Homework 3 Solution

Problem 1.

1) $x^2 y'' - 2x y' + 2y = x^3 \sin x$

Homogeneous ODE: $x^2 y'' - 2xy' + 2y = 0$ (*)

This is an Euler-Cauchy equation.

$$y = x^m, \; y' = mx^{m-1}, \; y'' = m(m-1)x^{m-2}$$

substitute into (*) : $x^2 m(m-1)x^{m-2} - 2x\, m x^{m-1} + 2x^m = 0$

$$m(m-1)x^m - 2m x^m + 2x^m = 0$$

$$m^2 - m - 2m + 2 = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0 \Rightarrow m_1 = 1, \; m_2 = 2$$

$$y_h = c_1 \underbrace{x}_{y_1} + c_2 \underbrace{x^2}_{y_2}$$

standard form of the nonhomogeneous ODE:

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = \boxed{x \sin x} \; r(x)$$

$$y_p = u(x)\, y_1 + v(x)\, y_2$$

$$u'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-\overset{x^2}{y_2}\, r}{\underset{x}{y_1}\underset{2x}{y_2'} - \underset{1}{y_1'}\underset{x^2}{y_2}} = -r(x) = -x \sin x$$

$$v'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\overset{x}{y_1}\, r}{y_1 y_2' - y_1' y_2} = \frac{r}{x} = \sin x$$

$$y_p = \left(\int -x\sin x\,dx\right) y_1 + \left(\int \sin x\,dx\right) y_2$$

$$= \Big(\underbrace{x\cos x - \int \cos x\,dx}_{\text{integration by parts}}\Big) x + (-\cos x)(x^2)$$

$$= (x\cos x - \sin x)x - x^2\cos x$$

$$= -x\sin x$$

$$y = y_h + y_p = C_1 x + C_2 x^2 - x\sin x$$

2) $y'' + 6y' + 8y = 40\cos 2t$

Homogeneous ODE: $y'' + 6y' + 8y = 0$

characteristic equation: $\lambda^2 + 6\lambda + 8 = 0$

$$\lambda = \frac{-6 \pm \sqrt{36-32}}{2} = \frac{-6 \pm 2}{2} = -3 \pm 1$$

$$\lambda_1 = -4, \quad \lambda_2 = -2$$

$$y_h = C_1 e^{-4t} + C_2 e^{-2t}$$

$y_p = B_1\cos 2t + B_2\sin 2t$ (method of undetermined coefficients)

$$y_p' = -2B_1\sin 2t + 2B_2\cos 2t$$

$$y_p'' = -4B_1\cos 2t - 4B_2\sin 2t$$

Substitute back into the nonhomogeneous ODE:

$$-4B_1\cos 2t - 4B_2\sin 2t - 12B_1\sin 2t + 12B_2\cos 2t$$
$$+ 8B_1\cos 2t + 8B_2\sin 2t = 40\cos 2t$$

$$\begin{cases} -4B_1 + 12B_2 + 8B_1 = 40 \\ -4B_2 - 12B_1 + 8B_2 = 0 \end{cases} \Rightarrow \begin{cases} B_1 = 1 \\ B_2 = 3 \end{cases}$$

$$y_p = \cos 2t + 3\sin 2t$$

$$y = y_h + y_p = c_1 e^{-4t} + c_2 e^{-2t} + \cos 2t + 3\sin 2t$$

Problem 2.

$$m y'' = -ky - cy' + F_0 \cos \omega t$$

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{F_0}{m} \cos \omega t$$

$$y_1 = y$$

$$y_2 = y' = y_1' \quad ①$$

$$y_2' + \frac{c}{m} y_2 + \frac{k}{m} y_1 = r(t) \quad ②$$

$$\left.\begin{matrix} ① \\ ② \end{matrix}\right\} \Rightarrow \underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{\underset{\sim}{y}'} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{k}{m} & \frac{c}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\underset{\sim}{y}} + \underbrace{\begin{bmatrix} 0 \\ r(t) \end{bmatrix}}_{\underset{\sim}{u}(t)}$$

$$\underset{\sim}{y} = e^{At} \underset{\sim}{y_0} + \int_0^t e^{A(t-\tau)} \underset{\sim}{u}(\tau)\, d\tau$$

where $e^{At} = P e^{Dt} P^{-1}$, $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

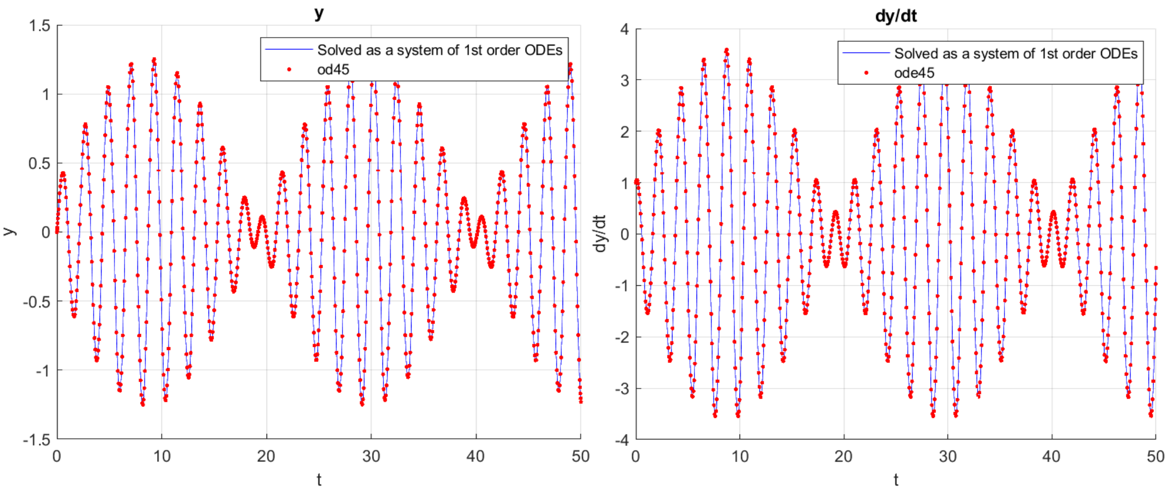
for case 1), 2), 3), 5)

$[P, D] = eig(A)$

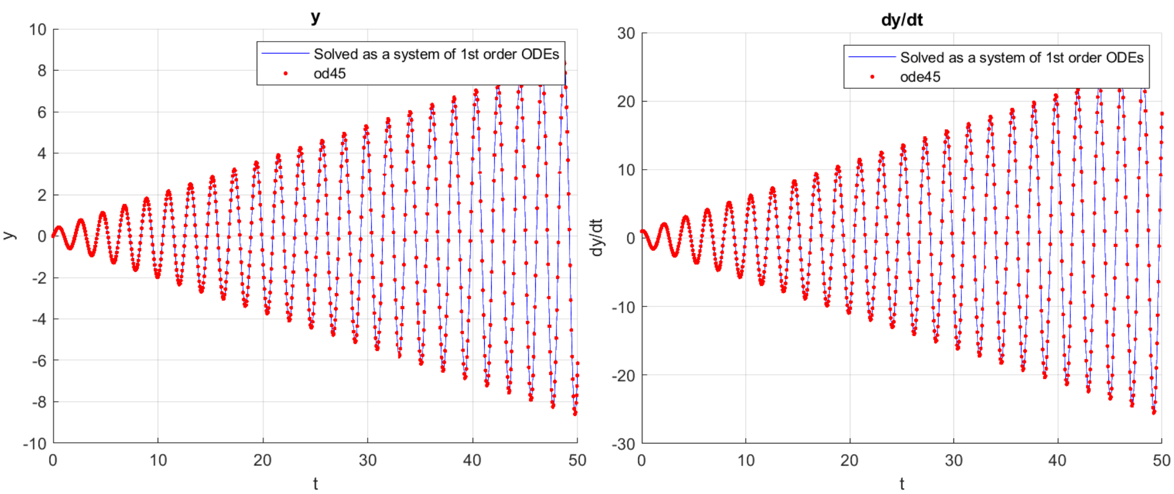
$$e^{At} = H e^{Jt} H^{-1}, \quad e^{Jt} = \begin{bmatrix} e^{\lambda_i t} & t e^{\lambda_i t} \\ 0 & e^{\lambda_i t} \end{bmatrix}$$

$[H, J] = \text{jordan}(A)$ for case 4).

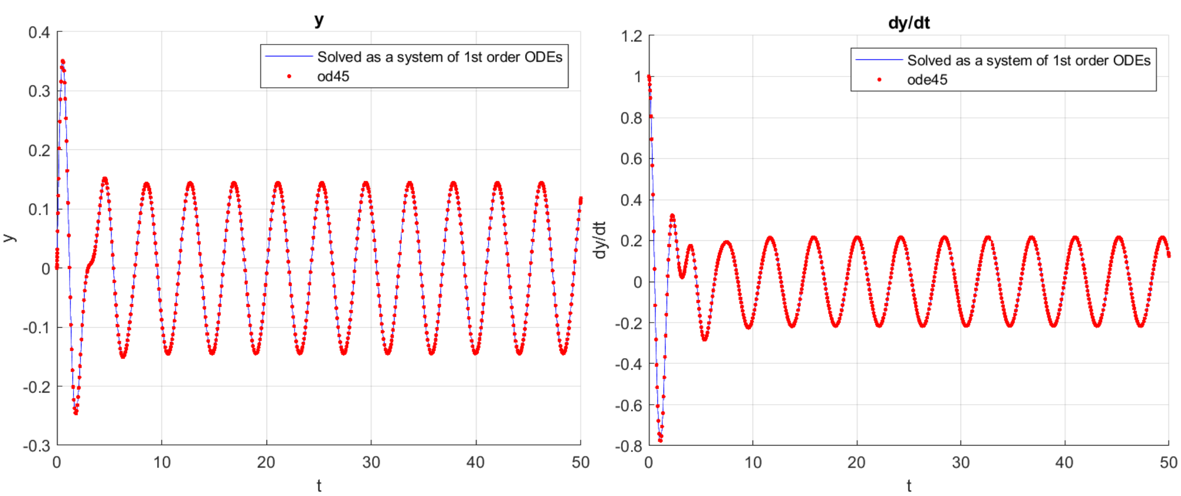
1) Undamped, beats



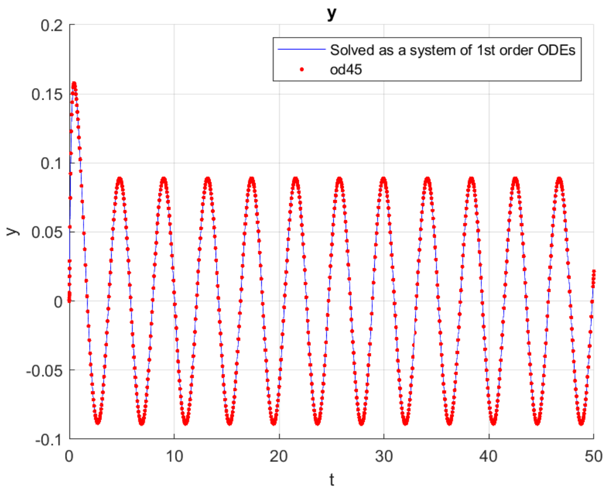
2) Resonance

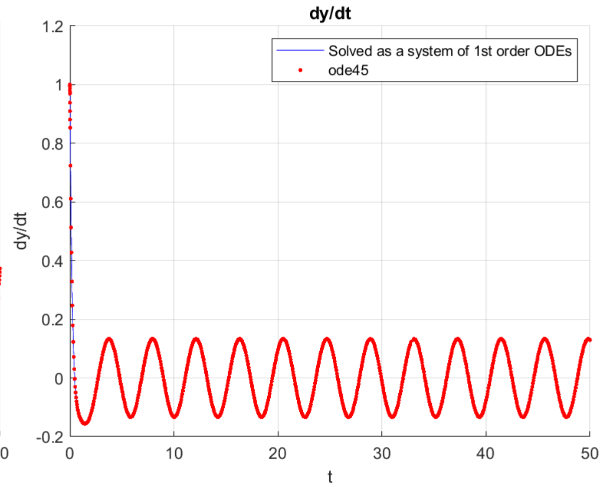


3) Underdamping, forced motion



4) Critical damping, forced motion





5) Overdamping, forced motion