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1.According to differential form of Ito's formula for Brownian Motion

$$f(x) = x^3, x = W(t)$$

$$f(x)' = 3x^2, f(x)''=6x$$

$$\begin{aligned} df(x) &= f_t(x)dt + f(x)'dx + \tfrac{1}{2} f(x)'' dt \\ &= 0 + 3x^2 dx + 3x dt \\ &= 3W(t)^2 dW(t) + 3W(t)dt \end{aligned}$$

Integrate it:

$$f(x) = 3\int_0^t W(t)^2 dW(t) + 3\int_0^t W(t)dt$$

2.

Let $Y = \mathrm{S(t)}^p = S(0)^p \exp(p\sigma W(t) + p(\alpha - \tfrac{1}{2}\sigma^2)t)$

Y(X,t) ,X=W(t)

$$\begin{aligned} dY &= Y_t dt + Y_x dx + \tfrac{1}{2} Y_{xx} dt \\ &= p(\alpha - \tfrac{1}{2}\sigma^2)(S(0)^p \exp(p\sigma W(t) + p(\alpha - \tfrac{1}{2}\sigma^2)t))dt \\ &+ p\sigma(S(0)^p \exp(p\sigma W(t) + p(\alpha - \tfrac{1}{2}\sigma^2)t))dW(t) \\ &+ \tfrac{1}{2}(p\sigma)^2(S(0)^p \exp(p\sigma W(t) + p(\alpha - \tfrac{1}{2}\sigma^2)t))dt \\ &= (p(\alpha - \tfrac{1}{2}\sigma^2)dt + p\sigma dW(t) + \tfrac{1}{2}(p\sigma)^2 dt)Y \\ &= (\tfrac{1}{2}(p\sigma)^2 + p(\alpha - \tfrac{1}{2}\sigma^2))Y dt + p\sigma dW(t) Y \end{aligned}$$

3.

(1)Let $Y = X^4$

W(t)=X;

Y(X,t)

$$\begin{aligned} dY &= Y_t dt + Y_x dx + \tfrac{1}{2} Y_{xx} dt \\ &= 0 + 4W(t)^3 dW(t) + 6W(t)^2 dt \end{aligned}$$

Integrate it

$$\mathrm{Y} = 4\int_0^t W(t)^3 dW(t) + 6\int_0^t W(t)^2 dt$$

(2)

$$\mathrm{E(Y)=4E(}\int_0^{\mathrm{t}} W(t)^3 dW(t)) + 6\int_0^t E(W(t)^2)dt$$
$$= 0 + 6\int_0^t t dt$$
$$= 3t^2$$

Since $\int_0^{\mathrm{t}} W(t)^3 dW(t)$ is a Ito integra with mean=0;

(3)

Let $Y = X^6$

Y(w(t),t)

$$dY = Y_t dt + Y_x dx + \tfrac{1}{2} Y_{xx} dt$$
$$=0+6\mathrm{W(t)}^5 dW(t) + 15W(t)^4 dt$$

Integrate it:

$$\mathrm{Y=6}\int_0^t \mathrm{W(t)}^5 dW(t) + 15\int_0^t W(t)^4 dt$$

$$\mathrm{EY=6E(}\int_0^t \mathrm{W(t)}^5 dW(t)) + 15\int_0^t E(W(t)^4)dt$$
$$= 0 + 15 \times \int_0^t 3t^2$$
$$= 15t^3$$

4.

Y(t, x), x=w(t);

$$dY = Y_t dt + Y_x dx + \tfrac{1}{2} Y_{xx} dt$$
$$= Y(cdt + \tfrac{1}{2}\alpha^2 dt + \alpha dW(t))$$
$$= Y(c + \tfrac{1}{2}\alpha^2)dt + Y\alpha\, dW(t)$$

So

$$dX_{\mathrm{t}} = X_t(c + \tfrac{1}{2}\alpha^2)dt + X_t\alpha\, dW(t)$$

5.

Use Ito's formula for Ito process

Y(t, x)=log(S(t)) x=S(t)

$$dY = Y_t dt + Y_x dX_t + \tfrac{1}{2} Y_{xx} dX_t dX_t$$

$$= 0 + \tfrac{1}{S(t)} \alpha_t dS(t) - \tfrac{1}{2S(t)^2} \sigma_t^2 S(t)^2 dt$$

$$= \alpha_t dt + \sigma_t dW(t) - \tfrac{1}{2} \sigma_t^2 dt$$

$$= \left(\alpha_t - \tfrac{1}{2} \sigma_t^2 \right) dt + + \sigma_t dW(t)$$