a_3 = -\frac{1}{3}a_1 \\ 6a_2 + 12a_4 = 0 & a_4 = 0 \\ 12a_3 + 20a_5 = 0 & a_5 = -\frac{3a_3}{5} = \frac{1}{5}a_1 \\ 20a_4 + 30a_6 = 0 & a_6 = 0 \\ 30a_5 + 42a_7 = 0 & a_7 = \frac{-5a_5}{7} = -\frac{1}{7}a_1 \end{array}$$

$$y = a_0 + a_1x + \left(-\frac{1}{3!}\right)a_1x^3 + \left(\frac{1}{5!}\right)a_1x^5 - \frac{1}{7!}a_1x^7 \cdots\cdots$$

$$y(0)=0 \Rightarrow a_0 = 0$$

$$y = a_1x + \frac{1}{3}(-1)a_1x^3 + \frac{1}{5}a_1x^5 + \frac{1}{7}(-1)a_1x^7 \cdots\cdots$$

$$y'(0) = 1 \Rightarrow a_1 = 1$$

$$\boxed{y = x + \frac{1}{3}(-1)x^3 + \frac{1}{5}x^5 + \frac{1}{7}(-1)x^7 \cdots = \tan^{-1}x}$$

P2

(a) $\theta'' = \sin\theta$

Let $y_1 = \theta$

$y_2 = \theta' = y_1'$ $\Big\}$ $y_2' - \sin y_1 = 0$

$y_1' = y_2 = f_1 \qquad f_1 = 0$ when $\boxed{y_2 = 0 = \theta'}$

$y_2' = \sin y_1 = f_2 \qquad f_2 = 0$ when $\boxed{y_1 = \pm n\pi = \theta}$, $n = \boxed{0}, 1, 2, 3 \cdots$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos y_1 & 0 \end{bmatrix}$$

when $n = 1, 3, 5 \cdots\cdots$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$n =$ odd, critical points are centers.

when $n = 0, 2, 4, 6 \cdots\cdots$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0, \quad \lambda^2 - 1 = 0, \quad \lambda = \pm 1$$

$\lambda_1 = 1$

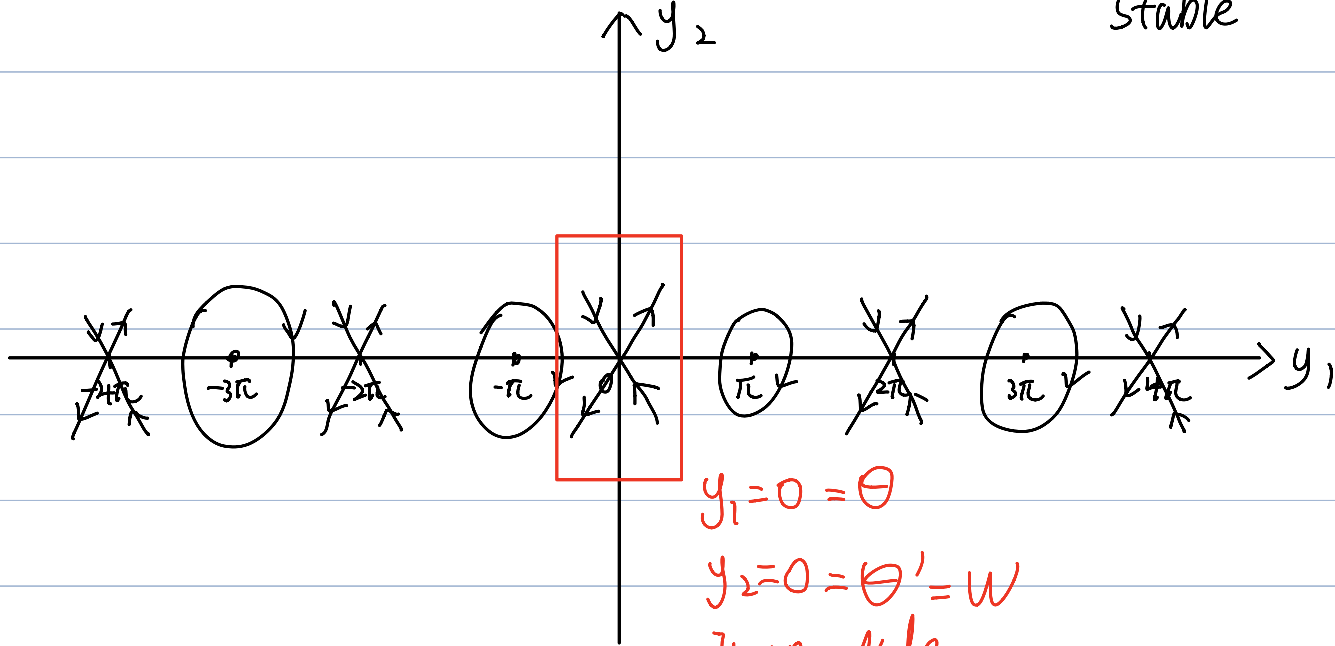
unstable

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^{I} \\ x_2^{I} \end{bmatrix} = 0 \quad \begin{cases} -x_1^{I} + x_2^{I} = 0 \\ x_1^{I} - x_2^{I} = 0 \end{cases} \Rightarrow \underset{\sim}{x}^{I} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$

$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1^{II}\\ x_2^{II}\end{bmatrix}=0 \quad \begin{cases} x_1^{II}+x_2^{II}=0\\ x_1^{II}+x_2^{II}=0\end{cases} \Rightarrow \underset{\sim}{x}^{II}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

Stable



$y_1 = 0 = \theta$

$y_2 = 0 = \theta' = w$

Improper Node

(b)

$$\begin{bmatrix}y_1'\\ y_2'\end{bmatrix}=\begin{bmatrix}0 & 1\\ \sin y_1 - 2 & -1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$\theta'' = \sin\theta - 2\theta - \theta'$

$\theta = y_1$

$\theta' = y_2 = y_1'$

$y_1' = y_2 \qquad = f_1$

$y_2' = \sin y_1 - 2y_1 - y_2 \quad = f_2$

critical point

when $\boxed{\theta = y_1 = 0 \ , \ \theta' = y_2 = 0}$

critical point $\begin{cases} f_1 = 0\\ f_2 = 0\end{cases}$

when $\boxed{y_2 = 0}$ $\quad f_1 = 0$

when $y_2 = 0$, $\boxed{y_1 = n\pi}$, $n = \boxed{0}, 1, 2, 3 \ldots$

$f_2 = 0$

$$A = \begin{bmatrix}\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2}\\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ \cos y_1 - 2 & -1\end{bmatrix}$$

When $n = 1, 3, 5 \cdots$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -3 & -1-\lambda \end{vmatrix} = 0, \quad \lambda^2+\lambda+3=0$$

$$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$$

when $n = 0, 2, 4, 6 \cdots$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2+\lambda+1=0$$

$$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

when $y_1 = 0 = \theta$, $y_2 = 0 = \theta'$, the critical point is a spiral point

$$p = \lambda_1 + \lambda_2 = -\frac{1}{2} \cancel{+ i\frac{\sqrt{3}}{2}} - \frac{1}{2} \cancel{- i\frac{\sqrt{3}}{2}} = -1 < 0$$

$$q = \lambda_1\lambda_2 = 1 > 0$$

$$\Delta = (\lambda_1 - \lambda_2)^2 = -3 < 0$$

$$\boxed{\text{Stable attractive spiral point}}$$