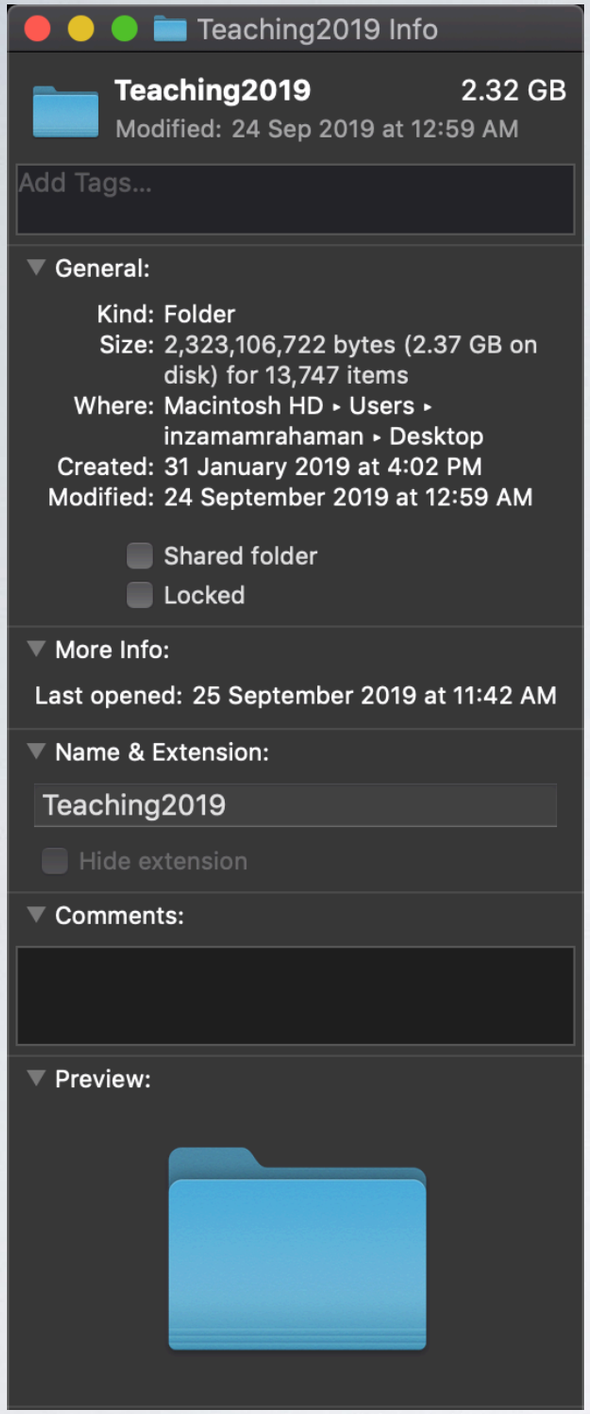
Trees and Binary Trees

COMP2611: Data Structures

Sept 2019

Outline

- Trees
  - Basic definitions and properties
  - Notion of a traversal on a tree
    - BFS (Breadth-first traversal)
    - DFS (Depth-first traversal)
- Binary Trees
  - Pre-order, in-order, post-order
  - Properties and how to prove them



- How does OS calculate size of directories?

What is a Tree?

- A lot of data and information can be organised hierarchically
- Such data can be stored in a tree
- Tree consists of
  - nodes with parent/child relationship
  - formally, a connected acyclic graph (DAG) where every node is connected to every other node by a single path (definition will become clearer towards end of course)
- Examples
  - Files/folders (Windows, MacOSX, …,)
  - Structure of Source code in compiler or parser
  - Decision Trees

Tree "Anatomy"

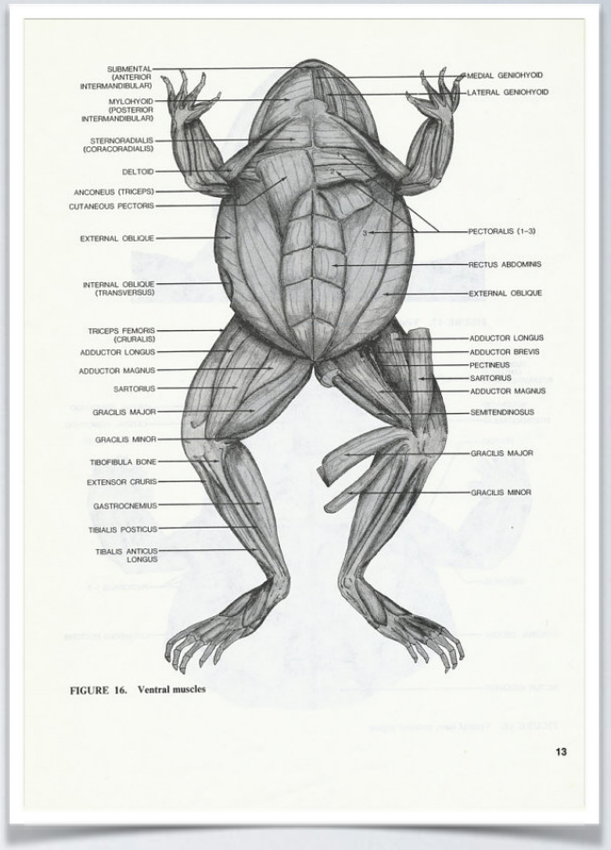


FIGURE 16. Ventral muscles

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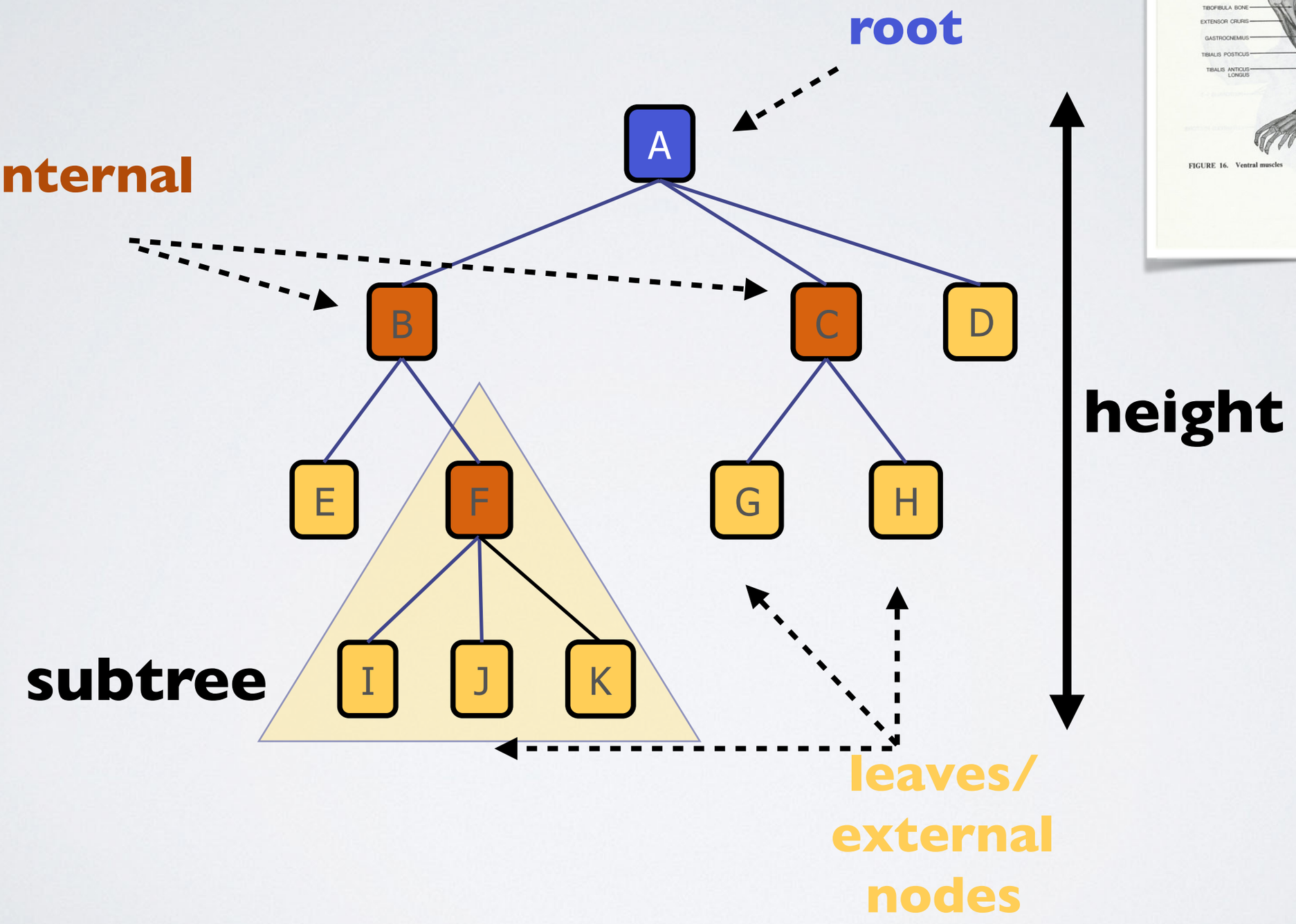


Tree Terminology

- **Root:** node without a parent (A)
- **Internal node:** node with at least one child (A, B, C, F)
- **Leaf (external node):** node without children (E, I, J, K, G, H, D)
- **Parent node:** node immediately above a given node (parent of C is A)
- **Child node:** node(s) immediately below a given node (children of C are G and H)
- **Ancestors of a node:**
  - parent, grandparent, grand-grandparent, etc. (ancestors of G are C, A)
- **Descendant of a node:** child, grandchild, grand-grandchild, etc.
- **Depth of a node:** number of ancestors (I has depth 3)
- **Height of a tree:**
  - maximum depth of any node (tree with just a root has height 0, this tree has height 3)
- **Subtree:** tree consisting of a node and its descendants

Example Tree ADT

- Tree methods:
  - int **size**( ): returns the number of nodes
  - boolean **isEmpty**( ): returns true if the tree is empty
  - Node **root**( ): returns the root of the tree
- Node methods:
  - Node **parent**( ): returns the parent of the node
  - Node[ ] **children**( ): returns the children of the node
  - boolean **isInternal**( ): returns true if the node has children
  - boolean **isExternal**( ): returns true if the node is a leaf
  - boolean **isRoot**( ): returns true if the node is the root
  - int **numDescendents**(): returns the number of descendants of this node

Implementation

- Usually implemented similarly to linked lists
  - Some special trees can also be implemented using arrays

Implementation (Python 3.x)

```
class TreeNode:
    def __init__(self, data, parent=None, children=[]):
        self.data = data
        self.parent = parent
        self.children = children

    def is_root(self):
        return self.parent is None

    def is_internal(self):
        return len(self.children) > 0

    def is_external(self):
        return not self.is_internal()

    def get_children(self):
        return self.children

    def add_child(self, child):
        child.parent = self
        self.children.append(child)

    def get_num_descendants(self):
        count = 0
        for child in self.children:
            count += 1
            count += child.get_num_descendants()
        return count
```

Implementation (Python 3.x)

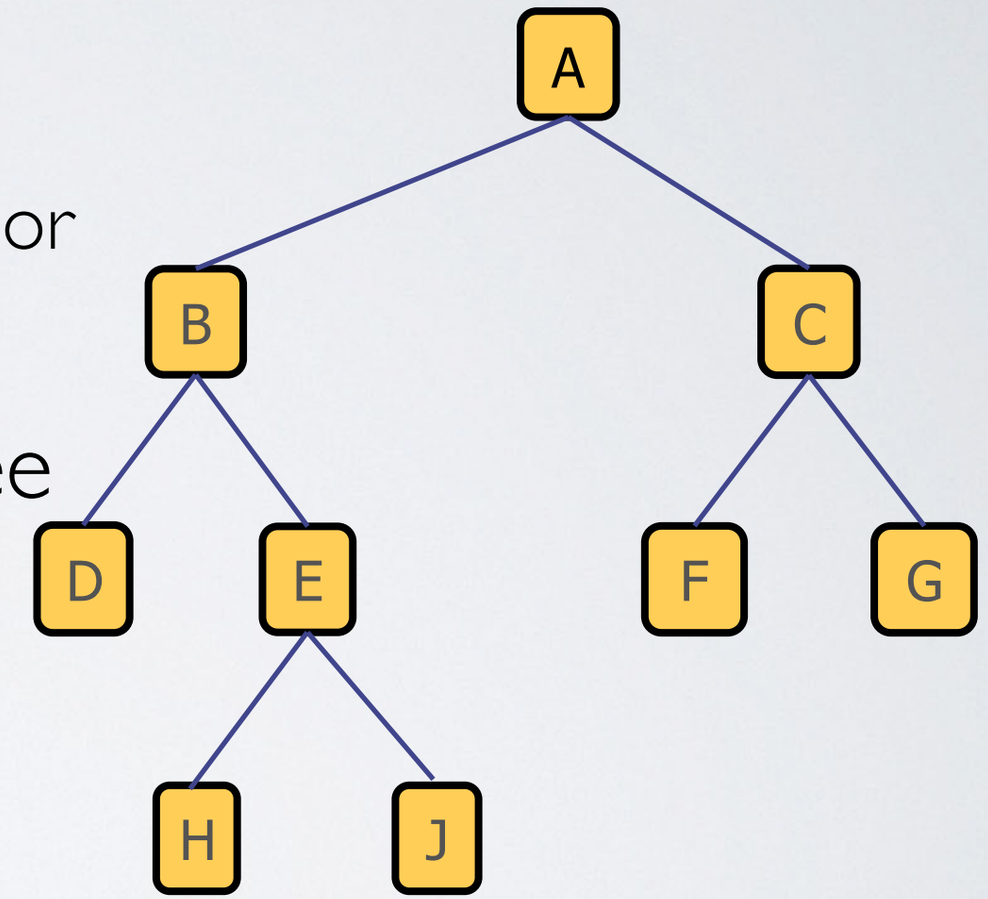
```
class Tree:
    def __init__(self, root=None):
        self.root = root

    def is_empty(self):
        return self.root is None

def size(self):
    if self.root is not None:
            count = 1 + self.root.get_num_descendants()
        return count
    return 0
```

General Trees

- General Tree structures are more difficult to analyse
  - Will look at better ways of implementing and analysis when doing Graphs towards the end of the course
- Will focus on special case called binary trees
  - But note that some (though not all) results generalise
  - In particular with traversals

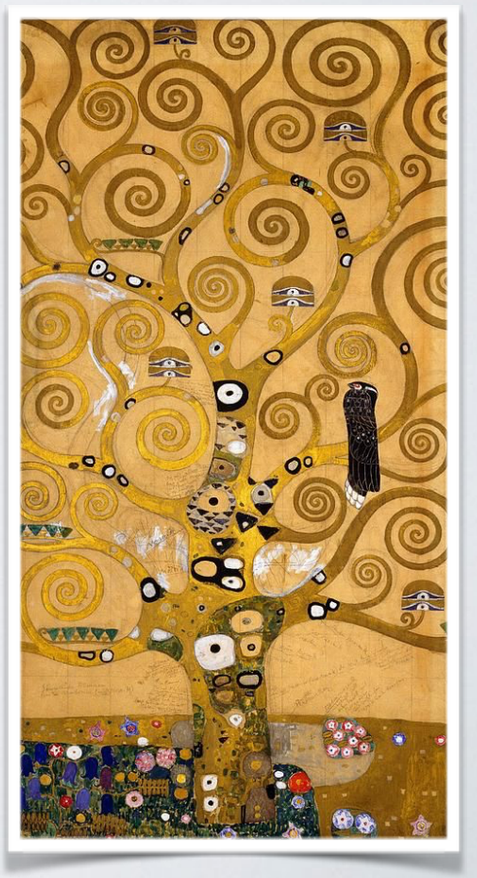
Binary Trees

- Internal nodes have at most **2** children
  - left and right
  - if only **1** child, still need to specify if left or right
- Recursive definition of a Binary Tree
  - a single node
  - or a root node with at most 2 children
    - each of which is a binary tree



Binary Tree ADT

- In addition to Tree methods *binary* trees have:
  - Node **left**( ): returns the left child if it exists, else NIL
  - Node **right**( ): returns the right child if it exists, else NIL
  - Node **hasLeft**( ): returns TRUE if node has left child
  - Node **hasRight**( ): returns TRUE if node has right child



How to implement?

- Can use linked structure
  - Pointers or references to left child, right child, or parent
- Can use array
  - Start storing nodes at index `1`
  - Node at index `i` has left child at `i × 2` and `i × 2 + 1`
  - Node at index `i` has parent `⌊i / 2⌋`
- Most analysis applied to both implementations
- Should be able to implement either one!

Trade offs

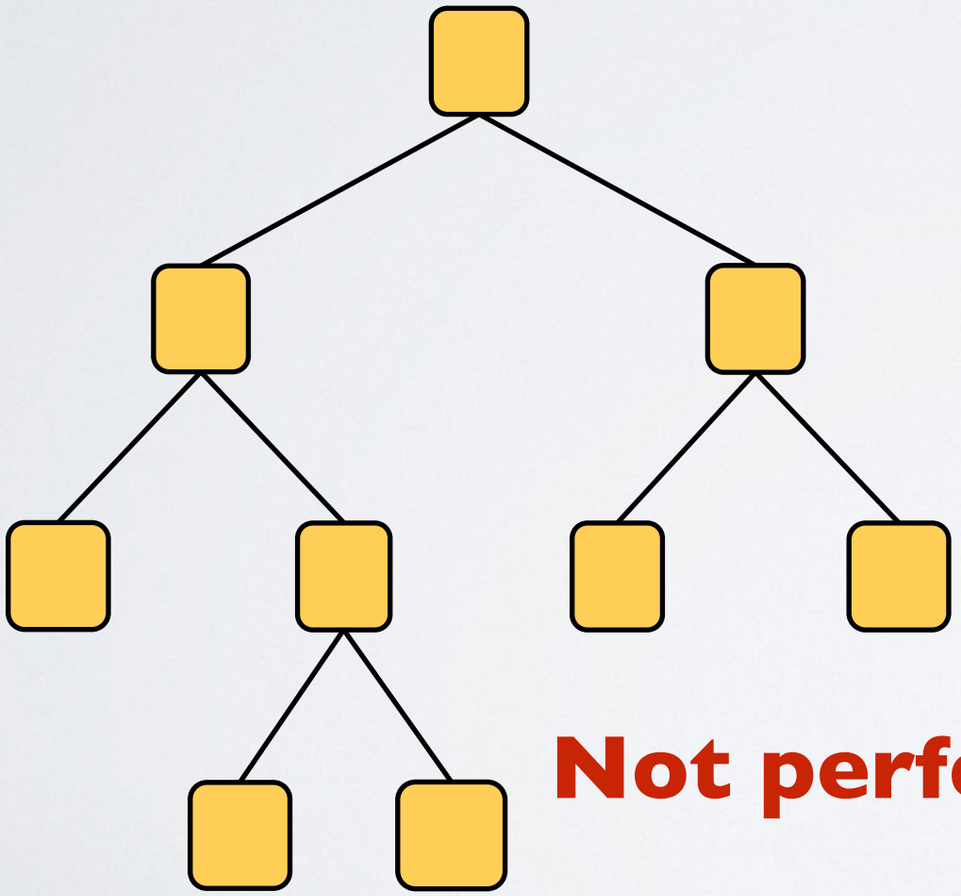
- Linked structures require dynamic memory allocation
  - Not a good idea for most embedded systems….
- Array access is quicker
  - Exploits cache locality
- Linked structure matches intuition more; hence "easier" to implement
  - Tip remember the formulae for left children, right children, and parents!
  - Then just "swap" out access through pointers

Implementation of Linked (Python 3.x)

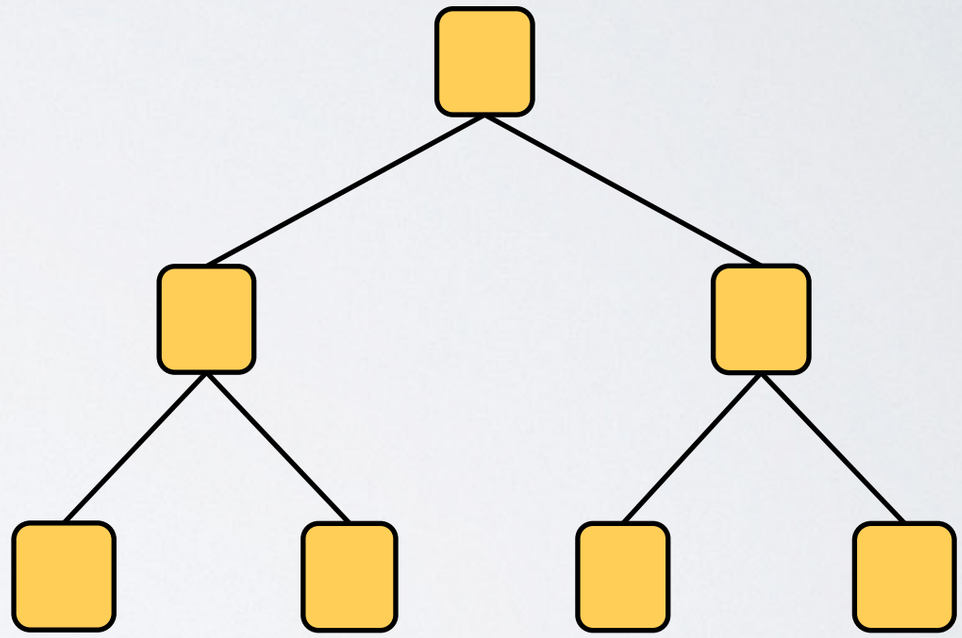
```
class TreeNode:
    def __init__(self, data, parent=None, left=None, right=None):
        self.data = data
        self.parent = parent
        # direct access to modify children
        self.left = left
        self.right = right

    def is_root(self):
        return self.parent is None

    def has_left(self):
        return self.left is not None

    def has_right(self):
        return self.right is not None

    def is_internal(self):
        return not (self.left is None and self.right is None)

    def is_external(self):
        return not self.is_internal()

    def get_num_descendants(self):
        count = 0
        if self.has_left():
            count += 1 + left.get_num_descendants()
        if self.has_right():
            count += 1 + right.get_num_descendants()
        return count
```

Perfection

- A binary tree is **perfect** if
  - every level is completely full



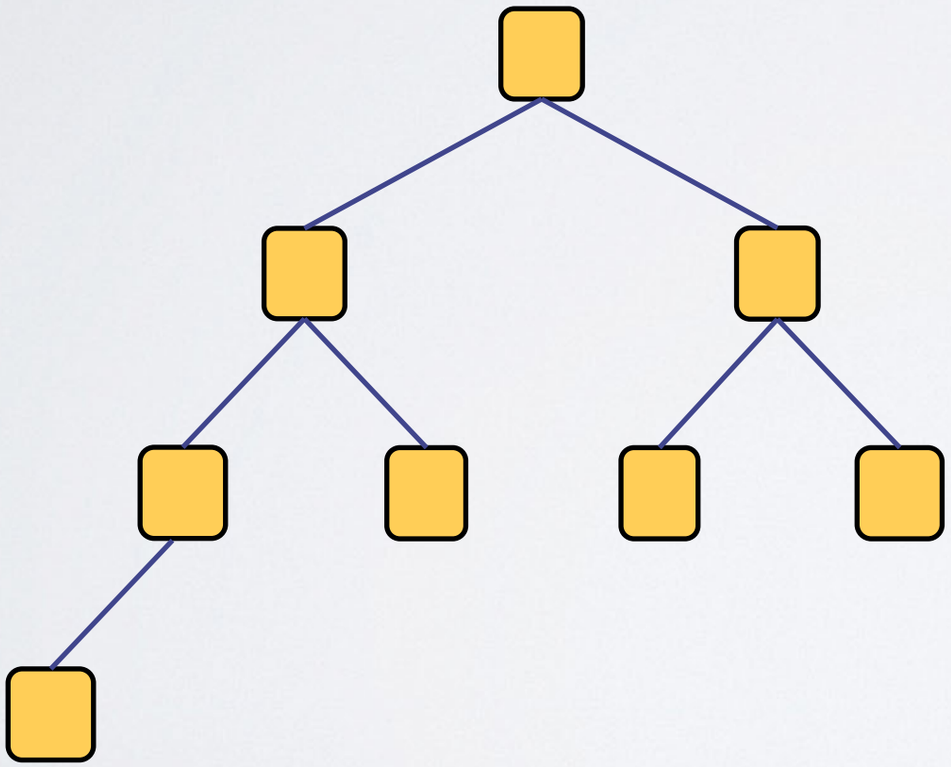
Not perfect



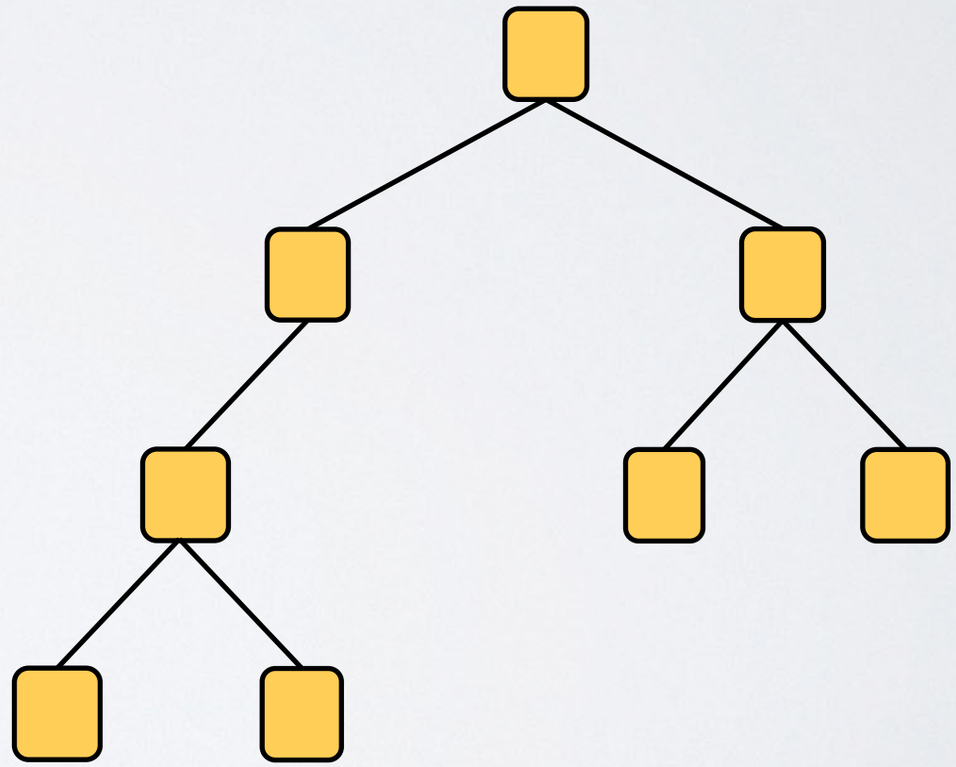
Perfect!

Completeness

- A binary tree is **left-complete** if
  - every level is completely full, possibly excluding the lowest level
  - all nodes are as far left as possible



Left-complete!



Not left-complete

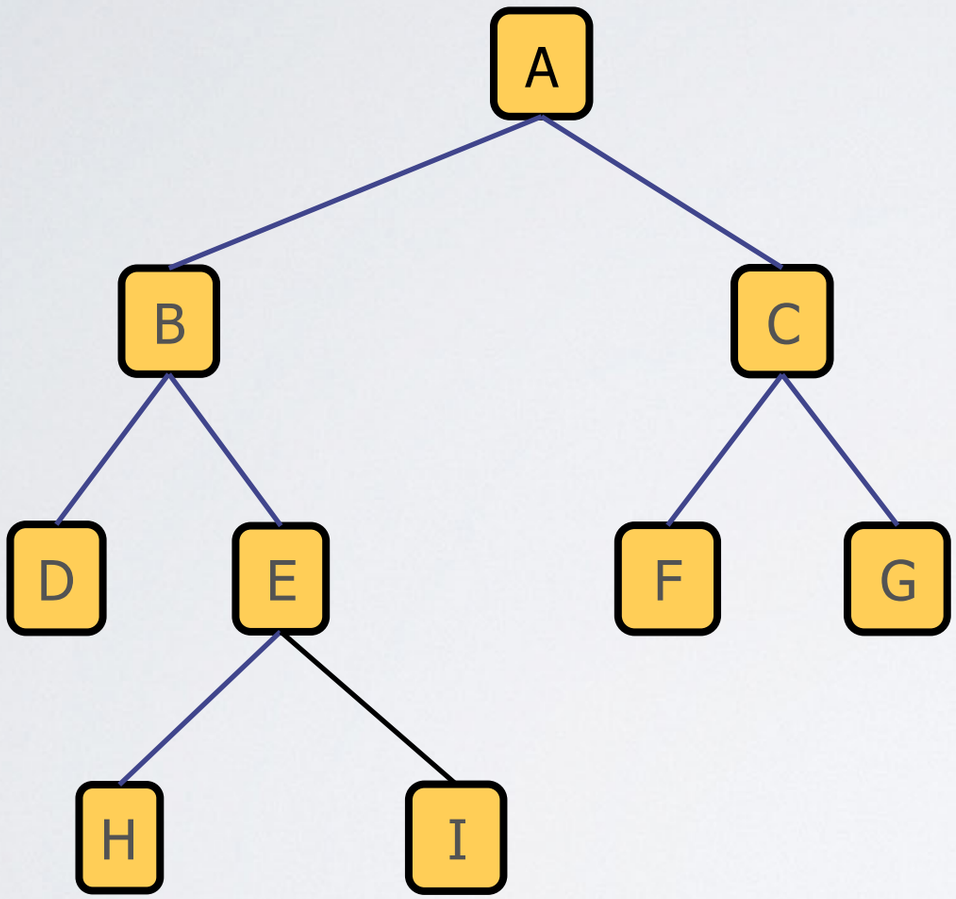
Aside: Decorations

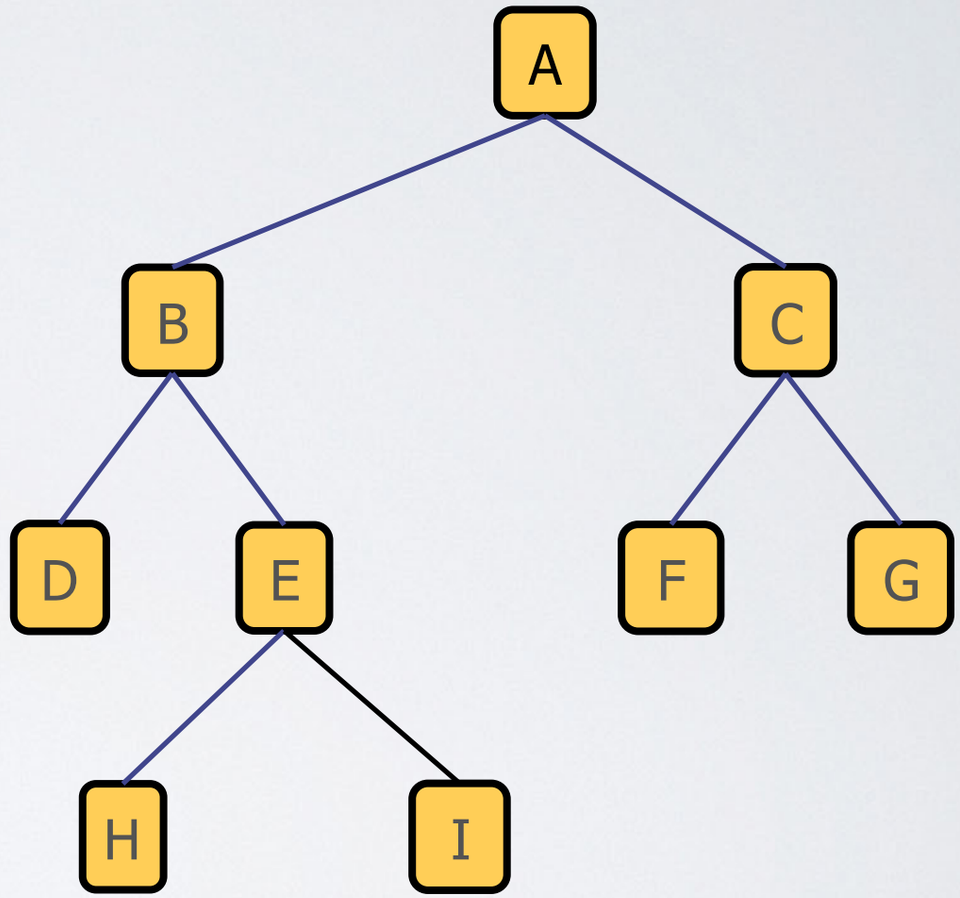
- Decorating a node
  - associating a value to it
- Two approaches
  - Add new attribute to each node
    - ex: **`node.numDescendants = 5`**
  - Maintain dictionary that maps nodes to decoration
    - do this if you can't modify tree
    - ex: **`descendantDict[node] = 5`**

Tree Traversals

- How would you enumerate every item in an array?
  - use a for loop from `i = 1` to `n` and read `arr[i]`
- How would you enumerate every item in a (linked) Tree?
  - not obvious…
  - because Trees don't have an "obvious" order like arrays
- Tree traversal
  - algorithm that visits every node of a tree
- *Many* possible tree traversals
  - each kind of traversal visits nodes in different order

Breadth- vs. Depth-First Traversals



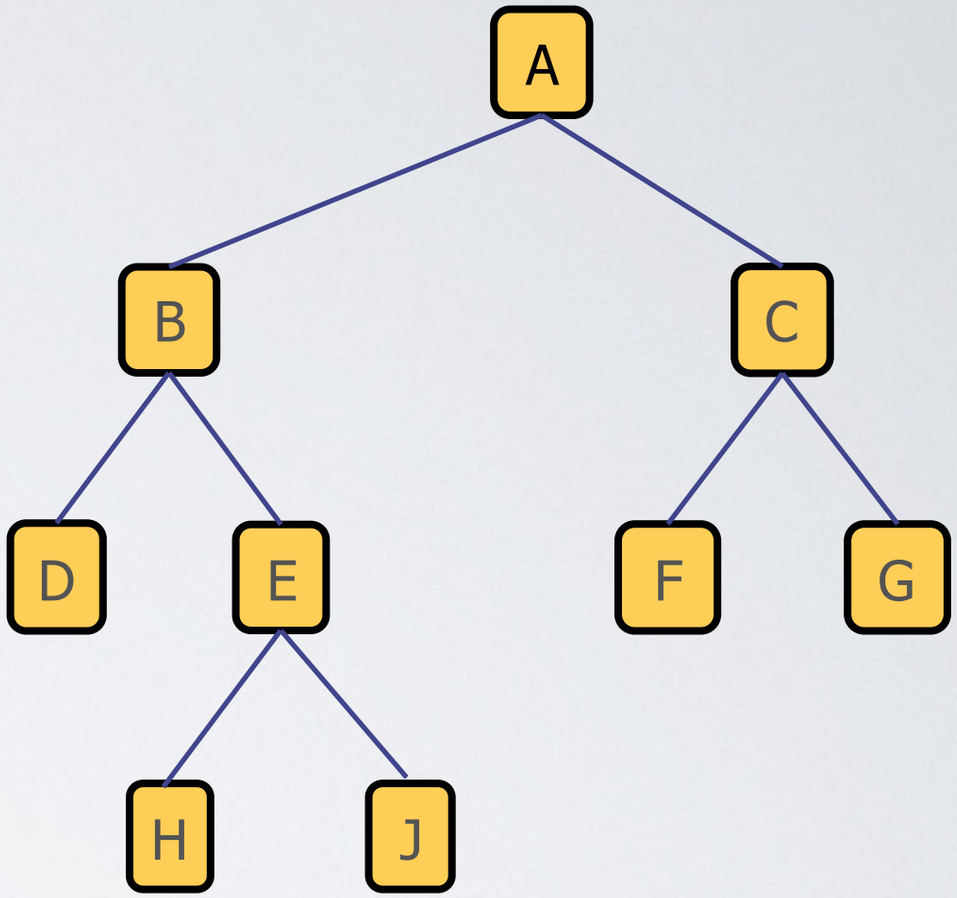


Traversals

- Can't use for loop
  - Don't know number of items apriori
  - Might need to loop for an undefined number of iterations
    - Might need to adjust direction every iteration
      - Would effect number of iterations
- Should use while loop instead

Traversal Strategy

```
function traversal(root):
  Store root in S
  while S is not empty
     get node from S
     do something with node
     store children in S
```



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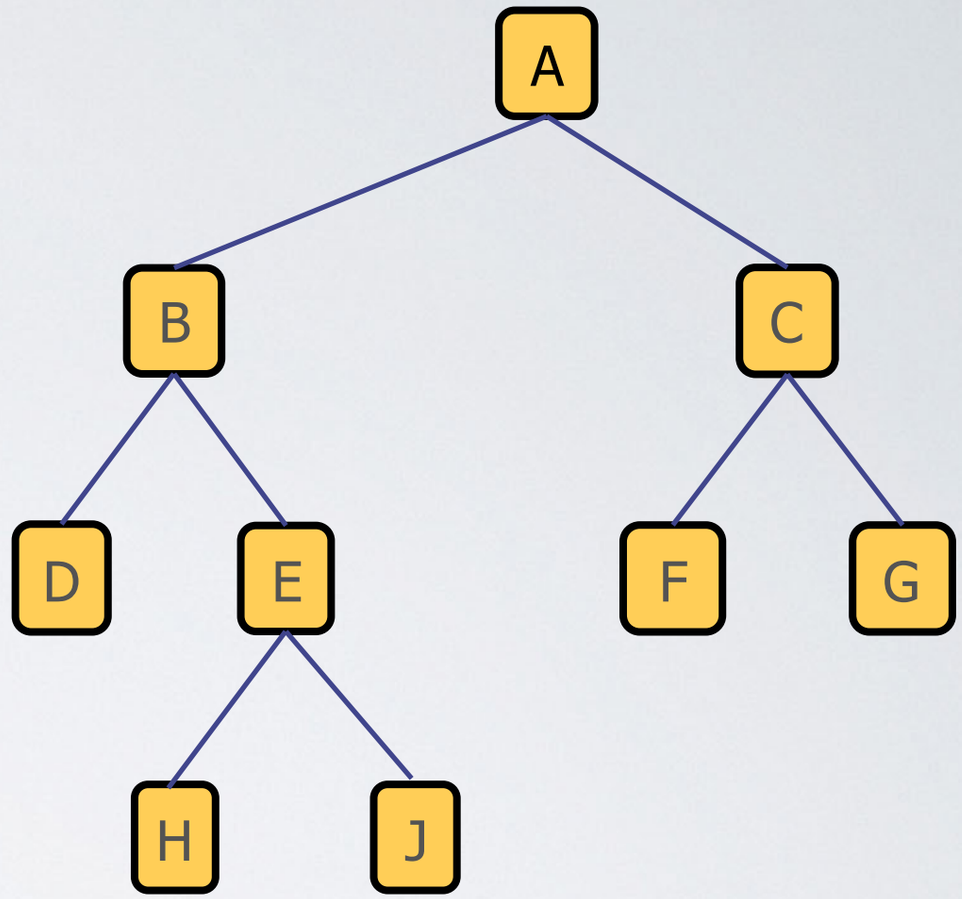

- What is **S** exactly?
  - A place we store nodes until we can process them
- Which node of **S** should we process next?
  - the first? the last?

Traversal Strategy — Grab Oldest Node

```
function traversal(root):
  Store root in S
  while S is not empty
     get node from S
     do something with node
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```

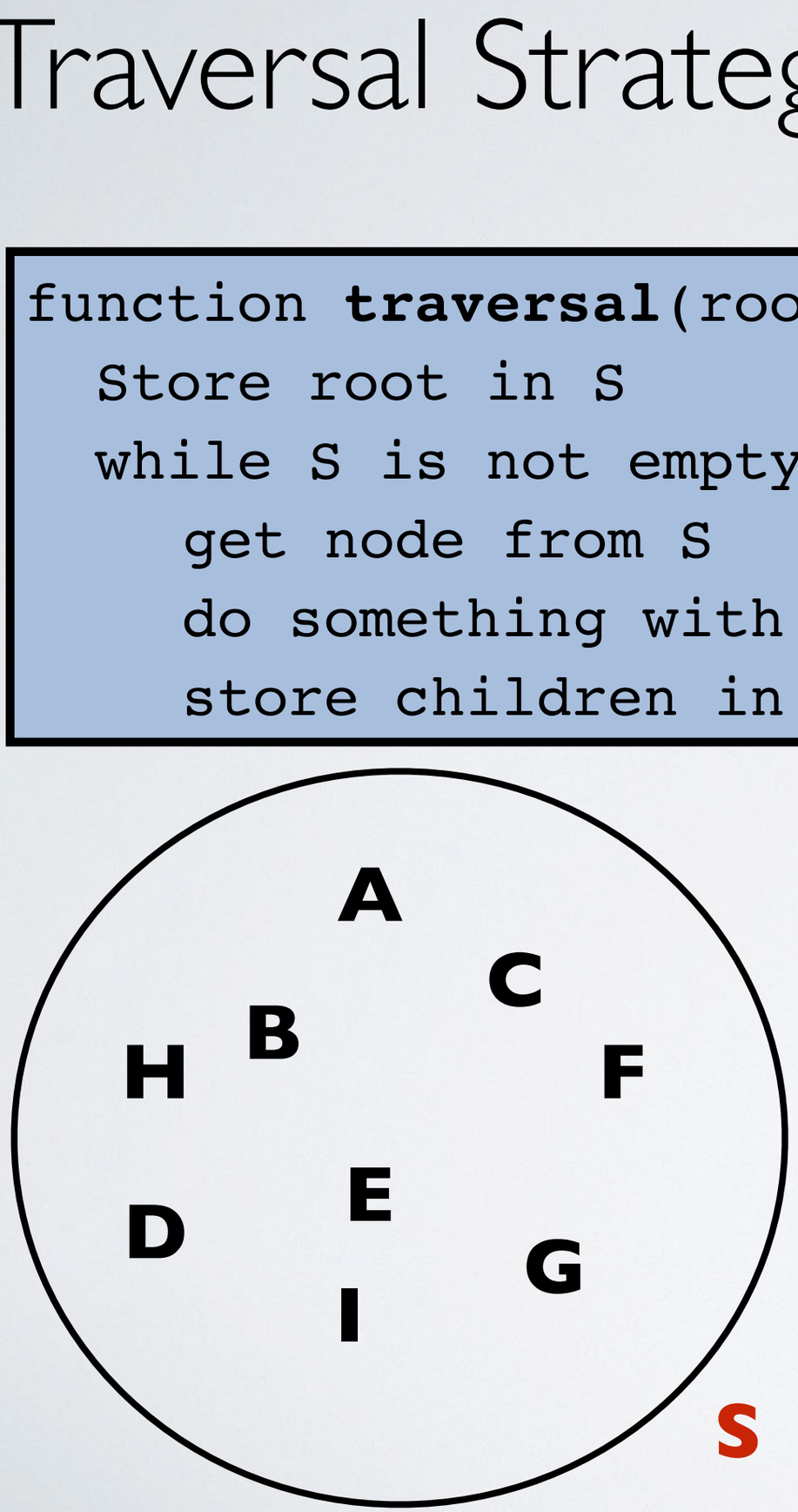

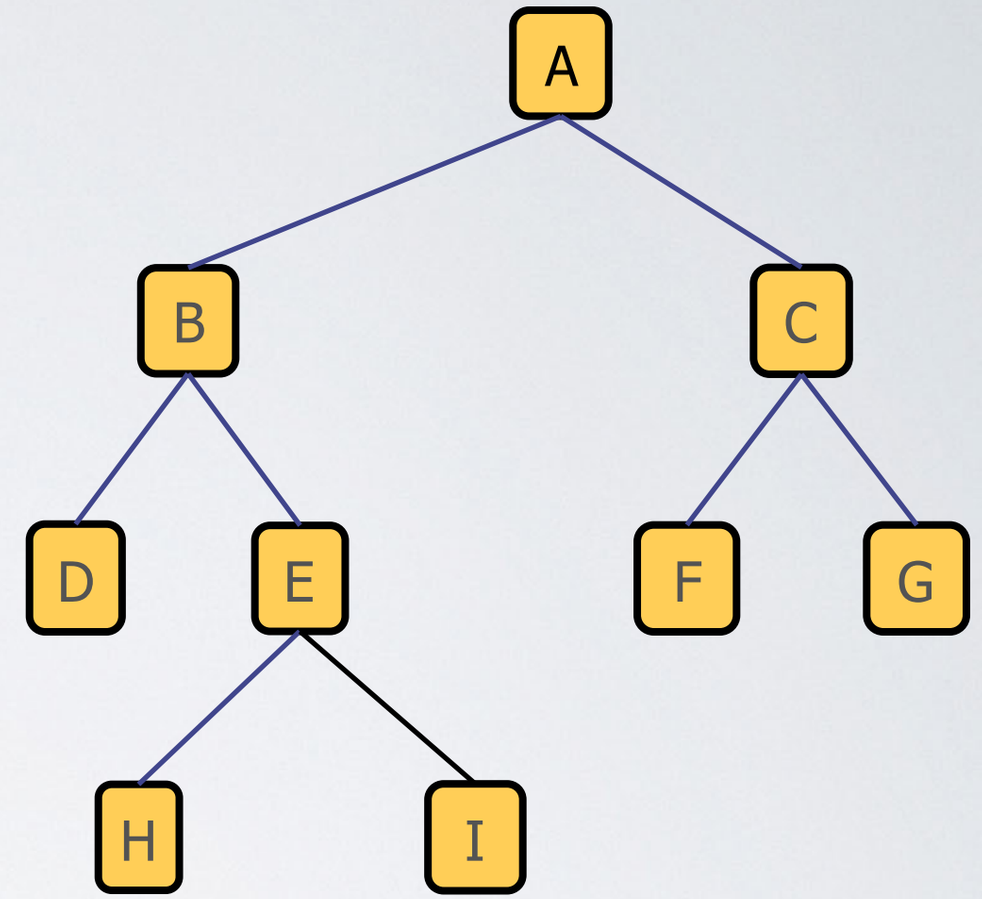

A B C D E F G H I

Traversal Strategy — Grab Oldest Node

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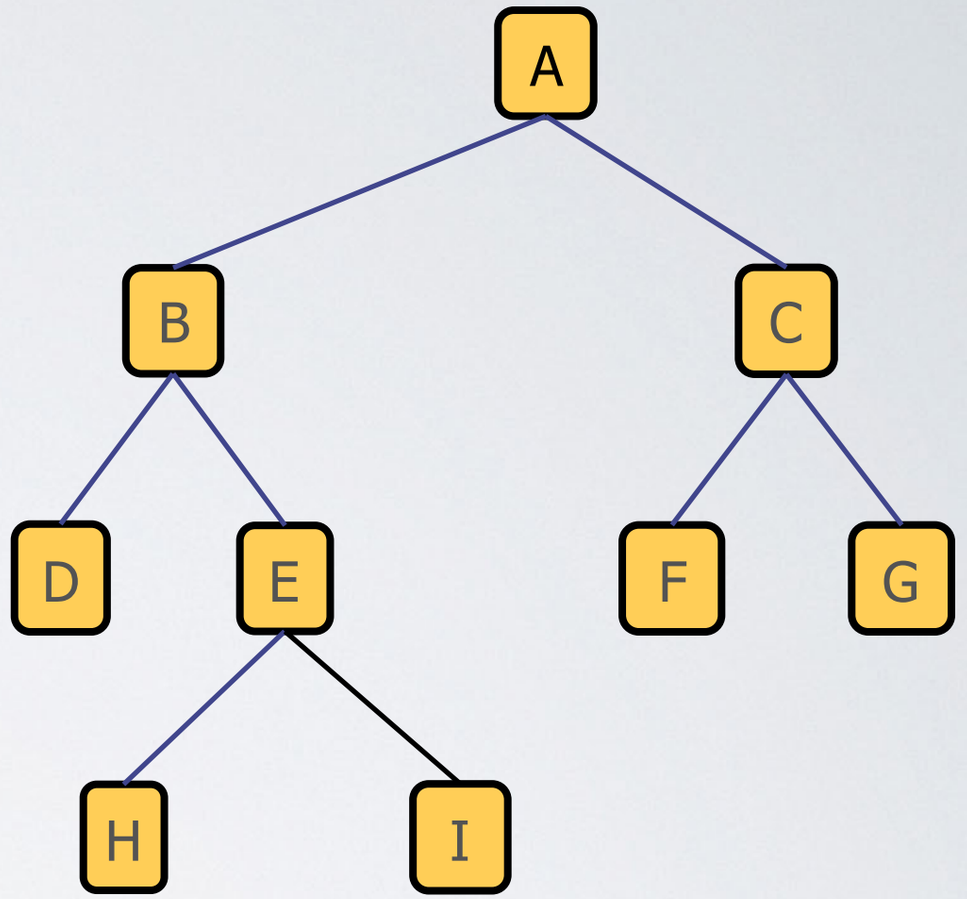
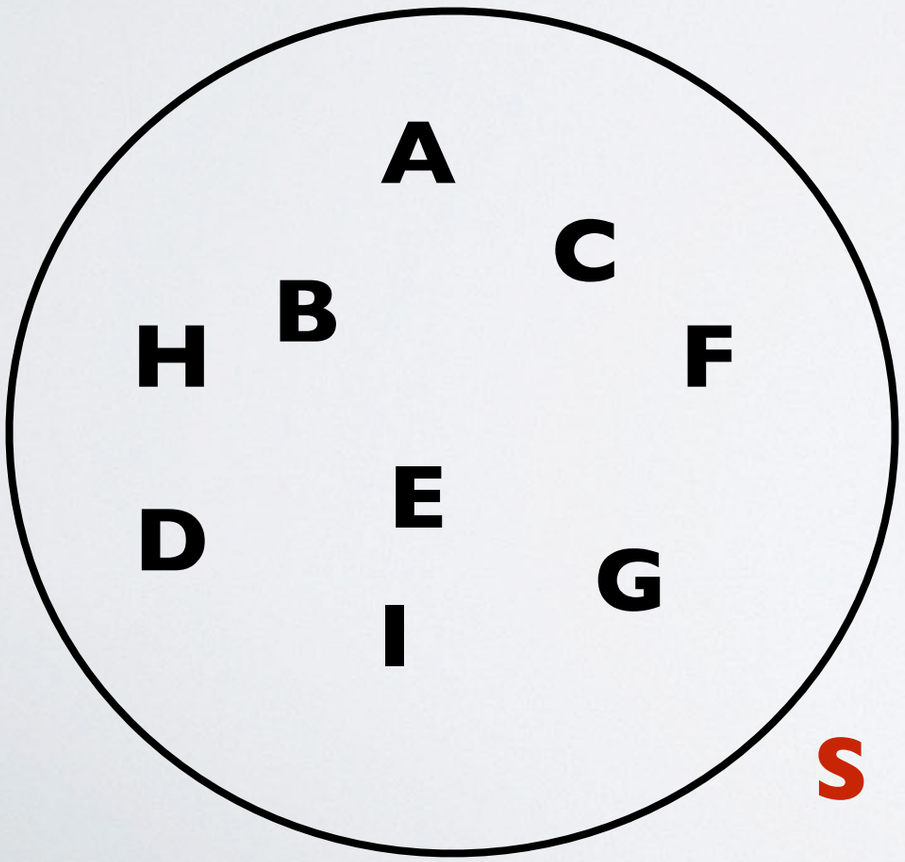




Does S remind you of something?

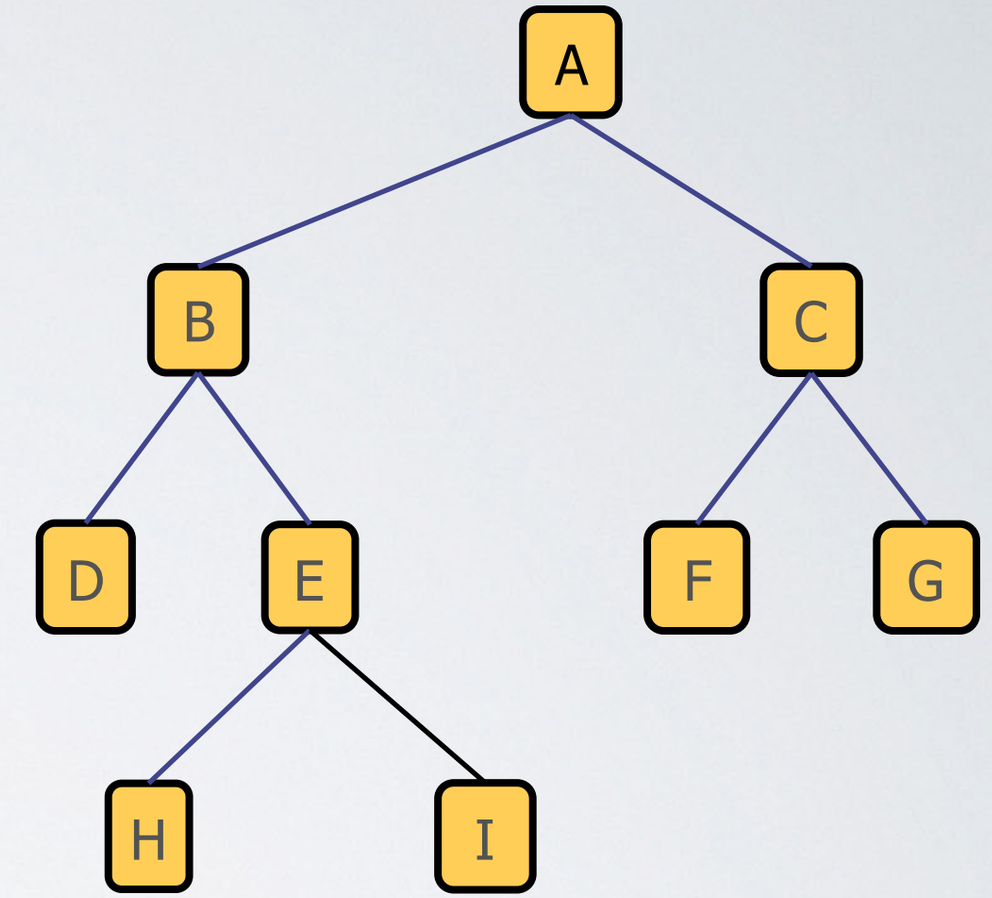
Traversal Strategy — Grab Oldest Node

- If we grab the oldest node in **S**
  - we're doing FIFO…
  - so **S** is a queue!
  - Traversal w/ Queue gives breadth-first traversal
- Why?
  - Queue guarantees a node is processed before its children
- Children can be inserted in any order

```
function bft(root):
  Q = new Queue()
  enqueue root
  while Q is not empty
     node = Q.dequeue()
     visit(node)
     enqueue node's left &
     right children
```

Breadth-First Traversal

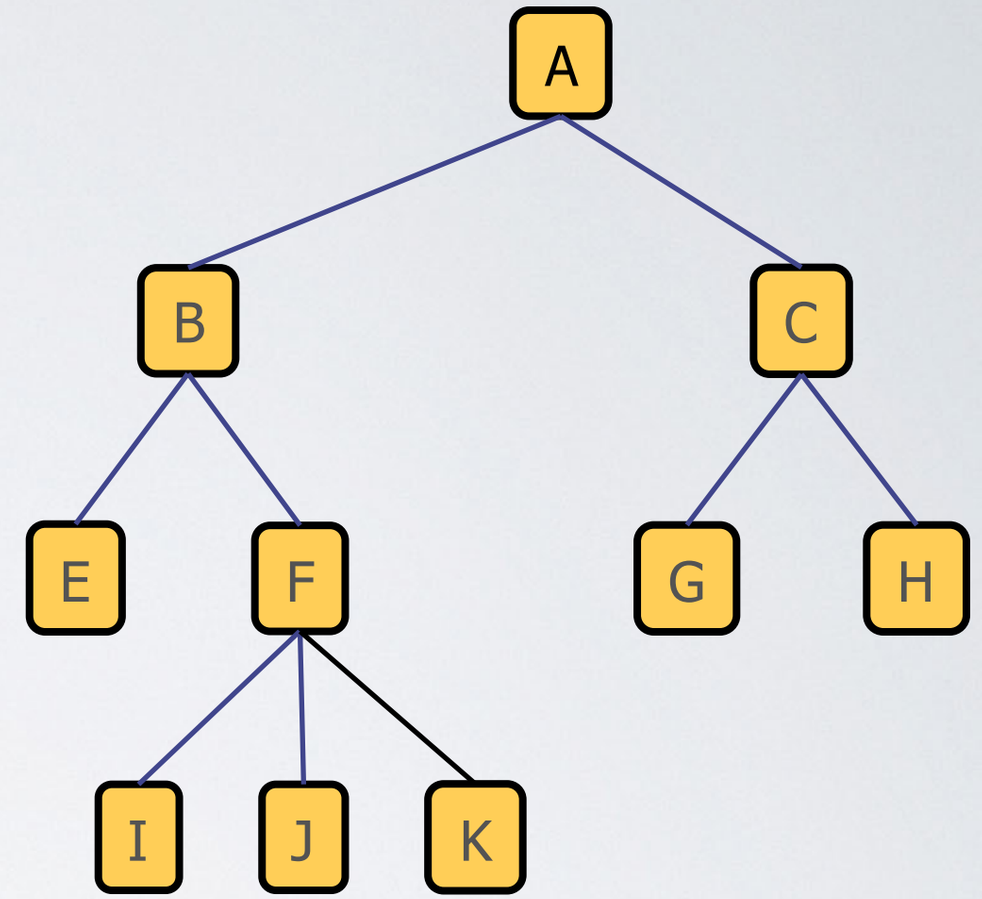
- Start at root
  - Visit both of its children first,
    - Then all of its grandchildren,
      - Then great-grandchildren
        - etc…
- Also known as
  - level-order traversal



A B C D E F G H I

Depth-First Traversal

- What if we grab youngest node in **S**?
  - we're doing LIFO…
  - so **S** is a stack!
  - Traversal w/ Stack gives us…
- Depth-first search
  - start from root
  - traverse each branch before backtracking
  - can produce different orders



A C H G B F K J I E

A B E F I J K C G H

Depth-First Traversal

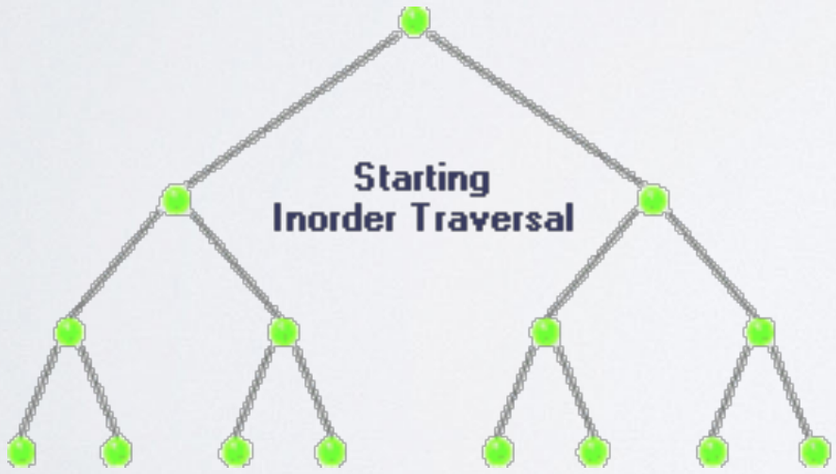
```
function dft(root):
  S = new Stack()
  push root
  while S is not empty
     node = S.pop()
     visit(node)
     push node's left & right children
```

- Why does Stack give DFT?
  - Stack guarantees entire branch will be visited before visiting another branch
- Children can be pushed on stack in any order

Recursive Depth-First Traversal

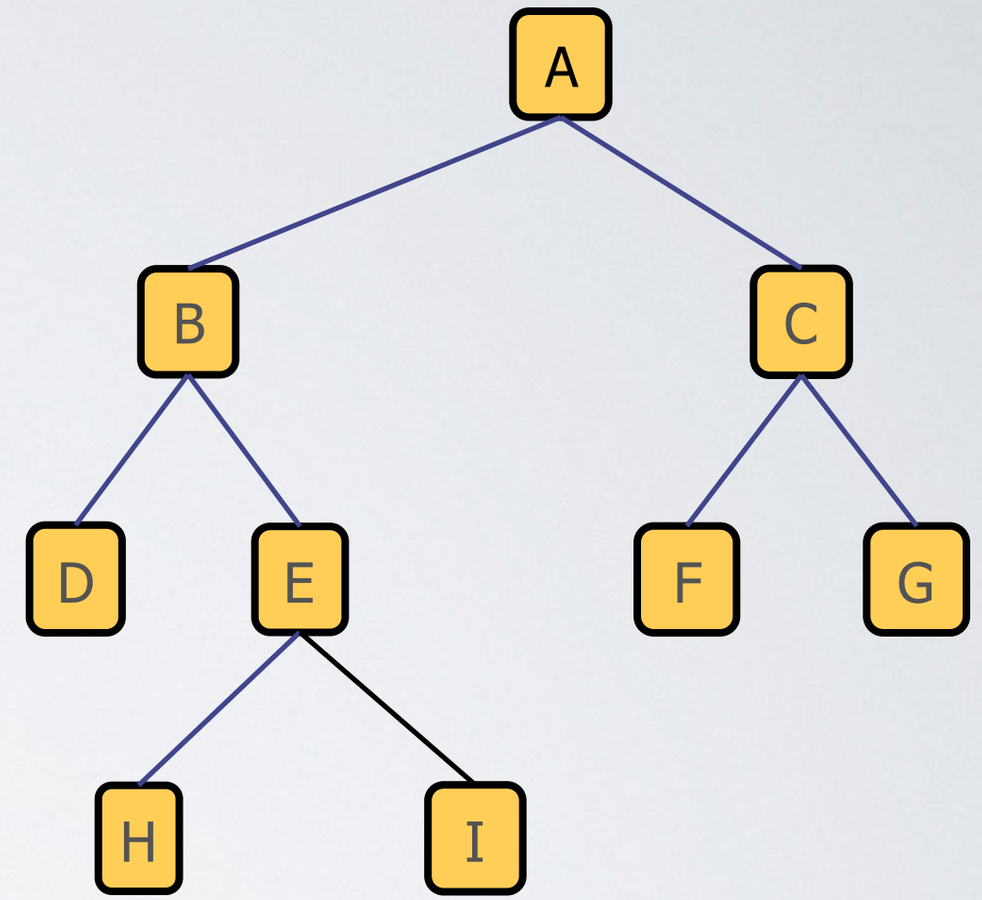
- DFT can be implemented recursively
- With recursion we can have *3* different orders
  - **pre-order:** visits node before visiting left and right children
    - V-L-R
  - **post-order:** visits each child before visiting node
    - L-R-V
  - **in-order:** visits left child, node and then right child
    - L-V-R
- Notice that left child is always visited before right child

Depth-First Visualizations



Pre-order Traversal

```
function preorder(node):
  visit(node)
  if node has left child
    preorder(node.left)
  if node has right child
    preorder(node.right)
```

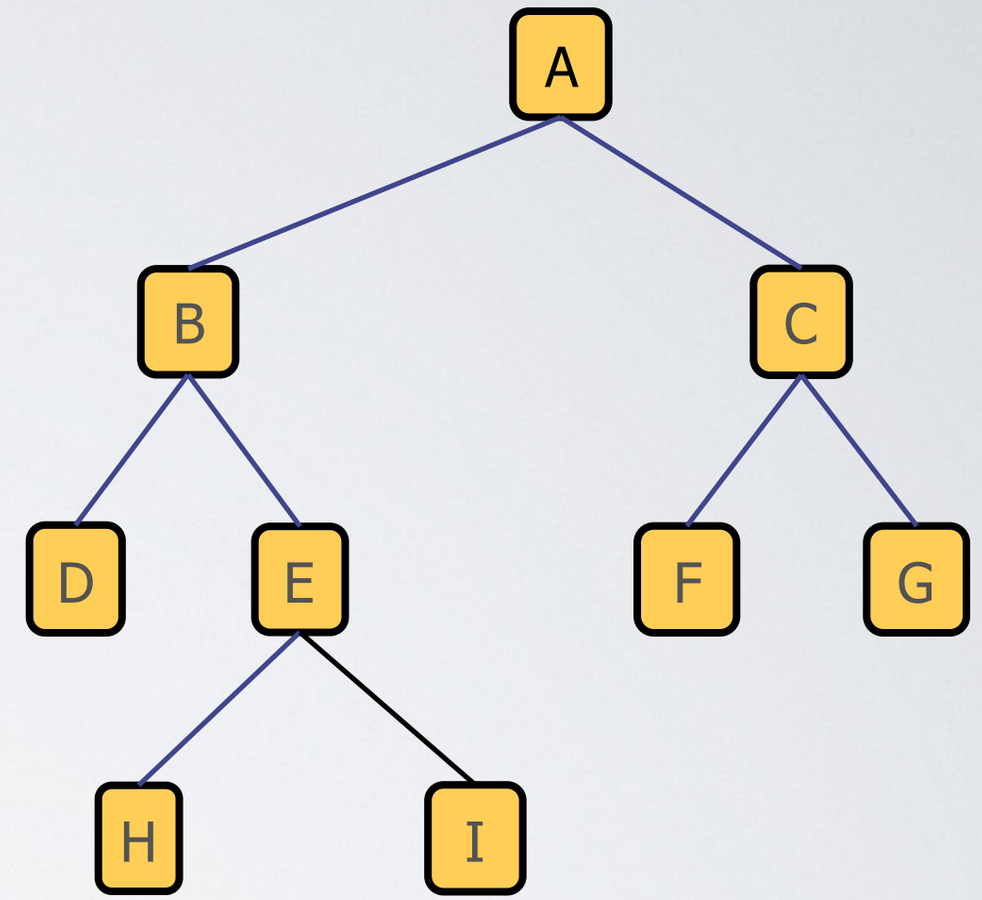


A B D E H I C F G

Note: like iterative DFT

Post-order Traversal

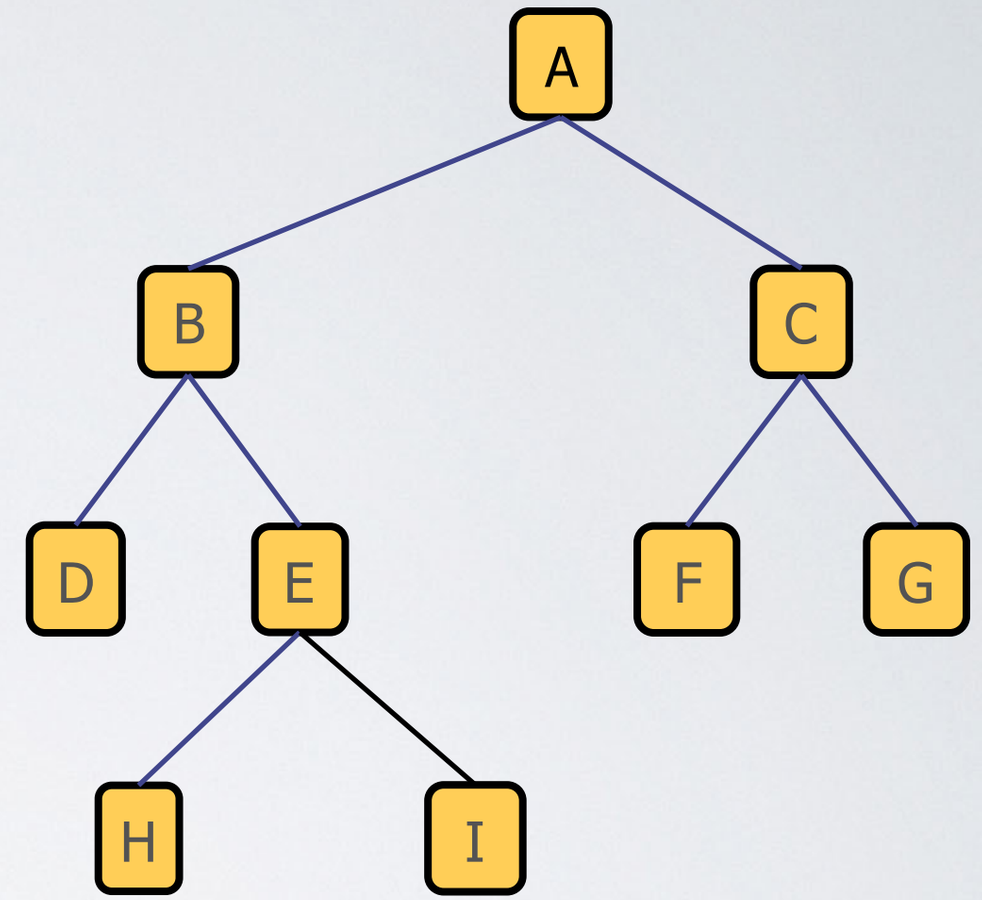
```
function postorder(node):
  if node has left child
    postorder(node.left)
  if node has right child
    postorder(node.right)
  visit(node)
```



D H I E B F G C A

In-order Traversal

```
function inorder(node):
  if node has left child
    inorder(node.left)
  visit(node)
  if node has right child
    inorder(node.right)
```



D B H E I A F C G

When to Use What Traversal?

- How do you know which traversal to use?
- Sometimes it doesn't matter
- Often one traversal makes solving problem easier

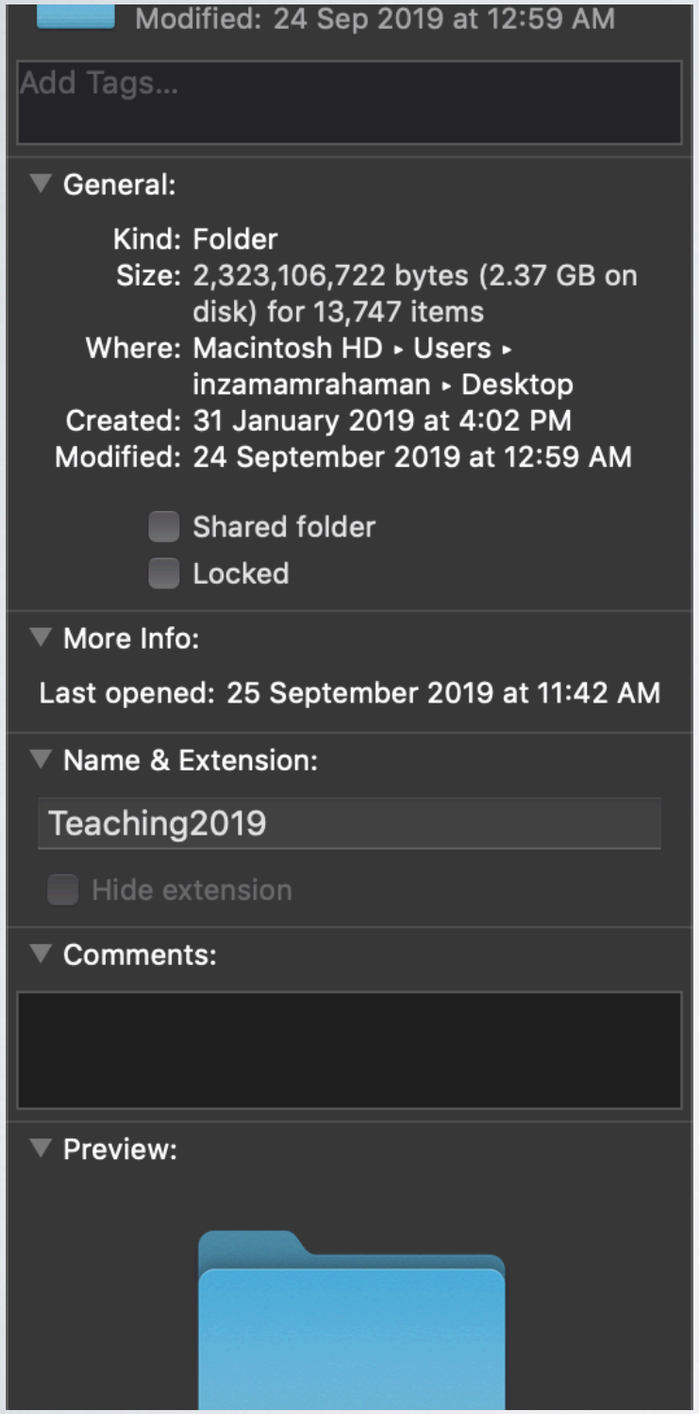
Tree Traversal Problem

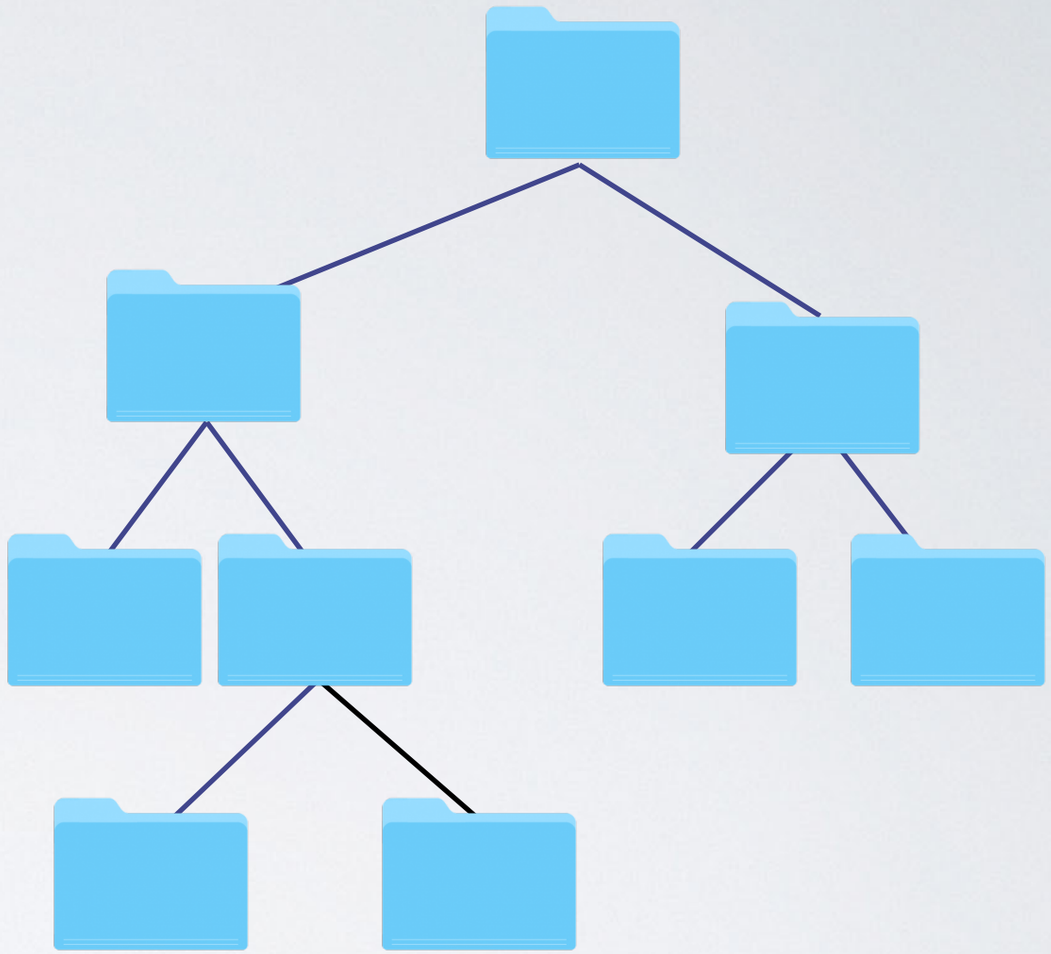
- Decorating with number of descendants?
- **Post-order**
  - visits both children before node
  - easy to calculate # of descendants if you know # of descendants of both children
  - try writing pseudo-code for this based on code on the Github repo

Tree Traversal Problem

- Testing if tree is perfect
- **Breadth-first**
  - traverses tree level by level
  - keep track of how many nodes at level
  - each level should have twice as many as previous level
  - should try writing pseudocode for this

Tree Traversal Problem



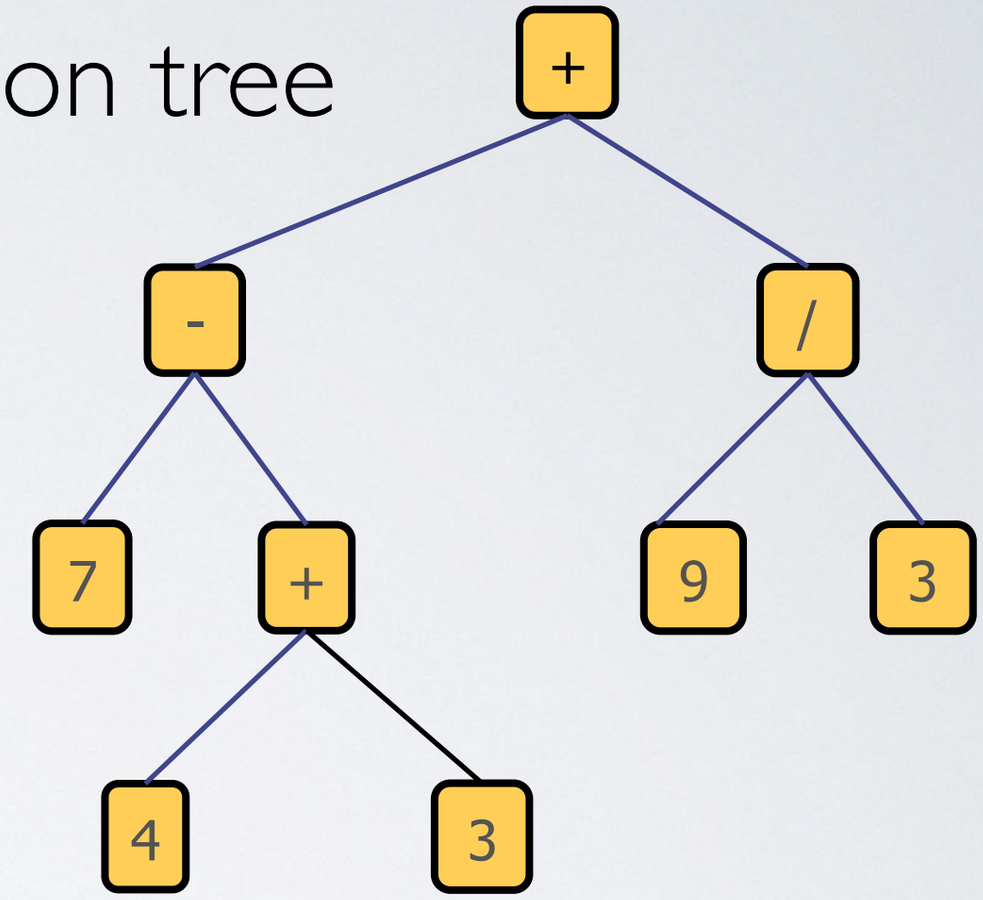


- Best traversal?
  - **post-order:** need to know size of subfolders before you can compute size of a folder

Tree Traversals Problems

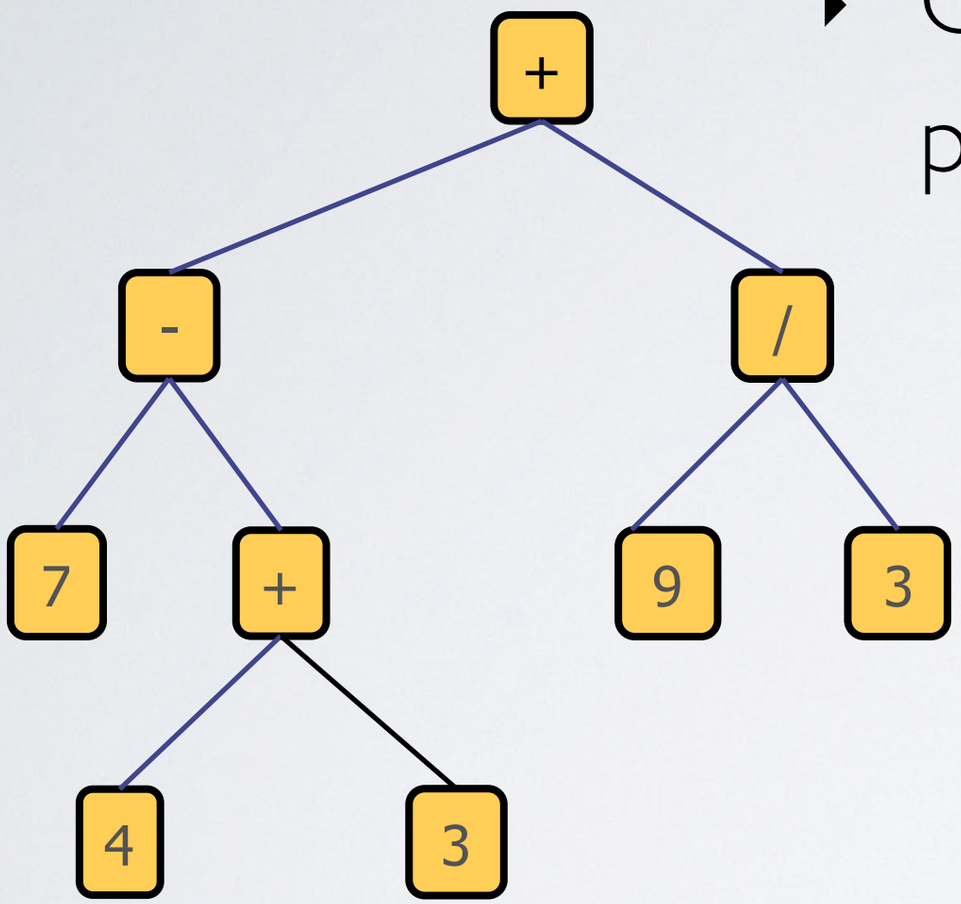
- *Evaluate* arithmetic expression tree

$$(7 - (4 + 3)) + (9 / 3) \quad =$$



- Best traversal?
  - **post-order:** to evaluate operation, you first need to evaluate sub-expression on each side
  - What should you do when you get to a leaf?

Tree Traversals Problems



$= \quad 7 - 4 + 3 + 9 / 3$

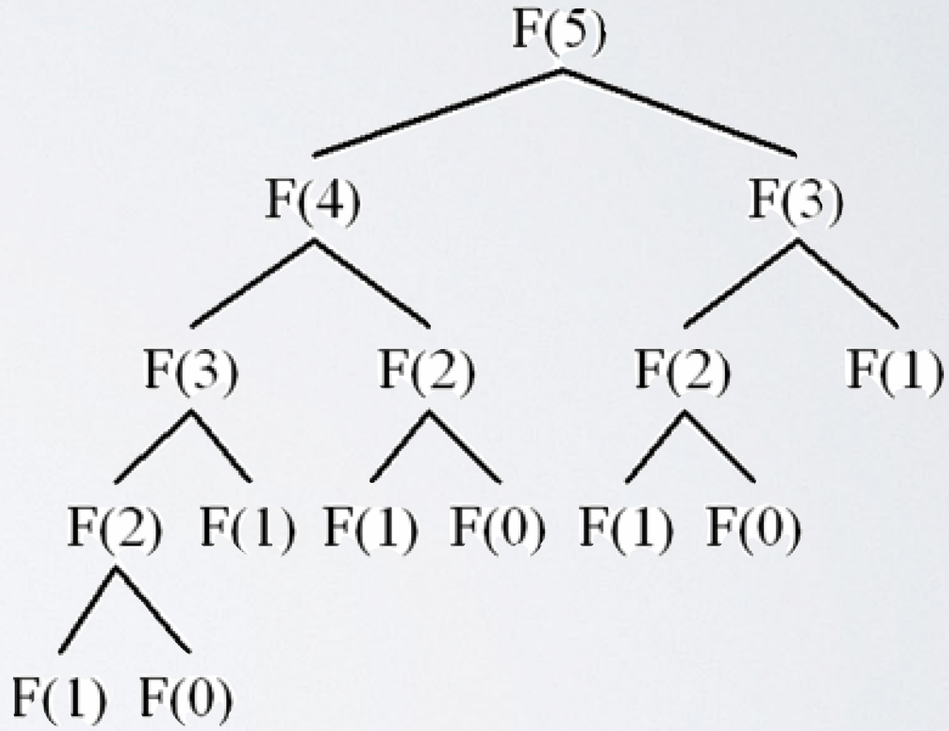
- Given tree, *print out* expression w/o parentheses

- Best traversal?
  - **in-order:** gives nodes from left to right

Analyzing Binary Trees

- Many things can be modeled as binary trees
  - ex: Fibonacci recursive tree

$$F(n) = F(n-1) + F(n-2)$$



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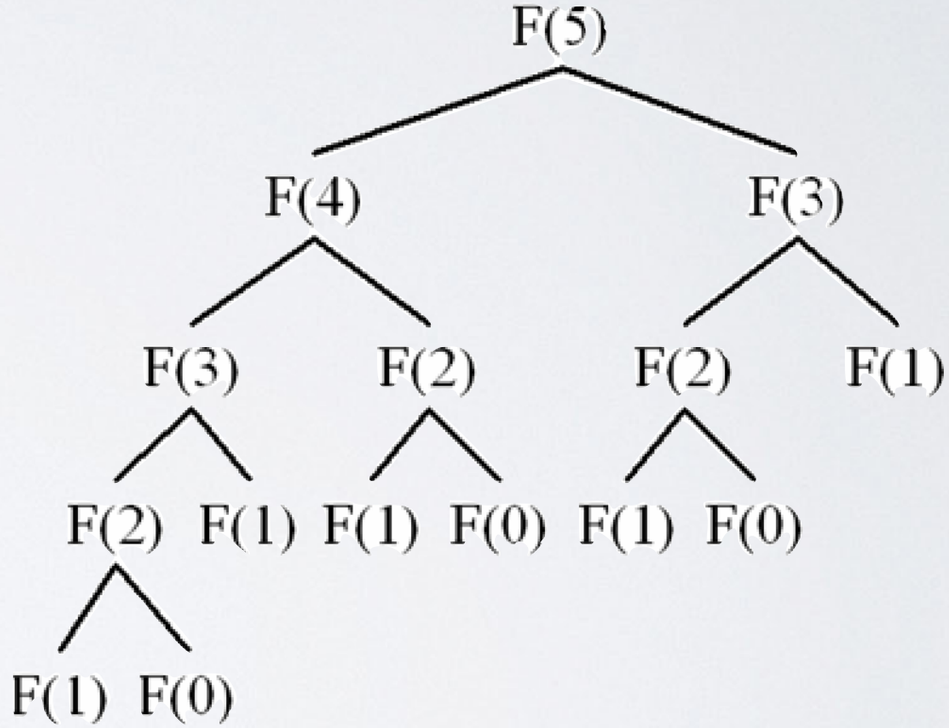

Analyzing *Perfect* Binary Trees

- Number of nodes in perfect binary tree of height **`h`**:
  - $2^{h+1} - 1$
- Height of a perfect binary tree with **`n`** nodes:
  - $\log_2(n+1)-1$
- Number of leaves in perfect binary tree of height **`h`**:
  - 2^{h}
- Number of nodes in perfect binary tree with **`L`** leaves:
  - `2L-1`

Induction on Perfect Binary Trees

- Can use (structural) induction to prove things about PBTs
- Using recursive definition of perfect binary trees
- Tree T is a perfect binary tree if
  - it has only one node
  - has root with left and right subtrees which are both perfect binary trees of same height
  - (if subtrees have height **`h`**, then T has height **`h+1`**)

Induction on non-Perfect Binary Trees

- Can use same techniques for balanced, though not perfect, binary trees
- However, instead of exact values, we often get bounds.

- Property **`x`** of binary tree with **`n`** nodes is no less than or equal to …

Example Inductive Proof on PBTs

- Prove **`P(n)`**:
  - number of nodes in a perfect binary tree of height **`n`** is **$f(n)=2^{n+1}-1$**
- Base case **`P(0)`**:
  - number of nodes in perfect binary tree of height **`0`** is **`1`** (by definition)
  - **$f(0) = 2^{0+1}-1 = 2-1 = 1$**
- Inductive hypothesis:
  - assume **`P(k)`** is true (for some **$k \geq 0$**)
  - in words: the number of nodes in perfect binary tree of height **`k`** is **$f(k)=2^{k+1}-1$**

Example Inductive Proof on PBTs

- Then prove that **`P(k+1)`** is true:
  - Let **`T`** be any perfect binary tree of height **`k+1`**
  - By definition, **`T`** consists of root with two subtrees, **`L`** and **`R`**, which are both perfect binary trees of height **`k`**
  - By inductive hypothesis, **`L`** and **`R`** both have $2^{k+1}-1$ nodes
    - So total number of nodes in **`T`** is:
      - $2*(2^{k+1}-1)+1 = 2^{k+2}-2+1 = 2^{(k+1)+1}-1$
- Since we've proved
  - **`P(0)`** is true
  - **`P(k)`** implies **`P(k+1)`** (for any **`k≥0`**)
  - It follows by induction that **`P(n)`** is true for all **`n≥0`**

Example Proof on Perfect Binary Trees

- Sometimes, you don't need induction
- Consider that the height of a perfect binary tree with `n` nodes is $\log_2(n+1)-1$
- Can you think of a way to prove this without induction?
- Hint: Add up all the number of nodes on each level of the tree to get the total number of nodes

References

- Some slides pulled from Brown's CS16
- Sedgewick