Algorithmic Graph Theory

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Assignment 1

Problem 1

Suppose we are given a directed graph G with n vertices, and let M be the nxn adjacency matrix corresponding to G.

a. Let the product of M with itself (M^2) be defined, for $1 \leq i,\ j \leq n$, as follows

$$M^2(i,j) = M(i,1) \odot M(1,j) \oplus \cdots \oplus M(i,n) \odot M(n,j)$$

where " $\oplus$ " is the Boolean or operator and " $\odot$ " is Boolean and. Given this definition, what does $M^2(i,j) = 1$ imply about the vertices i and j ? What if $M^2(i,j) = 0$?

b. Suppose $M^4$is the product of M^2 with itself. What do the entries of M^4 signify? How about the entries of $M^5 = (M^4)\ (M)$? In general, what information is contained in the matrix M^p ?

c. Now suppose that G is weighted and assume the following:

1. For $1 \leq i \leq n$, $M^2(i,i) = 0$.
2. For $1 \leq i,j \leq n$, $M(i,j)$ = weight (i,j) if (i,j) is in E.
3. For for $1 \leq i,j \leq n$, $M(i,j) = \infty$ if (i,j) is not in E.

Also, let M^2 be defined, for $1 \leq i,j \leq n$, as follows

$$M^2(i,j) = min M(i,1)+M(1,j), \ldots, M(i,n)+M(n,j)$$

If $M^2(i,j) = 1$, what may we conclude about the relationship between vertices i and j ?

Solution

a) Case I : $M(i,i) = 0 \ \forall$ i ϵ V $\Longrightarrow$ Graph G does not have any self loop

$$M(i,k) \odot M(k,j) = 1$$

means $\exists$ a path from i to j with atleast 2 or less edge length through k.

$$M(i,k) \odot M(k,j) = 0$$

means there does not exist any path of edge length 2 through k.

$$M^2(i,j) = 1$$

means $\exists$ atleast one path with edge length 2.

$$M^2(i,j) = 0$$

means there does not exist any path which has path length 2.

Case II : $M(i,i) = 1 \; \forall$ i ϵ V $\implies$ it has a self loop at every node
$M^2(i,j)$=1 means $\exists$ a path from i to j with path length 2 or less.
$M^2(i,j)$=0 means there does not exist any path of path length 2 or less.

b)

$$M^2(i,k) \odot M^2(k,j) = 1$$

means $\exists$ a path from i to j that has a edge length 4 or less going through k.

$$M^2(i,k) \odot M^2(k,j) = 1$$

(assuming graph does not contain any self-loop) means $\exists$ a path from i to j through k of path length 4 .
Assuming Graph contains self-loop for every node $\implies$ that $\exists$ a path from i to j through k of path length 4 or less .

$$M^2(i,k) \odot M^2(k,j) = 0$$

Assuming Graph does not contain any self-loop $\implies$ There does not exist any path of path length 4.
Assuming Graph contains self-loop at every node $\implies$there does not exist any path from i to j through k.

$M^4(i,j)$=1 Assuming graph does not contain any self-loop $\implies$ $\exists$ one path of path length 4 from i to j. But assuming Graph contain self loop at every node$\implies$ $\exists$ atleast one path from i to j with path length 4 or less.

$M^4(i,j)$=0 $\implies$ there does not exist any path of path length strictly equal to 4.

Infromation contained in $M^p(i,j)$:
$M^p(i,j)$=1 assuming graph does not contain any self loop $\implies$ $\exists$ atleast one path from i to j with path length p .
Assuming graph contains self-loop $\implies$ $\exists$ a path from i to j with path length p or less .

$M^p(i,j)$=0 assuming graph do not contain self loop $\implies$ there does not exist any path with path length strictly equal to p.

c) M^2*(i,j)*=k means the minimum cost to reach from i to j through a single vertex is k.

Problem 3

Consider a diagram of a telephone network, which is a graph G whose vertices represent switching centers and whose edges represent communication lines joining pairs of centers. Edges are marked by their bandwidth, and the bandwidth of a path is the bandwidth of its lowest bandwidth edge. Give an algorithm that, given a diagram and two switching centers a and b, outputs the maximum bandwidth of a path between a and b.

Solution

PSEUDO-CODE

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MAX-DIJKSTRA(G, s)
b[s] = ∞
for each v ϵ V \ {s}
b[v] = 0
S = ϕ
Q = V // Initialize Q, a max-priority queue
while Q ≠ ϕ
u = EXTRACT-MAX(Q)
S = S ∪ {u}
for each v ϵ Adj[u]
if b[v] < min(b[u], w(u, v))
b[v] = min(b[u], w(u, v))
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Why this solution Work?

First of all we maintain a piority queue of Pairs(V , B) $\forall$ v ϵ V and intially we set the source bandwidth to infinity *B(s)*=∞. and *B(v)*=0 $\forall$ v ϵ V $\setminus$ s
Lets first start from source node , and going to the next step it chooses all the nodes that can be reached from it and assigns the Bandwidth and pop the source vertex from Set V. And at the next step it chooses the node with greatest bandwidth call it X, We claim that everytime the node is choosen from the priority queue the bandwidth assigned to it is B_{max}. It is true for the intial case , suppose if it wasn't true then $\exists$ a path whose intial node is Y after the source. This path minimum edge bandwidth would be greater than B_{max}. But that leads to a contradiction that since $x > Y$.And any minimum edge bandwidth along this path is less than *B(Y)*. Hence it cant be greater than B_{max}.

Now Assume at n^{th} step every vertex not in the piority queue is assigned MAX Bandwidth. Now , we claim the that the next node F choosen from priority queue is having MAX Bandwidth , assume to the contrary then $\exists$ a path S which leads to maximum bandwidth , on that path consider the first leading node T which is in priority queue. Bandwidth along the path must be lower than *B(T)*. Since we chose F before T in the piority queue , which means *B(F)* > *B(T)* and Hence *B(F)* > Bandwidth along S. Hence ,the contradiction that path S have Maximum Bandwidth.This completes the induction that every step taken from the priority queue has the maximum n=bandwidth.