**HW00**

**Xiaotian Zhu**

**Q1:Let random variable X and Y denote the returns on a stock for year 1 and year 2. Assume that we know X and Y are distributed as N(0:06;0:3); and N(0:1;0:2) respectively. Also assume that the correlation between X and Y is 0:5.**

**(i) Write down the joint density of X and Y:**

**(ii) What is the conditional distribution of Y given X = 0:2: That is, nd out the condition density of Y|X = 0:2:**

**(iii) From the condition density of Y|X = 0:2 in (ii), and use it calculate the condition expectation E[Y|X = 0:2]:**

**(iv) Calculate the condition expectation of Y 2 conditional on X = 0:2: That is calculate E[Y^2|X = 0:2].**

1.

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp[-\frac{1}{2(1-\rho^2)}(\frac{(x-\mu_X)^2}{\sigma_X^2}-\frac{2\rho(x-\mu_X)(y-\mu_X)}{\sigma_X\sigma_y}+\frac{(y-\mu_y)}{\sigma_y^2})]$$

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sqrt{0.3}\times\sqrt{0.2}\times\sqrt{1-0.5^2}}\exp[-\frac{1}{2(1-0.5^2)}(\frac{(x-0.06)^2}{0.3}-\frac{1\times(x-0.06)(y-0.1)}{\sqrt{0.3}\times\sqrt{0.2}}+\frac{(y-0.1)}{0.2})]$$

$$=\frac{10}{3\sqrt{2}\pi}\exp[-\frac{2}{3}(\frac{(x-0.06)^2}{0.3}-\frac{1\times(x-0.06)(y-0.1)}{\sqrt{0.06}}+\frac{(y-0.1)}{0.2})]$$

2. the condition density of Y given X=x is

$$f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)},\ f_X(x)=\int f_{X,Y}(x,y)dy$$

$$f_{Y|X}(y|x=0.2)=\frac{f_{X,Y}(x=0.2,y)}{f_X(x=0.2)}=\frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp[-\frac{1}{2(1-\rho^2)}(\frac{(x-\mu_X)^2}{\sigma_X^2}-\frac{2\rho(x-\mu_X)(y-\mu_X)}{\sigma_X\sigma_y}+\frac{(y-\mu_y)}{\sigma_y^2})]}{\frac{1}{\sqrt{2\pi}\sigma_X}\exp(-\frac{(x-\mu_X)^2}{2\sigma_X^2})}$$

$$=\frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}\exp[-\frac{1}{2(1-p^2)}(p^2\frac{(x-\mu_X)^2}{\sigma_X^2}-\frac{2p(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}+\frac{(y-\mu_y)}{\sigma_Y^2})]$$

$$=\frac{1}{\sqrt{2\pi}\times\sqrt{0.2}\sqrt{1-0.5^2}}\exp[-\frac{1}{2\times0.75}(0.25\times\frac{0.14^2}{0.3}-1\times\frac{0.14\times(y-0.1)}{\sqrt{0.06}}+\frac{y-0.1}{0.2})]$$

$$=\frac{1}{\sqrt{\frac{3}{10\pi}}}\exp[-\frac{(y-\frac{30+7\sqrt{6}}{300})^2}{0.3}]$$

3.The conditional expectation of Y given X=x is

$$E[Y|X=x]=\int yf_{Y|X}(y|x)dy$$

From question 2, we get

$$f_{Y|X}(y|x)=\frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}\exp[-\frac{[y-(\mu_y+p\frac{\sigma_Y}{\sigma_X}(x-\mu_X)^2]}{2(1-p^2)\sigma_Y^2}]$$

So

$$Y|X\sim N(\mu_y+p\frac{\sigma_Y}{\sigma_x}(x-\mu_X),(1-p^2)\sigma_Y^2)$$

$$E[Y|X=0.2]=\mu_y+p\frac{\sigma_X}{\sigma_Y}(x-\mu_X)=0.1+0.5\times\sqrt{\frac{0.2}{0.3}}\times(0.2\text{-}0.06)\approx0.15715$$

4.

$$D(Y|X)=E(Y^2|X)\text{-}E^2(Y|X)$$

$$E(Y^2|\text{X=0.2})=E^2(Y|\text{X=0.2})+D(Y|X=0.2)=\mu_y+p\frac{\sigma_Y}{\sigma_x}(x-\mu_X)+\sigma_Y^2(1-p^2)=(\frac{30+7\sqrt{6}}{300})^2+0.2\times(1-0.5^2)\approx0.174698$$

**Q2:Assume that random variable X has normal distribution with mean 2 and standard deviation of 5:**

**(1) Find the density of random variable Y = X3:**

**(2) Find the mean and variance of random variable Y defined above in (1):**

1. $PDF: f_X(x), f_Y(y), CDF: F_X(x), F_Y(y)$

$$F_Y(y) = p(Y \le y) = p(X^3 \le y) = p(X \le \sqrt[3]{y}) = F_X(\sqrt[3]{y})$$

Derive it :

$$f_Y(y) = \frac{f_X(\sqrt[3]{y})}{3\sqrt[3]{y^2}}$$

Since X~N(2,5^2);

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp(-\frac{(x-\mu_X)^2}{2\sigma_X^2}) = \frac{1}{5\sqrt{2\pi}}\exp(-\frac{(x-2)^2}{50})$$

So

$$f_Y(y) = \frac{1}{15\sqrt{2\pi}\sqrt[3]{y^2}}\exp(-\frac{(\sqrt[3]{y}-2)^2}{50})$$

2. let z~N(0,1), So x=5z+2 , y=125Z^3+150Z^2+60Z+8

E(Y)=158

**Q3:Assume random variable X has continuous density f(x) on the whole real line, nd the density of**

**(i) Y = X$^2$**

**(ii) Z = 1/x**

1.

$PDF: f_X(x), f_Y(y), f_Z(z) CDF: F_X(x), F_Y(y) F_Z(z)$

If y>0;

$$F_Y(y) = p(Y \le y) = p(X^2 \le y) = p(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$y \le 0$;

$F_Y(y)$ =0;

Derive it

$f_Y(y) = \frac{1}{2} f(\sqrt{y}) y^{-\frac{1}{2}} + \frac{1}{2} f(-\sqrt{y}) y^{-\frac{1}{2}}$ For y>0;

$f_Y(y) = 0$ $y \le 0$;

2

$F_Z(z) = p(\frac{1}{X} \le z) = p(X \ge \frac{1}{z}) = 1 - F_X(\frac{1}{z})$ when Z>0

$F_Z(z) = p(\frac{1}{X} \le z) = p(\frac{1}{z} \le X \le 0) = F_x(0) - F_X(\frac{1}{z})$ when $z \le 0$

Derive it:

$f_Z(z) = f(\frac{1}{z}) z^{-2}$ for $z \ne 0$

$f_Z(z) = 0$ for Z=0;