P1

① $x^2y'' - 2xy' + 2y = x^3\sin x$

Homogeneous part : $x^2y'' - 2xy' + 2y = 0$

Let $\begin{cases} y = x^m \\ y' = mx^{m-1} \\ y'' = m(m-1)x^{m-2} \end{cases}$

$$x^2 m(m-1)x^{m-2} - 2x\,m x^{m-1} + 2x^m = 0$$

$$\cancel{x^2} m^2 x^{\cancel{m-2}} - \cancel{x^2} m x^{\cancel{m-2}} - 2\cancel{x} m x^{\cancel{m-1}} + 2x^m = 0$$

$$x^m(m^2 - m - 2m + 2) = 0 \qquad \begin{cases} m_1 = 2 \\ m_2 = 1 \end{cases}$$

$$x^m(m^2 - 3m + 2) = 0$$

$y_1 = x^2 \qquad y_2 = x \qquad \boxed{y_h = c_1x^2 + c_2x}$

$y_1' = 2x \qquad y_2' = 1$

$$W = \det\begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

$$y_p = -x^2\int \frac{x\,x\sin x}{-x^2}\,dx + x\int \frac{x^2\,x\sin x}{-x^2}\,dx$$

$$y_p = -x\sin x$$

$$\boxed{y(x) = \underbrace{c_1x^2 + c_2x}_{y_h} \underbrace{- x\sin x}_{y_p}}$$

② $y'' + 6y' + 8y = 40\cos 2t$

Homo part $y'' + 6y' + 8y = 0$

char eqn : $\lambda^2 + 6\lambda + 8 = 0$

$$\begin{cases} \lambda_1 = -2 \\ \lambda_2 = -4 \end{cases}$$

$$y_h = C_1 e^{-2t} + C_2 e^{-4t}$$

$40\cos 2t \to k\cos wt$

$$y_p = K\cos 2t + M\sin 2t$$

Put y_p back in eqn

$-4K\cos(2t) - 4M\sin(2t)$
$-12K\sin(2t) + 12M\cos(2t)$
$+8K\cos(2t) + 8M\sin(2t) = 40\cos 2t$

$(4m - 12K)\sin 2t + (4K + 12m)\cos 2t$
$= 40\cos 2t$

$4m - 12K = 0$
$4K + 12m = 40$
$\begin{cases} m = 3 \\ K = 1 \end{cases}$

$$\boxed{y = \underbrace{C_1 e^{-2t} + C_2 e^{-4t}}_{y_h} + \underbrace{1\cos 2t + 3\sin 2t}_{y_p}}$$

②

$my'' = -ky - cy' + F_0\cos wt$,
$my'' + cy' + ky = F_0\cos wt$, $y(0) = 0$ $y'(0) = 1$

$w_0 = \sqrt{\frac{K}{m}} = 3$, $m = 10$. $k = 90$
$F_0 = 10$

$10y'' + 90y = 10\cos 2.7t \Rightarrow y'' + 9y = \cos 2.7t$

$y_1 = y$ $\quad y_1' = y_2$

$y_2 = y' = y_1'$ $\quad y_2' = -9y_1 + \cos 2.7t$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ r(t) \end{bmatrix}$$

$\to$ matlab

→ For matlab code.

$$my'' + cy' + ky = F_0 \cos wt$$

$y(0)=0 \quad y'(0) = 1$

$y_1 = y$

$y_2 = y' = y_1'$

$w_0 = \sqrt{\frac{k}{m}} = 3 \quad , \quad m = 10 \, , \, k = 90$

$F_0 = 10$

$\Downarrow$

$$y_1' = y_2$$

$$y_2' = -\frac{c}{m} y_2 - \frac{k}{m} y_1 + \frac{F_0}{m} \cos wt$$

$$\Rightarrow \underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{\underset{\sim}{y}'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\underset{\sim}{y}} + \underbrace{\begin{bmatrix} 0 \\ \frac{F_0}{m} \cos wt \end{bmatrix}}_{\underset{\sim}{u}}$$

$$\underset{\sim}{y}(t) = e^{At} \underbrace{\underset{\sim}{y_0}}_{= \begin{bmatrix} 0 \\ 1 \end{bmatrix}} + e^{At} \int_{t_0=0}^{t=50} e^{-\tau A} \underset{\sim}{u}(\tau) d\tau$$

$$\boxed{= e^{At} y_0 + \int_0^t e^{A(t-\tau)} \underset{\sim}{u}(\tau) d\tau}$$

Use Matlab from here for all cases.