Shortest Paths in Graphs

Taken from Brown's CS16

Outline

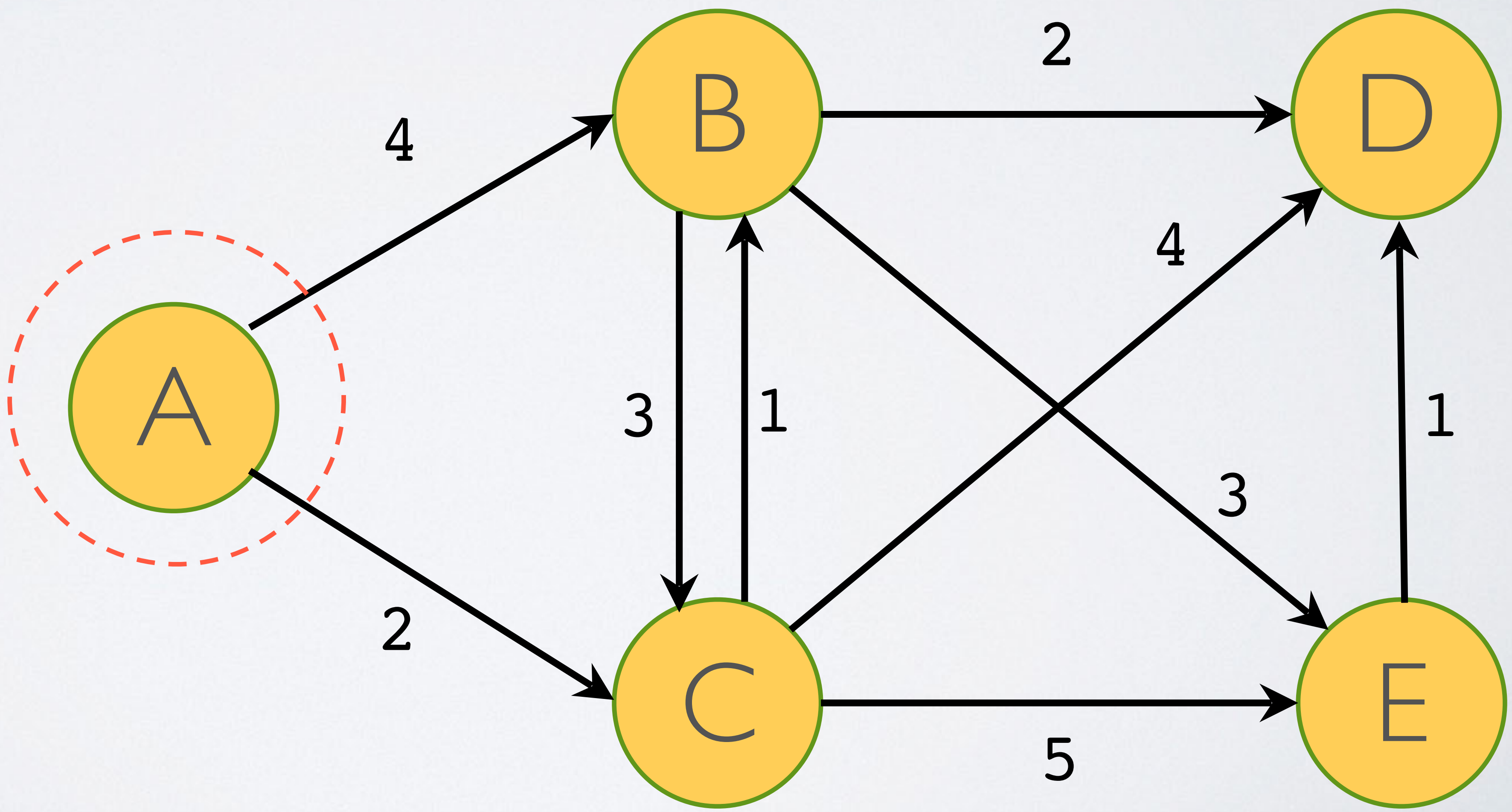
- Shortest Paths
- Breadth First Search
- Dijkstra's Algorithm

What is a Shortest Path?

- Given weighted graph `G` (weights on edges)…
- …what is shortest path from node `u` to `v`?
- Applications
  - Google maps
  - Routing packets on the Internet
  - Social networks

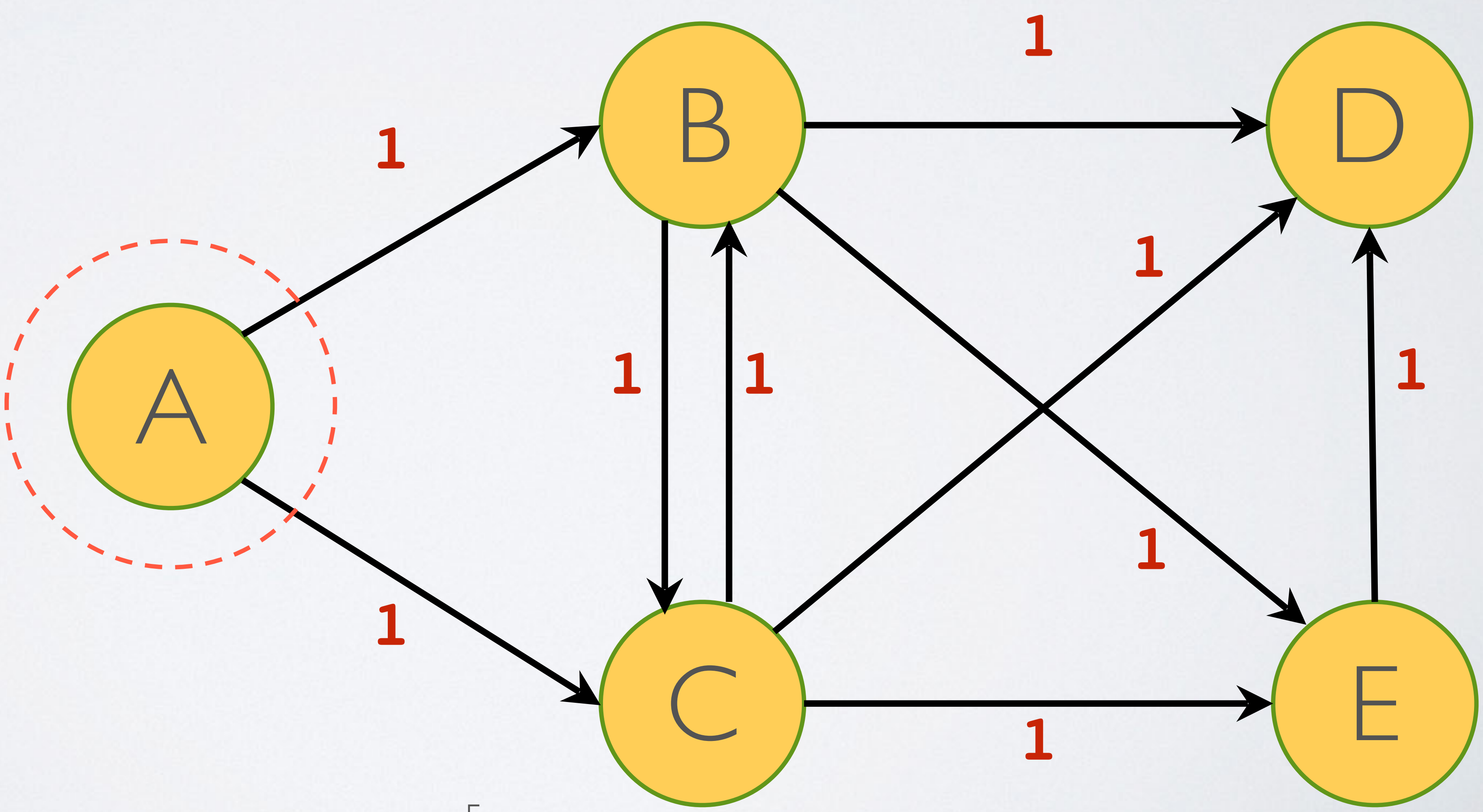
Single Source Shortest Paths (SSSP)

- Given a graph and a source node
  - find the shortest paths to all other nodes



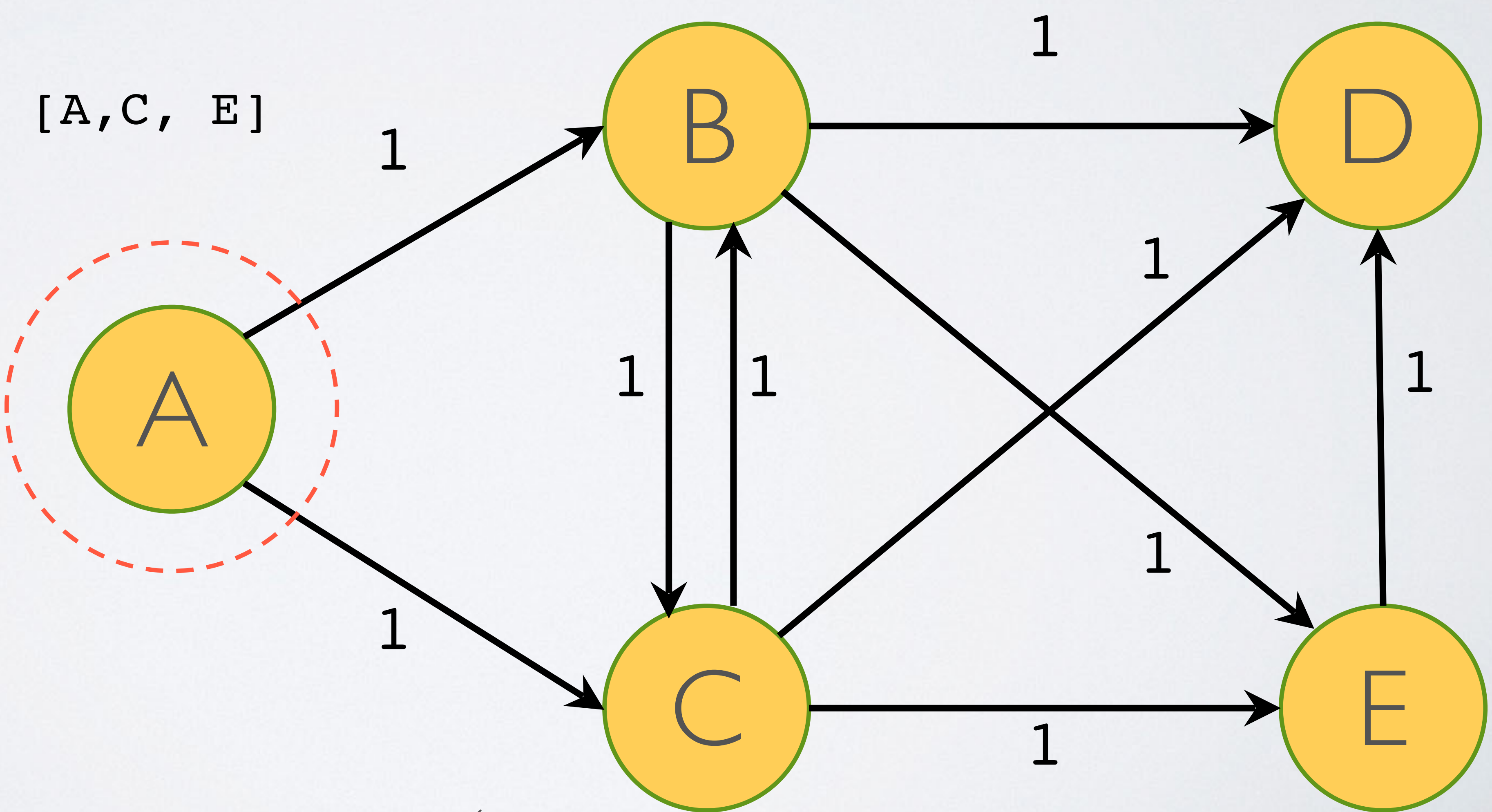
Simpler Problem: Unit Edges

- Let's start with simpler problem
- On graph where every edge has unit cost



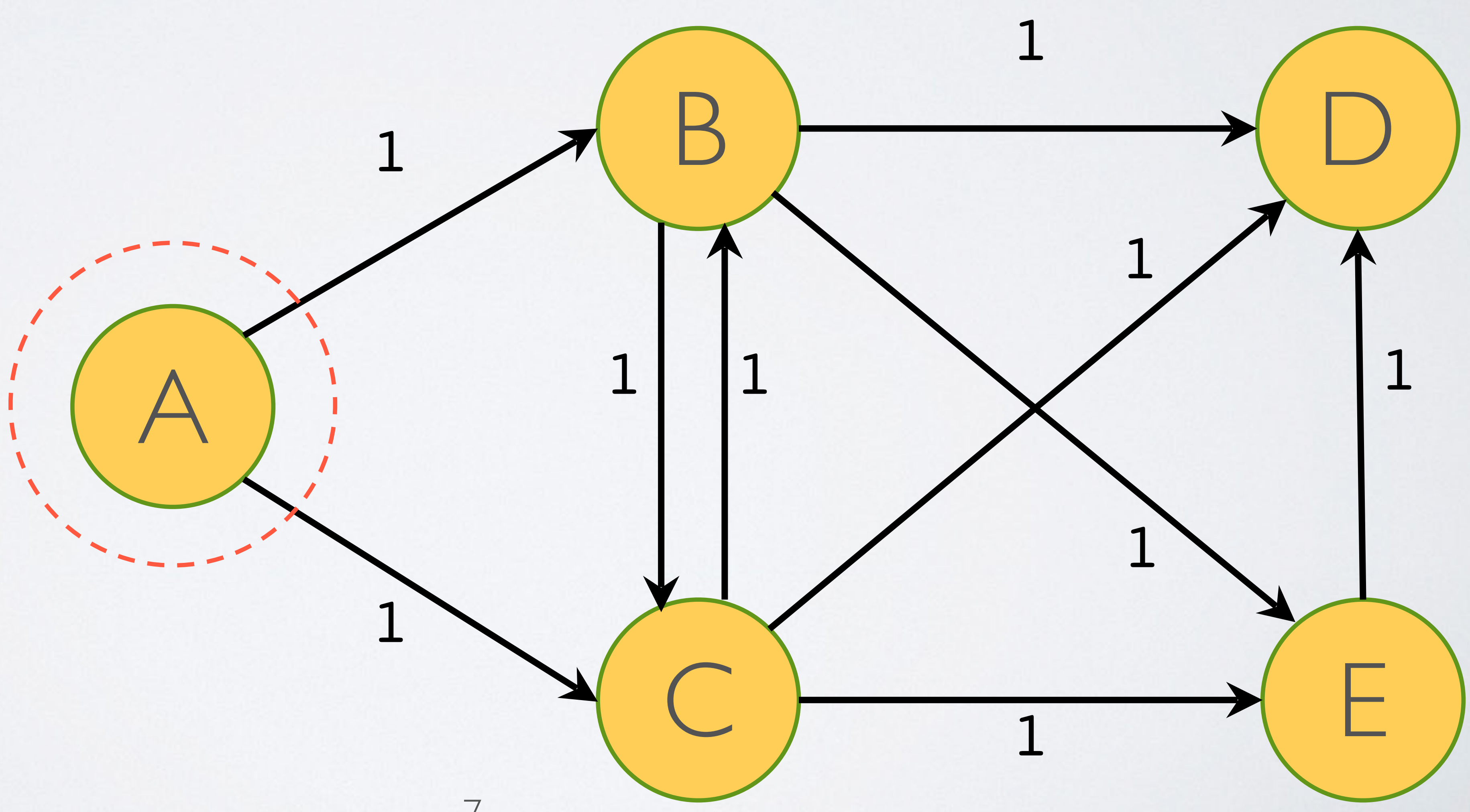
Simpler Problem: Unit Edges

- What is shortest path from **A** to each node?
  - `B:[A,B]`
  - `D:[A,B,D] or [A,C,D]`
  - `C:[A,C]`
  - `E:[A,B,E] or [A,C, E]`



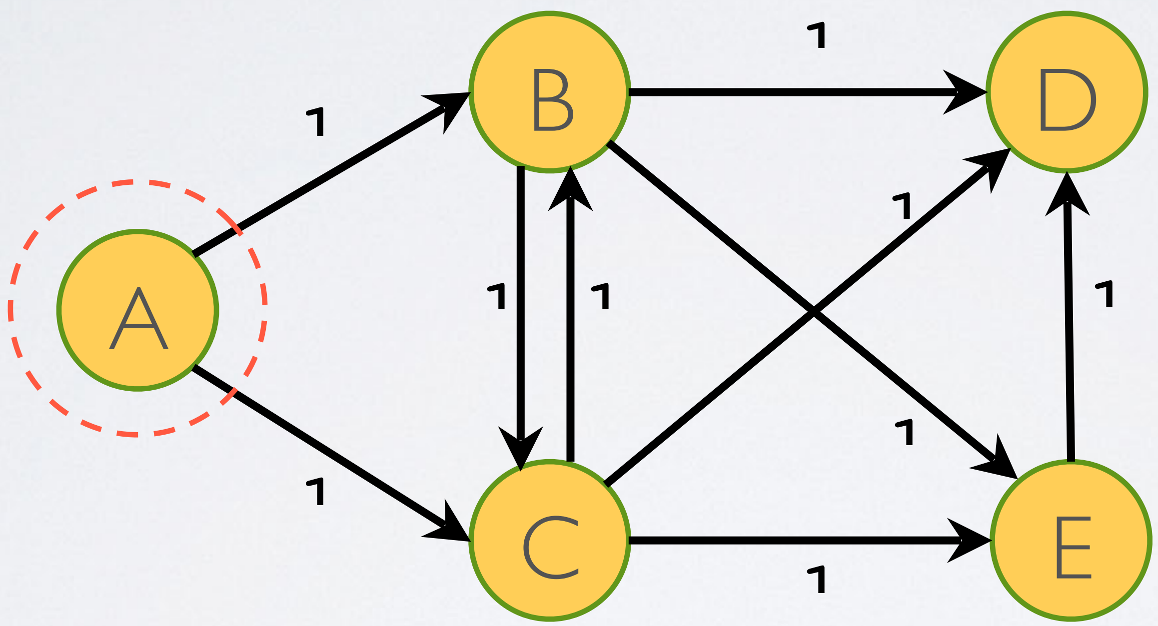
Simpler Problem: Unit Edges

- Is there an algorithm we've already seen that solves problem?
  - Hint: yes!
- What graph traversals have we learned?



Breadth-First Search

- Use BFS to find shortest path from **`A`** to **`E`**.
- Consider all steps of adding/removing nodes from queue …
- …and updating each node's 'previous' pointer.

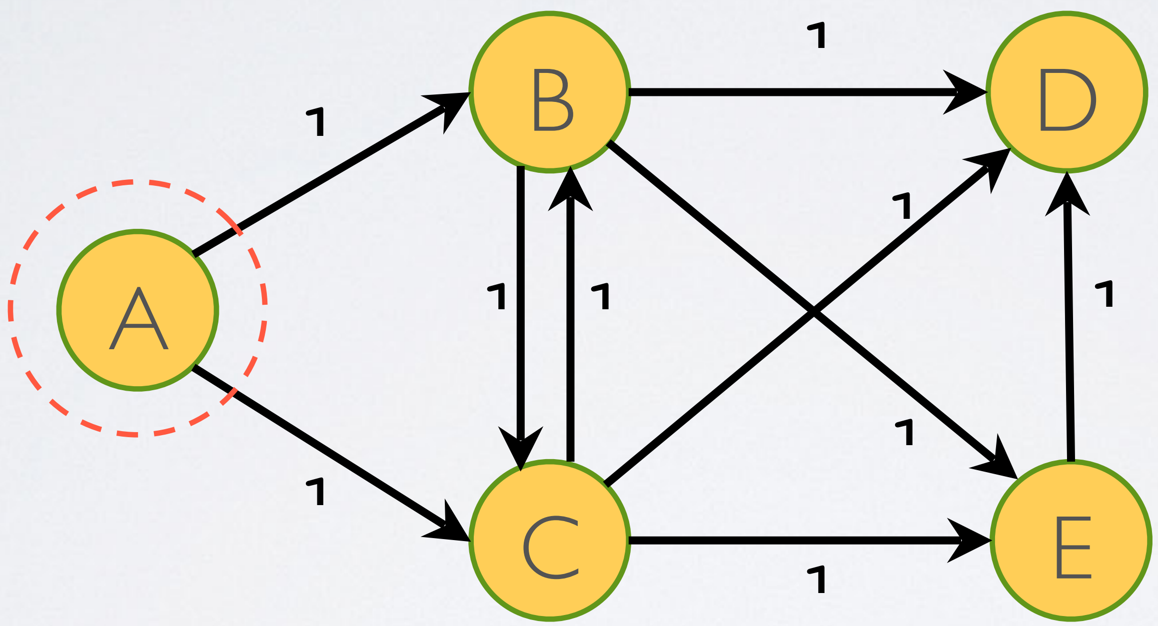


Activity #1

1 min

Breadth-First Search

- Use BFS to find shortest path from `A` to `E`.
- Consider all steps of adding/removing nodes from queue …
- …and updating each node's 'previous' pointer.

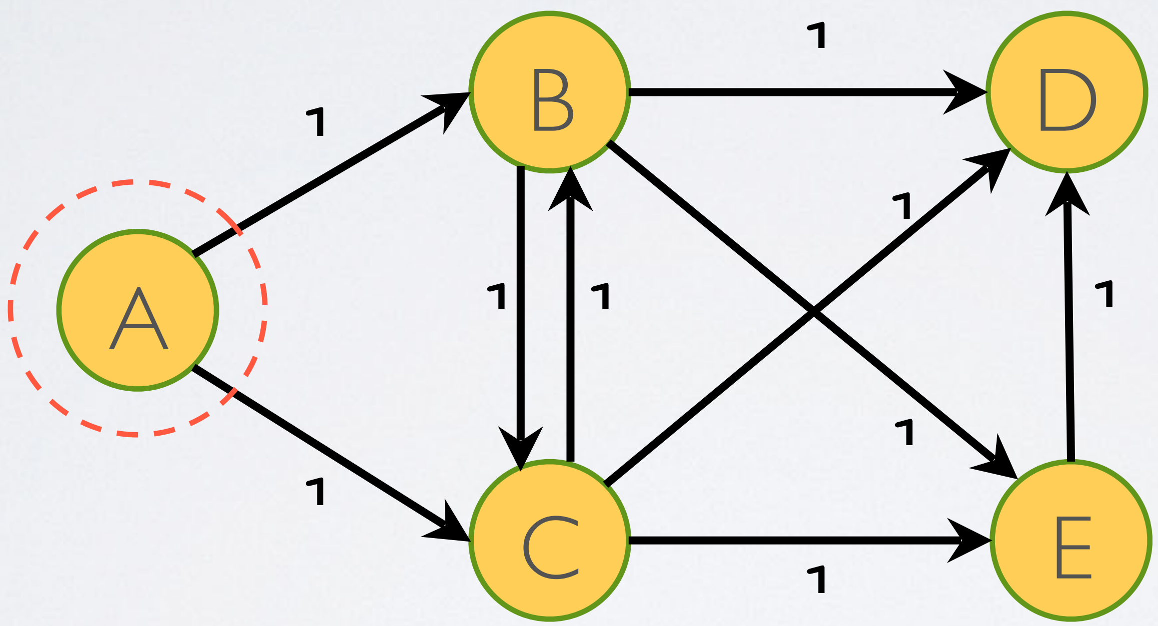


Activity #1

1 min

Breadth-First Search

- Use BFS to find shortest path from `A` to `E`.
- Consider all steps of adding/removing nodes from queue …
- …and updating each node's 'previous' pointer.

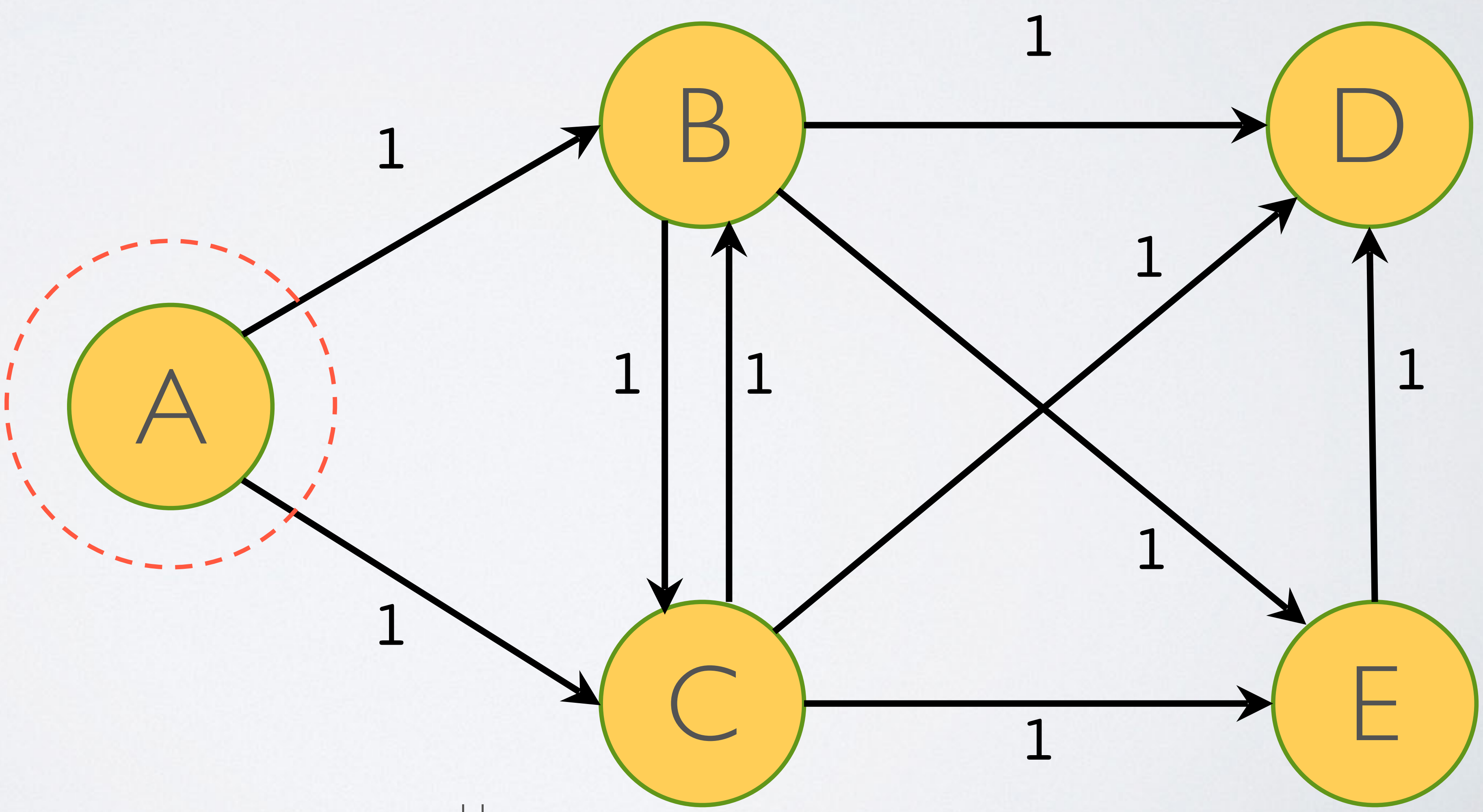


Activity #1

0 min

Breadth First Search

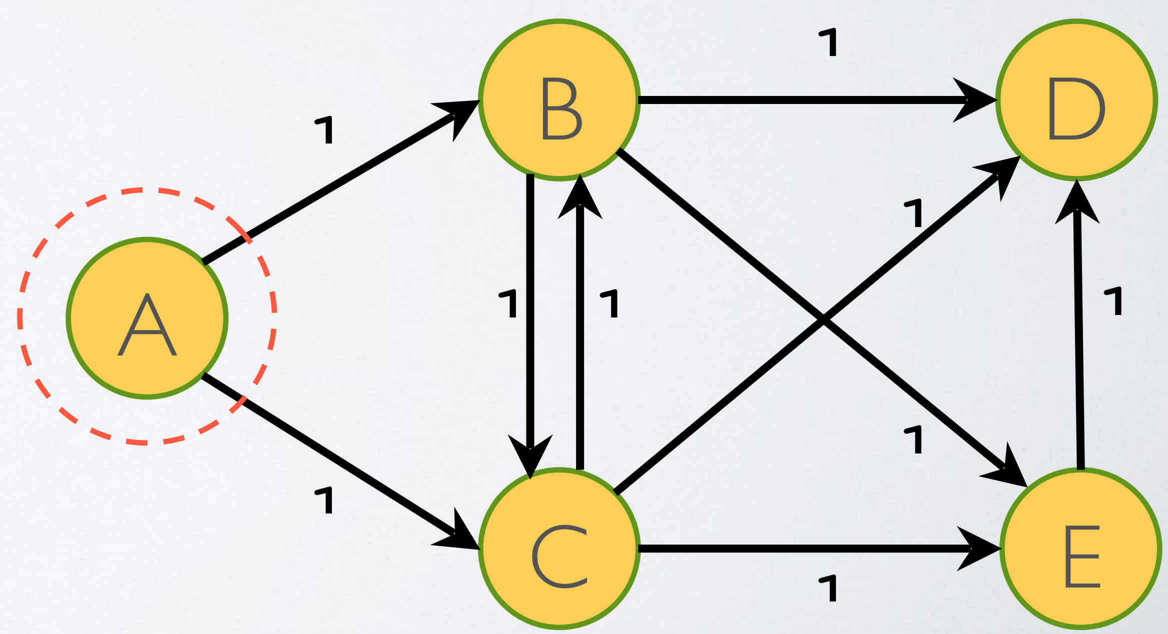
- BFS always reaches target node in fewest steps
- Let's look at path from **`A`** to **`E`**



Breadth First Search Simulation

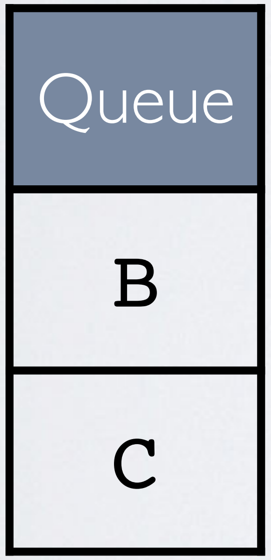
- Strategy
  - BFS uses queue to store nodes to visit
  - Enqueue start node
  - Decorate nodes w/ previous pointers to keep track of path

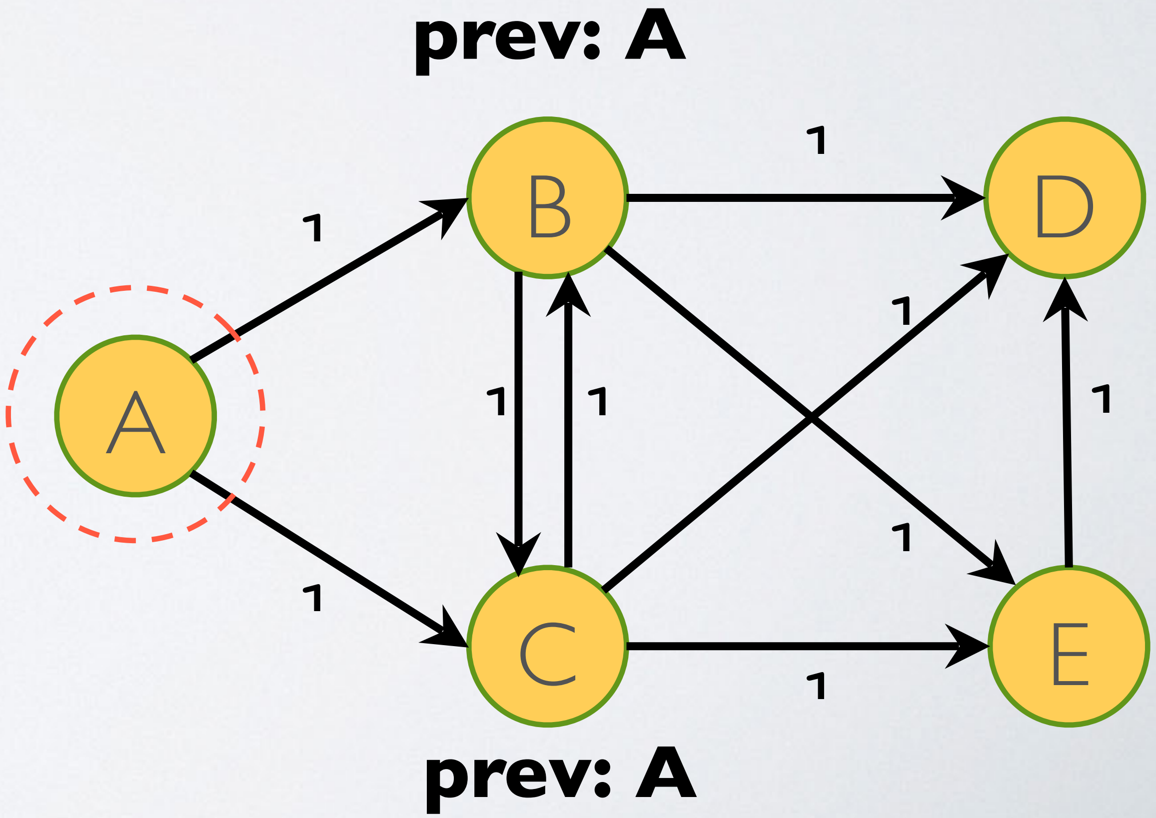
Queue
A



Breadth First Search Simulation

- Dequeue **`A`**
- Decorate its neighbors w/ "prev: **`A`**"
- Enqueue them

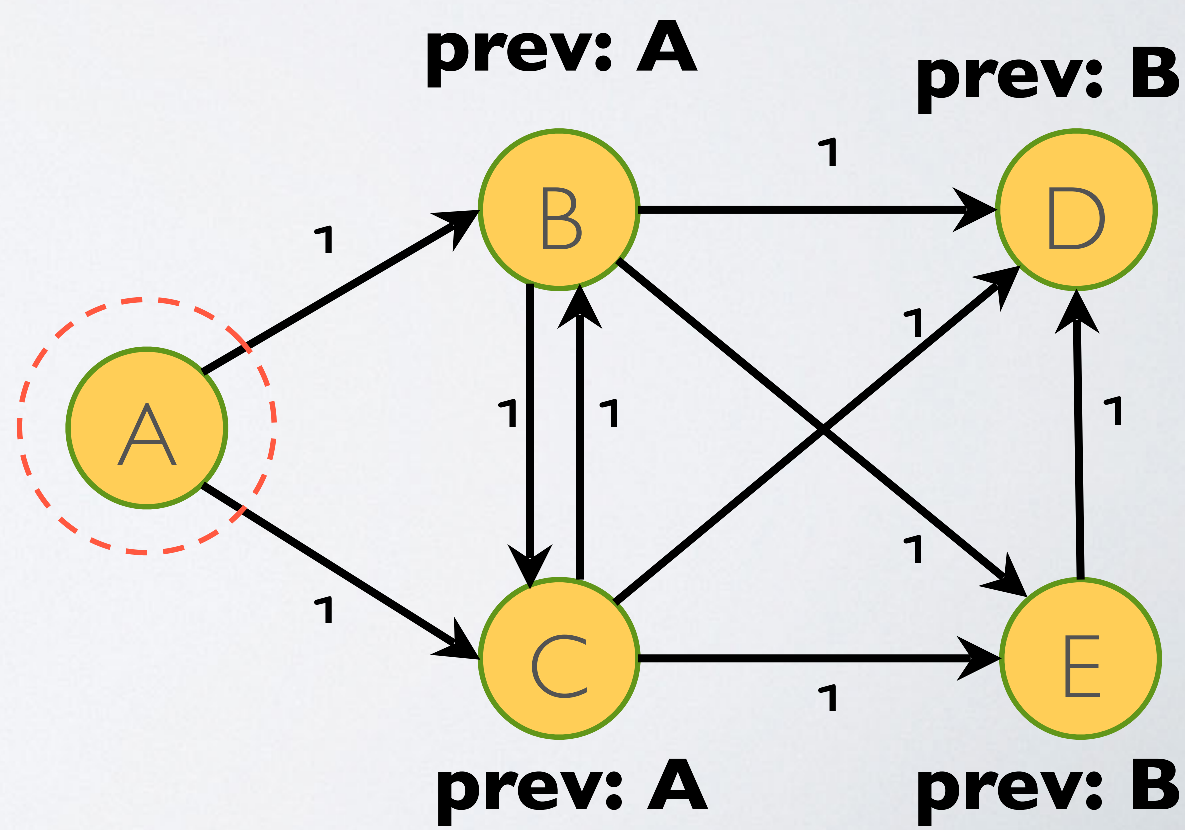




Breadth First Search Simulation

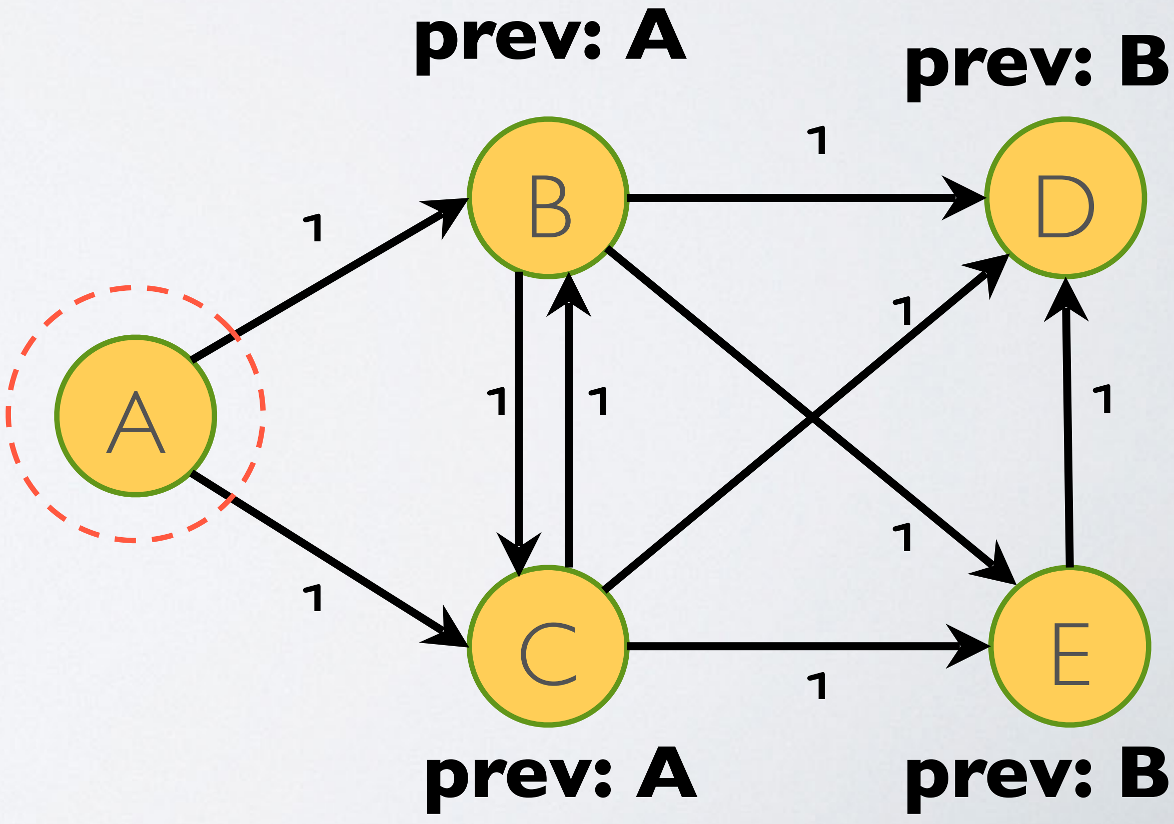
- Dequeue **B** and repeat…
- …but ignoring nodes that have been decorated

Queue
C
D
E



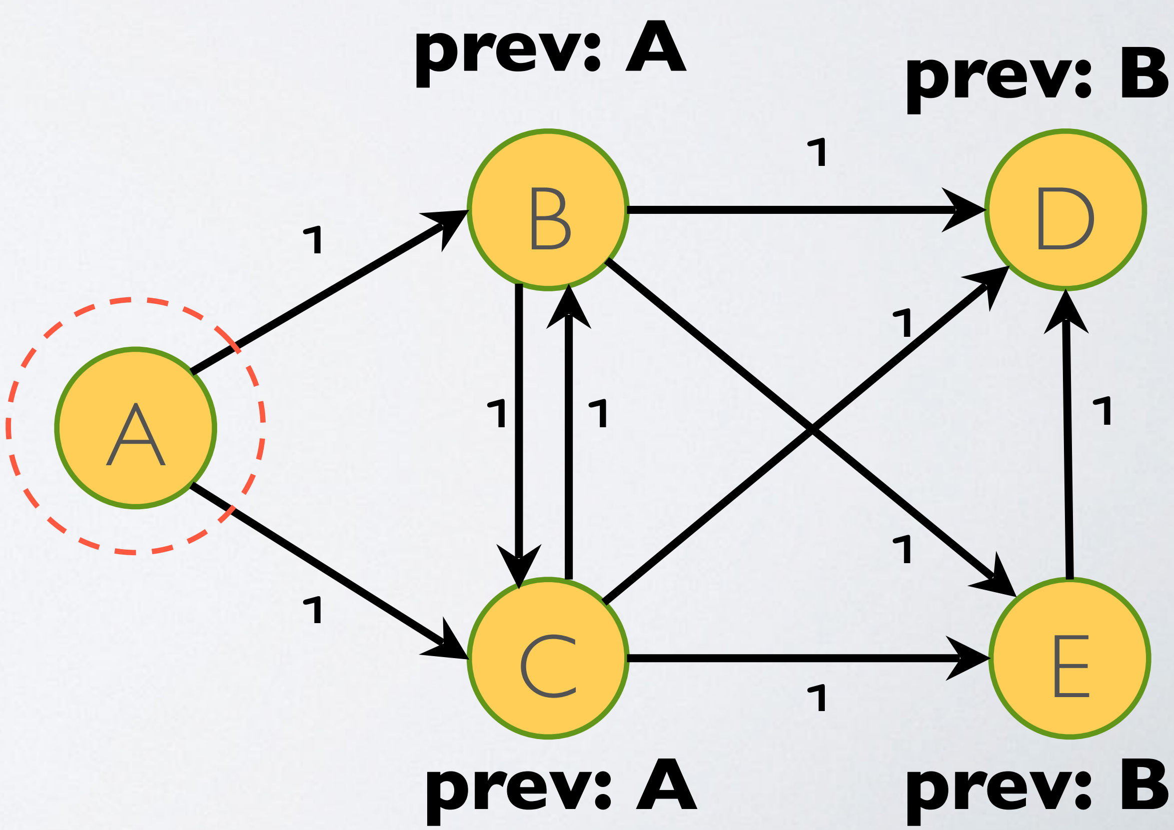
Breadth First Search Simulation

- Dequeuing **`C`** and **`D`** has no effect…
- …since their neighbors have been decorated



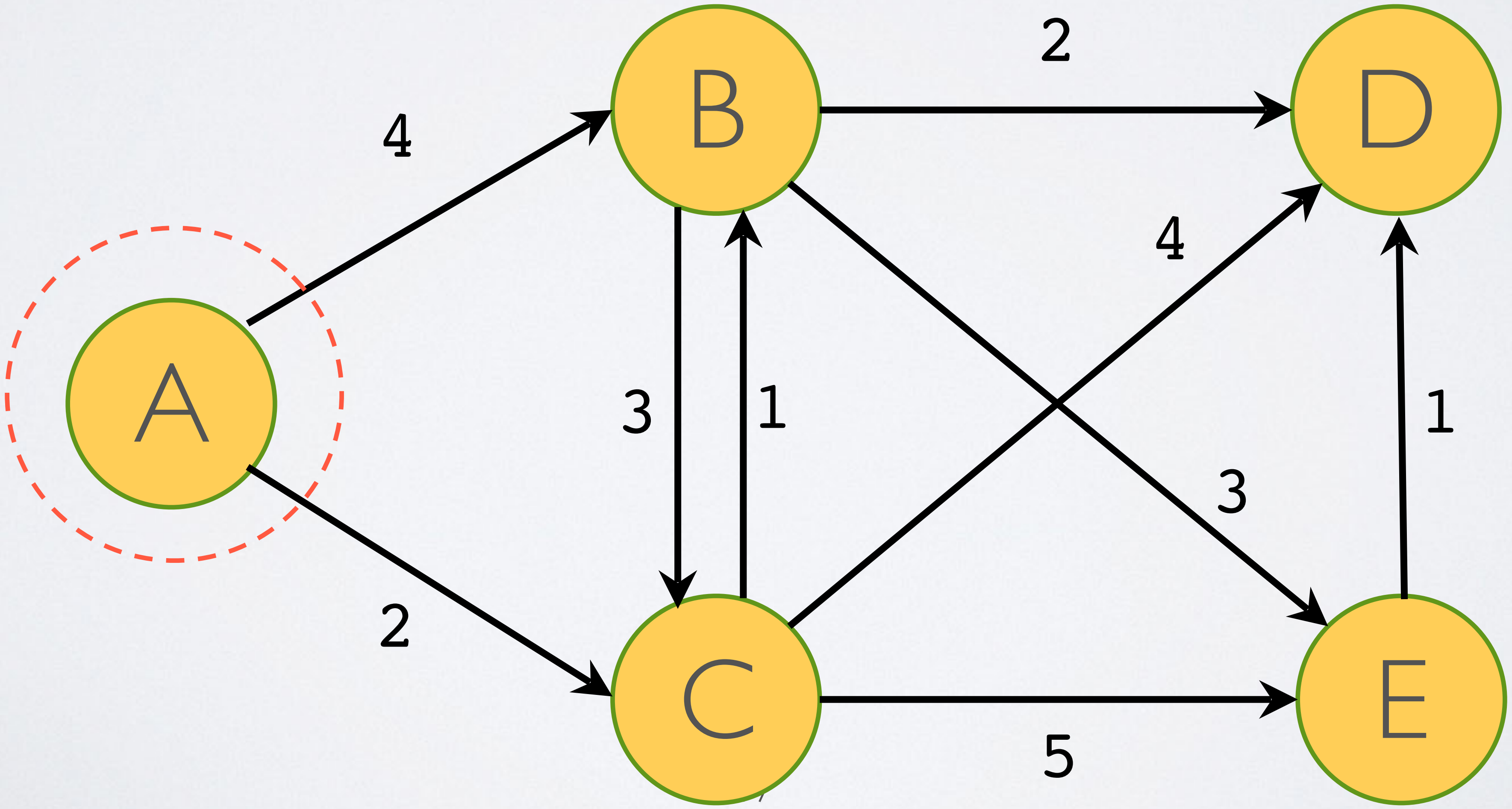
Breadth First Search Simulation

- When we dequeue **`E`**…
- …we traverse the prev pointers to return paths
  - shortest path to **`E`**: **`[A,B,E]`**



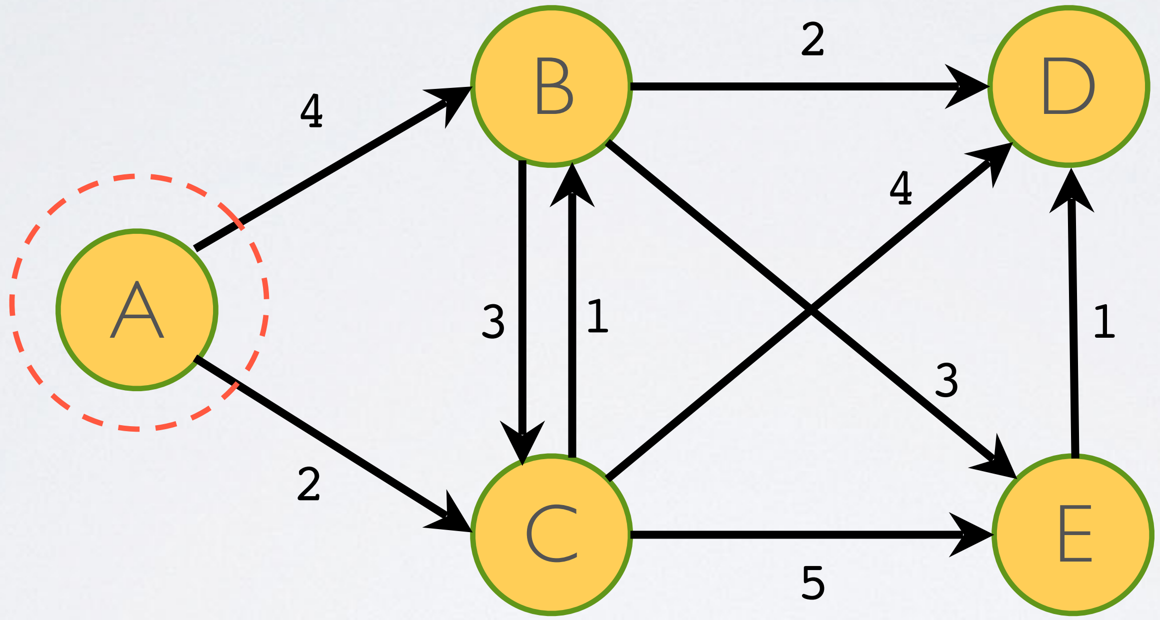
Non-Unit Edge Weights

- What if edge weights are not 1?
- More complicated



Shortest Path

- Fill in missing spaces using graph below
- Use **`A`** as source vertex

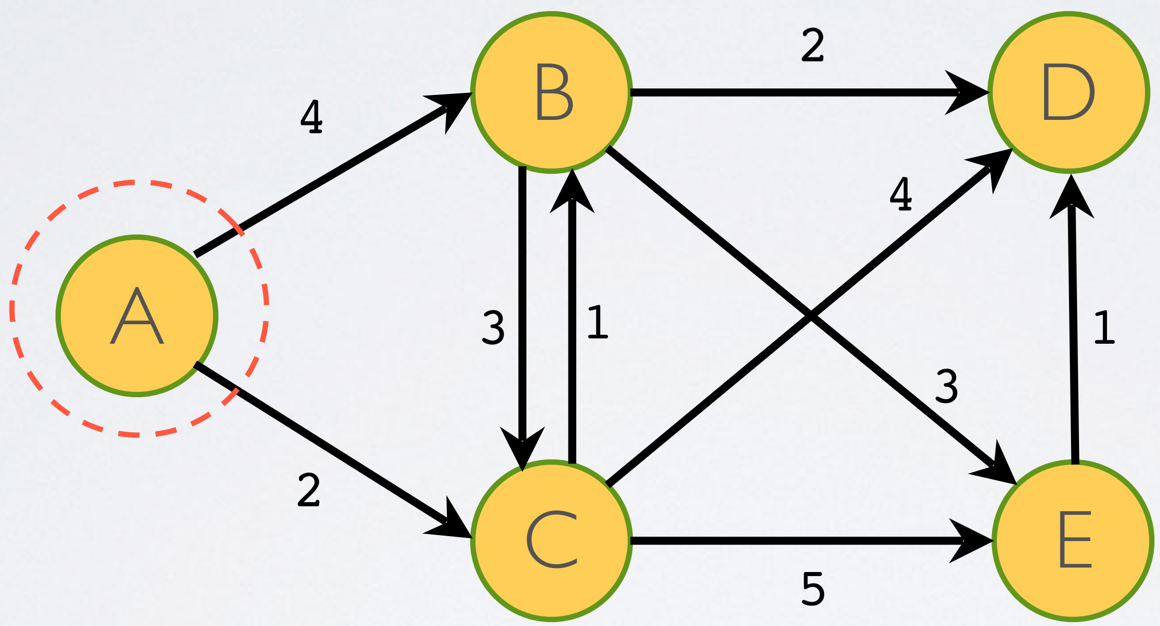


Activity #2

2 min

Shortest Path

- Fill in missing spaces using graph below
- Use `A` as source vertex

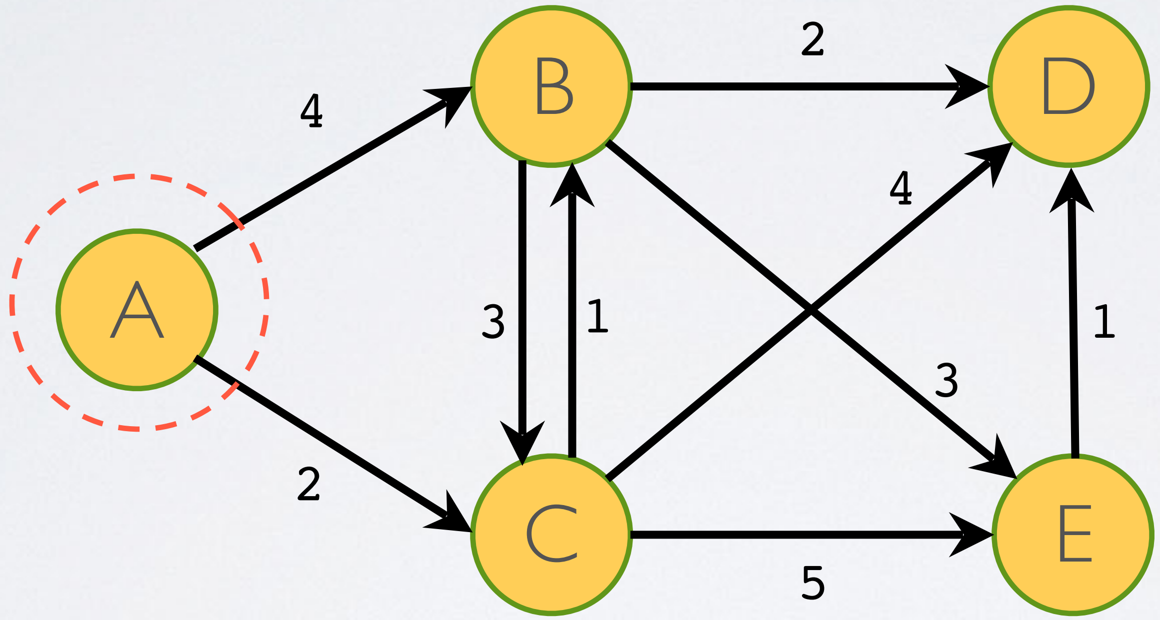


Activity #2

2 min

Shortest Path

- Fill in missing spaces using graph below
- Use **`A`** as source vertex

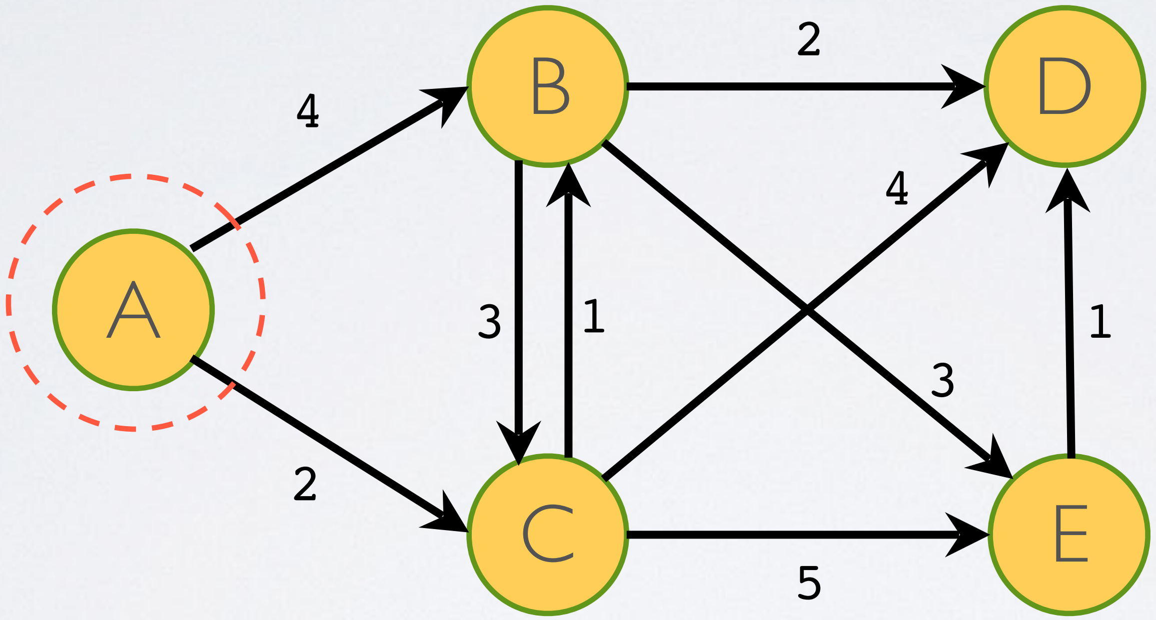


Activity #2

1 min

Shortest Path

- Fill in missing spaces using graph below
- Use **`A`** as source vertex

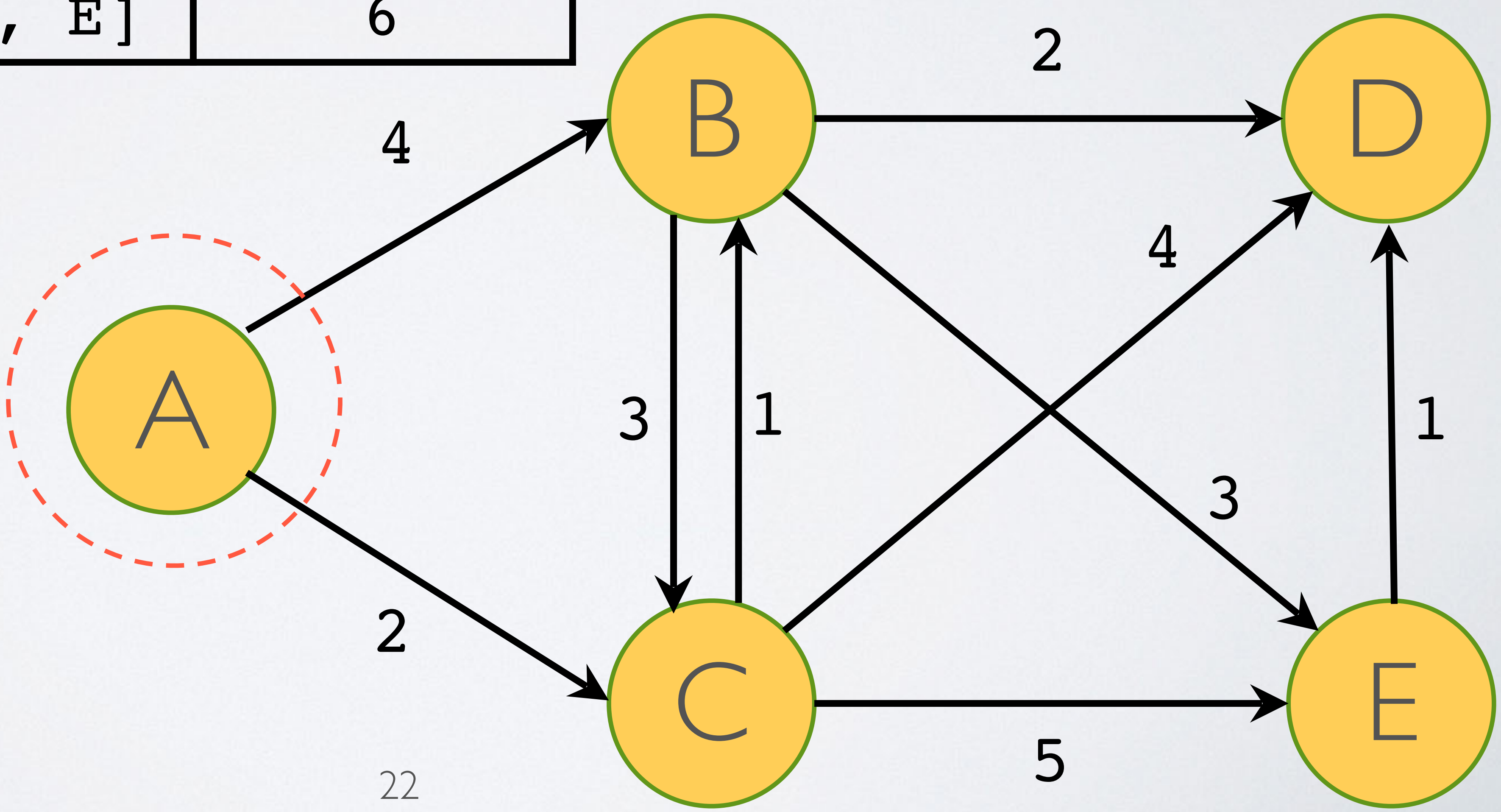


Activity #2

0 min

Non-unit Edge Weights

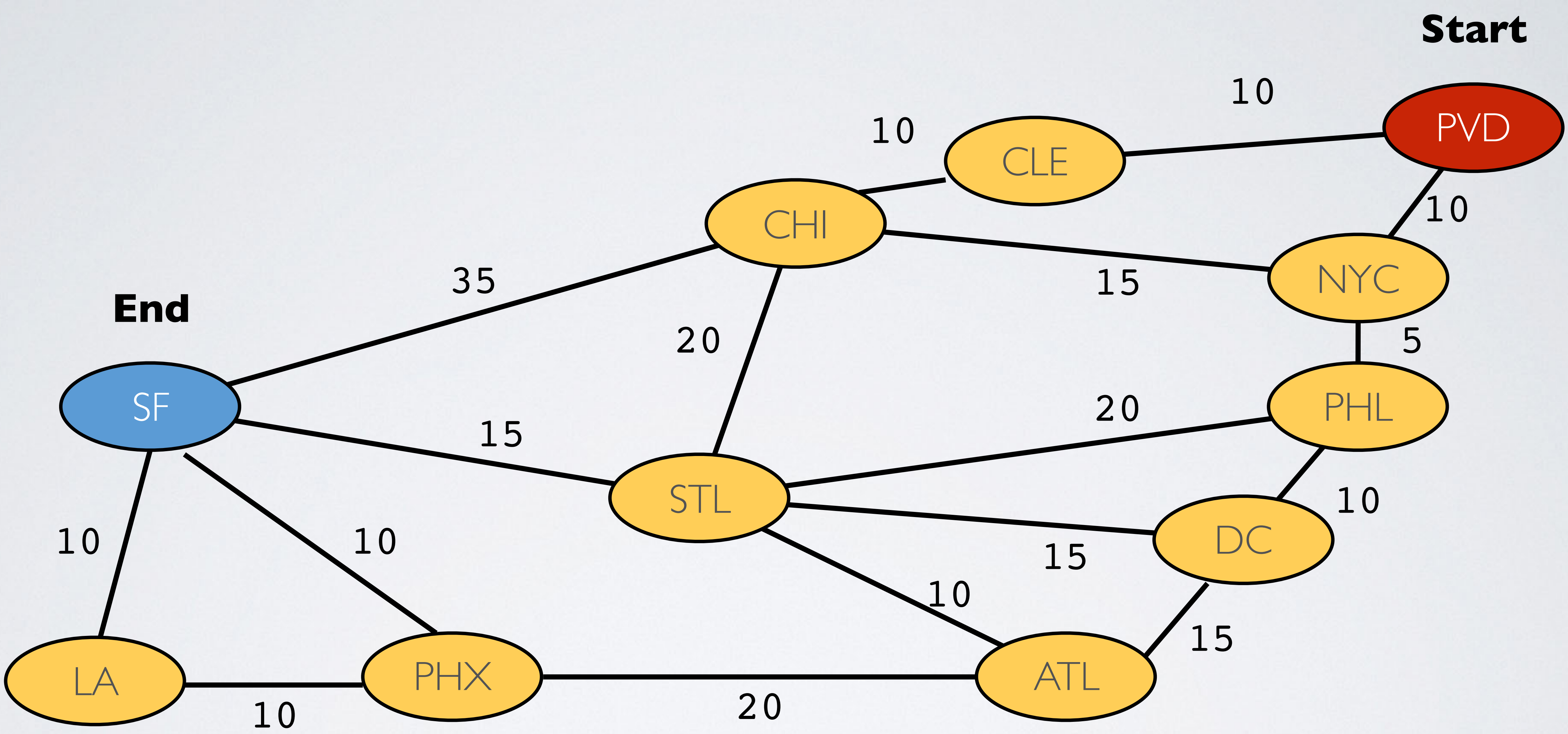
Goal Node	Shortest Path	Shortest Distance
B	[A, C, B]	3
C	[A, C]	2
D	[A, C, B, D]	5
E	[A, C, B, E]	6



Shortest Path Application

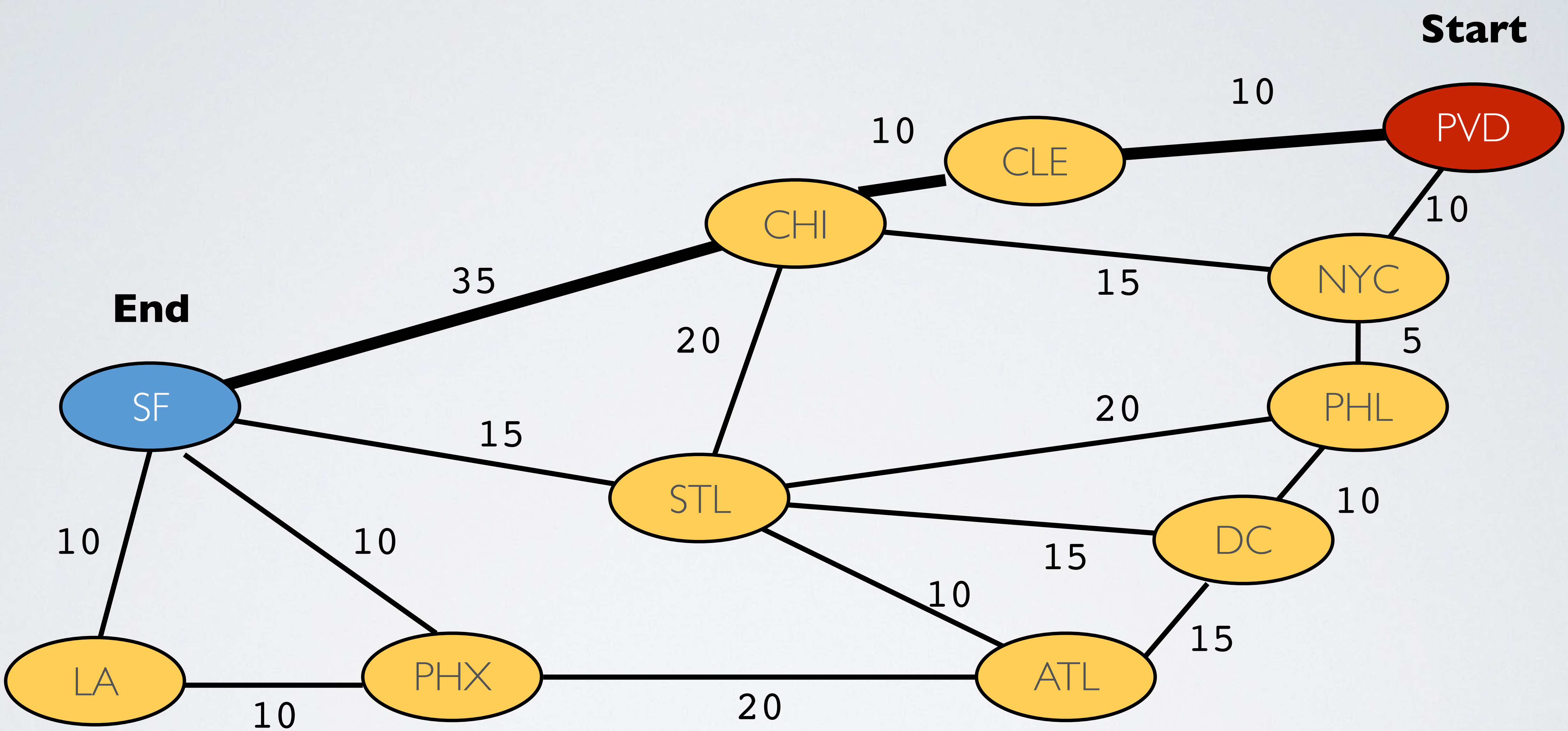
- Road trip
- Alina, Maggie, Prakrit & Stephanie want to get from PVD to SF…
- …following limited set of highways
- Cities are nodes and highways are edges
- Get to SF using shortest path

Our Graph



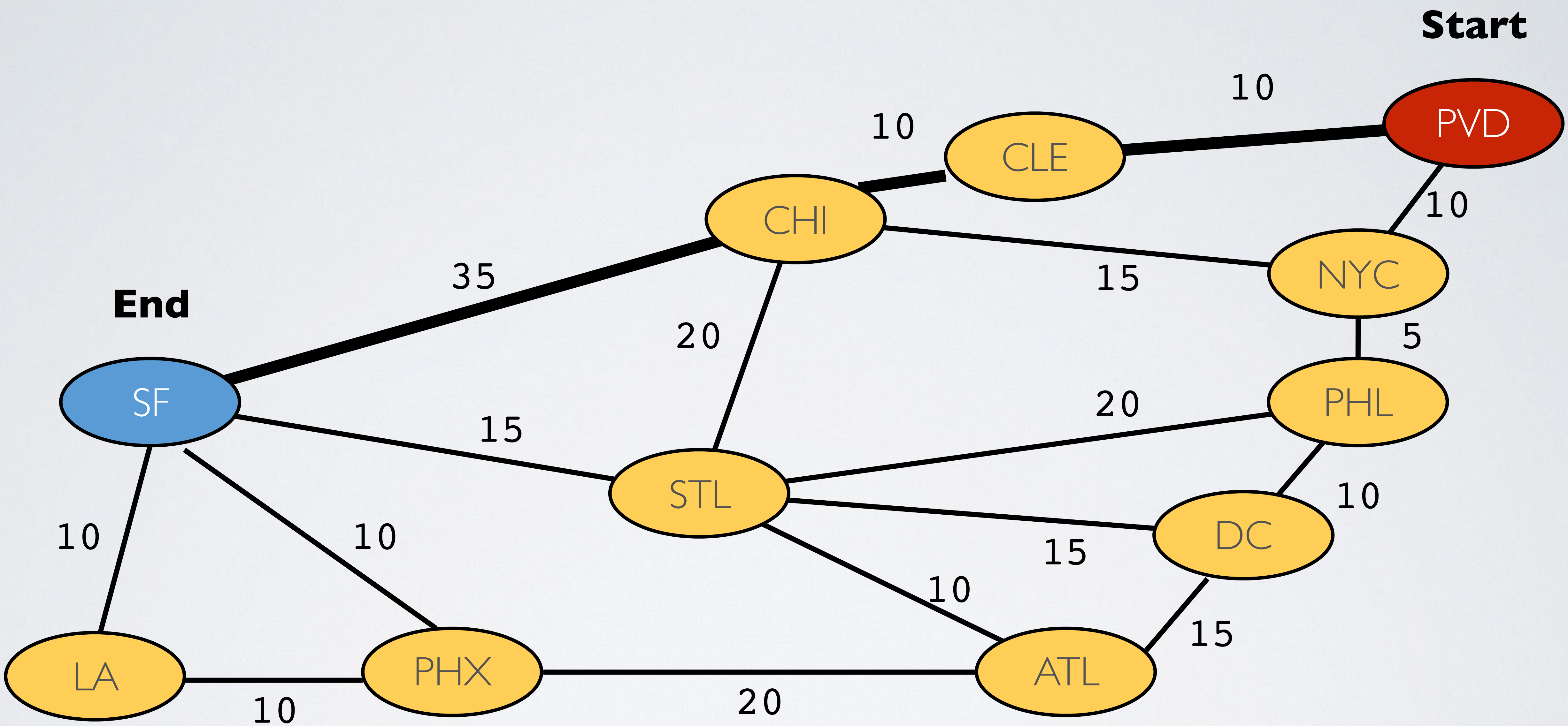
Start
End
PVD
CLE
CHI
NYC
PHL
SF
STL
DC
LA
PHX
ATL
10
10
10
35
15
20
5
20
15
10
10
15
10
15
10
20
10

Our Graph



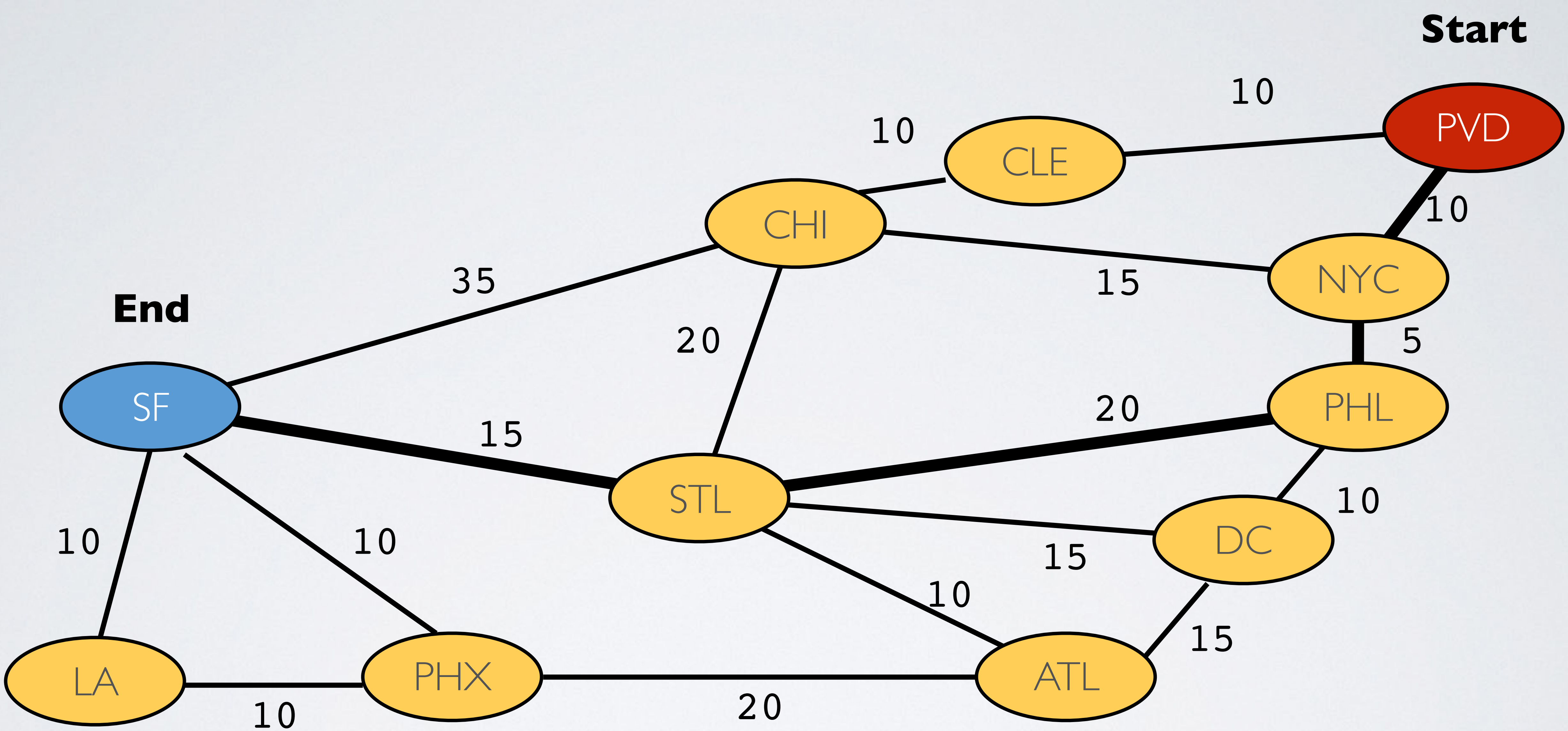
What is the cost of this path?

Our Graph



What is the cost of this path?
Is there a shorter path?

Our Graph



What is the cost of this path?
Is there a shorter path?

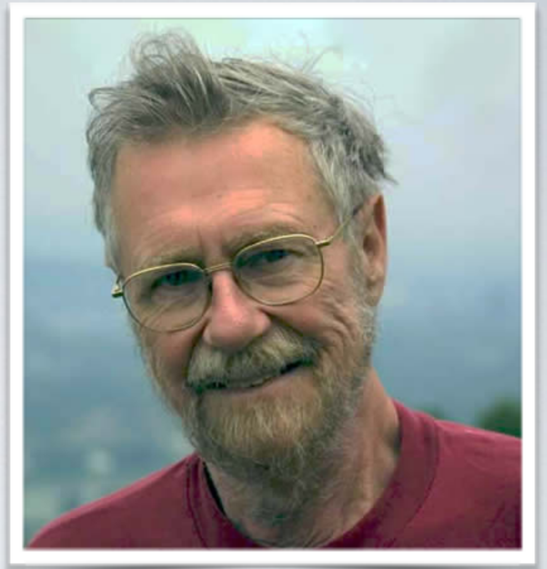
Shortest Path

- Why does BFS work with unit edges?
  - Nodes visited in order of total distance from source
- We need way to do the same even when edges have distinct weights!
- How can we do this?
  - Hint: we'll use a data structure we've already seen

Shortest Path

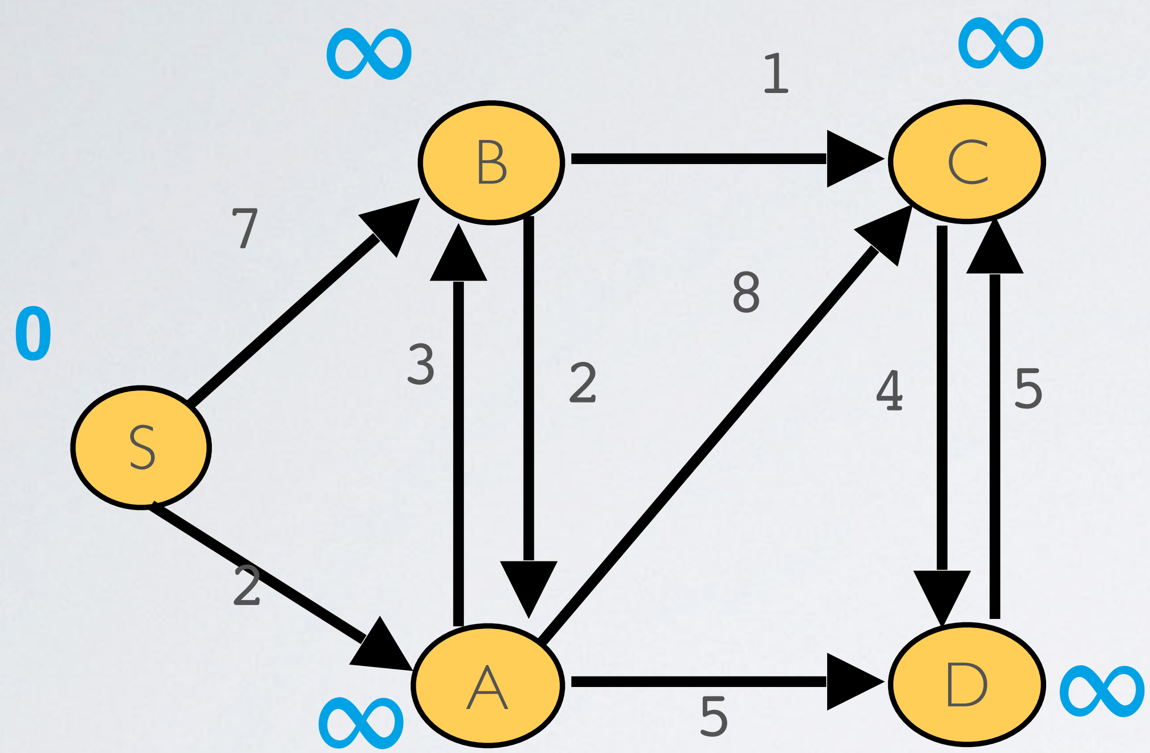
- Use a priority queue!
  - where priorities are total distances from source
  - By visiting nodes in order returned by `removeMin()`…
  - …you visit nodes in order of how far they are from source
- You guarantee shortest path to node because…
  - …you don't explore a node until all nodes closer to source have already been explored

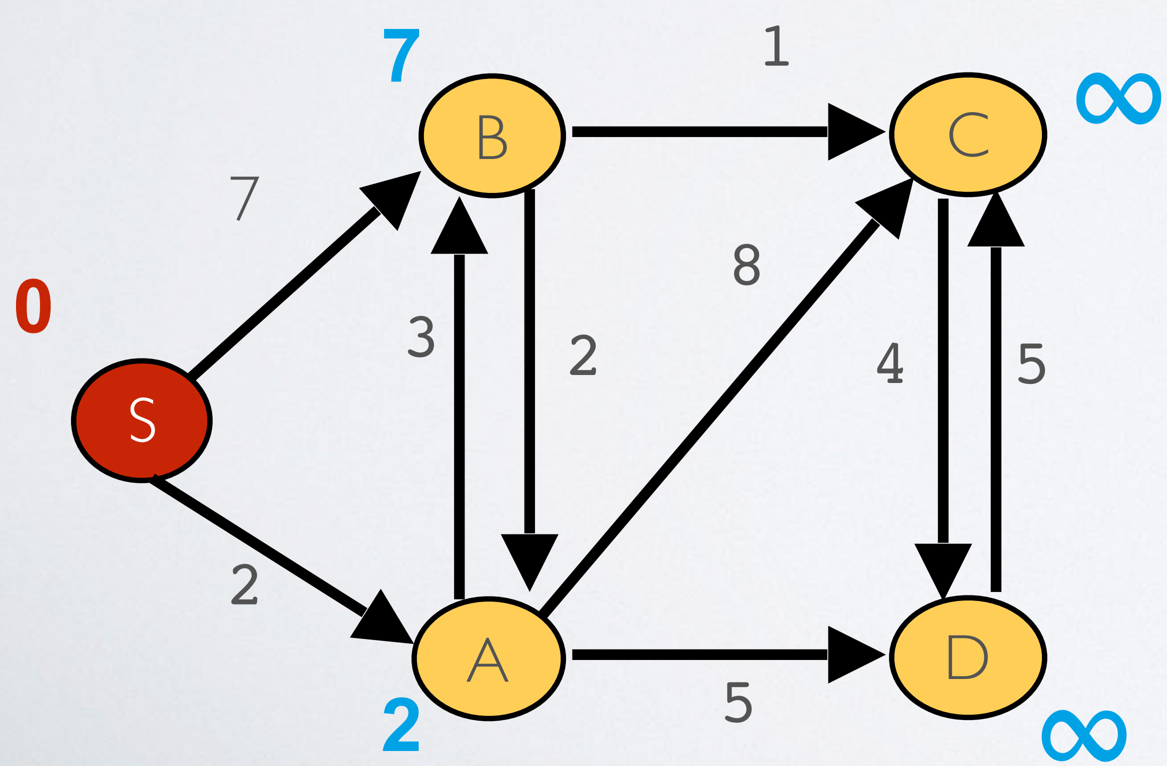
Dijkstra's Algorithm



- The algorithm is as follows:
  - Decorate source with distance 0 & all other nodes with ∞
  - Add all nodes to priority queue w/ distance as priority
  - While the priority queue isn't empty
    - Remove node from queue with minimal priority
    - Update distances of the removed node's neighbors if distances decreased
- When algorithm terminates, every node is decorated with minimal cost from source

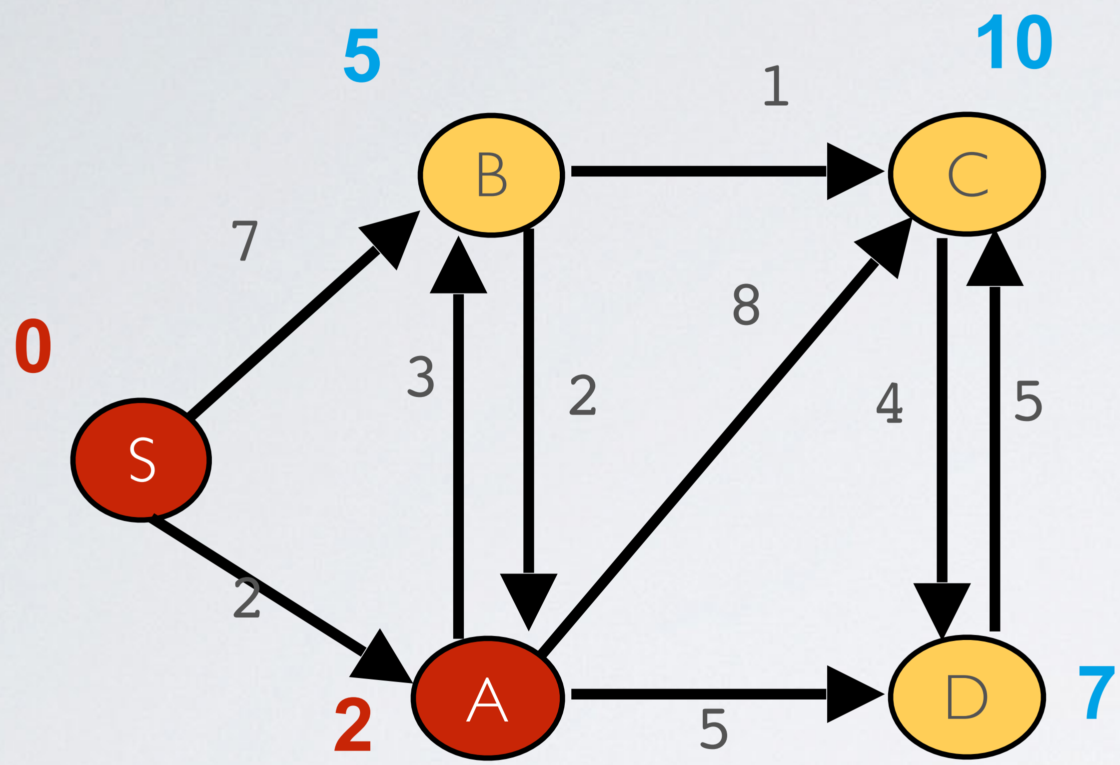
Dijkstra's Algorithm Example

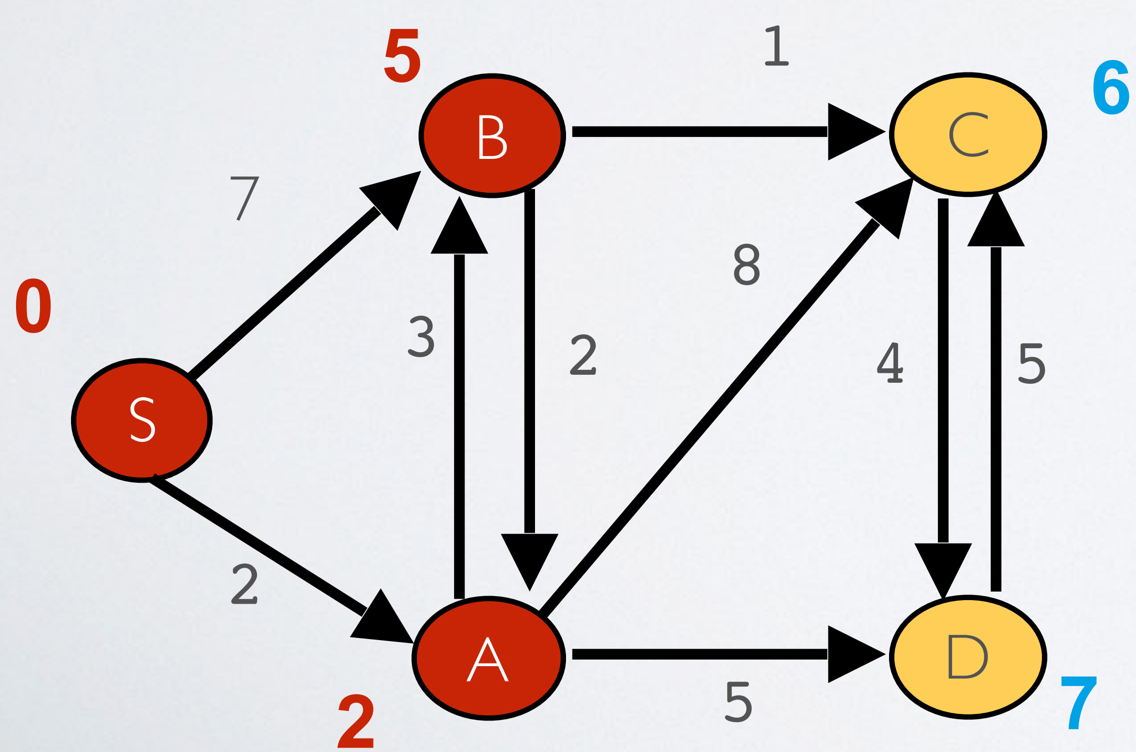




- Step 1
  - Label source w/ dist. 0
  - Label other vertices w/ dist. ∞
  - Add all nodes to `Q`
- Step 2
  - Remove node with min. priority from `Q` (`S` in this example).
  - Calculate dist. from source to removed node's neighbors…
  - …by adding adjacent edge weights to `S`'s dist.

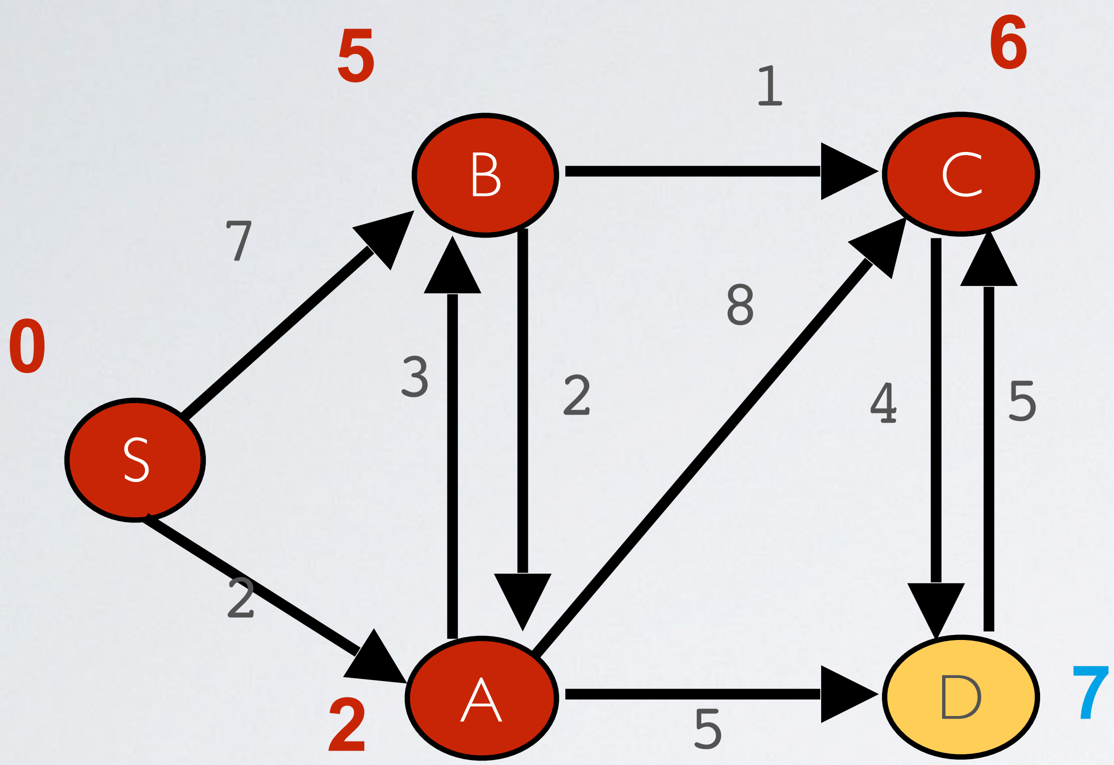
Dijkstra's Algorithm Example

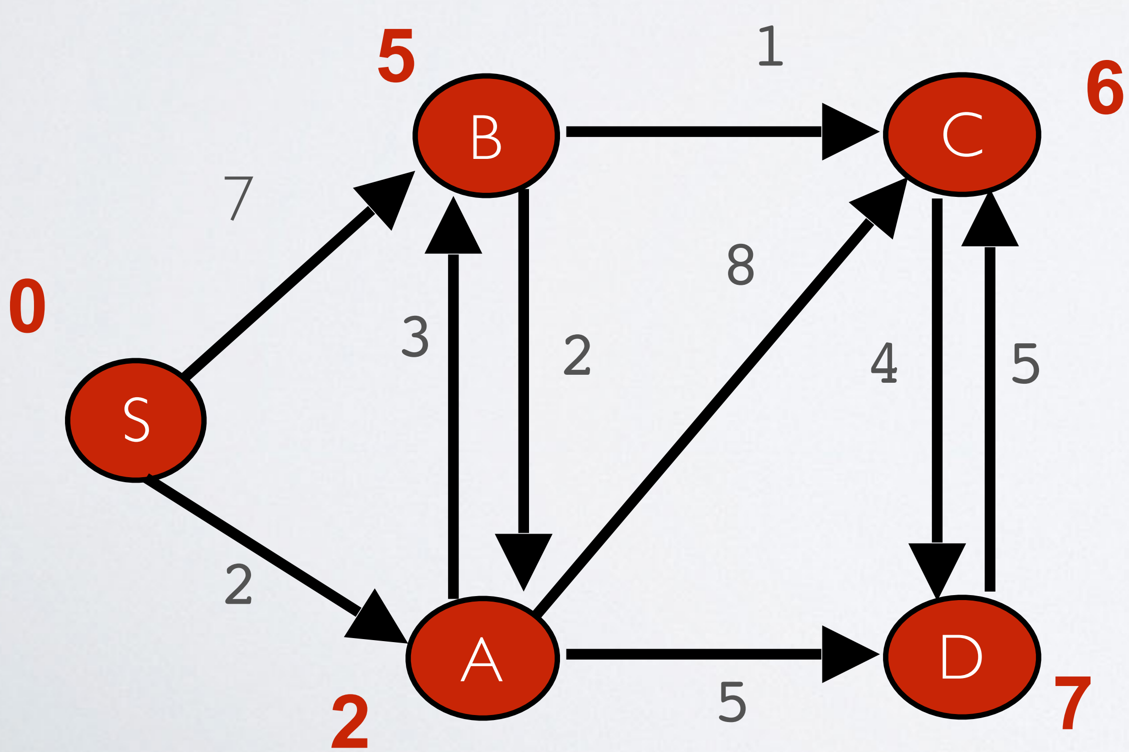




- Step 3
  - While `Q` isn't empty,
    - repeat previous step
    - removing `A` this time
  - Priorities of nodes in `Q` may have to be updated
    - ex: `B`' s priority
- Step 4
  - Repeat again by removing vertex `B`
  - Update distances that are shorter using this path than before
  - ex: C now has a distance `6` not `10`

Dijkstra's Algorithm Example



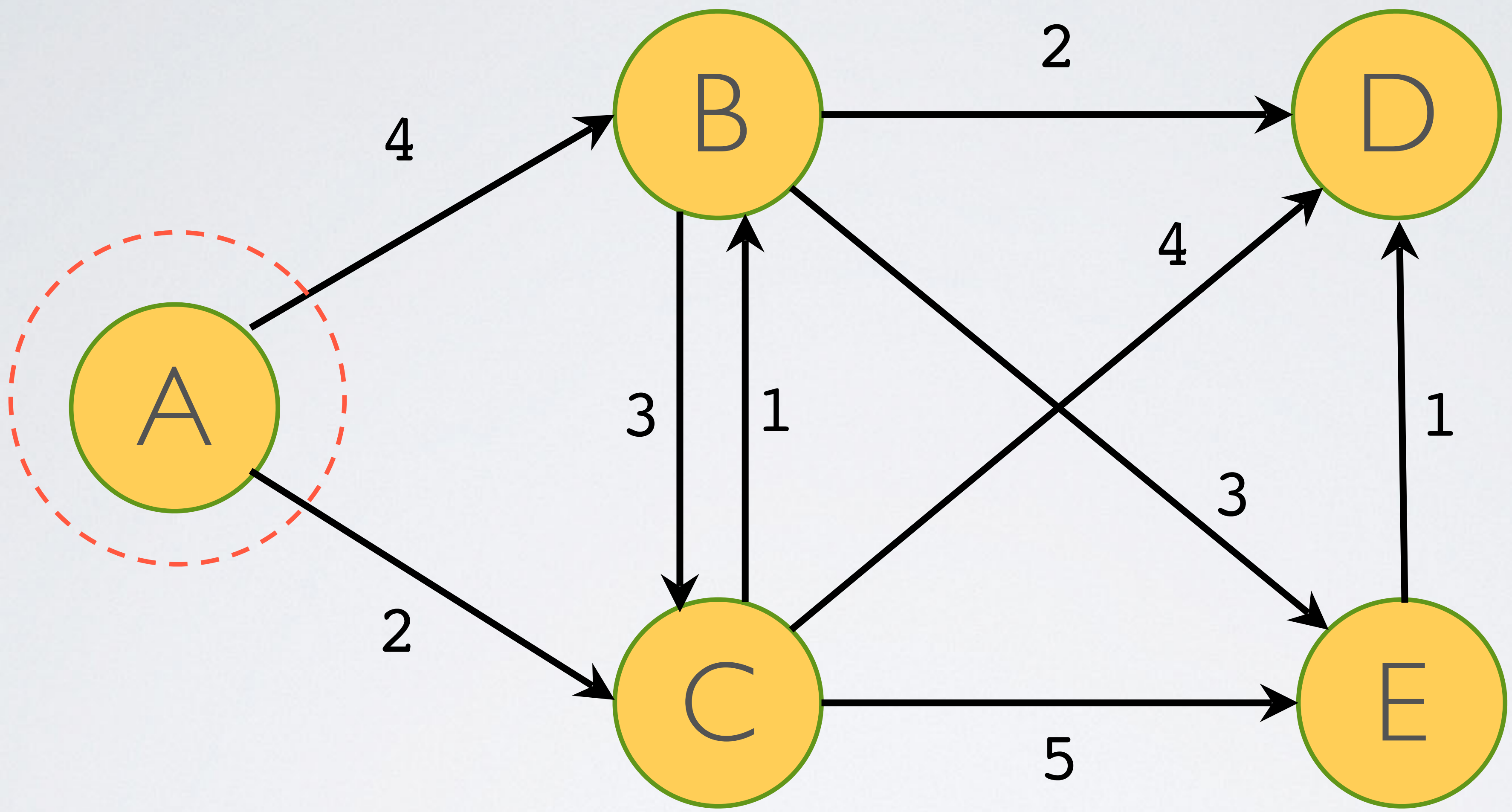


- Step 5
  - Repeat
    - this time removing C
- Step 6
  - After removing D…
  - …every node has been visited…
  - …and decorated w/ shortest dist. to source

Dijkstra's Algorithm Example

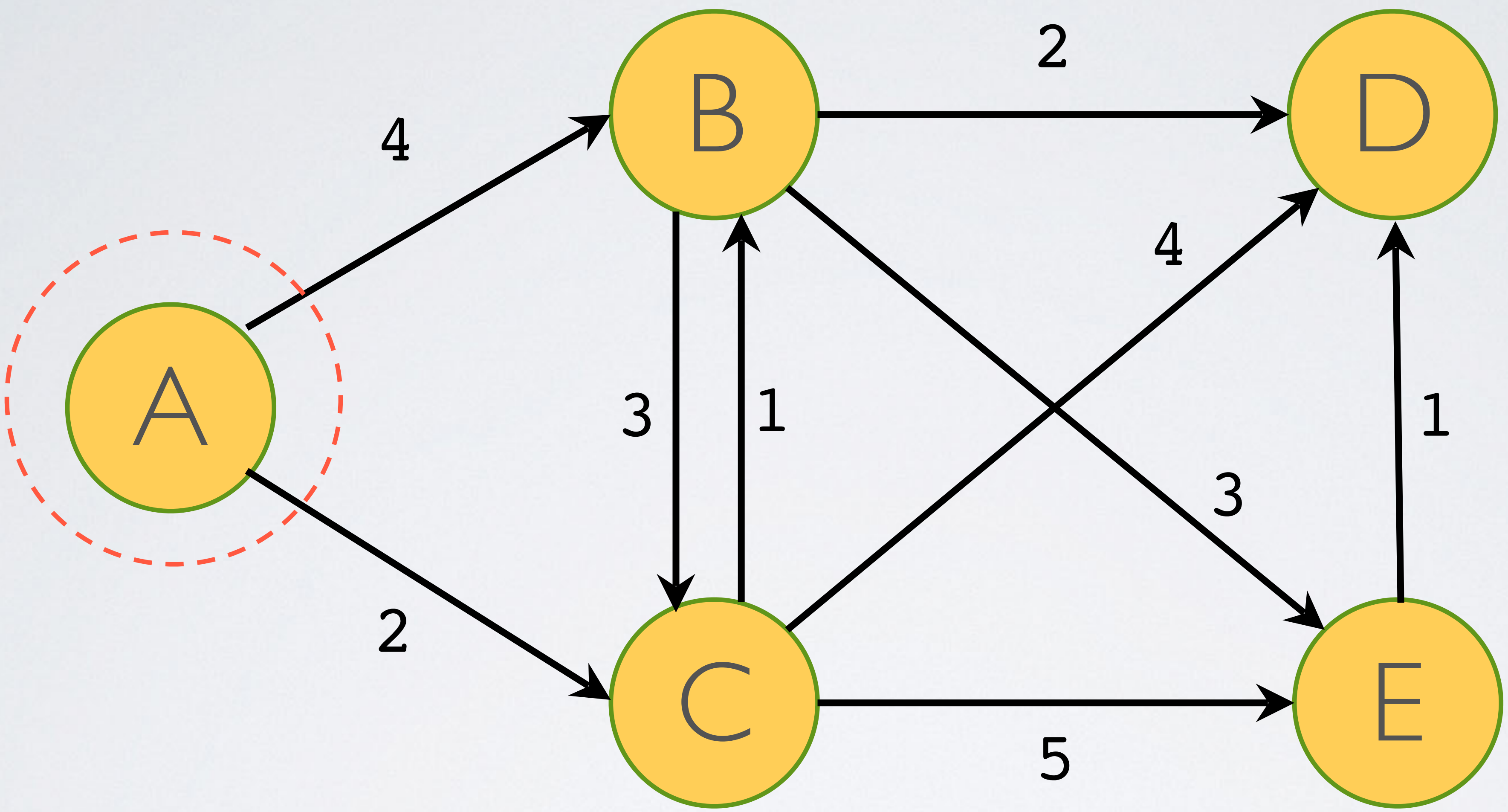
- Previous example decorated nodes with shortest *distance* but did not "create" paths
- How could you enhance algorithm to return the shortest path to a particular node?
  - Previous pointers!
- Let's do another example
  - but this time without explanation
  - try to explain what the algorithm is doing at each step

Dijkstra's Example



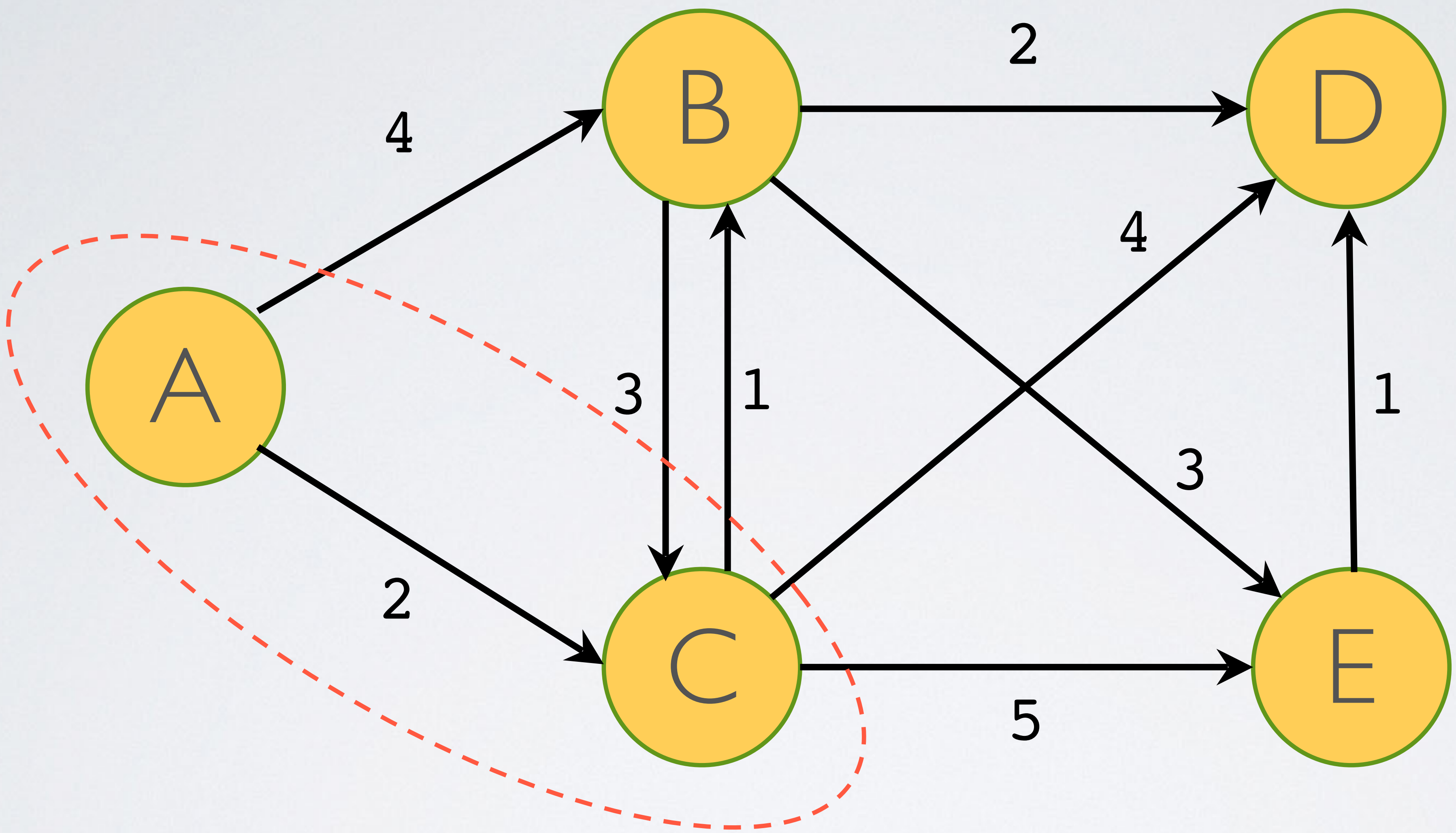
A	B	C	D	E
0	∞	∞	∞	∞

Dijkstra's Example



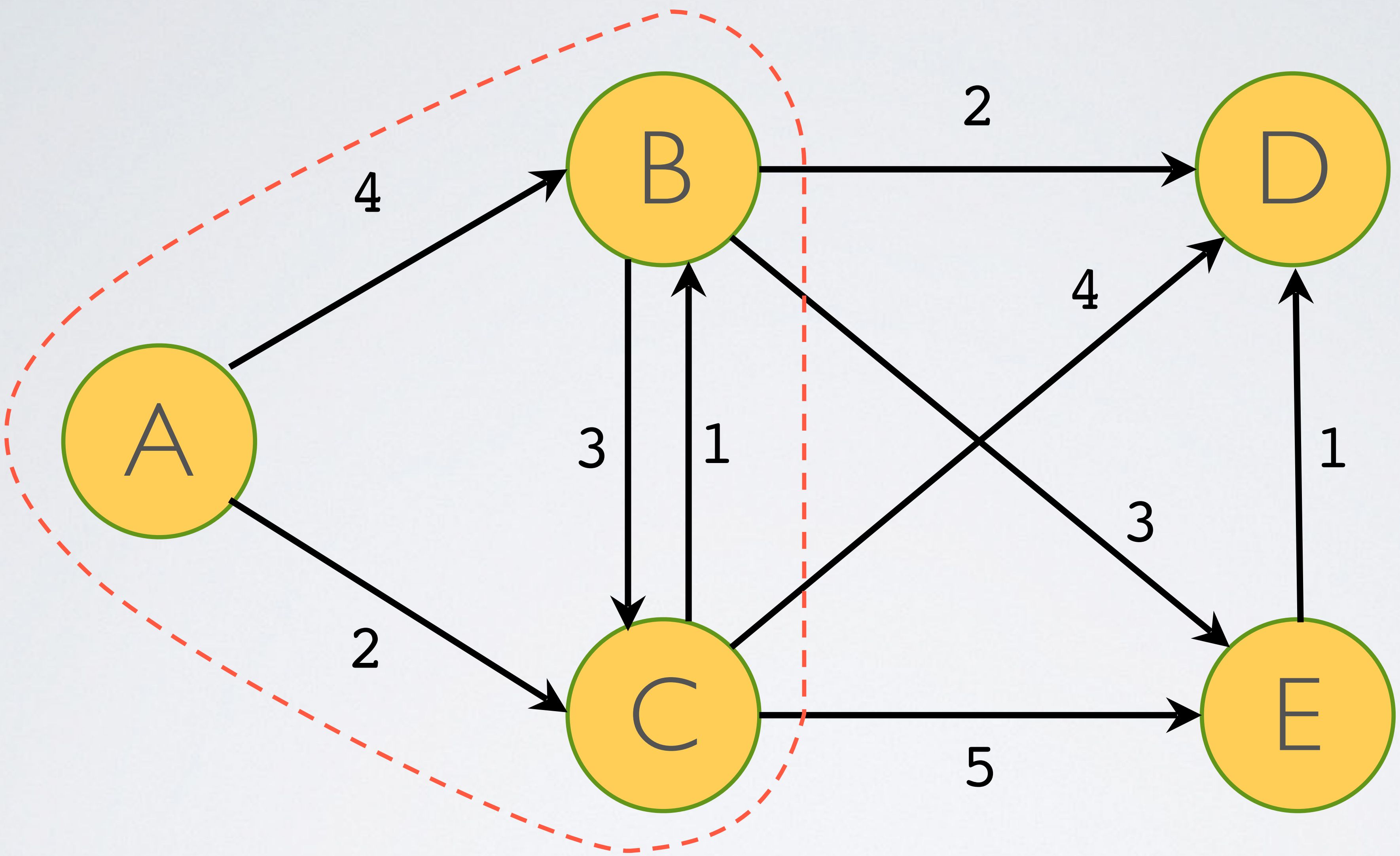
A	B	C	D	E
0	4	2	∞	∞

Dijkstra's Example



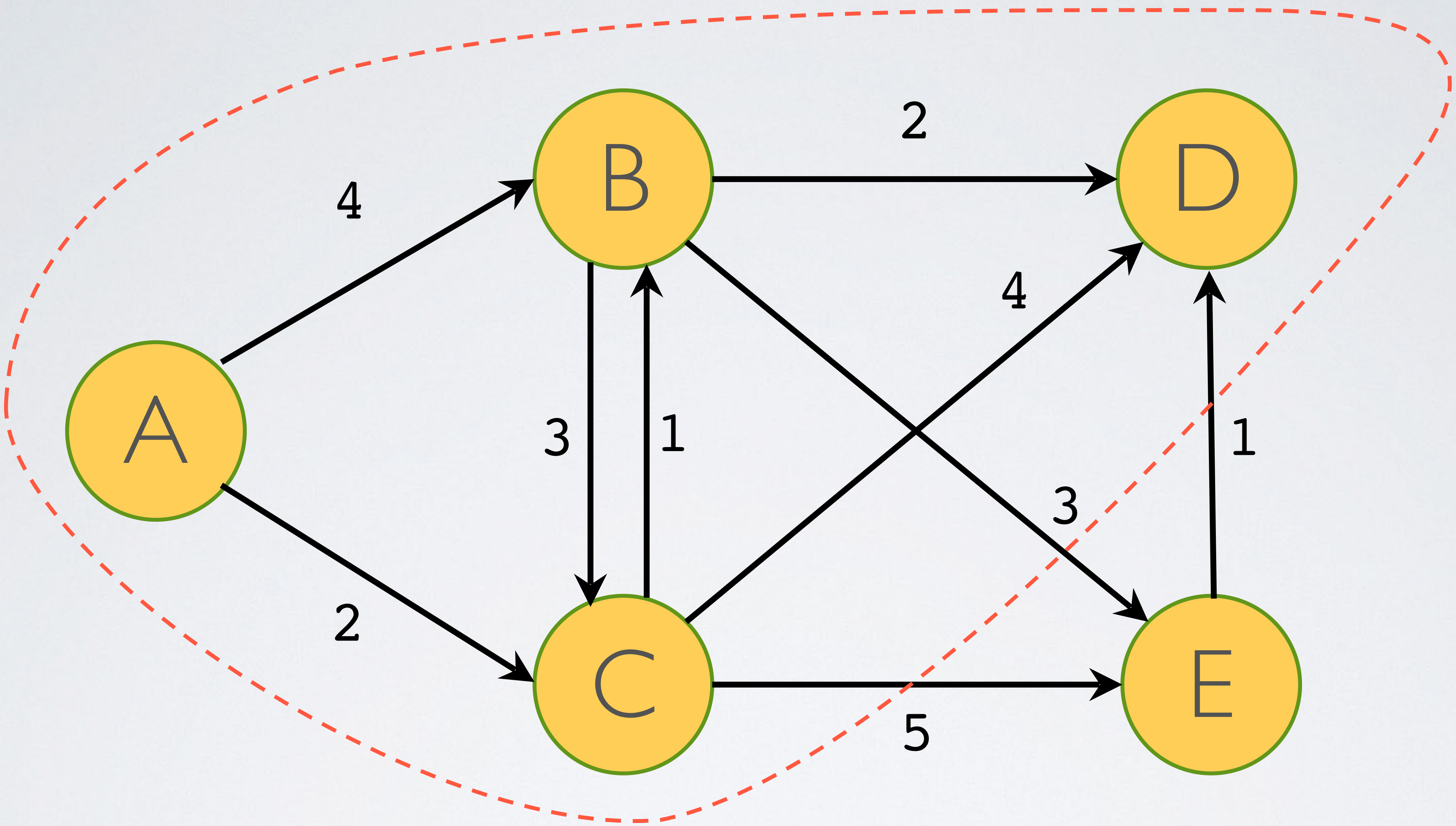
A	B	C	D	E
0	**3**	2	**6**	**7**

Dijkstra’s Example



A	B	C	D	E
0	3	2	**5**	**6**

Dijkstra's Example



A	B	C	D	E
0	3	2	5	6

Simulate Dijkstra's

Activity #3

2 min

Simulate Dijkstra's

Activity #3

2 min

Simulate Dijkstra's

Activity #3

1 min

Simulate Dijkstra's

Activity #3

0 min

Dijkstra's Algorithm

- Dijkstra's algorithm is an example of a class of algorithms we previously mentioned
- Since it uses a priority queue,
  - at each step of iteration…
  - …we consider next closest node given the information we have
- What algorithm paradigm does this fall under?

Dijkstra Pseudo-Code

```
function dijkstra(G, s):
   // Input: graph G with vertices V, and source s
   // Output: Nothing
   // Purpose: Decorate nodes with shortest distance from s
   for v in V:
     v.dist = infinity  // Initialize distance decorations
     v.prev = null      // Initialize previous pointers to null
   s.dist = 0          // Set distance to start to 0

   PQ = PriorityQueue(V)    // Use v.dist as priorities
   while PQ not empty:
      u = PQ.removeMin()
      for all edges (u, v):  //each edge coming out of u
         if u.dist + cost(u, v) < v.dist: // cost() is weight
            v.dist = u.dist + cost(u,v)   // Replace as necessary
            v.prev = u       // Maintain pointers for path
            PQ.decreaseKey(v, v.dist)
```

Runtime of Dijkstra w/ Heap

```
function dijkstra(G, s):
   for v in V:                // 1. O(__)
    v.dist = infinity
    v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)     // 2. O(__)
   while PQ not empty:       // 3. O(__)
      u = PQ.removeMin()     // 4. O(__)
      for all edges (u, v): // 5. O(__)
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)   // 6. O(__)
```

Total:

7. O(__)

Activity #3

2 min

Runtime of Dijkstra w/ Heap

```
function dijkstra(G, s):
   for v in V:                // 1. O(__)
    v.dist = infinity
    v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)     // 2. O(__)
   while PQ not empty:       // 3. O(__)
      u = PQ.removeMin()     // 4. O(__)
      for all edges (u, v): // 5. O(__)
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)   // 6. O(__)
```

Total:

7. O(__)

Activity #3

2 min

Runtime of Dijkstra w/ Heap

```
function dijkstra(G, s):
   for v in V:                // 1. O(__)
    v.dist = infinity
    v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)      // 2. O(__)
   while PQ not empty:        // 3. O(__)
      u = PQ.removeMin()      // 4. O(__)
      for all edges (u, v):   // 5. O(__)
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)   // 6. O(__)
```

Total:

7. O(__)

Activity #3

1 min

Runtime of Dijkstra w/ Heap

```
function dijkstra(G, s):
   for v in V:                // 1. O(__)
    v.dist = infinity
    v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)     // 2. O(__)
   while PQ not empty:       // 3. O(__)
      u = PQ.removeMin()     // 4. O(__)
      for all edges (u, v): // 5. O(__)
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)   // 6. O(__)
```

Total:

7. O(__)

Activity #3

0 min

Dijkstra Runtime

```
function dijkstra(G, s):
   for v in V:
     v.dist = infinity
     v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)
   while PQ not empty:
      u = PQ.removeMin()
      for all edges (u, v):
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)
```

O(|V|)

depends on PQ

O(|V|)

depends on PQ

O(|E|) total

depends on PQ

Dijkstra Runtime

- Depends on priority queue implementation
- If PQ implemented with Array or Linked List
  - `insert()` is `O(1)`
  - `removeMin( )` is `O(|V|)`
    - you have to scan to find min-priority element
  - `decreaseKey()` is `O(1)`
    - you already have node when you change its key

Dijkstra Runtime w/ Array or List

```
function dijkstra(G, s):
   for v in V:
     v.dist = infinity
     v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)
   while PQ not empty:
      u = PQ.removeMin()
      for all edges (u, v):
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)
```

O(|V|)

O(|V|)

O(|V|)

O(|V|)

O(|E|)
total

O(1)

Dijkstra Runtime w/ Array or List

- If PQ implemented with Array or Linked List
  - $$\begin{aligned} O(|V| + |V| + |V|^2 + |E|) &= O(|V|^2 + |E|) \\ &= O(|V|^2) \end{aligned}$$
  - since $|\mathtt{E}| \leq |\mathtt{V}|^2$

Dijkstra Runtime w/ Heap

- If PQ implemented with Heap
  - `insert( )` is `O(log|V|)`
    - you may need to upheap
  - `removeMin( )` is `O(log|V|)`
    - you may need to downheap
  - `decreaseKey()` is `O(log|V|)`
    - assume we have dictionary that maps vertex to heap entry in `O(log|V|)` time (so no need to scan heap to find entry)
    - you may need to upheap after decreasing the key

Dijkstra Runtime w/ Heap

```
function dijkstra(G, s):
   for v in V:                                  ← O(|V|)
      v.dist = infinity
      v.prev = null
   s.dist = 0

   PQ = PriorityQueue(V)                        ← O(|V|log|V|)
   while PQ not empty:                          ← O(|V|)
      u = PQ.removeMin()                        ← O(log|V|)
      for all edges (u, v):                     ← O(|E|) total
         if v.dist > u.dist + cost(u, v):
            v.dist = u.dist + cost(u,v)
            v.prev = u
            PQ.decreaseKey(v, v.dist)           ← O(log|V|)
```

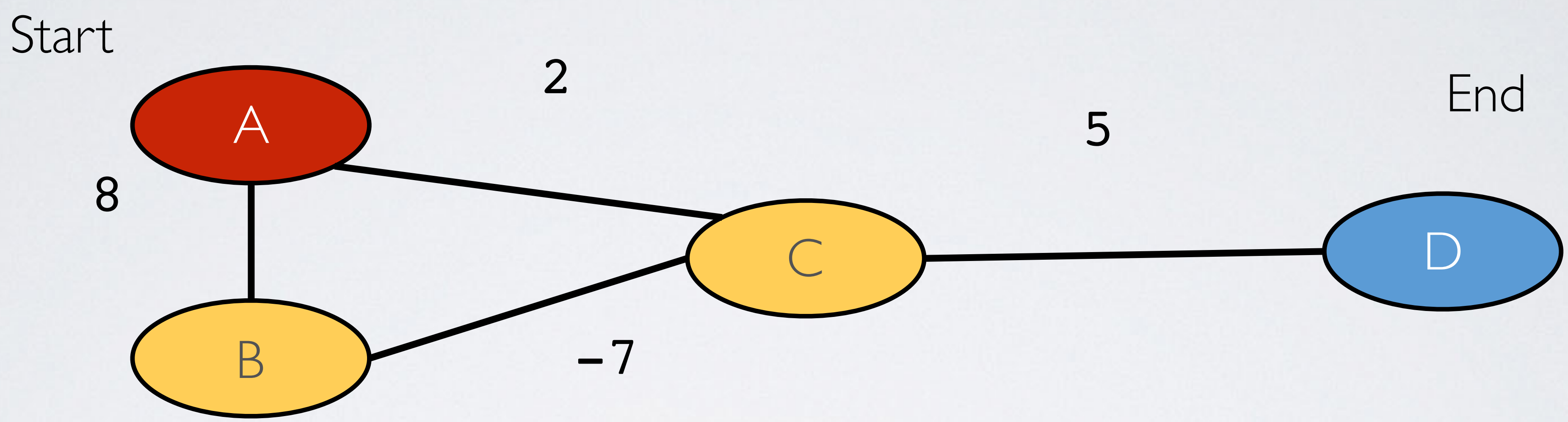
Dijkstra Runtime w/ Heap

- If PQ implemented with Heap

$$O(|V| + |V|\log|V| + |V|\log|V| + |E|\log|V|)$$
$$= O(|V| + |V|\log|V| + |E|\log|V|)$$
$$= O\Big((|V| + |E|)\cdot\log|V|\Big)$$

- Note
  - though the `O(|E|)` loop is nested in the `O(|V|)` loop
  - we visit each edge at most twice rather than `|V|` times
  - That's why while loop is $O\Big((V\log|V|) + (|E|\log|V|)\Big)$

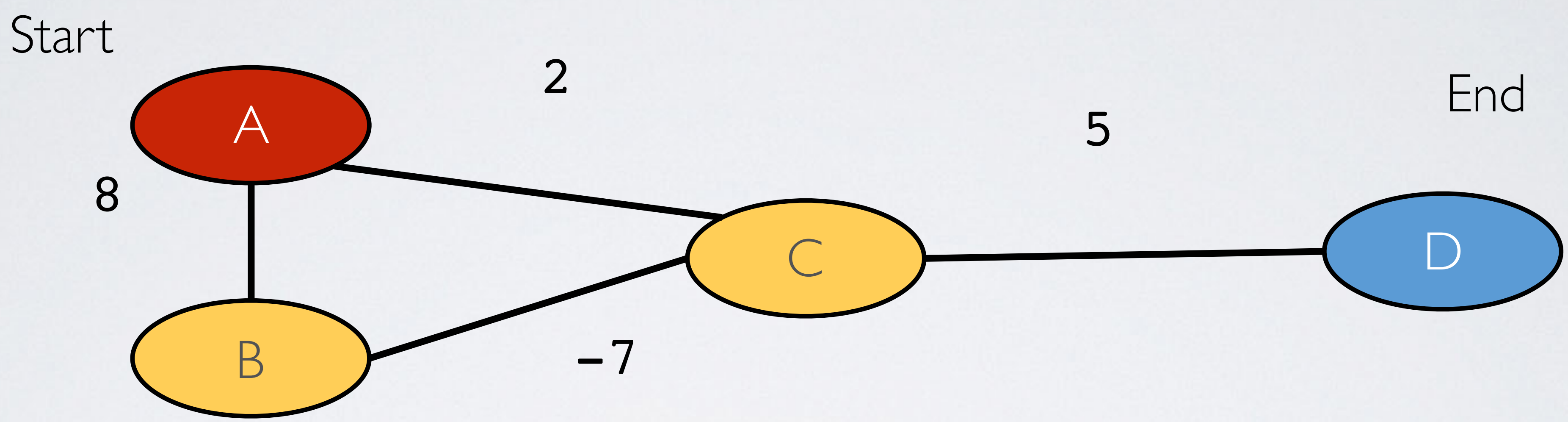
Dijkstra's on Graph with Negative Edges



Activity #5

1 min

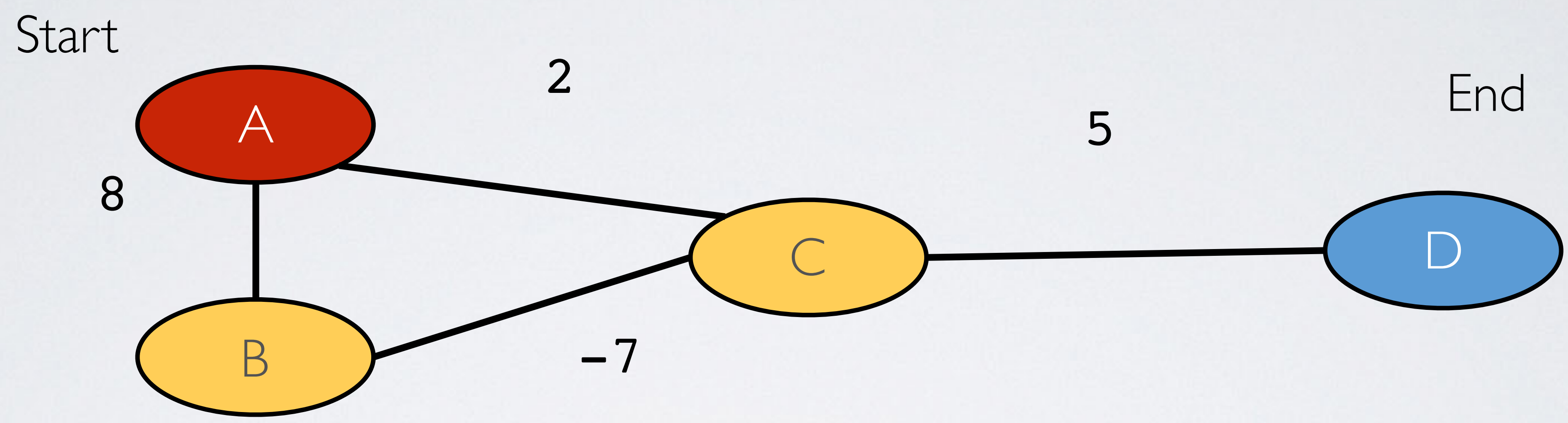
Dijkstra's on Graph with Negative Edges



Activity #5

1 min

Dijkstra's on Graph with Negative Edges

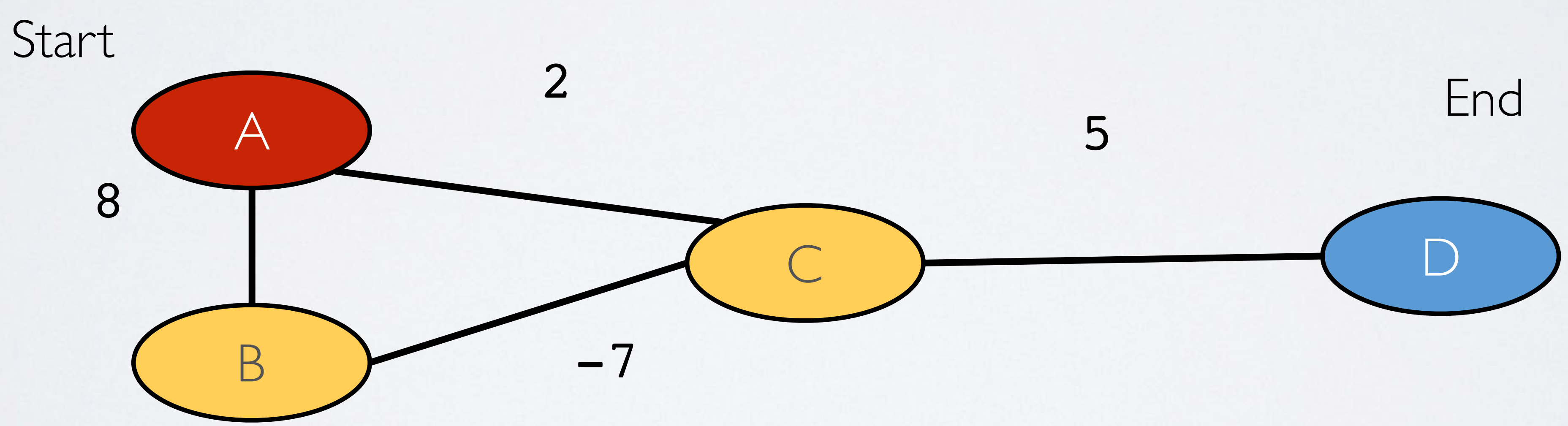


Activity #5

0 min

Dijkstra isn't perfect!

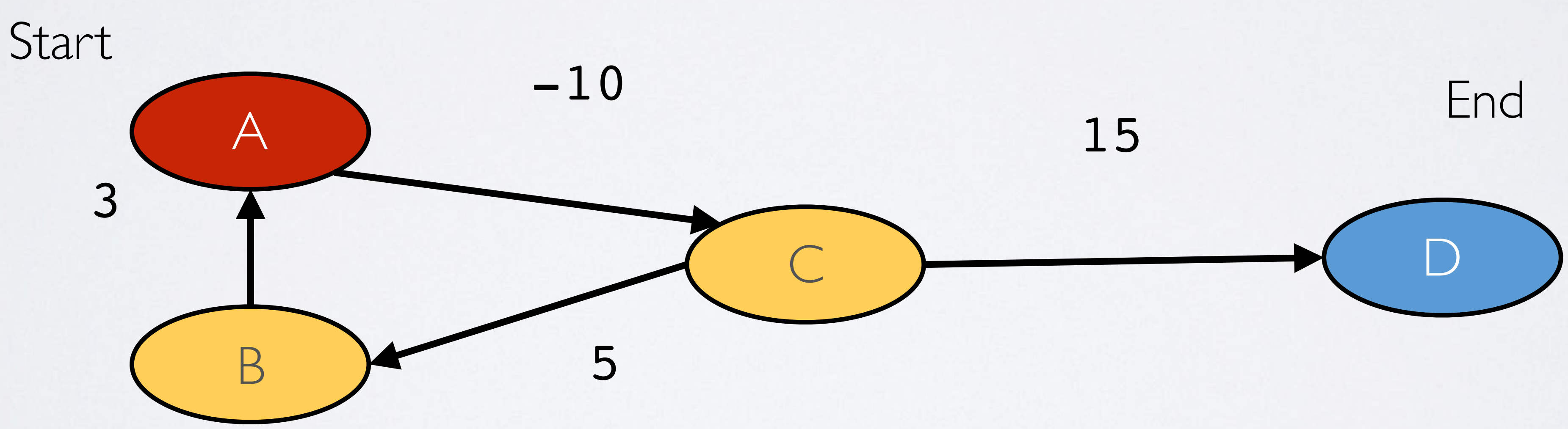
- We can find shortest path on weighted graph in
  - `O((|V|+|E|)×log|V|)`
  - or can we…
- Dijkstra fails with negative edge weights



- Returns `[A,C,D]` when it should return `[A,B,C,D]`

Negative Edge Weights

- Negative edge weights are problem for Dijkstra
- But negative cycles are even worse!
  - because there is no true shortest path!



Bellman-Ford Algorithm

- Algorithm that handles graphs w/ neg. edge weights
- Similar to Dijkstra's but more robust
  - Returns same output as Dijkstra's for any graph w/ only positive edge weights (but runs slower)
  - Returns correct shortest paths for graphs w/ neg. edge weights