HW5 Xiaotian Zhu

1.

Let

$\tau = T - t$

$$\Theta = \frac{\partial \Pi}{\partial t} = \frac{\partial \Pi}{\partial \tau}\frac{\partial \tau}{\partial t} = (-1)\frac{\partial \Pi}{\partial \tau}$$

$$\begin{aligned}
\Theta = -\frac{\partial C_t}{\partial \tau} &= -S_t \frac{\partial N(d_1)}{\partial \tau} + (-r)\cdot K \cdot e^{-r\tau} N(d_2) + Ke^{-r\tau}\frac{\partial N(d_2)}{\partial \tau} \\
&= -S_t \frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \tau} - rK \cdot e^{-r\tau} N(d_2) + ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \tau} \\
&= -S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(\frac{r + \frac{\sigma_s^2}{2}}{\sigma_s \sqrt{\tau}} - \frac{\ln\left(\frac{s_t}{K}\right)}{2\sigma_s \tau^{3/2}} - \frac{r + \frac{\sigma_s^2}{2}}{2\sigma_s\sqrt{\tau}} \right) - rK \cdot e^{-r\tau} N(d_2) \\
&\quad + Ke^{-r\tau} \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{S_t}{K} \cdot e^{r\tau} \right) \cdot \left(\frac{r}{\sigma_s\sqrt{\tau}} - \frac{\ln\left(\frac{s_t}{K}\right)}{2\sigma_s \tau^{3/2}} - \frac{r + \frac{\sigma_s^2}{2}}{2\sigma_s\sqrt{\tau}} \right) \\
&= -S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(\frac{r + \frac{\sigma_s^2}{2}}{\sigma_s \sqrt{\tau}} - \frac{\ln\left(\frac{s_t}{K}\right)}{2\sigma_s \tau^{3/2}} - \frac{r + \frac{\sigma_s^2}{2}}{2\sigma_s\sqrt{\tau}} \right) - rK \cdot e^{-r\tau} N(d_2) \\
&\quad + S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(\frac{r}{\sigma_s\sqrt{\tau}} - \frac{\ln\left(\frac{s_t}{K}\right)}{2\sigma_s \tau^{3/2}} - \frac{r + \frac{\sigma_s^2}{2}}{2\sigma_s\sqrt{\tau}} \right) \\
&= -S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(\frac{\frac{\sigma_s^2}{2}}{\sigma_s\sqrt{\tau}} \right) - rK \cdot e^{-r\tau} N(d_2) \\
&= -\frac{S_t \sigma_s}{2\sqrt{\tau}} \cdot N'(d_1) - rK \cdot e^{-r\tau} N(d_2)
\end{aligned}$$

$$\Delta = \frac{\partial C_t}{\partial S_t} = N(d_1) + S_t \frac{\partial N(d_1)}{\partial S_t} - Ke^{-r\tau} \frac{\partial N(d_2)}{\partial S_t}$$
$$= N(d_1) + S_t \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S_t} - Ke^{-r\tau} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial S_t}$$
$$= N(d_1) + S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{1}{S_t \sigma_s \sqrt{\tau}} - Ke^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{S_t}{K} \cdot e^{r\tau} \cdot \frac{1}{S_t \sigma_s \sqrt{\tau}}$$
$$= N(d_1) + S_t \frac{1}{S_t \sigma_s \sqrt{2\pi\tau}} e^{-\frac{d_1^2}{2}} - S_t \frac{1}{S_t \sigma_s \sqrt{2\pi\tau}} e^{-\frac{d_1^2}{2}}$$
$$= N(d_1)$$

$$\Gamma = \frac{\partial^2 C_t}{\partial {S_t}^2} = \frac{\partial \left(\frac{\partial C_t}{\partial S_t} \right)}{\partial S_t}$$
$$= \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial S_t}$$
$$= N'(d_1) \cdot \frac{\frac{1}{S_t}}{\sigma_s \sqrt{\tau}}$$
$$= \frac{1}{S_t \sigma_s \sqrt{\tau}} N'(d_1)$$

Plug in , we get

$$rSN(d_1) - rKe^{-r\tau} N(d_2) = r(SN(d_1) - Ke^{-r\tau} N(d_2)) = rc$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

When s>k, t take limit to T, $\sqrt{\tau}$ becomes positive infinity small and d1 becomes positive infinity. So does d2. Therefore, N(d1)=N(d2)=1;

When s<k, t take limit to T, $\sqrt{\tau}$ become s positive infinity small and d1 becomes 0. So does d2. Therefore N(d1)=N(d2)=0;

The payoff will be:

$$\lim_{t->T} C(S,t) = \begin{cases} S - K, s > k \\ 0, s \le k \end{cases}$$

Hence. The boundary condition is proved.

2.

$$C(s,t) = E^{s,t}(e^{-r(T-t)}(S_T - K)^+)$$

$$\log S_T \sim \log S_t + (r - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z$$

The first integral is:

$$\begin{aligned}
&E^{s,t}(e^{-r(T-t)}\log S_T) \\
&= e^{-r(T-t)}E^{\log s,t}e^{\log S_t} \\
&= e^{-r(T-t)}\int_{z<d_-} e^{\log S + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z}\,\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}dz \\
&= e^{-r(T-t)}\int_{z<d_+} e^{\log S + r(T-t)}\,\frac{e^{-\frac{(s+\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}dz \\
&= SN(d_+)
\end{aligned}$$

The second is easy to get, which is $Ke^{-r(T-t)}N(d_-)$

Add them up: $C(s,t) = SN(d_+) + Ke^{-r(T-t)}N(d_-)$

3.

Boundary condition is $C(s,t) = \max(S_T - K, 0)^2$

According to Feynman-Kac Theorem

The solution to PDE is $C(s,t) = E^{s,t}(e^{-r(T-t)}\max(S_T - K, 0))^2$

Since S is geometric BM, So we get

$$\begin{aligned}
dY_t &= (r - \frac{1}{2}\sigma^2)dt + \sigma W_t \\
Y_T - Y_t &= (r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t) \\
Y_T &= \log S_t + (r - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z \\
\log Y_T &\sim N(\log S_t + (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))
\end{aligned}$$

$$C(s,t) = E^{s,t}(e^{-r(T-t)}\max(S_T - K, 0))^2$$
$$= E^{s,t}(e^{-r(T-t)}\max(S_T - K, 0))^2$$
$$= e^{-r(T-t)}\int_{z=-\infty}^{d_-}(\exp(\log S_t + (r - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z) - K)^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$
$$= e^{-r(T-t)}\int_{z=-\infty}^{d} e^{2(\log S_t + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z)}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz - 2e^{-r(T-t)}\int_{z=-\infty}^{d} e^{(\log S_t + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z)}K\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$
$$+e^{-r(T-t)}\int_{z=-\infty}^{d}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}K^2 dz$$

The first integral:

$$e^{-r(T-t)}e^{2(\log(St) + (r-\frac{1}{2}\sigma^2)(T-t) + \sigma^2(T-t))\int_{z=-\infty}^{d_-}\frac{e^{-\frac{(z+2\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}dz}$$
$$== (S_T)^2 e^{\sigma\sigma(T-t)}\int_{n=-\infty}^{d_- + 2\sigma\sqrt{T-t}}\frac{e^{-\frac{(n)^2}{2}}}{\sqrt{2\pi}}dz$$
$$= (S_T)^2 e^{\sigma\sigma(T-t)}N(d_- + 2\sigma\sqrt{T-t})$$

Let $n = z + 2\sigma\sqrt{T-t}$

The second integral :

$$2e^{-r(T-t)}\int_{z=-\infty}^{d} e^{(\log S_t + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z)}K\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$
$$= 2Ke^{-r(T-t)}\int_{z=-\infty}^{d} e^{(\log S_t + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z)}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$
$$= 2Ke^{-r(T-t)}\int_{z=-\infty}^{d} e^{(\log S_t + (r-\frac{1}{2}\sigma^2)(T-t) + \frac{1}{2}\sigma\sigma(T-t))}\frac{e^{-\frac{(z+\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}dz$$
$$= 2Ke^{-r(T-t)}e^{(\log S_t + (r-\frac{1}{2}\sigma^2)(T-t) + \frac{1}{2}\sigma\sigma(T-t))}\int_{l=-\infty}^{d_- + \sigma\sqrt{T-t}}\frac{e^{-\frac{l^2}{2}}}{\sqrt{2\pi}}dz$$
$$= 2KS_T N(d_- + \sigma\sqrt{T-t})$$

Let $l = z + \sigma\sqrt{T-t}$

Plug in, we will get

$$C(s,t) = (S_T)^2 e^{\sigma\sigma(T-t)} N(d_- + 2\sigma\sqrt{T-t}) + 2KS_T N(d_- + \sigma\sqrt{T-t}) + K^2 e^{-r(T-t)} N(d_-)$$