

Minimum Spanning Trees: Prim-Jarnik & Kruskal

Taken from Brown's CS16

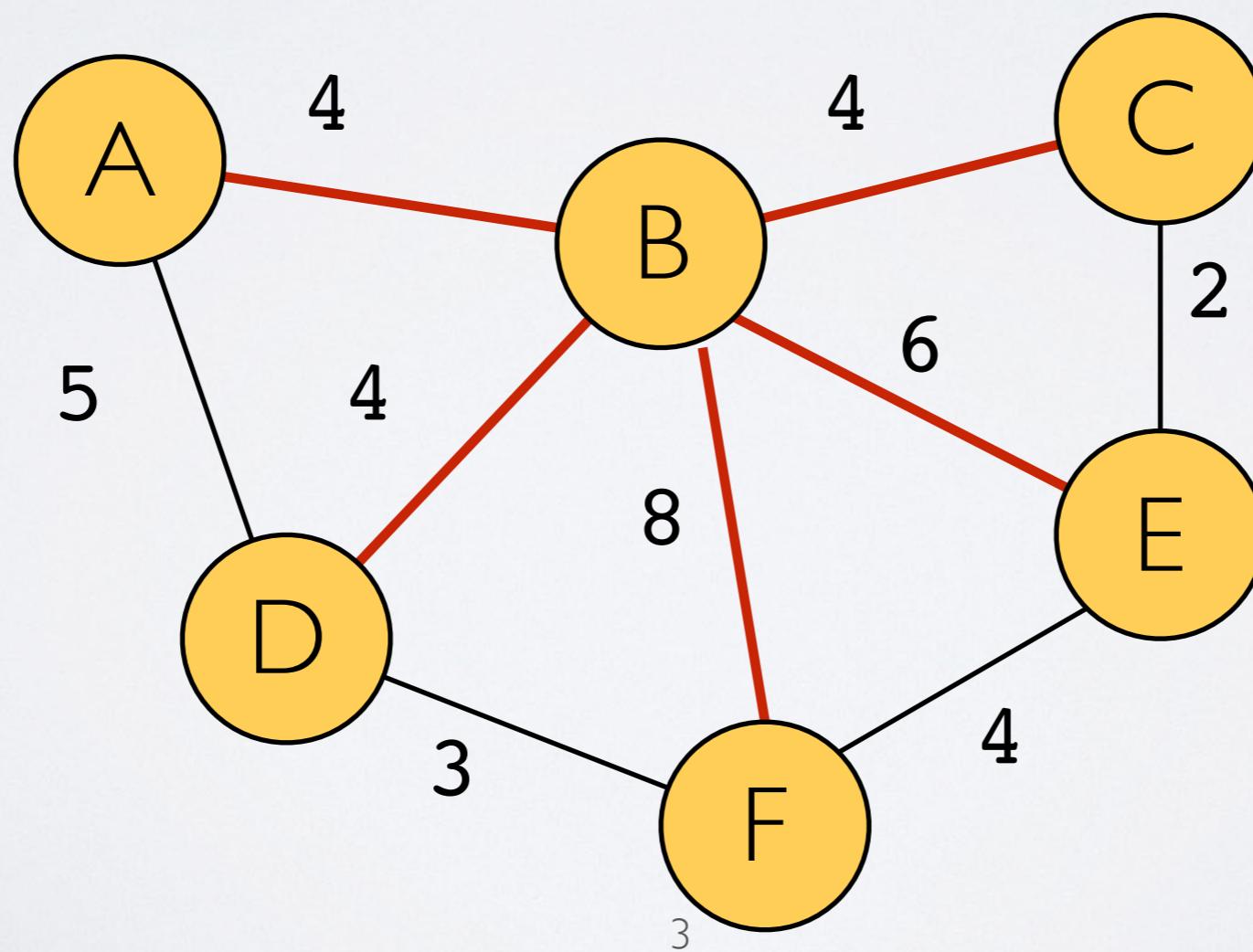
Outline

- ▶ Minimum Spanning Trees
- ▶ Prim-Jarnik Algorithm
 - ▶ Analysis
 - ▶ Proof of Correctness
- ▶ Kruskal's Algorithm
 - ▶ Union-Find
 - ▶ Analysis
 - ▶ Proof of Correctness



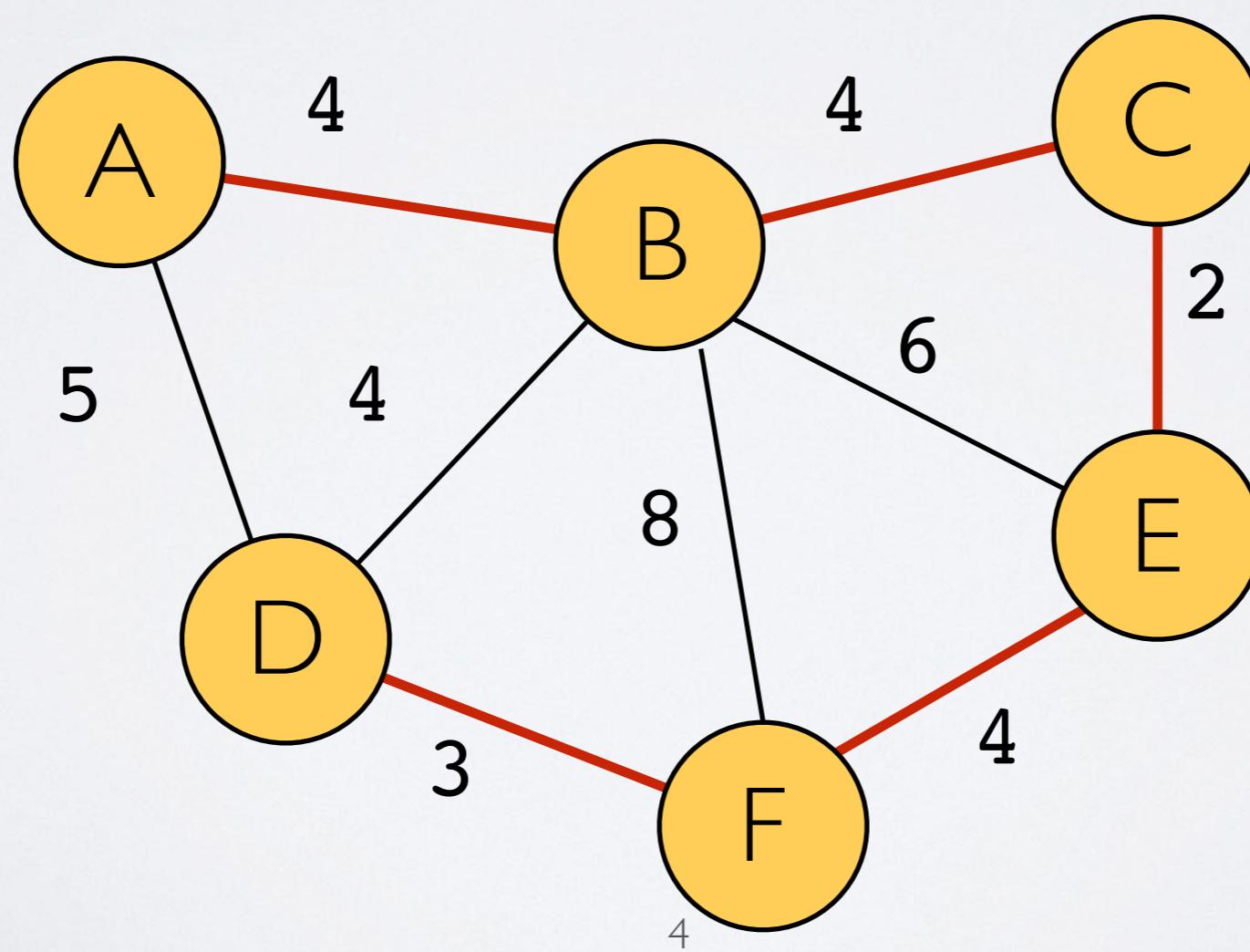
Spanning Trees

- ▶ A **spanning tree** of a graph is
 - ▶ subset of edges that form tree that span every vertex



Minimum Spanning Trees

- ▶ A **minimum spanning tree** (MST) is
 - ▶ spanning tree with minimum total edge weight



Applications

- ▶ Networks

- ▶ electric
- ▶ computer
- ▶ water
- ▶ transportation

- ▶ Computer vision

- ▶ Facial recognition
- ▶ Handwriting recognition
- ▶ **Image segmentation**

- ▶ Low-density parity check codes (LDPC)

Efficient Graph-Based Image Segmentation

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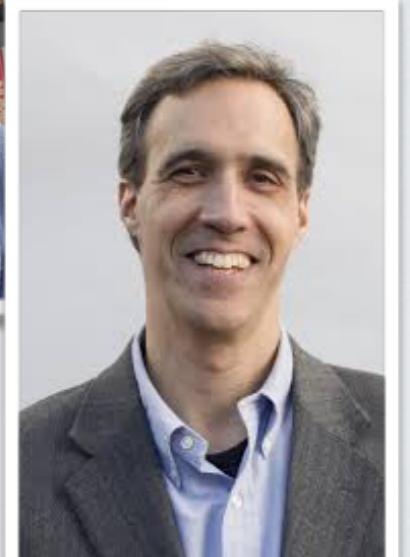
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Abstract

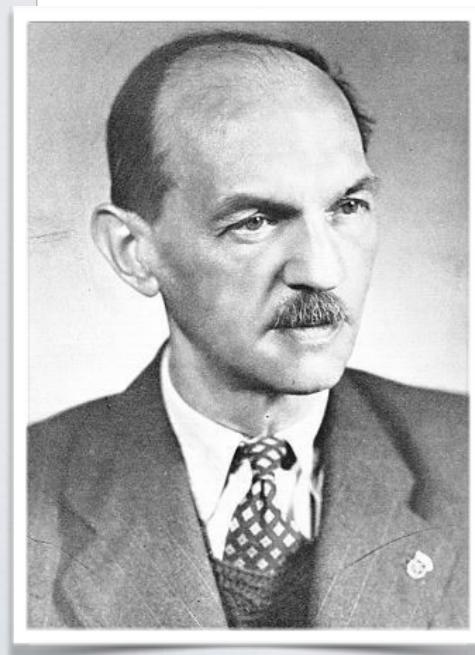
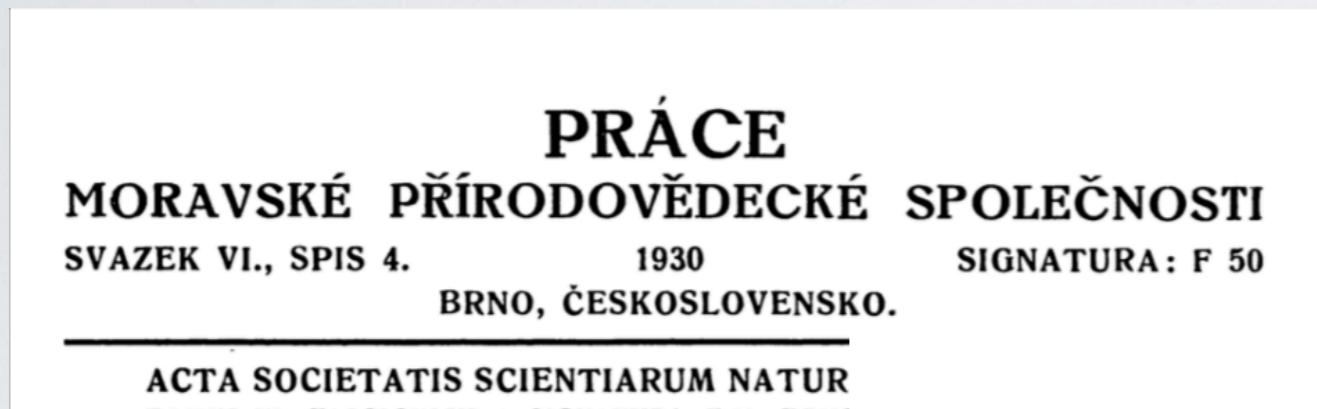
This paper addresses the problem of segmenting an image into regions. We define a predicate for measuring the evidence for a boundary between two regions using a graph-based representation of the image. We then develop an efficient segmentation algorithm based on this predicate, and show that, given a reasonable set of decisions it produces segmentations that are competitive with other algorithms to image segmentation using a greedy approach. We construct the graph, and illustrate the performance of the algorithm to image segmentation using a variety of images. The algorithm runs in time nearly linear in the number of pixels in the image, and is also fast in practice. An important characteristic of the algorithm is that it preserves detail in low-variability image regions while ignoring noise.

Keywords: image segmentation, clustering, graph-based representations



Minimum Spanning Tree Algos

► Prim-Jarník Algorithm



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Shortest Connection Networks And Some Generalizations

By R. C. PRIM

(Manuscript received May 8, 1957)

The basic problem considered is that of interconnecting a given set of terminals with a shortest possible network of direct links. Simple and practical procedures are given for solving this problem both graphically and computationally. It develops that these procedures also provide solutions for a much broader class of problems, containing other examples of practical interest.

Minimum Spanning Tree Algos

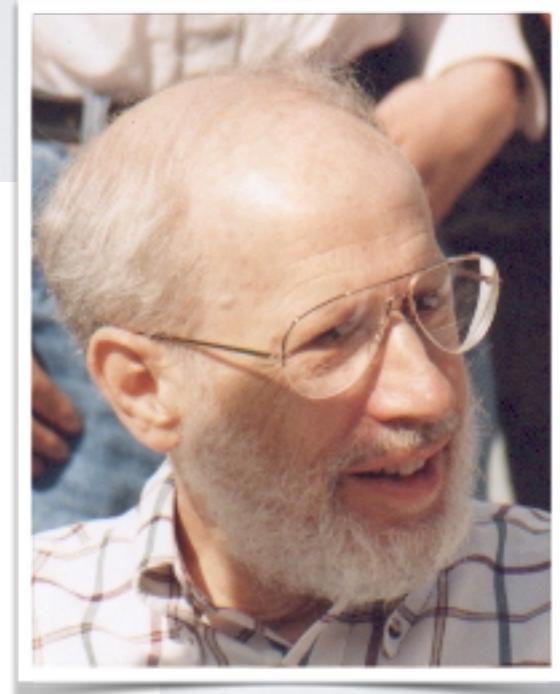
- ▶ Kruskal's algorithm (1956)

ON THE SHORTEST SPANNING SUBTREE OF A GRAPH AND THE TRAVELING SALESMAN PROBLEM

JOSEPH B. KRUSKAL, JR.

Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the *length* of the edge), and if these lengths are all distinct, then among the spanning¹ trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the “matrix of lengths” of the graph, that is, the matrix $\|a_{ij}\|$ where a_{ij} is the length of the edge connecting vertices i and j . Of course, it is assumed that $a_{ij} = a_{ji}$ and that $a_{ii} = 0$ for all i and j .)

The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.



Minimum Spanning Tree Algos

► Karger-Klein-Tarjan (1995)

A Randomized Linear-Time Algorithm to Find Minimum Spanning Trees

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AND

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Princeton University and NEC Research Institute, Princeton, New Jersey

Abstract. We present a randomized linear-time algorithm to find a minimum spanning tree in a connected graph with edge weights. The algorithm uses random sampling in combination with a recently discovered linear-time algorithm for verifying a minimum spanning tree. Our computational model is a unit-cost random-access machine with the restriction that the only operations allowed on edge weights are binary comparisons.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*computations on discrete structures*; G.2.2 [**Discrete Mathematics**]: Graph Theory—*graph algorithms, network problems, trees*; G.3 [**Probability and Statistics**]: *probabilistic algorithms (including Monte Carlo)*; I.5.3 [**Pattern Recognition**]: Clustering
General Terms: Algorithms

Additional Key Words and Phrases: Matroid, minimum spanning tree, network, randomized algorithm

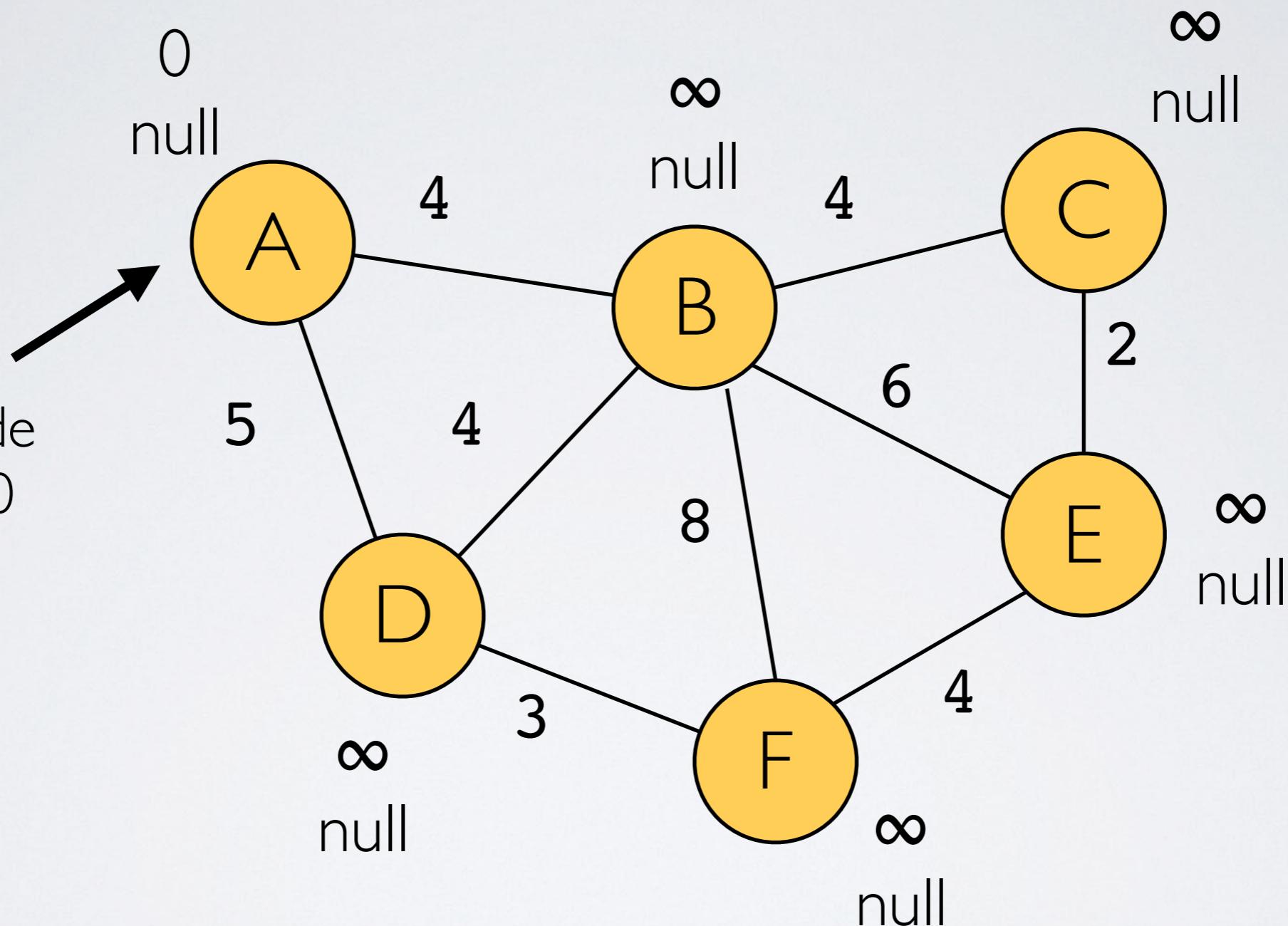


Prim-Jarnik Algorithm

- ▶ Traverse **G** starting at any node
 - ▶ Maintain priority queue of nodes
 - ▶ set priority to weight of the edge that connects them to MST
- ▶ Un-added nodes start with priority ∞
- ▶ At each step
 - ▶ Connect the node with lowest cost
 - ▶ Update (“relax”) neighbors as necessary
- ▶ Stop when all nodes added to MST

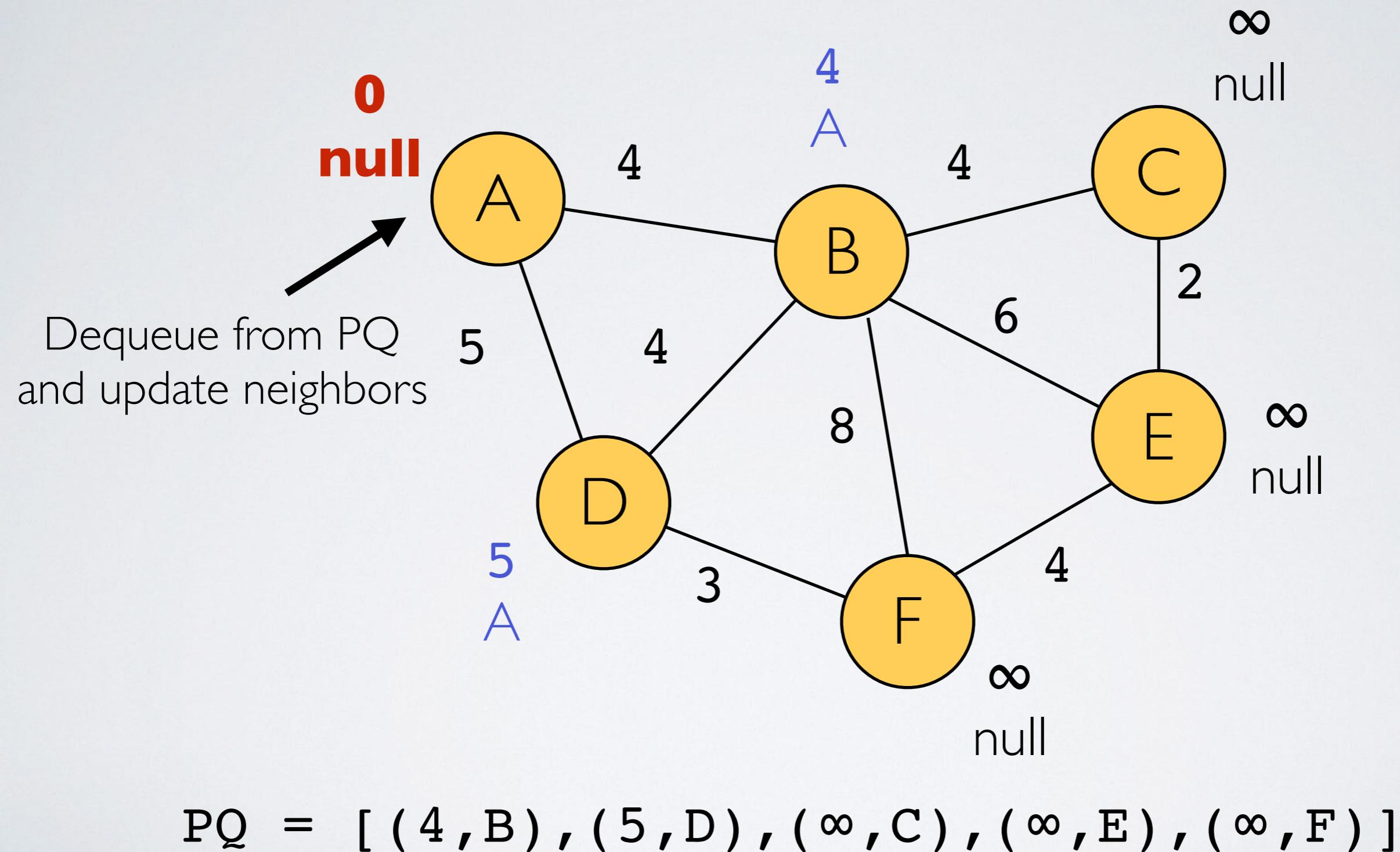
Example

Random node
set to cost 0



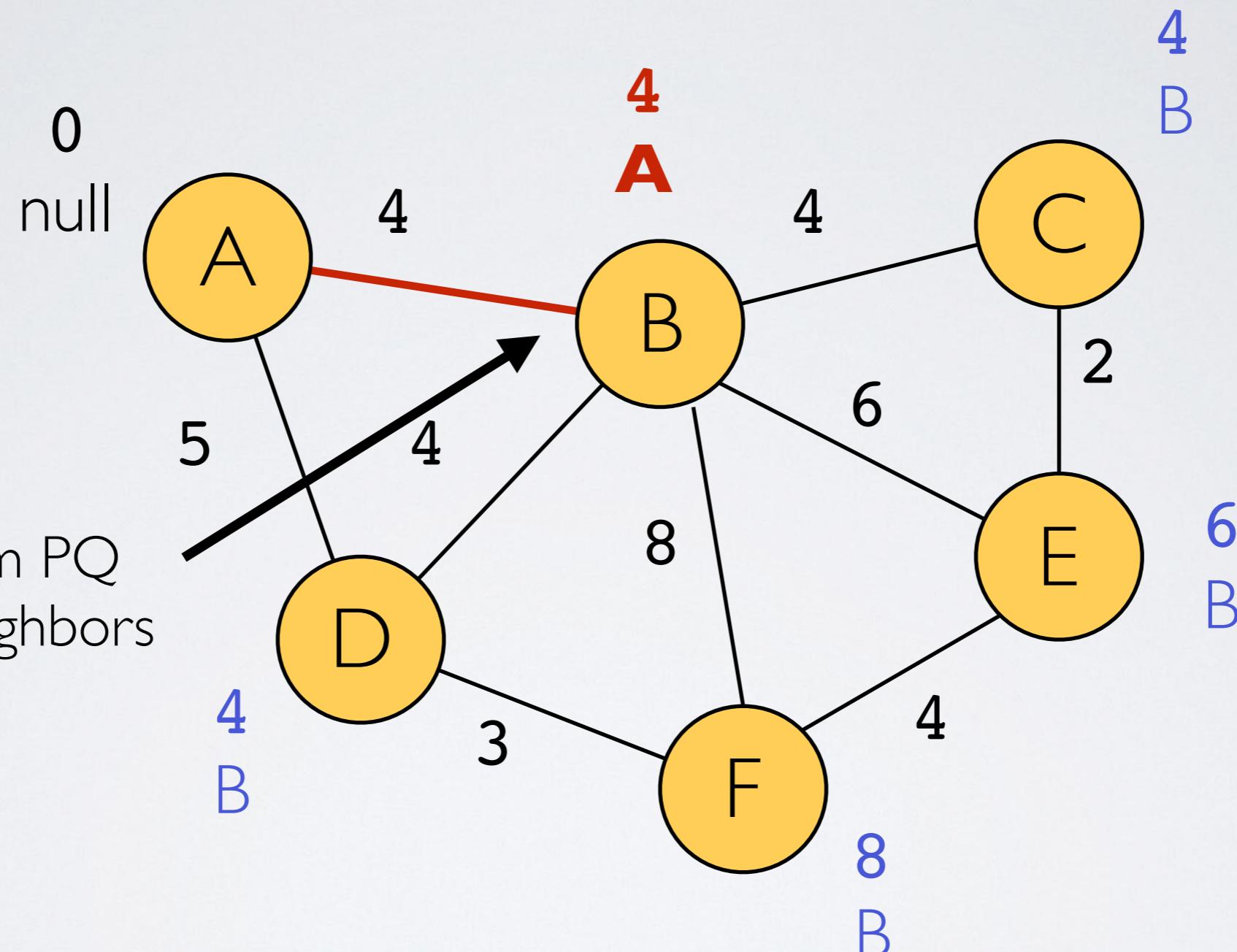
PQ = [(0, A), (∞ , B), (∞ , C), (∞ , D), (∞ , E), (∞ , F)]

Example



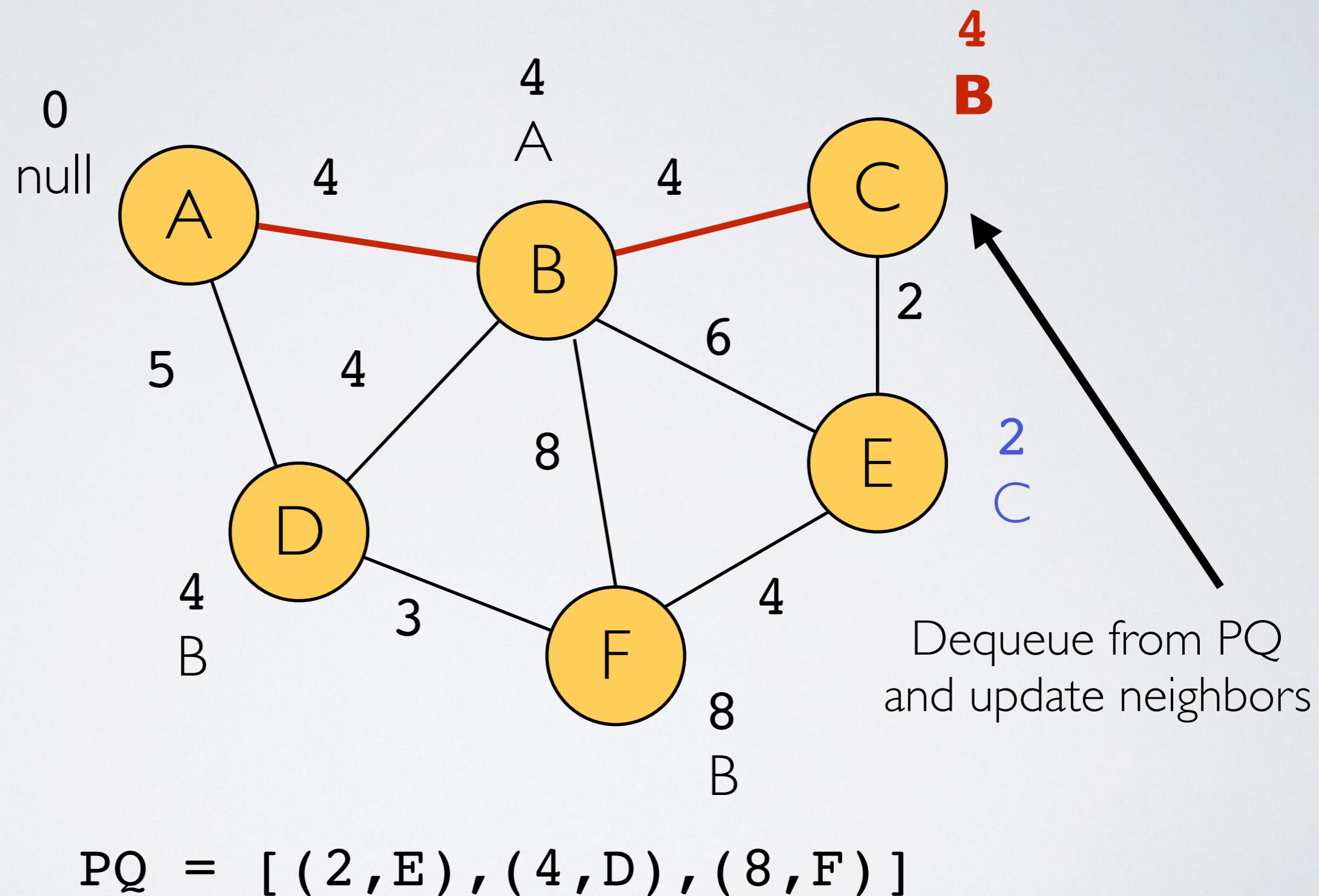
Example

Dequeue from PQ
and update neighbors

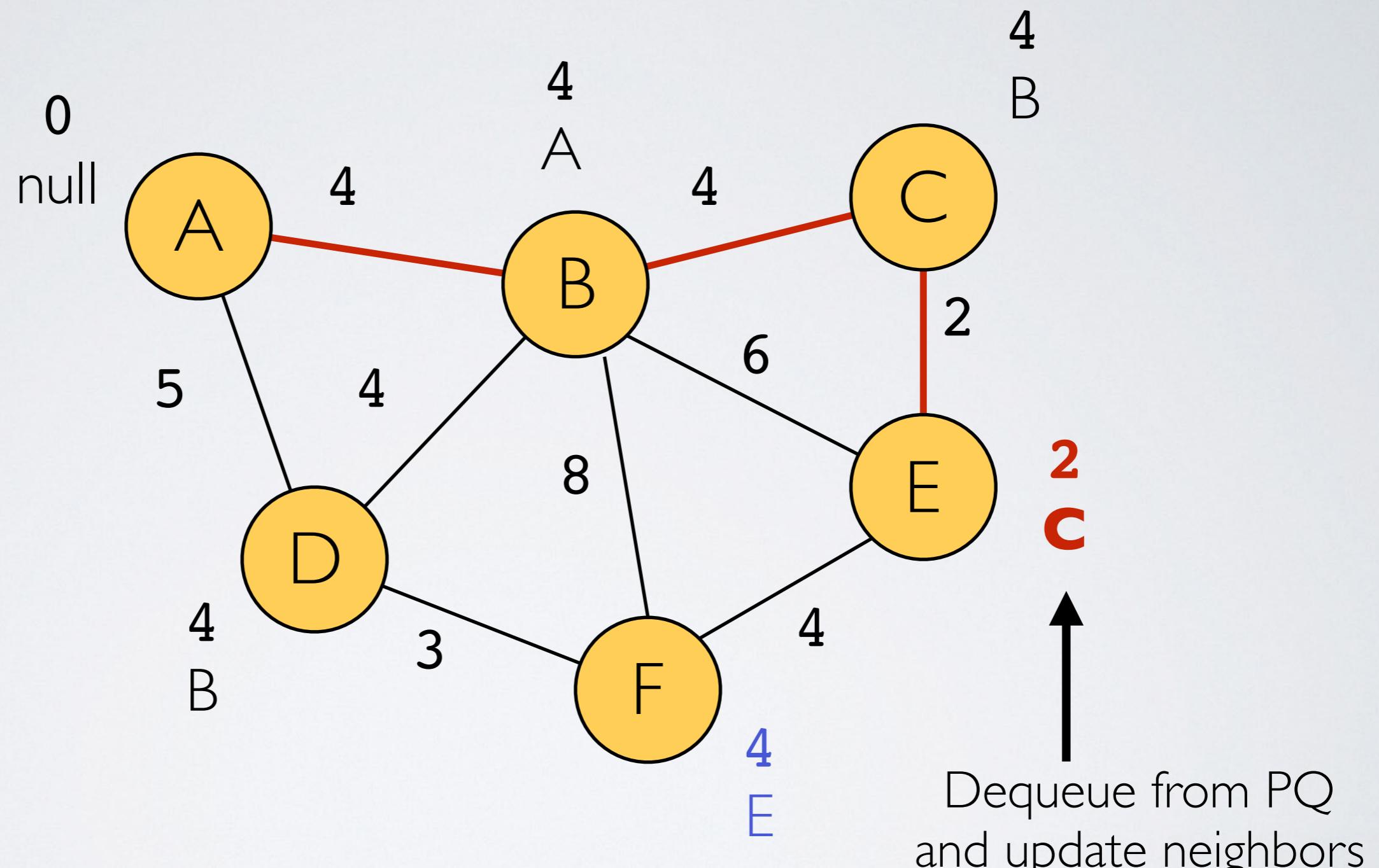


$$PQ = [(4, C), (4, D), (6, E), (8, F)]$$

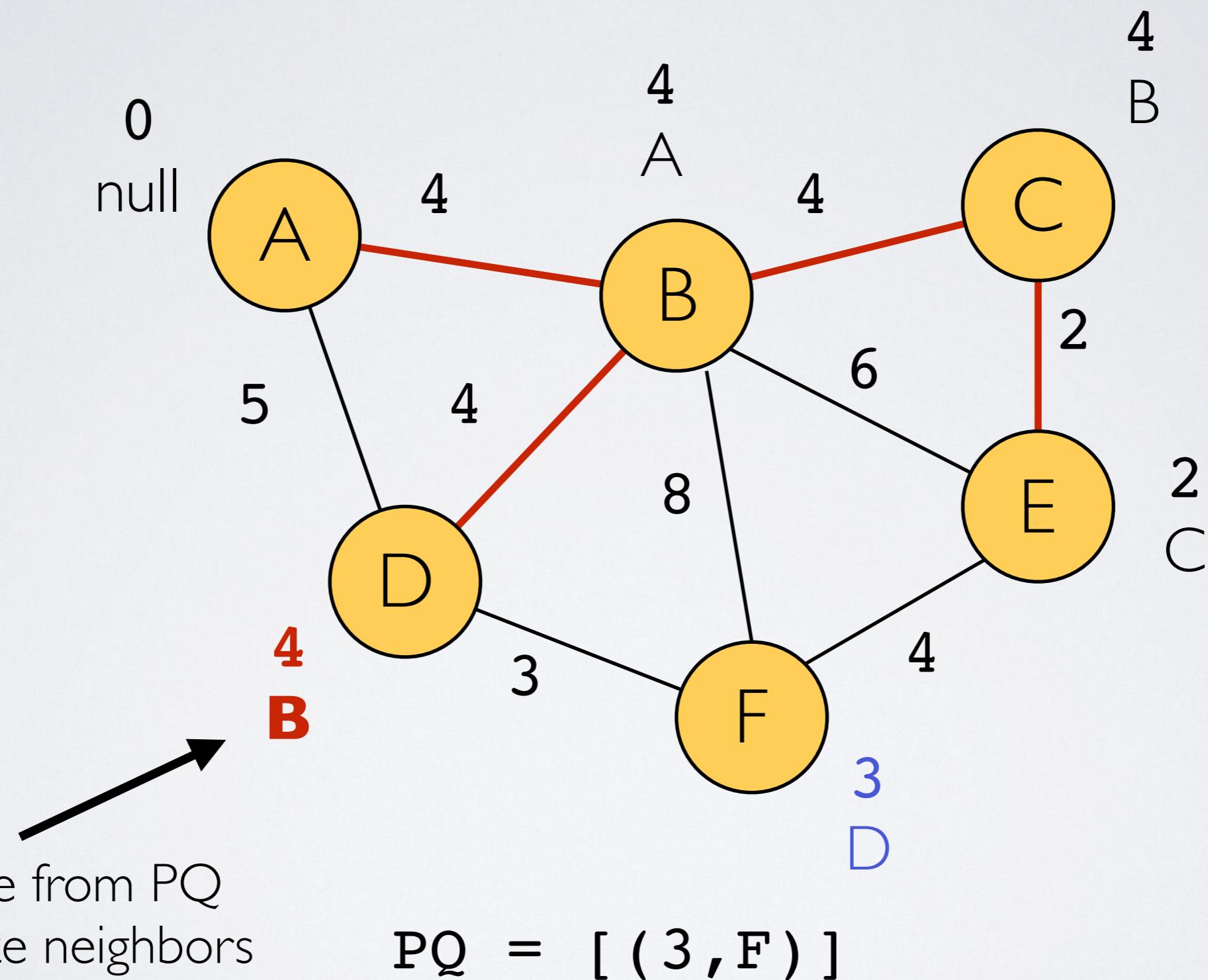
Example



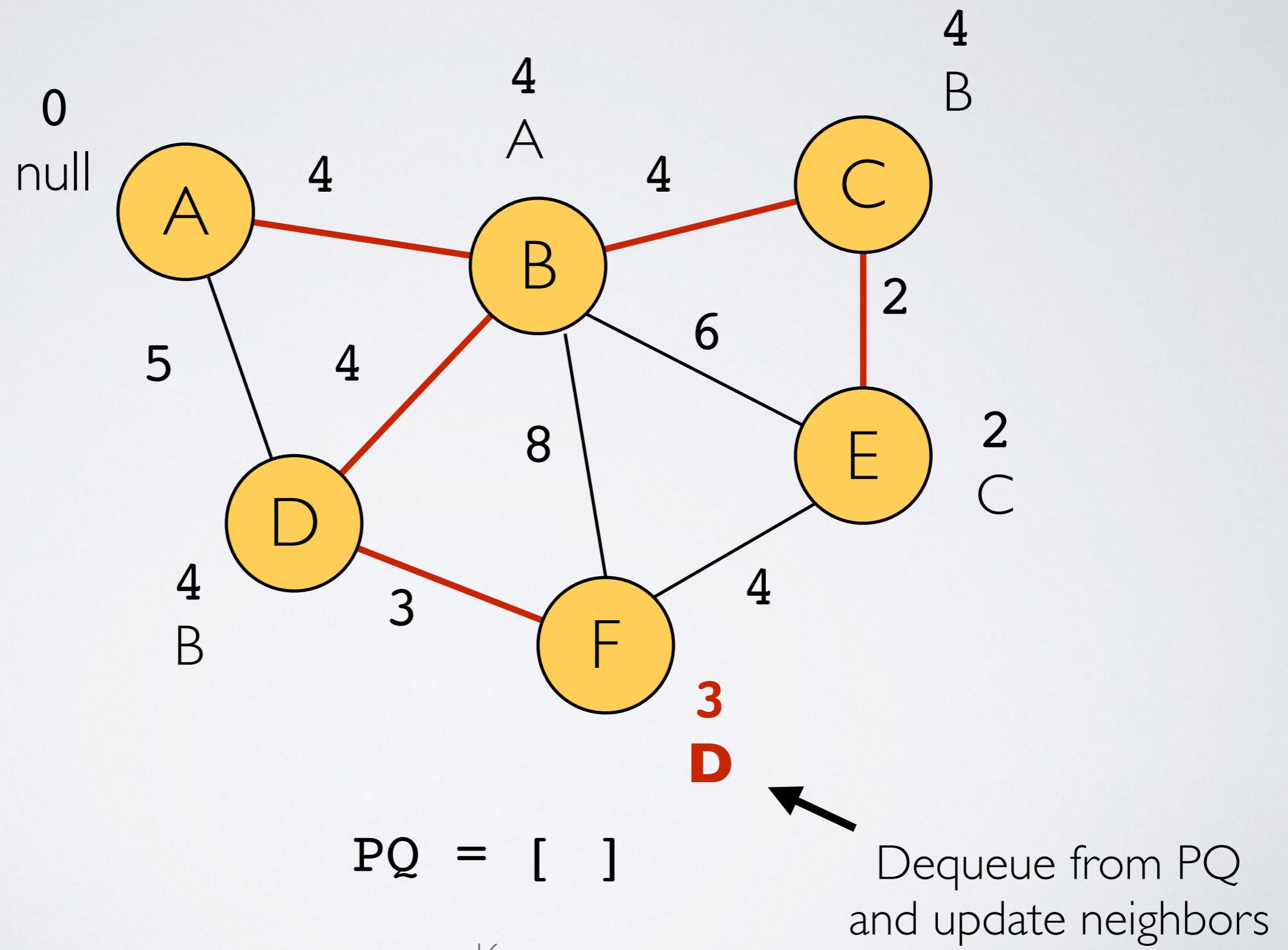
Example



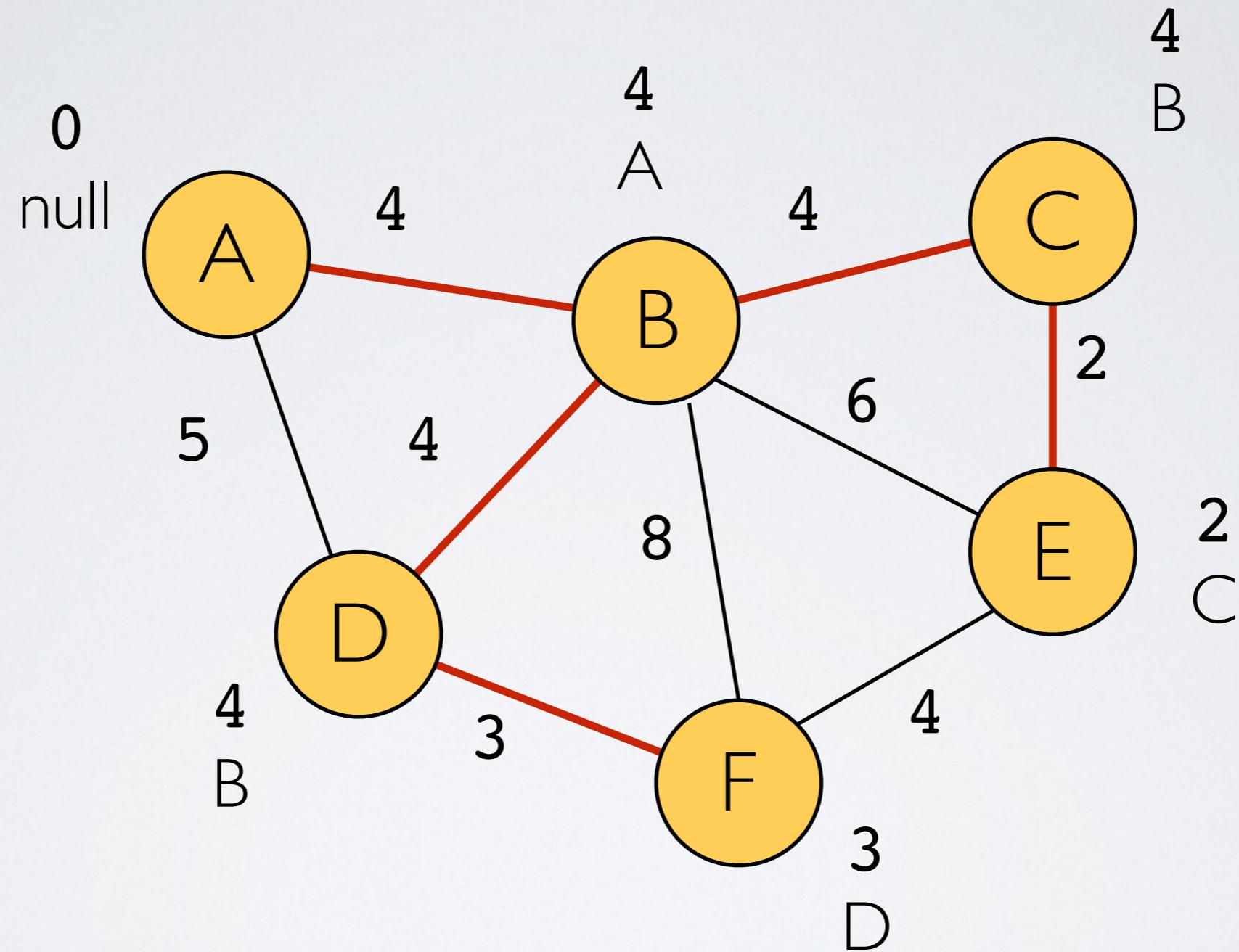
Example



Example



Example



Pseudo-code

```
function prim(G):
    // Input: weighted, undirected graph G with vertices V
    // Output: list of edges in MST
    for all v in V:
        v.cost = ∞
        v.prev = null
    s = a random v in V // pick a random source s
    s = 0
    MST = []
    PQ = PriorityQueue(V) // priorities will be v.cost values
    while PQ is not empty:
        v = PQ.removeMin()
        if v.prev != null:
            MST.append((v, v.prev))
        for all incident edges (v,u) of v such that u is in PQ:
            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```

Simulate Prim-Jarnik

```
function prim(G):
    // Input: weighted, undirected graph G with vertices V
    // Output: list of edges in MST
    for all v in V:
        v.cost = ∞
        v.prev = null
    s = a random v in V // pick a random source s
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    MST = []
    PQ = PriorityQueue(V) // priorities will be v.cost values
    while PQ is not empty:
        v = PQ.removeMin()
        if v.prev != null: //guarantees we don't add (s, s.prev)
            MST.append((v, v.prev))
        for all incident edges (v,u) of v such that u is in PQ:
            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```

3 min

Activity #1

Simulate Prim-Jarnik

```
function prim(G):
    // Input: weighted, undirected graph G with vertices V
    // Output: list of edges in MST
    for all v in V:
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            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```

3 min

Activity #1

Simulate Prim-Jarnik

```
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    // Input: weighted, undirected graph G with vertices V
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            MST.append((v, v.prev))
        for all incident edges (v,u) of v such that u is in PQ:
            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```

2 min

Activity #1

Simulate Prim-Jarnik

```
function prim(G):
    // Input: weighted, undirected graph G with vertices V
    // Output: list of edges in MST
    for all v in V:
        v.cost = ∞
        v.prev = null
    s = a random v in V // pick a random source s
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        if v.prev != null: //guarantees we don't add (s, s.prev)
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        for all incident edges (v,u) of v such that u is in PQ:
            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```

1 min

Activity #1

Simulate Prim-Jarnik

```
function prim(G):
    // Input: weighted, undirected graph G with vertices V
    // Output: list of edges in MST
    for all v in V:
        v.cost = ∞
        v.prev = null
    s = a random v in V // pick a random source s
    s = 0
    MST = []
    PQ = PriorityQueue(V) // priorities will be v.cost values
    while PQ is not empty:
        v = PQ.removeMin()
        if v.prev != null: //guarantees we don't add (s, s.prev)
            MST.append((v, v.prev))
        for all incident edges (v,u) of v such that u is in PQ:
            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```

Omin

Activity #1

Runtime of Prim-Jarnik

2 min

Activity #2

Runtime of Prim-Jarnik

2 min

Activity #2

Runtime of Prim-Jarnik

1 min

Activity #2

Runtime of Prim-Jarnik

O min.

Activity #2

Runtime Analysis

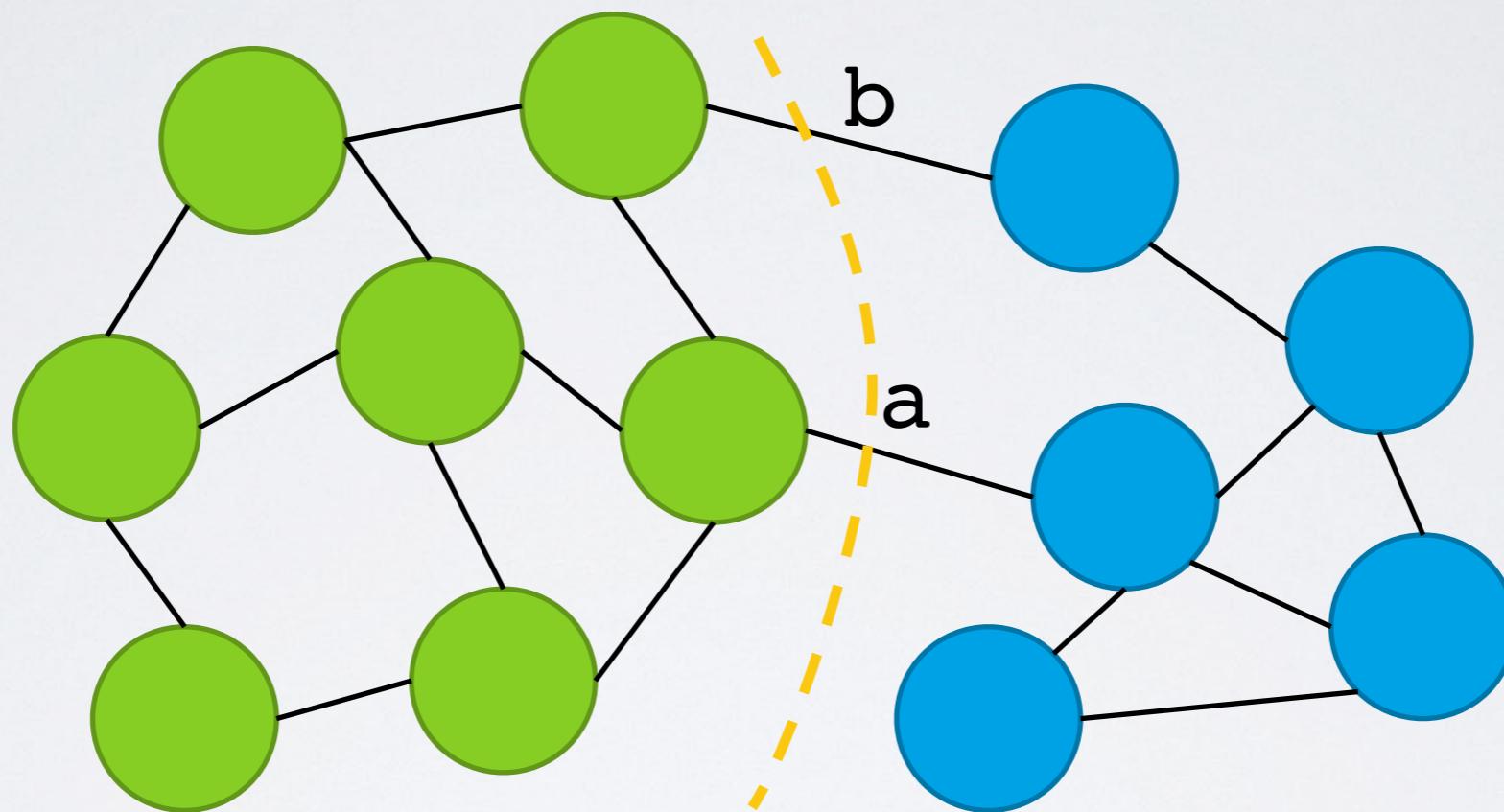
- ▶ Decorating nodes with distance and previous pointers is $O(|v|)$
- ▶ Putting nodes in PQ is $O(|v|\log|v|)$ (really $O(|v|)$ since ∞ priorities)
- ▶ While loop runs $|v|$ times
 - ▶ removing vertex from PQ is $O(\log|v|)$
 - ▶ So $O(|v|\log|v|)$
- ▶ For loop (in while loop) runs $|E|$ times **in total**
 - ▶ Replacing vertex's key in the PQ is $\log|v|$
 - ▶ So $O(|E|\log|v|)$
- ▶ Overall runtime
 - ▶ $O(|v| + |v|\log|v| + |v|\log|v| + |E|\log|v|)$
 - ▶ $= O((|E| + |v|)\log|v|)$

Proof of Correctness

- ▶ Common way of proving correctness of greedy algos
 - ▶ show that algorithm is always correct at every step
- ▶ Best way to do this is by induction
 - ▶ tricky part is coming up with the right invariant

Graph Cuts

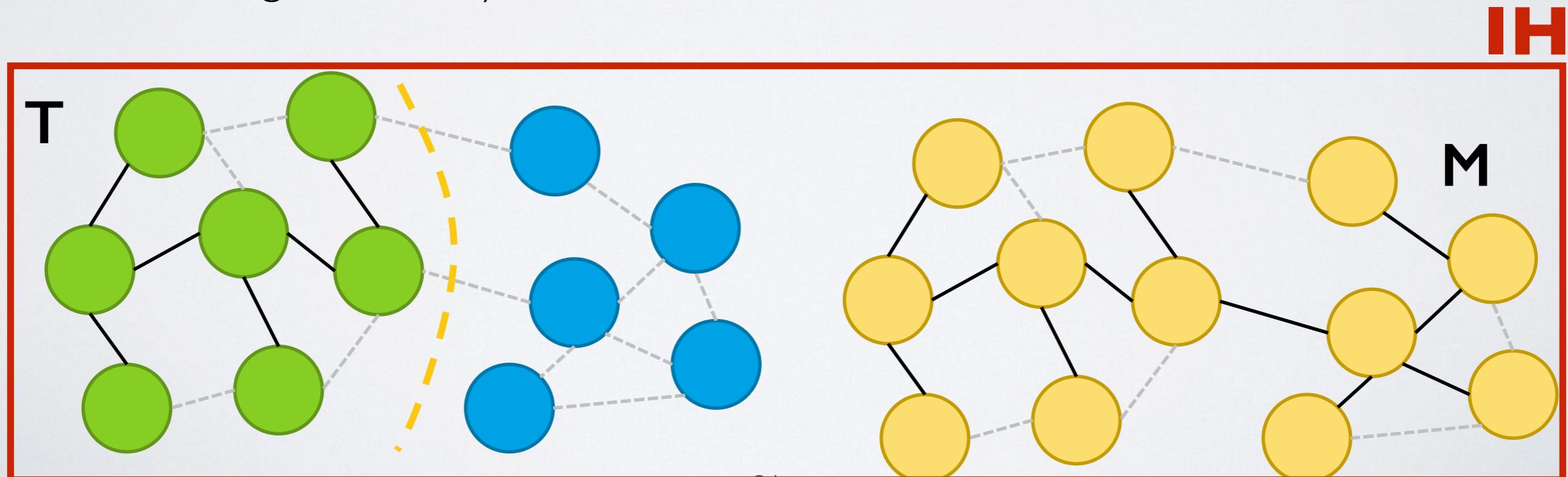
- ▶ A cut is any partition of the vertices into two groups



- ▶ Here G is partitioned in 2
 - ▶ with edges **b** and **a** joining the partitions

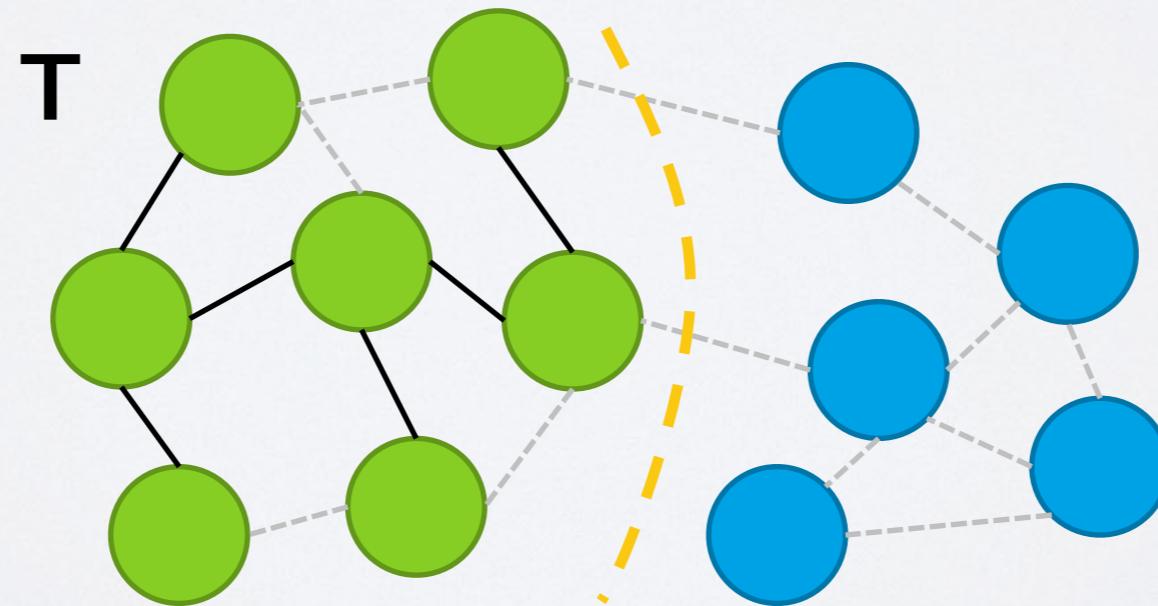
Proof of Correctness

- ▶ $P(n)$
 - ▶ first n edges added by Prim are a subtree of some MST
- ▶ Base case when $n=0$
 - ▶ no edges have been added yet so $P(0)$ is trivially true
- ▶ Inductive Hypothesis
 - ▶ first k edges added by Prim form a tree T which is subtree of some MST M



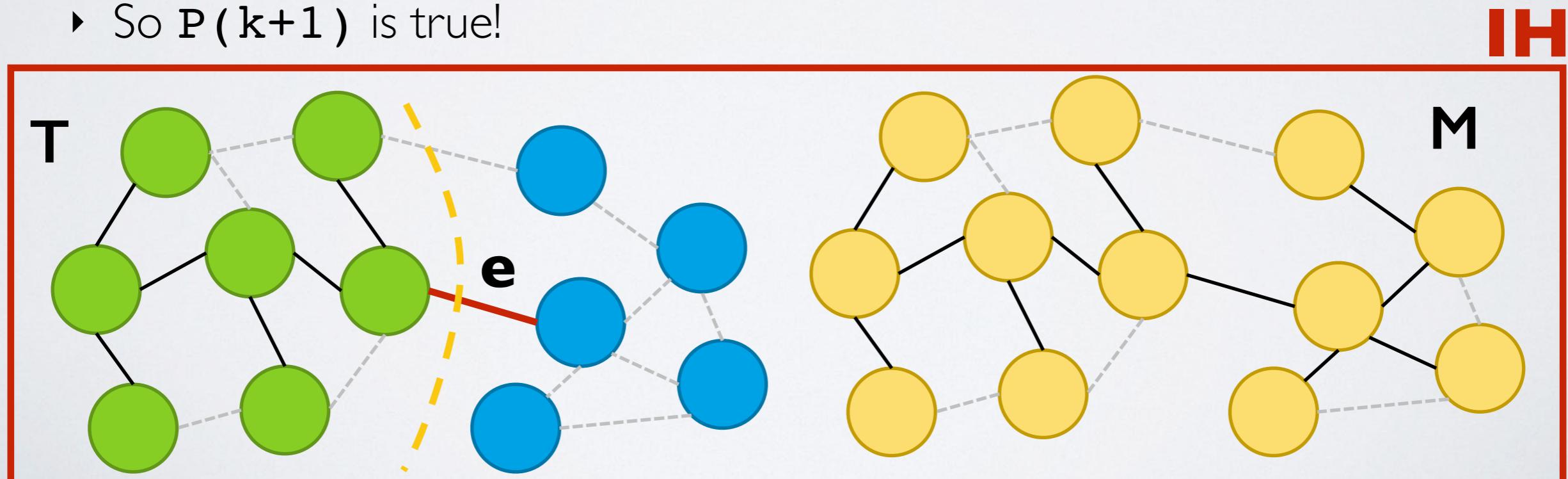
Proof of Correctness

- ▶ Inductive Step
 - ▶ Let e be the $(k+1)$ th edge that is added
 - ▶ e will connect T (green nodes) to an unvisited node (one of blue nodes)
 - ▶ We need to show that adding e to T
 - ▶ forms a subtree of some MST M'
 - ▶ (which may or may not be the same MST as M)



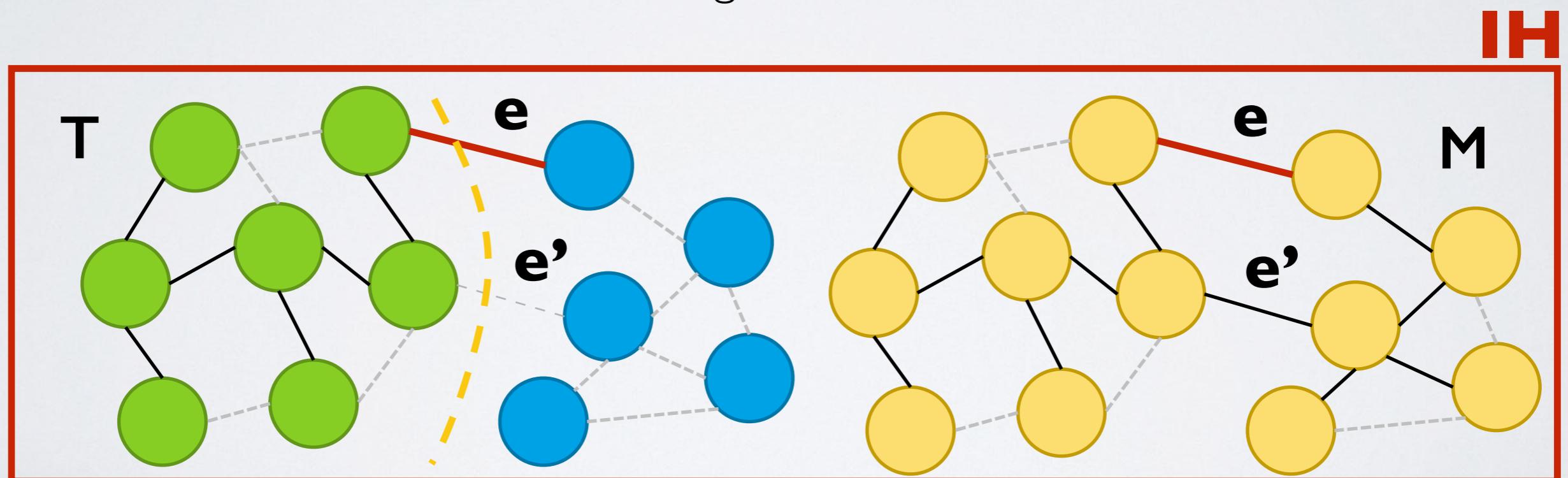
Proof of Correctness

- ▶ Two cases
 - ▶ e is in original MST M
 - ▶ e is not in M
- ▶ Case 1: e is in M
 - ▶ there exists an MST that contains first $k+1$ edges
 - ▶ So $P(k+1)$ is true!



Proof of Correctness

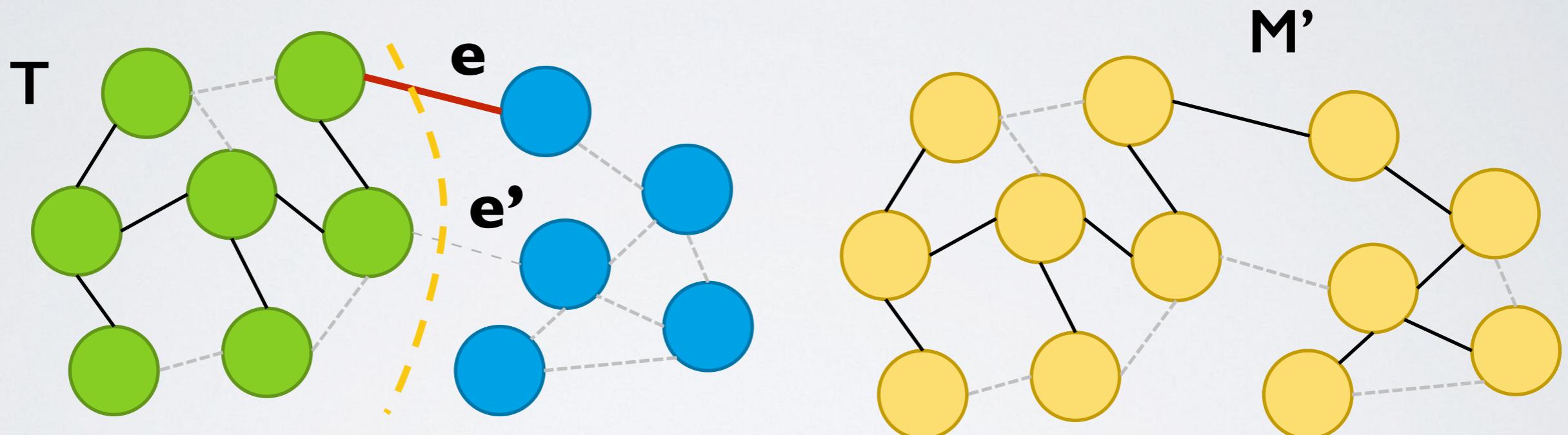
- ▶ Case 2: e is not in M
 - ▶ if we add $e = (u, v)$ to M then we get a cycle
 - ▶ why? since M is span. tree there must be path from u to v w/o e
 - ▶ so there must be another edge e' that connects T to unvisited nodes



- ▶ We know $e.\text{weight} \leq e'.\text{weight}$ because Prim chose e first

Proof of Correctness

- ▶ So if we add e to M and remove e'
 - ▶ we get a new MST M' that is no larger than M and contains T & e



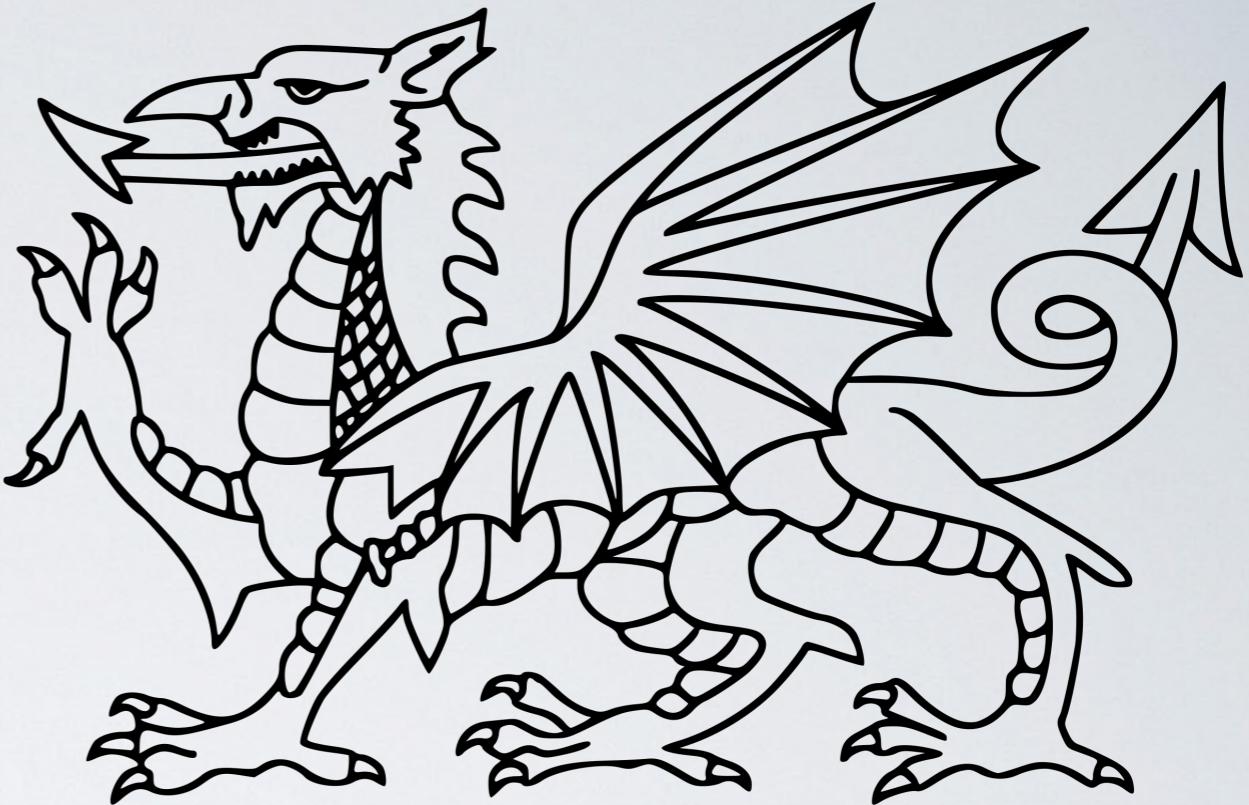
- ▶ $P(k+1)$ is true
 - ▶ because M' is an MST that contains the first $k+1$ edges added by Prim's

Proof of Correctness

- ▶ Since we have shown
 - ▶ $P(0)$ is true
 - ▶ $P(k+1)$ is true assuming $P(k)$ is true (for both cases)
 - ▶ The first n edges added by Prim form a subtree of some MST

Outline

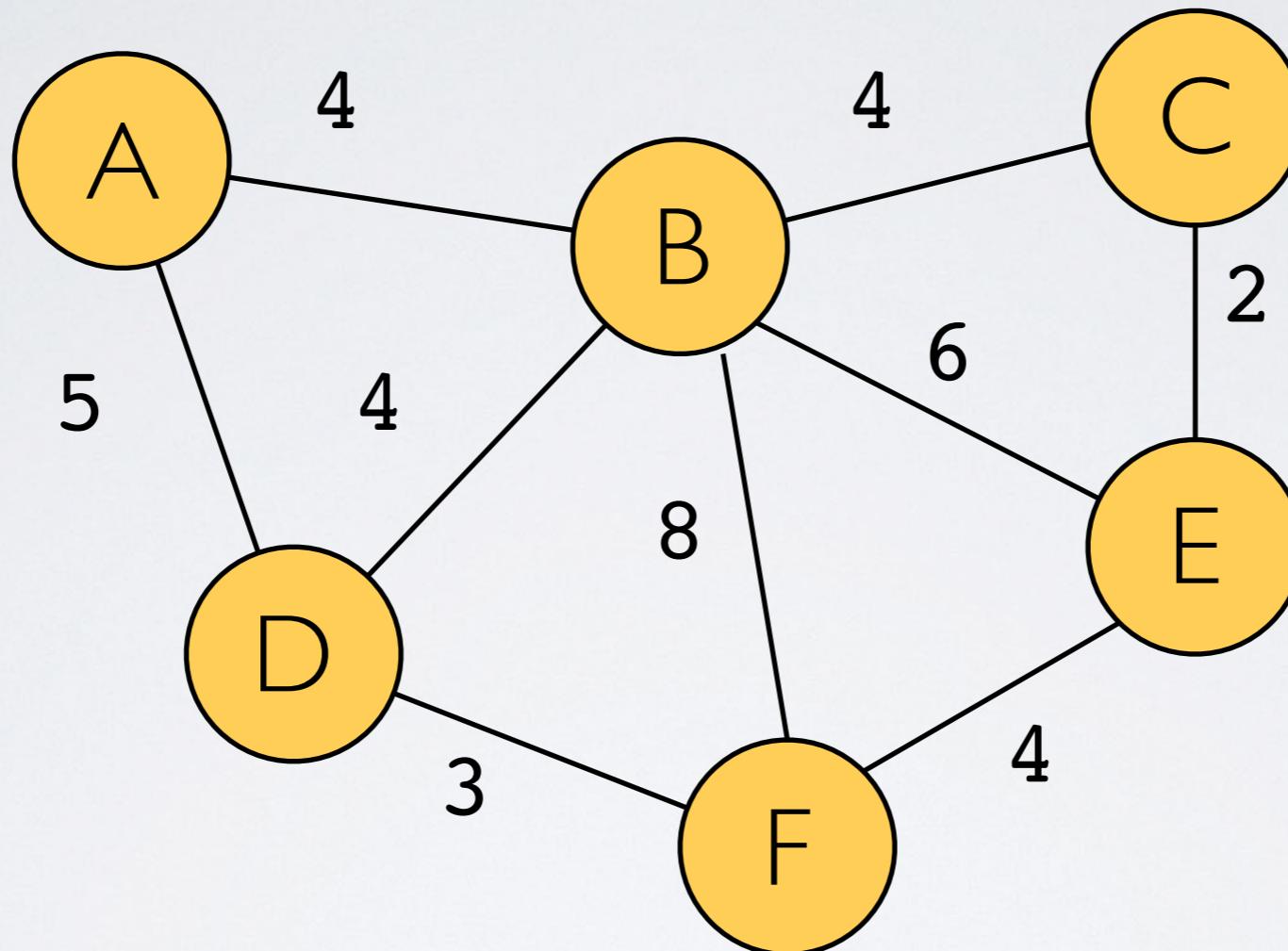
- ▶ Minimum Spanning Trees
- ▶ Prim-Jarnik Algorithm
 - ▶ Analysis
- ▶ Proof of Correctness
- ▶ Kruskal's Algorithm
 - ▶ Union-Find
 - ▶ Analysis



Kruskal's Algorithm

- ▶ Sort edges by weight in ascending order
- ▶ For each edge in sorted list
 - ▶ If adding edge does not create cycle...
 - ▶ ...add it to MST
- ▶ Stop when you have gone through all edges

Example



```
edges = [ (C,E), (D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Simulate Kruskal

2 min

Activity #3

Simulate Kruskal

2 min

Activity #3

Simulate Kruskal

1 min

Activity #3

Simulate Kruskal

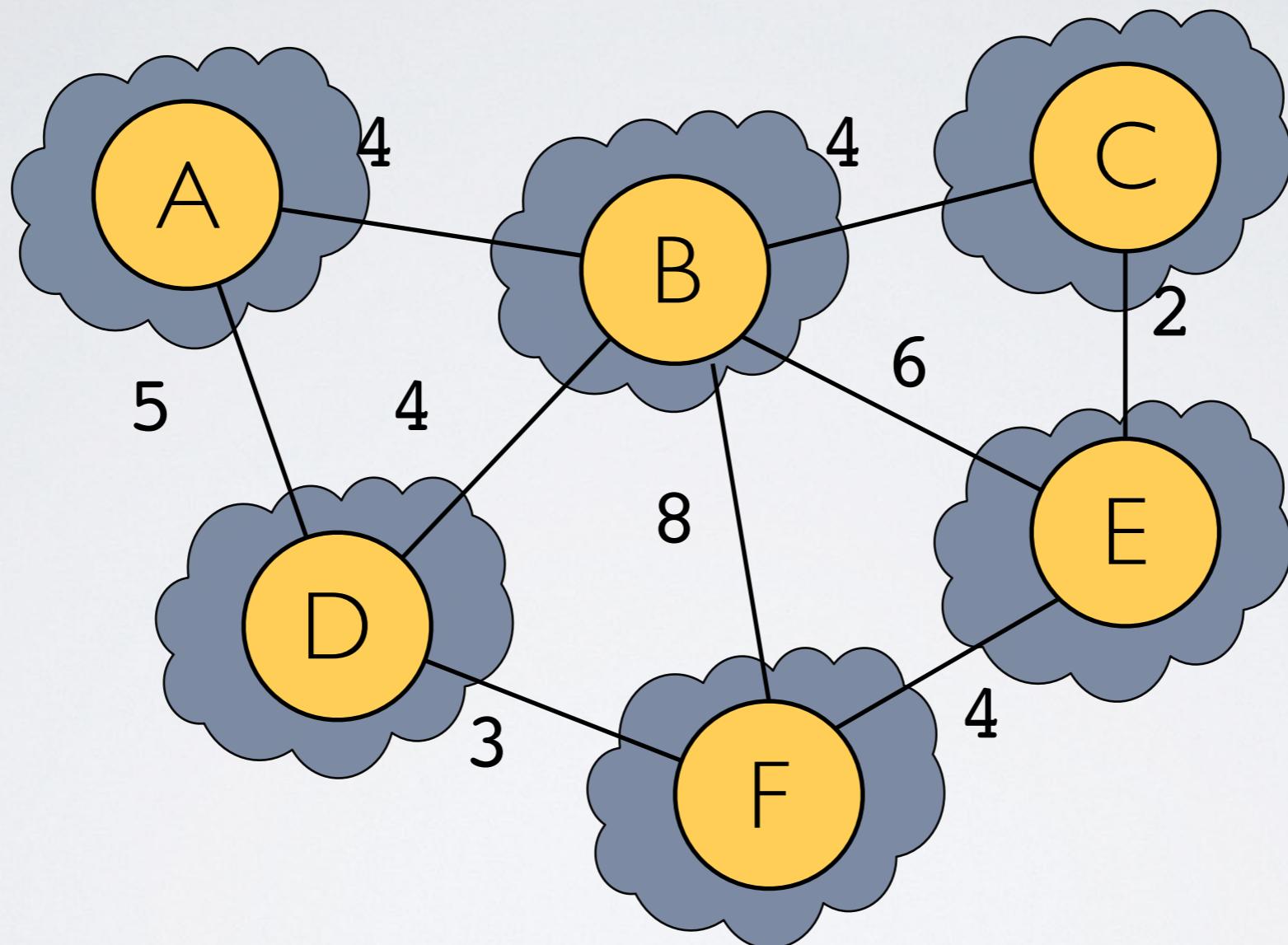
Omin

Activity #3

Kruskal

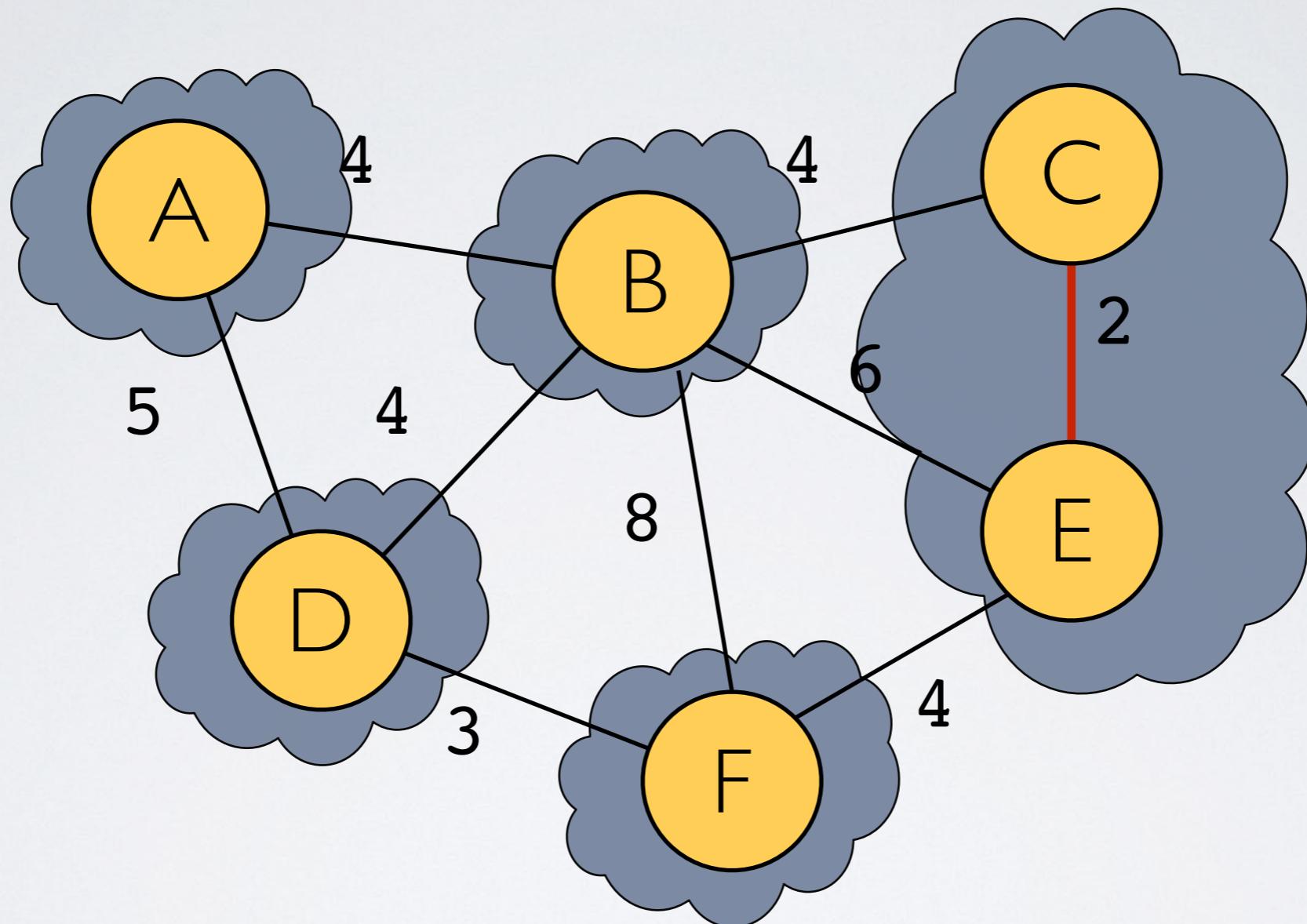
- ▶ How can we tell if adding edge will create cycle?
- ▶ Start by giving each vertex its own “cloud”
- ▶ If both ends of lowest-cost edge are in same cloud
 - ▶ we know that adding the edge will create a cycle!
- ▶ When edge is added to MST
 - ▶ merge clouds of the endpoints

Example



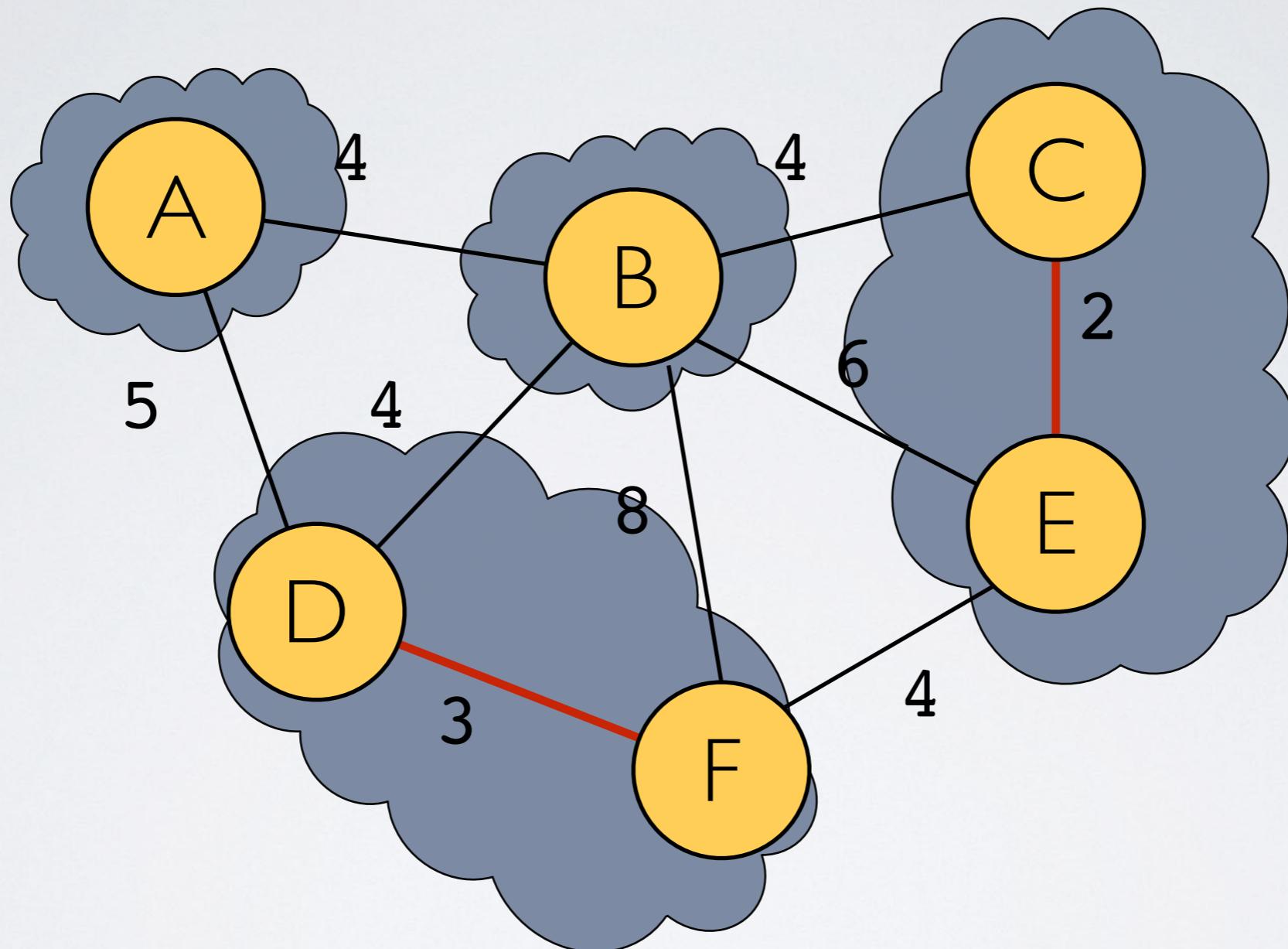
```
edges = [ (C,E) , (D,F) , (B,C) , (E,F) , (B,D) , (A,B) , (A,D) , (B,E) , (B,F) ]
```

Example



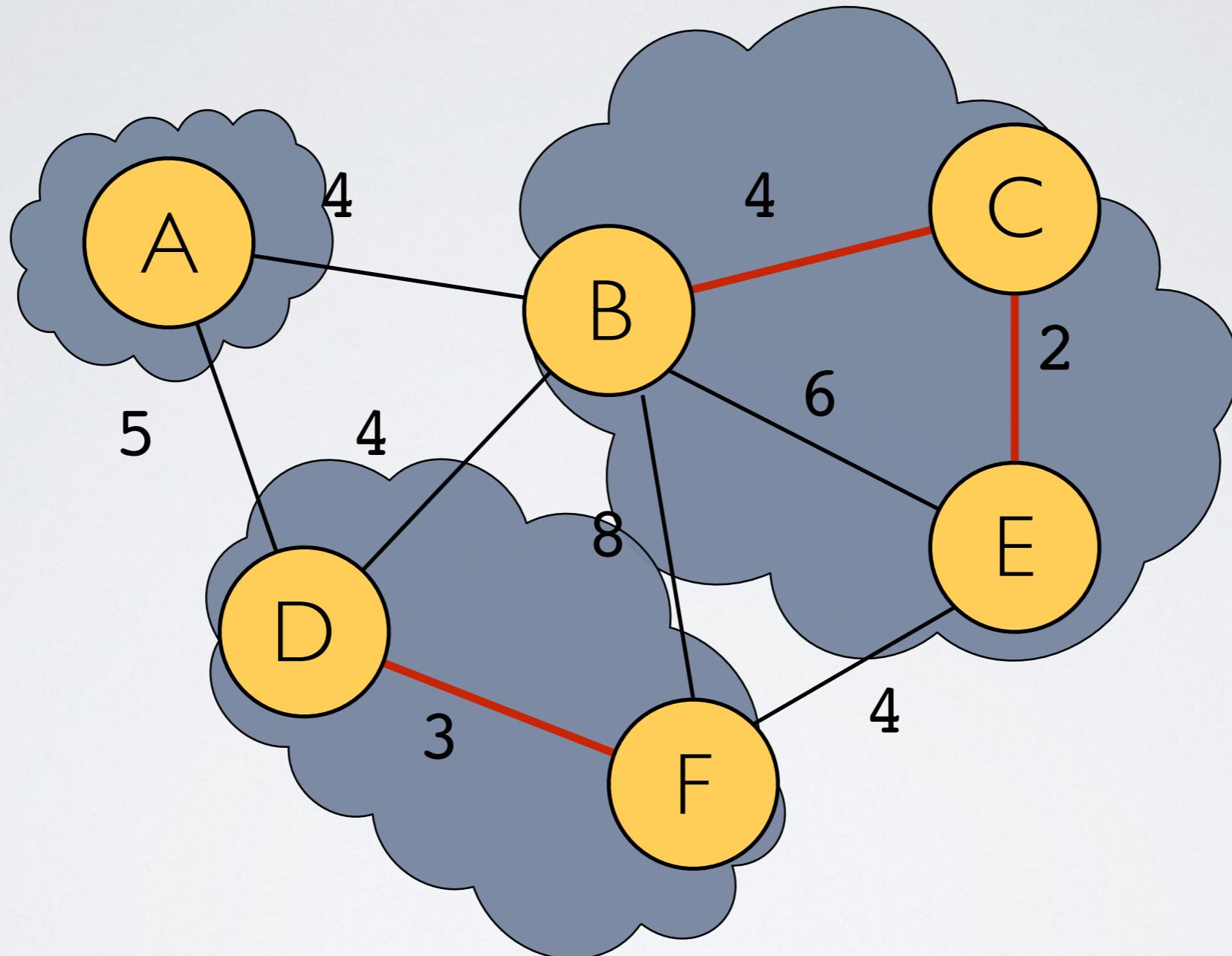
```
edges = [ (D,F) , (B,C) , (E,F) , (B,D) , (A,B) , (A,D) , (B,E) , (B,F) ]
```

Example



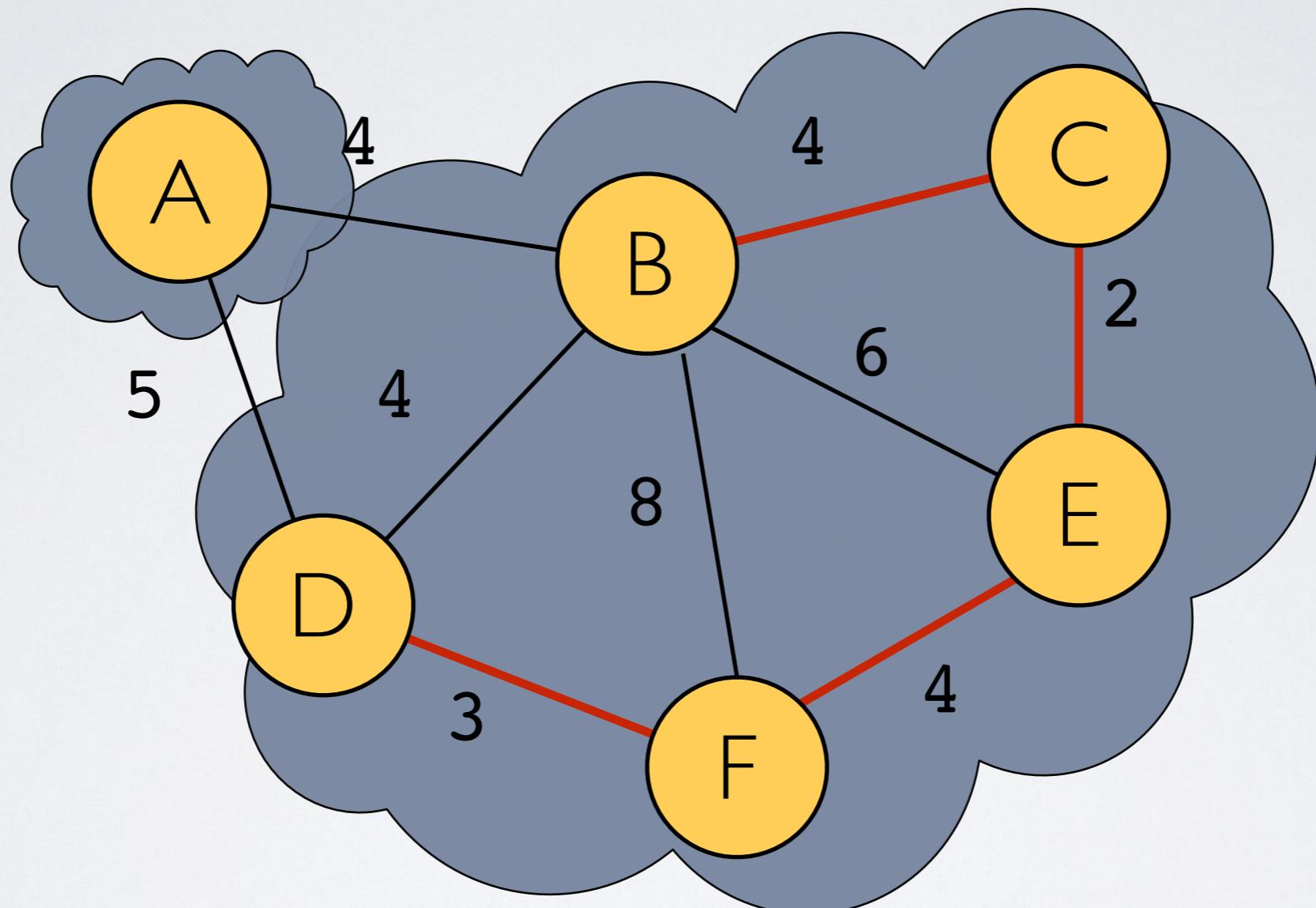
```
edges = [ (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Example



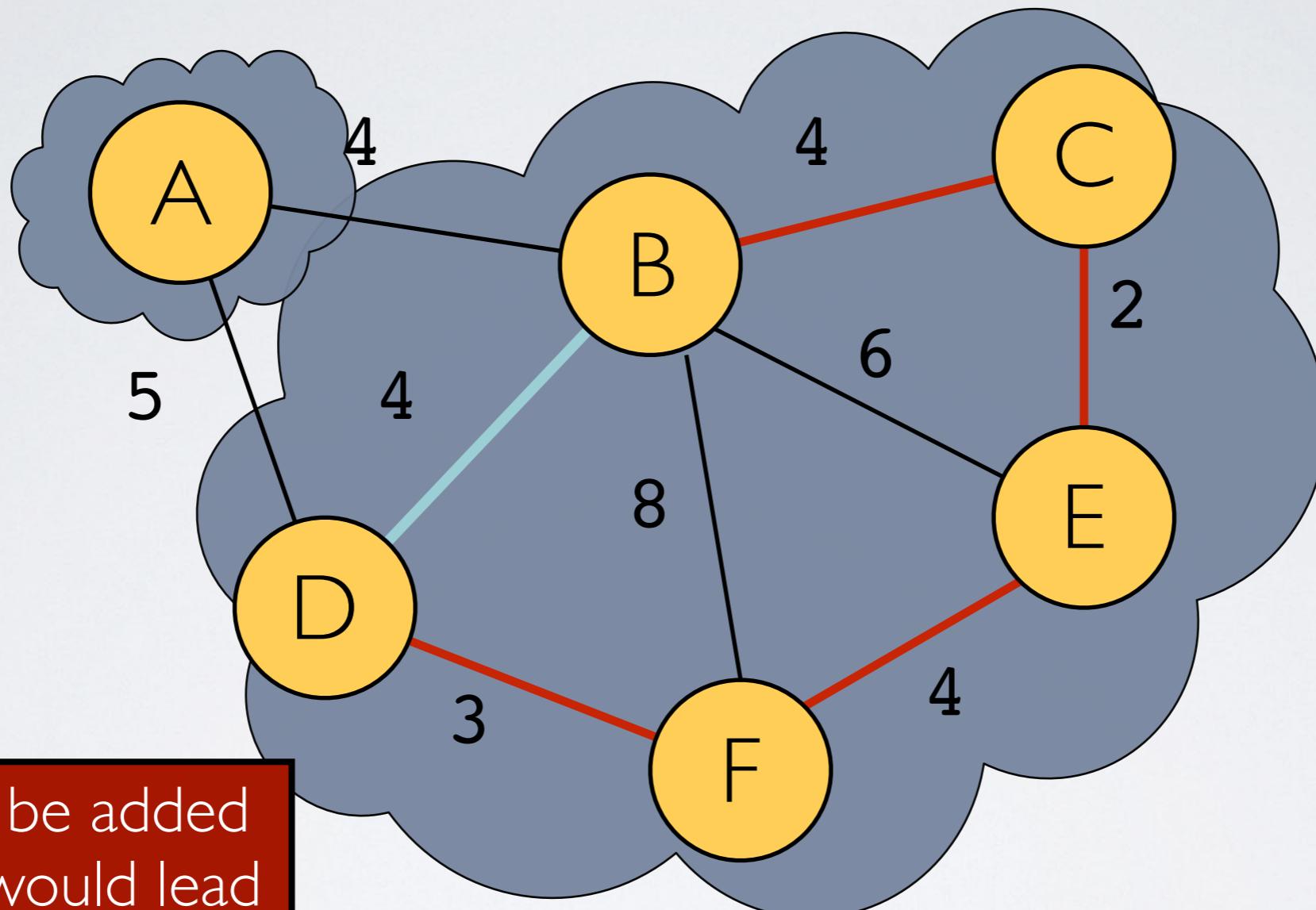
```
edges = [ (E,F), (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Example



```
edges = [ (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

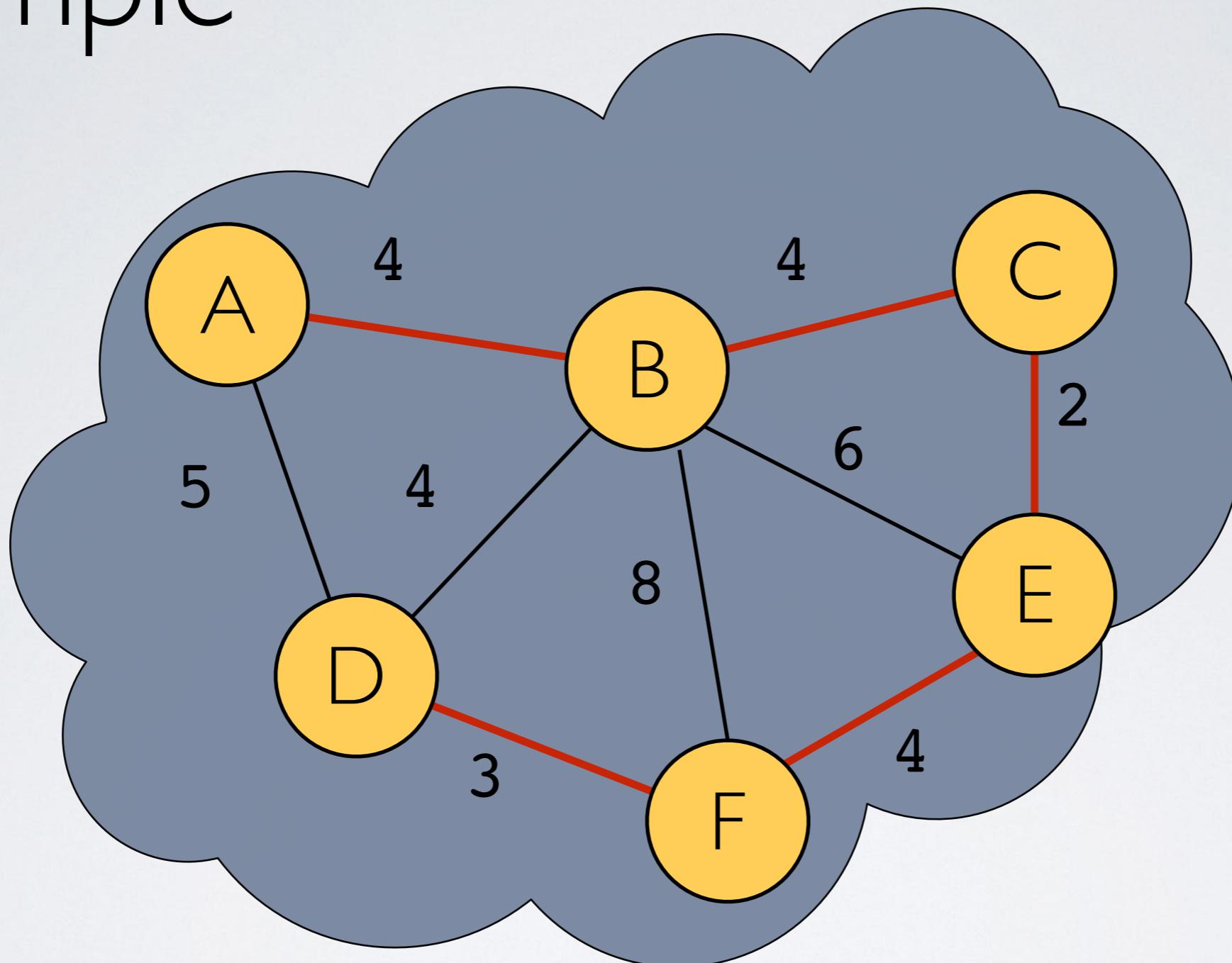
Example



BD cannot be added
because it would lead
to a cycle

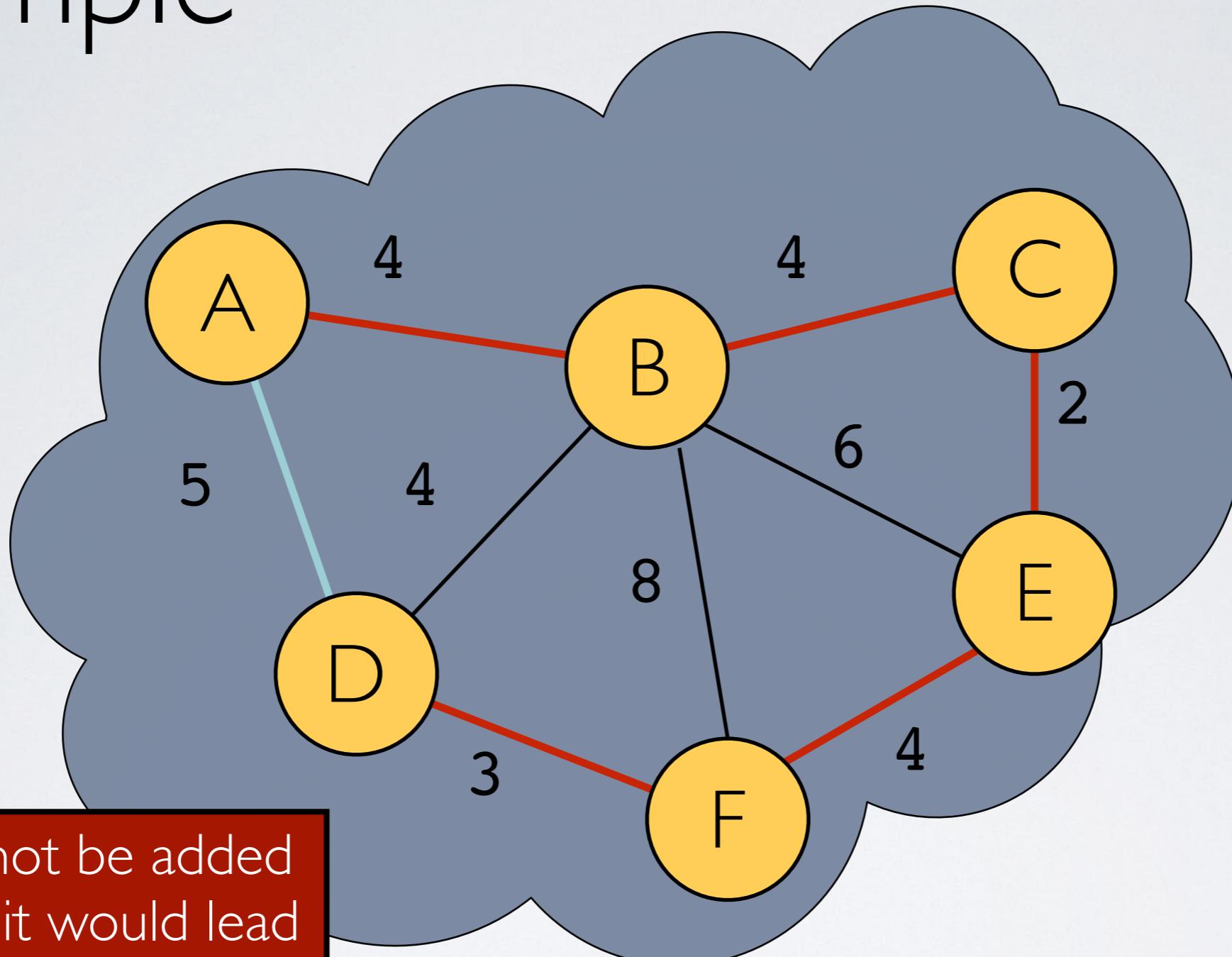
```
edges = [ (A,B), (A,D), (B,E), (B,F) ]
```

Example



```
edges = [ (A,D), (B,E), (B,F) ]
```

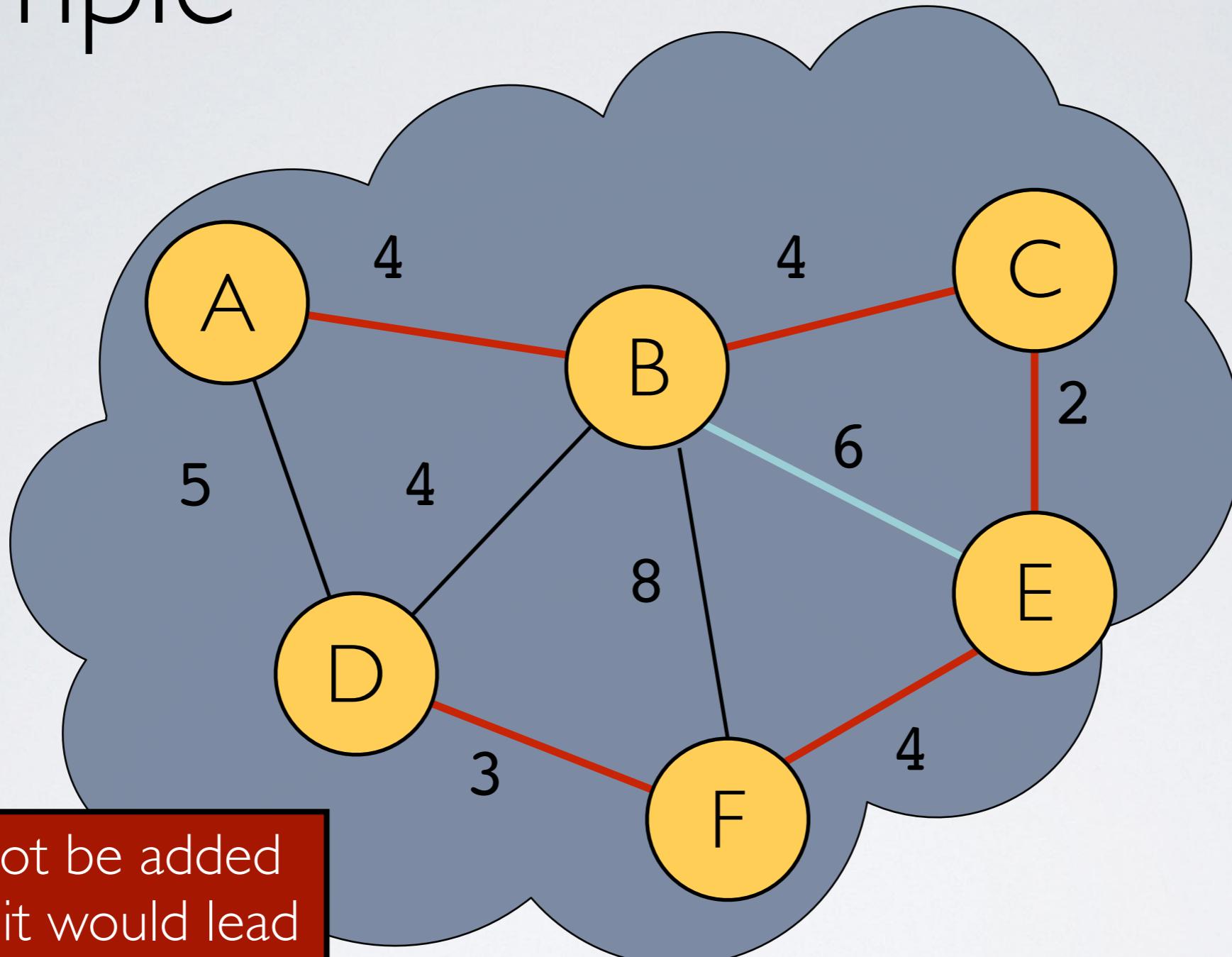
Example



AD cannot be added
because it would lead
to a cycle

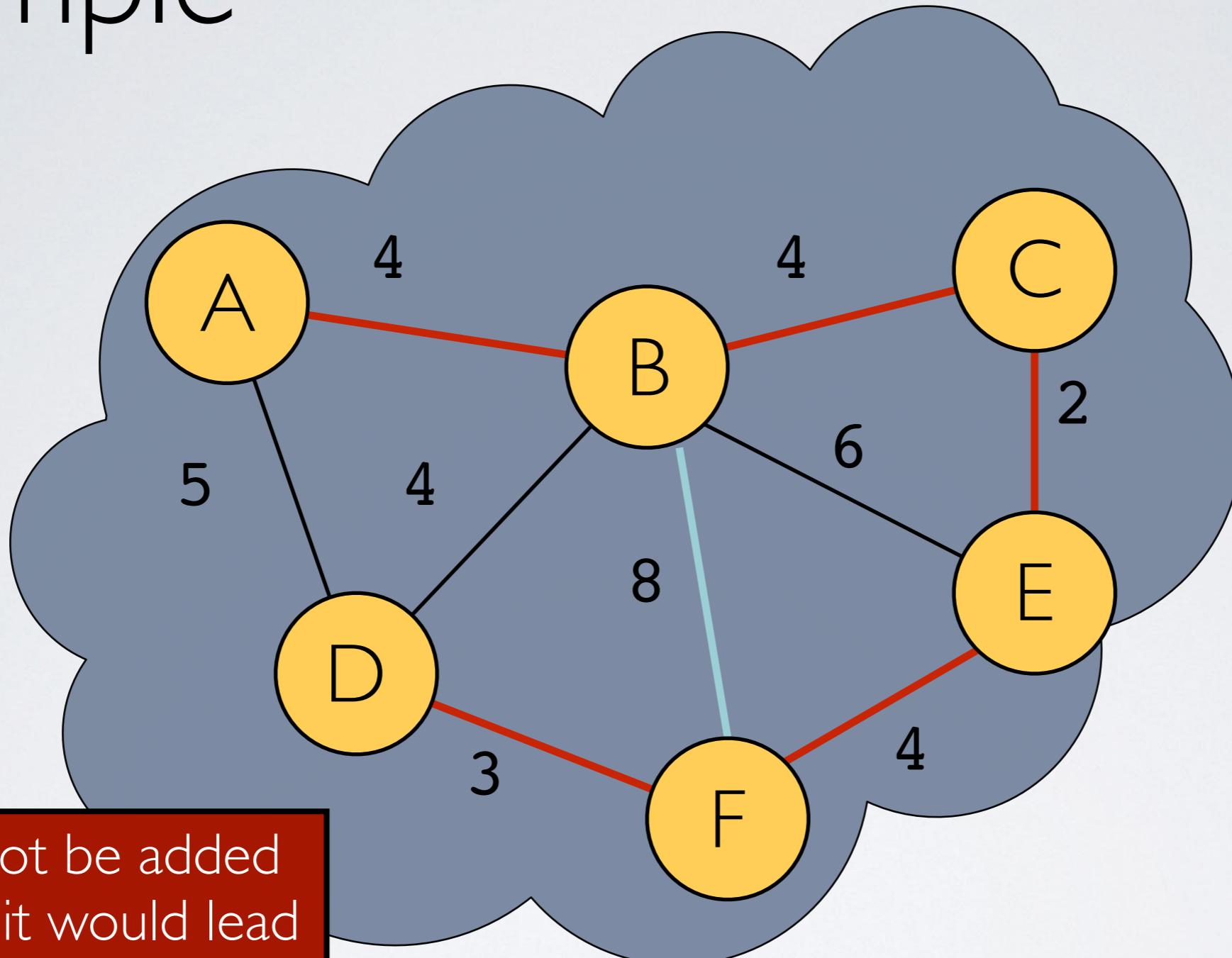
```
edges = [ (B,E) , (B,F) ]
```

Example



```
edges = [ (B, F) ]
```

Example



edges = []

Kruskal Pseudo-Code

```
function kruskal(G):
    // Input: undirected, weighted graph G
    // Output: list of edges in MST
    for vertices v in G:
        makeCloud(v) // put every vertex into its own set
    MST = []
    Sort all edges
    for all edges (u,v) in G sorted by weight:
        if u and v are not in same cloud:
            add (u,v) to MST
            merge clouds containing u and v
    return MST
```

Merging Clouds (Naive way)

- ▶ Assign each vertex a different number
 - ▶ that represents its initial cloud
- ▶ To merge clouds of **u** and **v**
 - ▶ Find all vertices in each cloud
 - ▶ Figure out which of the clouds is smaller
 - ▶ Redecorate all vertices in smaller cloud w/ bigger cloud's number

Merging Clouds (Naive way)

- ▶ Finding all vertices in u & v 's clouds is $O(|v|)$
 - ▶ because we have to iterate through each vertex...
 - ▶ ...and check if its cloud number matches u or v 's cloud number
- ▶ Figuring out smaller cloud is $O(1)$
 - ▶ as long as we keep track of cloud size as we find vertices in them
- ▶ Changing cloud numbers of nodes in smaller cloud is $O(|v|)$
 - ▶ because smallest cloud could be as big as $|v|/2$ vertices
- ▶ Total runtime to merge clouds
 - ▶ $O(|v| + 1 + |v|) = O(|v|)$

Runtime of Naive Kruskal

- ▶ Finding all vertices in u & v 's clouds is $O(|v|)$
 - ▶ because we have to iterate through each vertex...
 - ▶ ...and check if its cloud number matches u or v 's cloud number
- ▶ Figuring out smaller cloud is $O(1)$
 - ▶ as long as we keep track of cloud size as we find vertices in them
- ▶ Changing cloud numbers of vertices in smaller cloud is $O(|v|)$
 - ▶ because cloud could be as big as $|v|/2$ vertices
- ▶ Merge Runtime
 - ▶ $O(|v|) + O(1) + O(|v|) = O(|v|)$

2 min.

Activity #4

Runtime of Naive Kruskal

- ▶ Finding all vertices in u & v 's clouds is $O(|v|)$
 - ▶ because we have to iterate through each vertex...
 - ▶ ...and check if its cloud number matches u or v 's cloud number
- ▶ Figuring out smaller cloud is $O(1)$
 - ▶ as long as we keep track of cloud size as we find vertices in them
- ▶ Changing cloud numbers of vertices in smaller cloud is $O(|v|)$
 - ▶ because cloud could be as big as $|v|/2$ vertices
- ▶ Merge Runtime
 - ▶ $O(|v|) + O(1) + O(|v|) = O(|v|)$

2 min.

Activity #4

Runtime of Naive Kruskal

- ▶ Finding all vertices in u & v 's clouds is $O(|v|)$
 - ▶ because we have to iterate through each vertex...
 - ▶ ...and check if its cloud number matches u or v 's cloud number
- ▶ Figuring out smaller cloud is $O(1)$
 - ▶ as long as we keep track of cloud size as we find vertices in them
- ▶ Changing cloud numbers of vertices in smaller cloud is $O(|v|)$
 - ▶ because cloud could be as big as $|v|/2$ vertices
- ▶ Merge Runtime
 - ▶ $O(|v|) + O(1) + O(|v|) = O(|v|)$

Activity #4

1 min

Runtime of Naive Kruskal

- ▶ Finding all vertices in u & v 's clouds is $O(|v|)$
 - ▶ because we have to iterate through each vertex...
 - ▶ ...and check if its cloud number matches u or v 's cloud number
- ▶ Figuring out smaller cloud is $O(1)$
 - ▶ as long as we keep track of cloud size as we find vertices in them
- ▶ Changing cloud numbers of vertices in smaller cloud is $O(|v|)$
 - ▶ because cloud could be as big as $|v|/2$ vertices
- ▶ Merge Runtime
 - ▶ $O(|v|) + O(1) + O(|v|) = O(|v|)$

On min.

Activity #4

Kruskal Runtime w/ Naive Clouds

```
function kruskal(G):
    // Input: undirected, weighted graph G
    // Output: list of edges in MST
    for vertices v in G: ←  $O(|v|)$ 
        makeCloud(v)
    MST = []
    Sort all edges ←  $O(|E| \log |E|)$ 
    for all edges (u,v) in G sorted by weight: ←  $O(|E|)$ 
        if u and v are not in same cloud:
            add (u,v) to MST
            merge clouds containing u and v ←  $O(|v|)$ 
    return MST
```

Kruskal Runtime

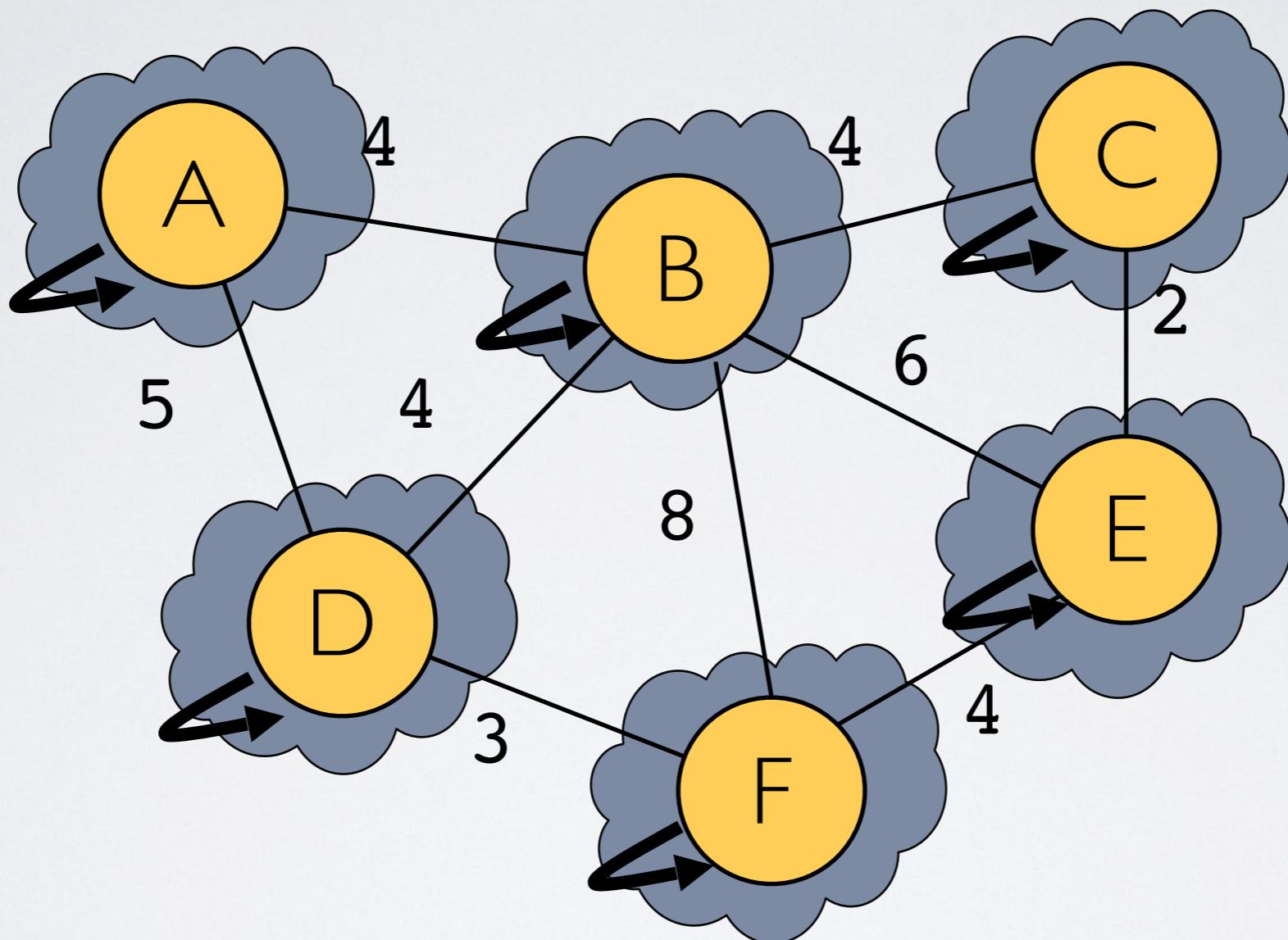
- ▶ $O(|V|)$ for iterating through vertices
- ▶ $O(|E| \log |E|)$ for sorting edges
- ▶ $O(|E| \times |V|)$ for iterating through edges and merging clouds naively
- ▶ $O(|V| + |E| \log |E| + |E| \times |V|)$
 - ▶ $= O(|E| \times |V|) = O(|V|^2 \times |V|) = O(|V|^3)$
- ▶ Can we do better?

since $|E| \leq |V|^2$

Union-Find

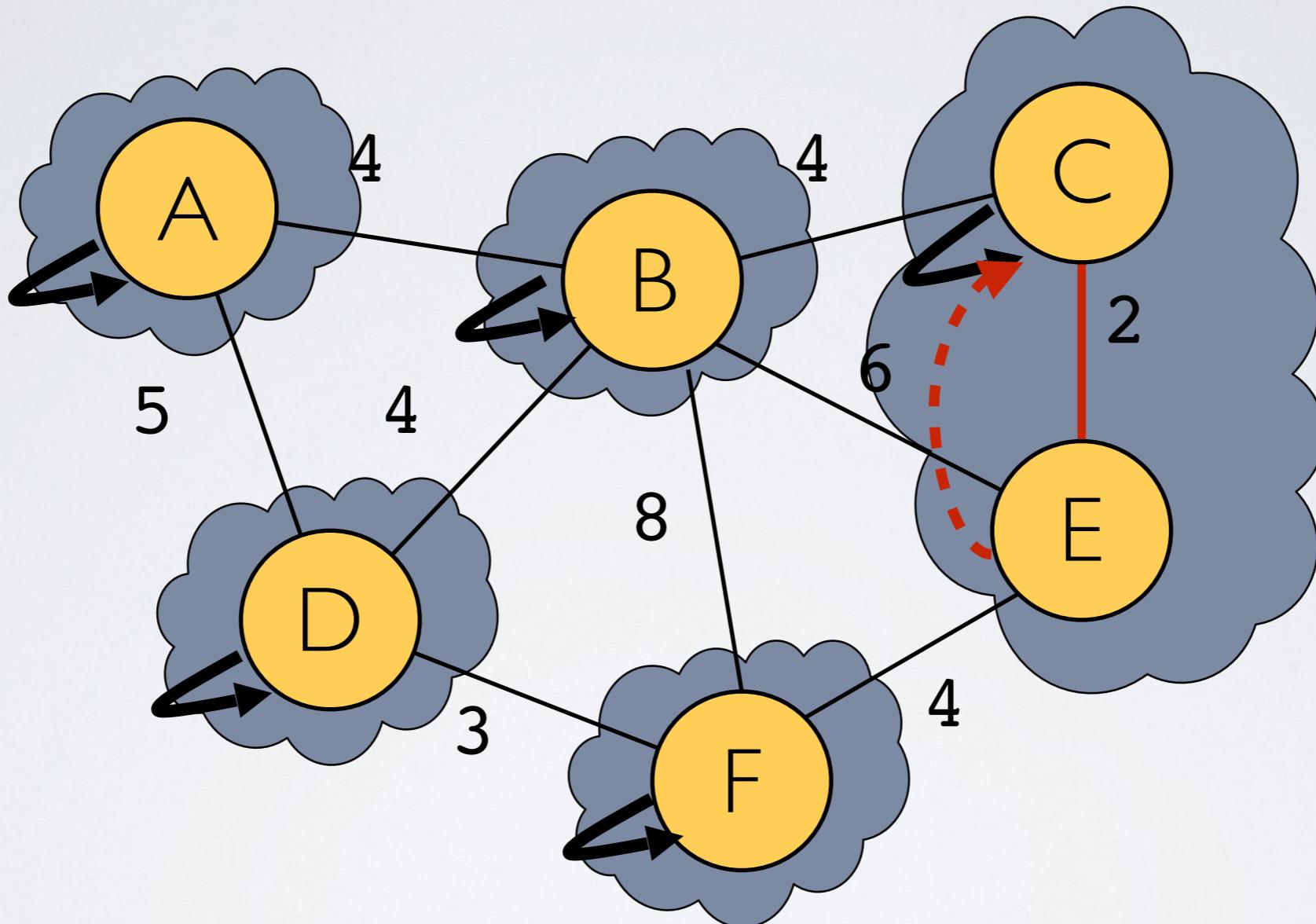
- ▶ Let's rethink notion of clouds
 - ▶ instead of labeling vertices w/ cloud numbers
 - ▶ think of clouds as small trees
- ▶ Every vertex in these trees has
 - ▶ a parent pointer that leads up to root of the tree
 - ▶ a rank that measures how deep the tree is

Example



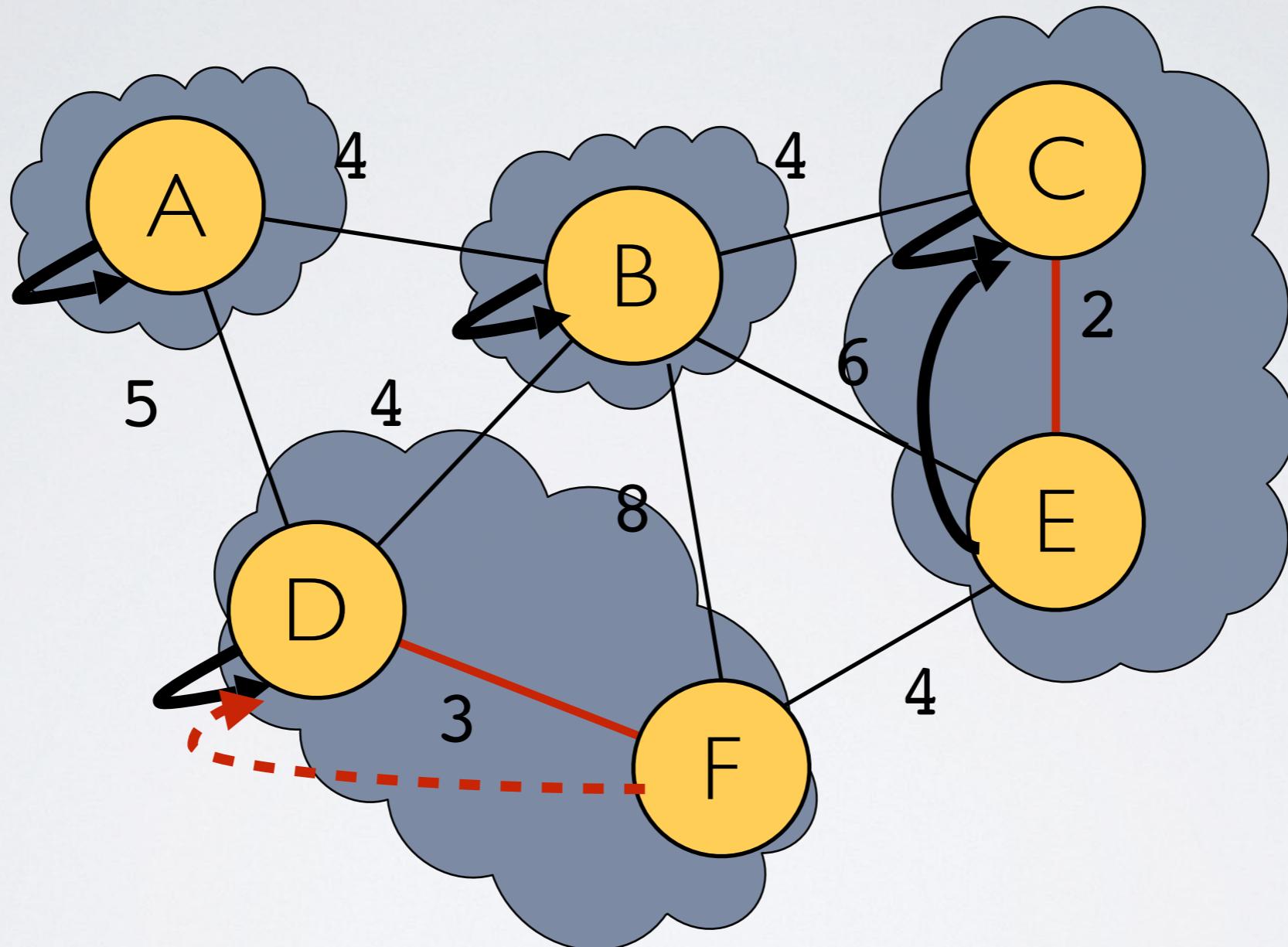
```
edges = [ (C,E), (D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Example



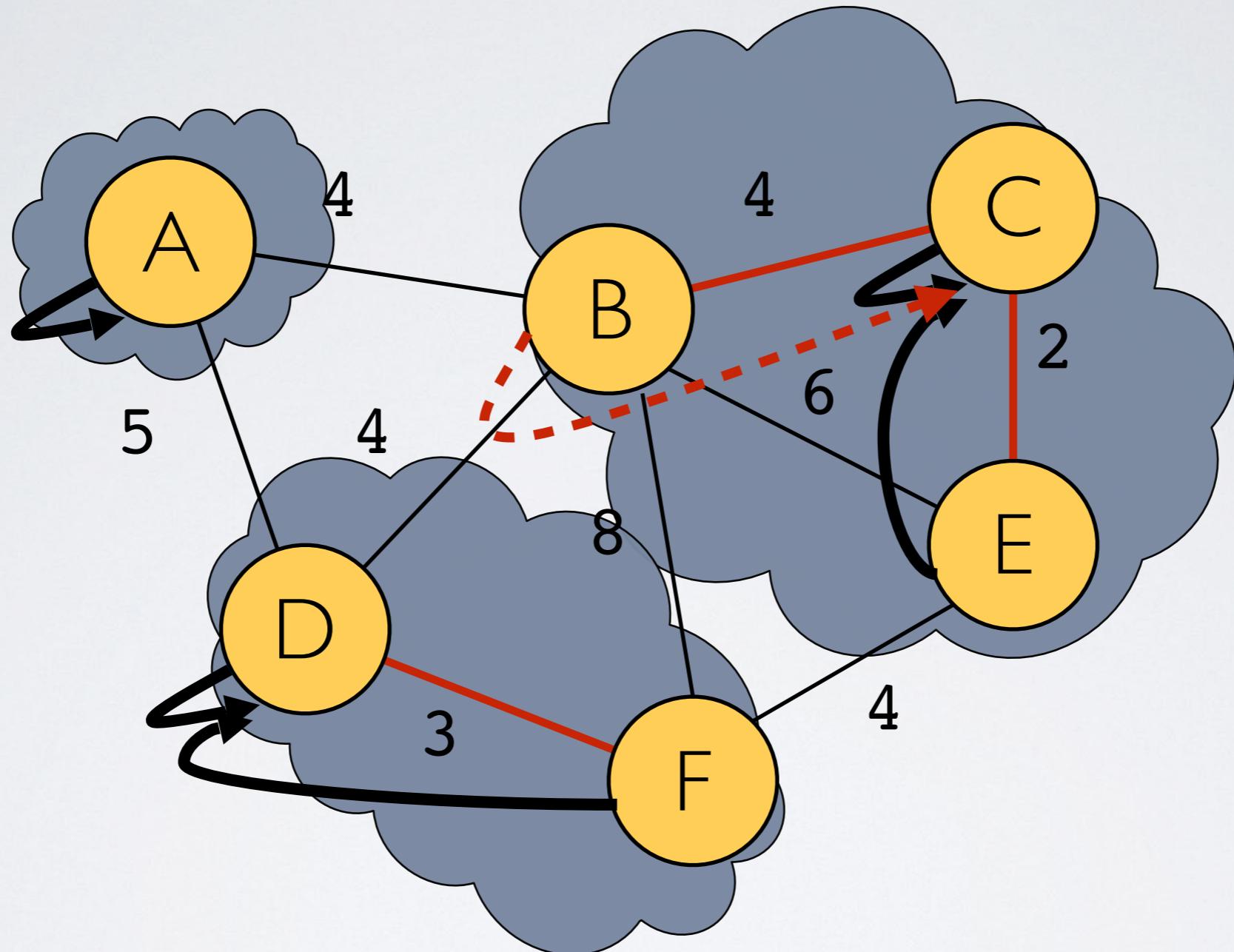
```
edges = [ (D,F) , (B,C) , (E,F) , (B,D) , (A,B) , (A,D) , (B,E) , (B,F) ]
```

Example



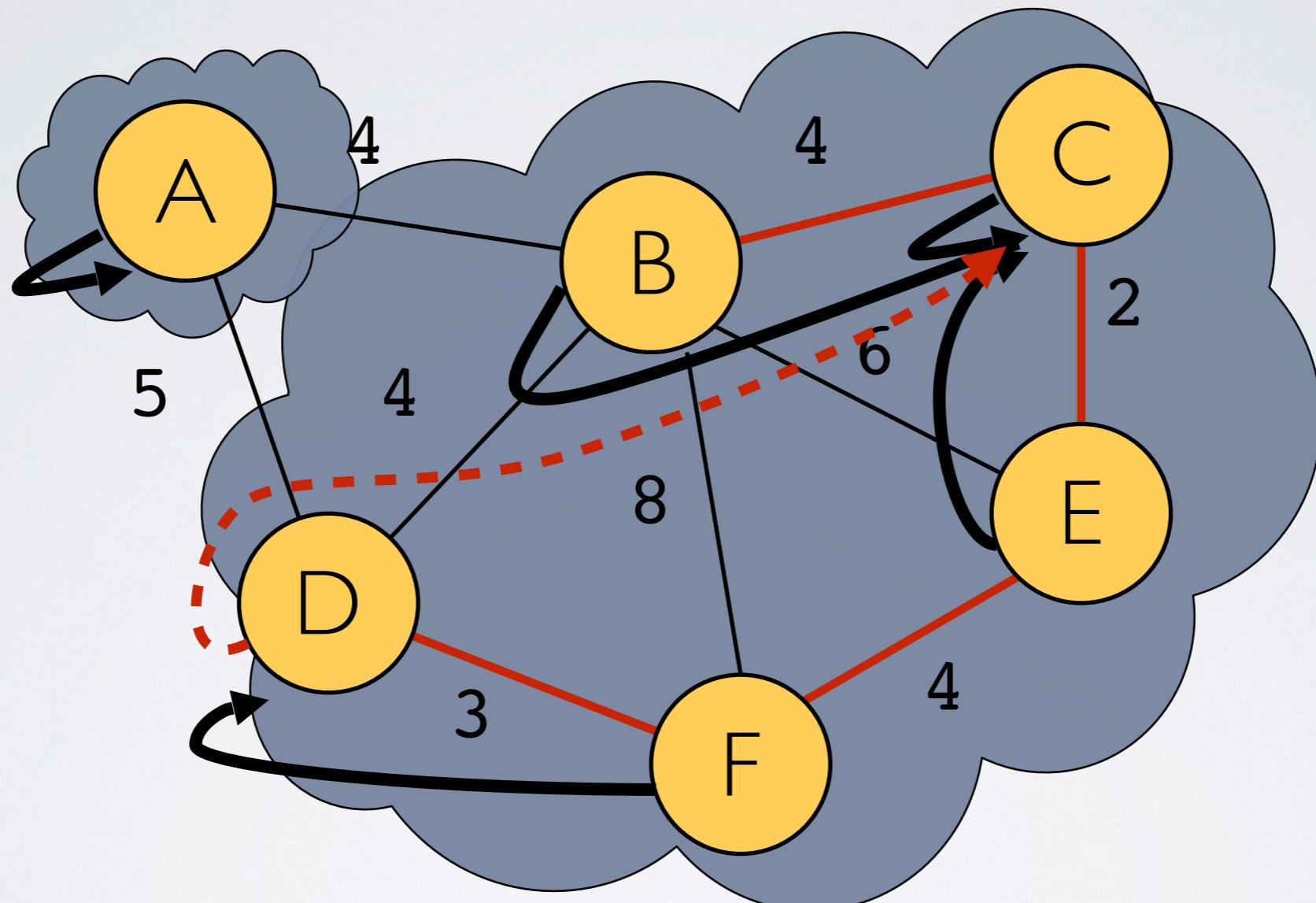
```
edges = [ (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Example



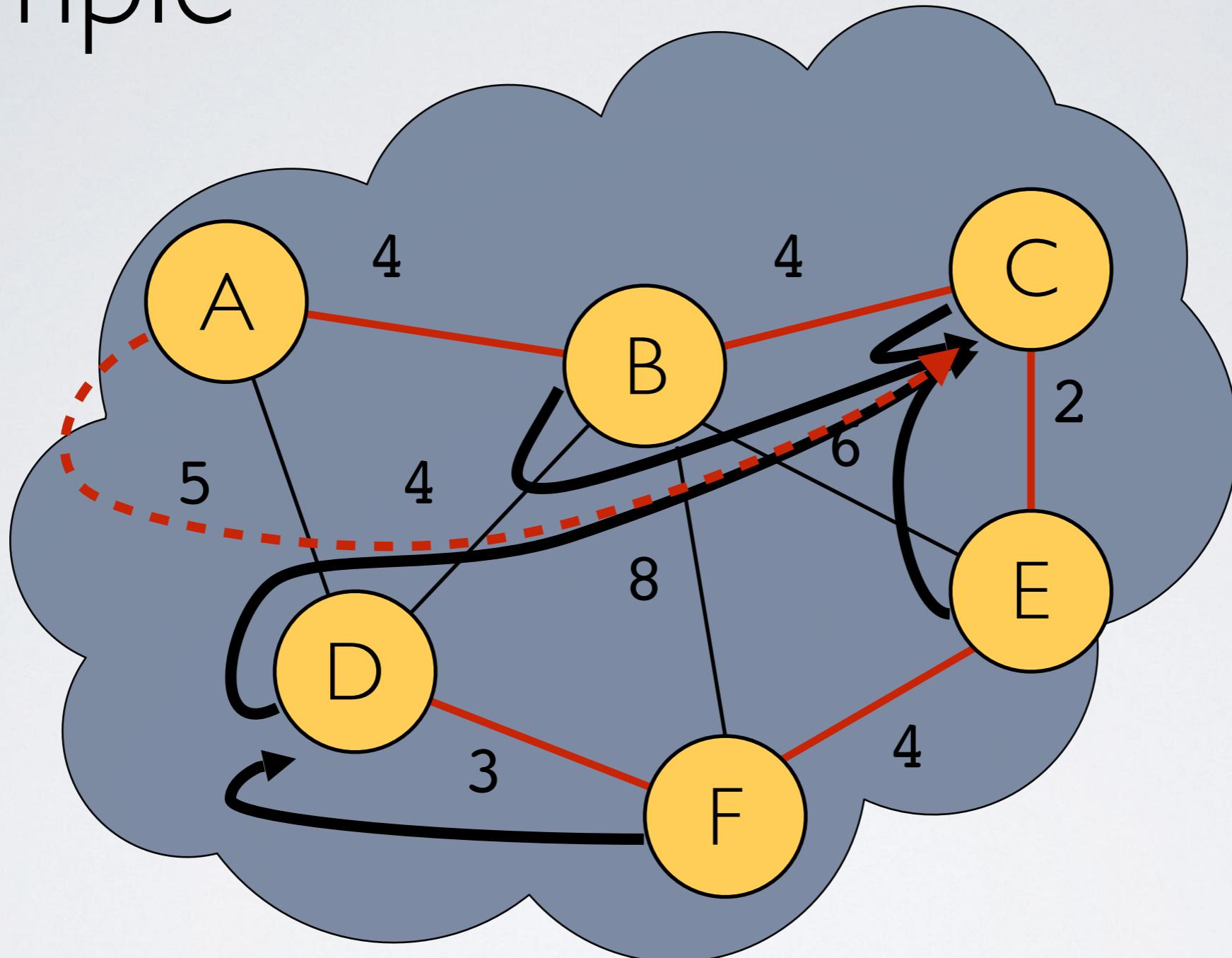
```
edges = [ (E,F), (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Example



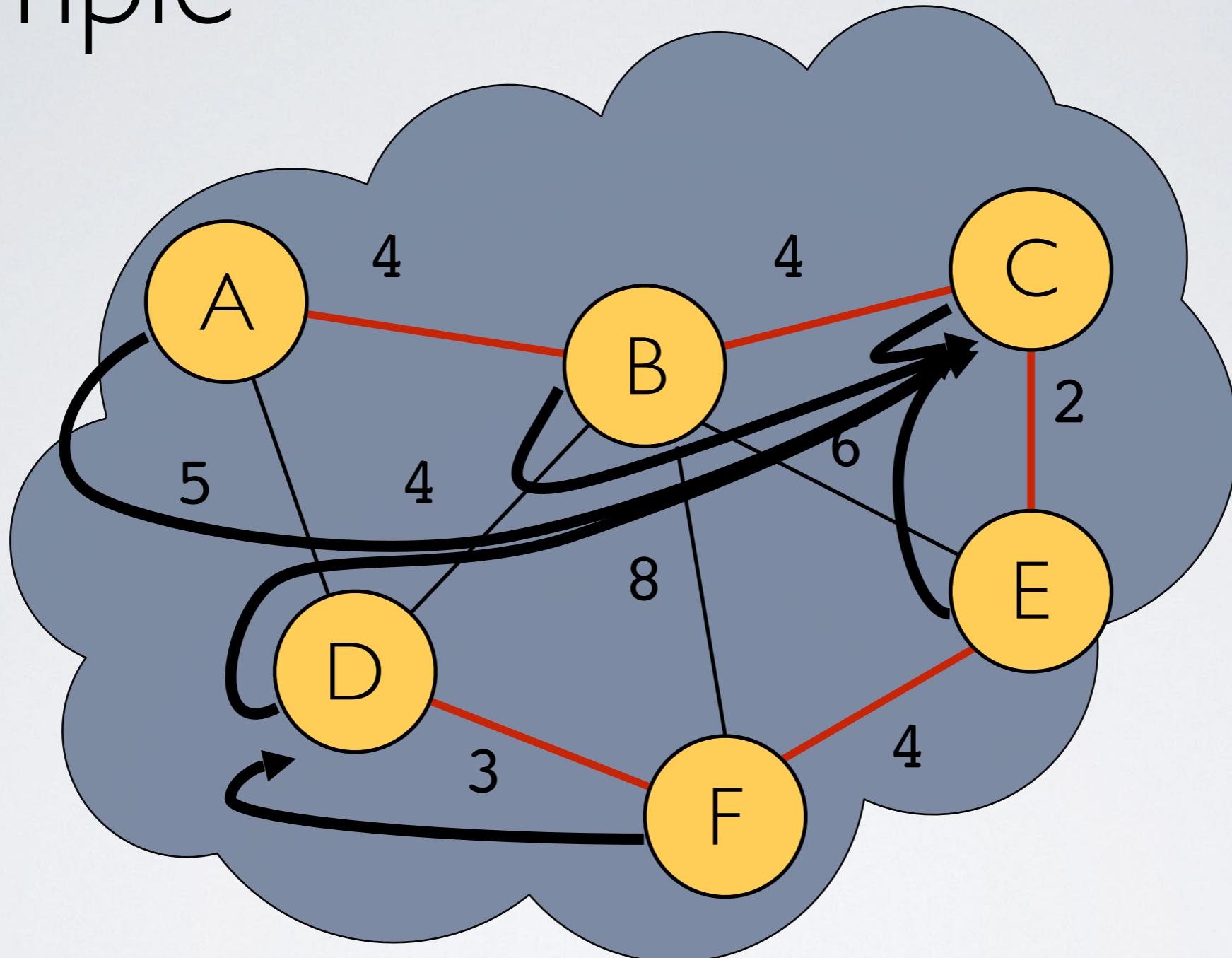
```
edges = [ (B,D), (A,B), (A,D), (B,E), (B,F) ]
```

Example



```
edges = [ (A,D), (B,E), (B,F) ]
```

Example



```
edges = [ (A,D), (B,E), (B,F) ]
```

Implementing Union-Find

- ▶ At start of Kruskal
 - ▶ every node is put into own cloud

```
// Decorates every vertex with its parent ptr & rank
function makeCloud(x):
    x.parent = x
    x.rank = 0
```



Implementing Union-Find

- ▶ Suppose **A** is in cloud 1 and **B** is in cloud 2
- ▶ Instead of relabeling **B** as cloud 1 make **B** point to **A**
 - ▶ Think of this as the union of two clouds



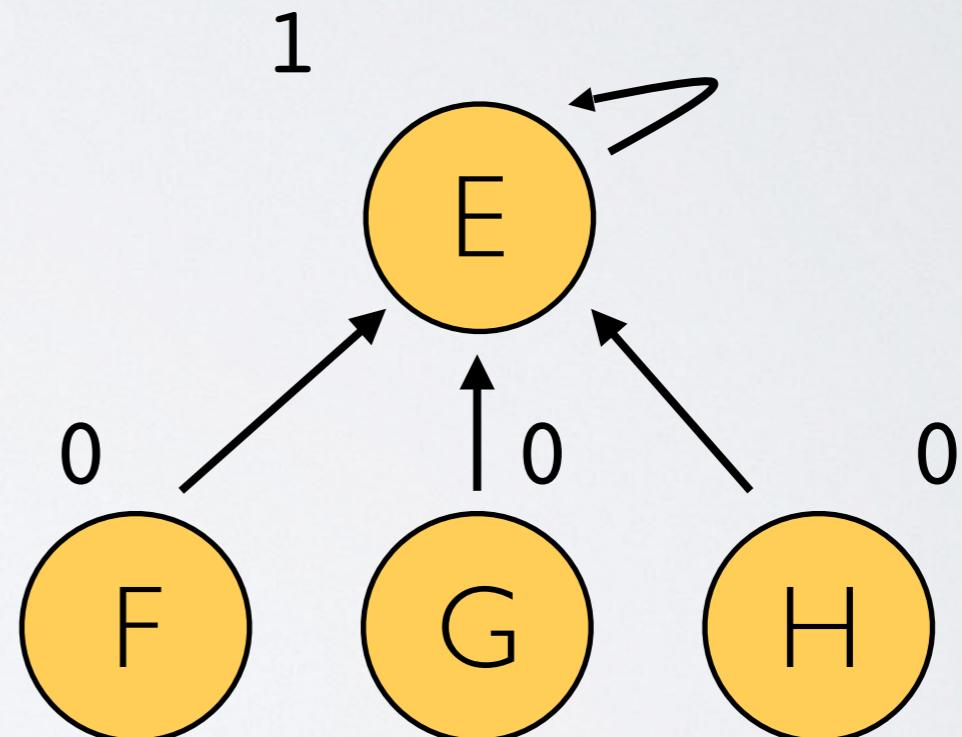
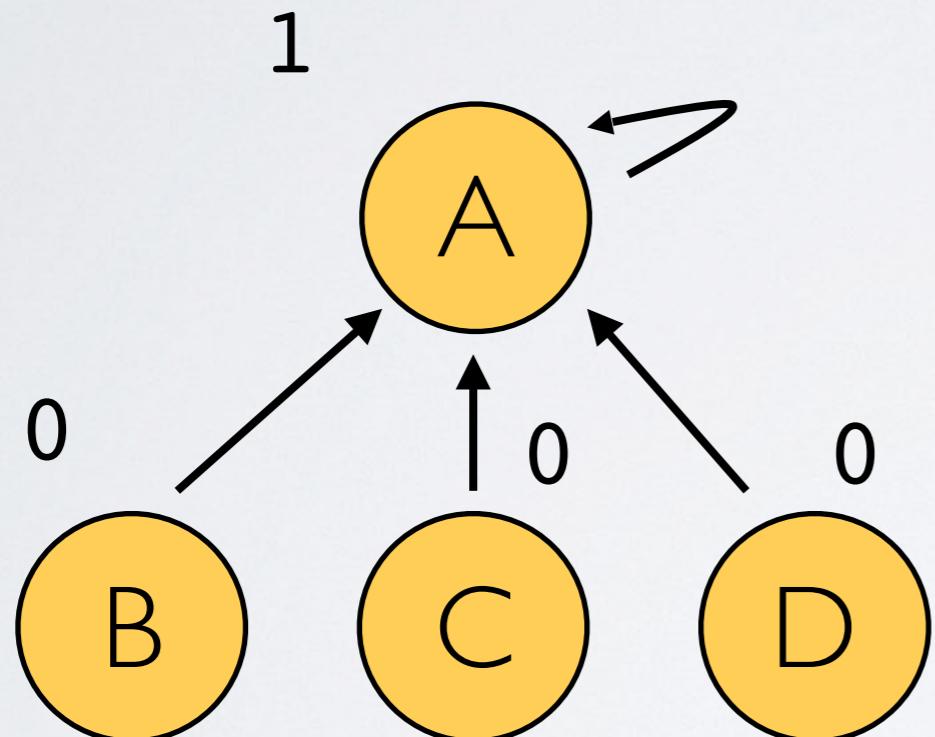
- ▶ Given two clouds which one should point to the other?

Implementing Union-Find

- ▶ We use the rank to decide
 - ▶ make lower-ranked root point to higher-ranked root
 - ▶ then update rank
- ▶ How do we update ranks?
 - ▶ For clouds of size 1 root always has rank 0
 - ▶ For clouds of size larger than 1 we increment rank only when merging clouds of same rank

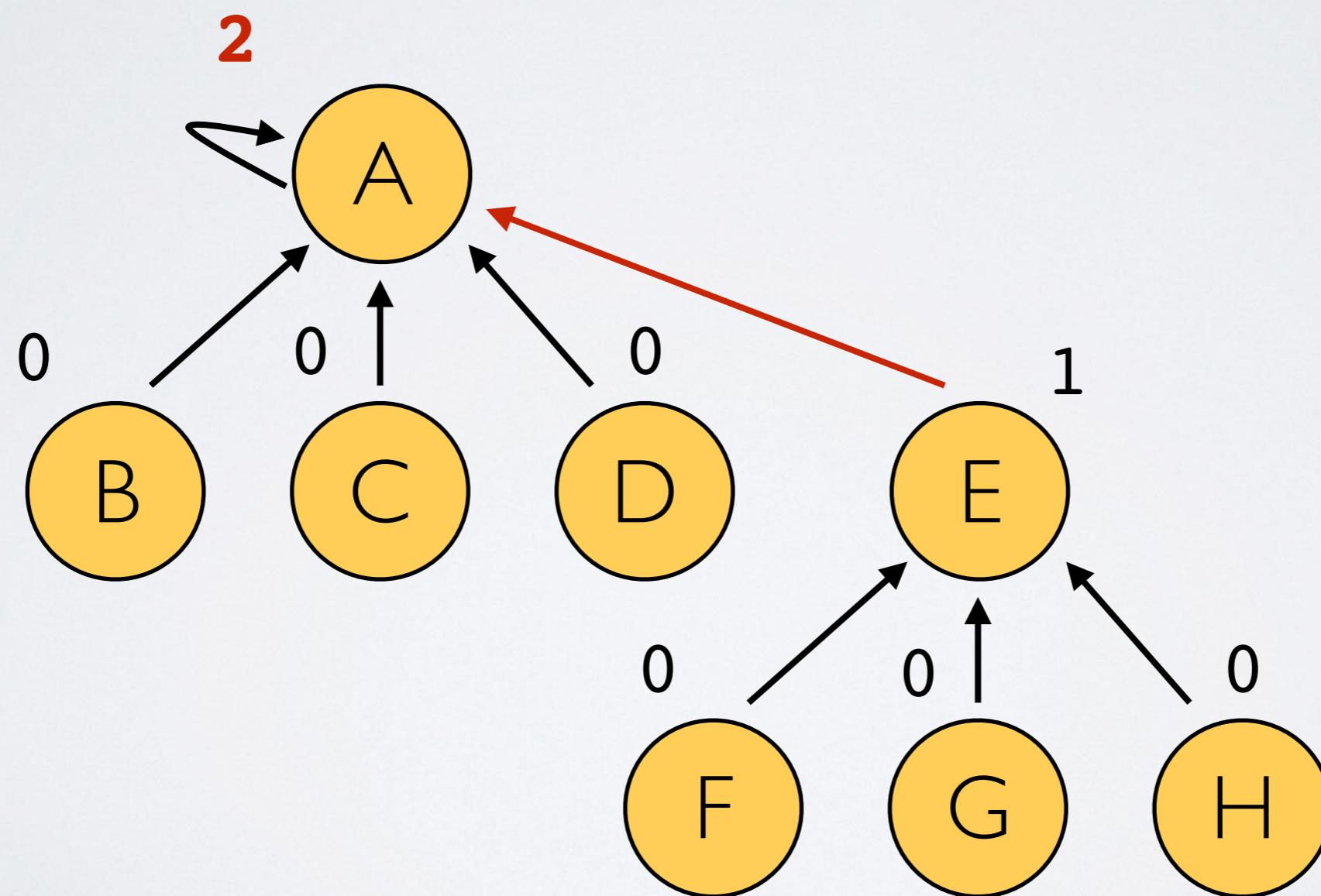
Implementing Union-Find

- ▶ Merging trees with same rank



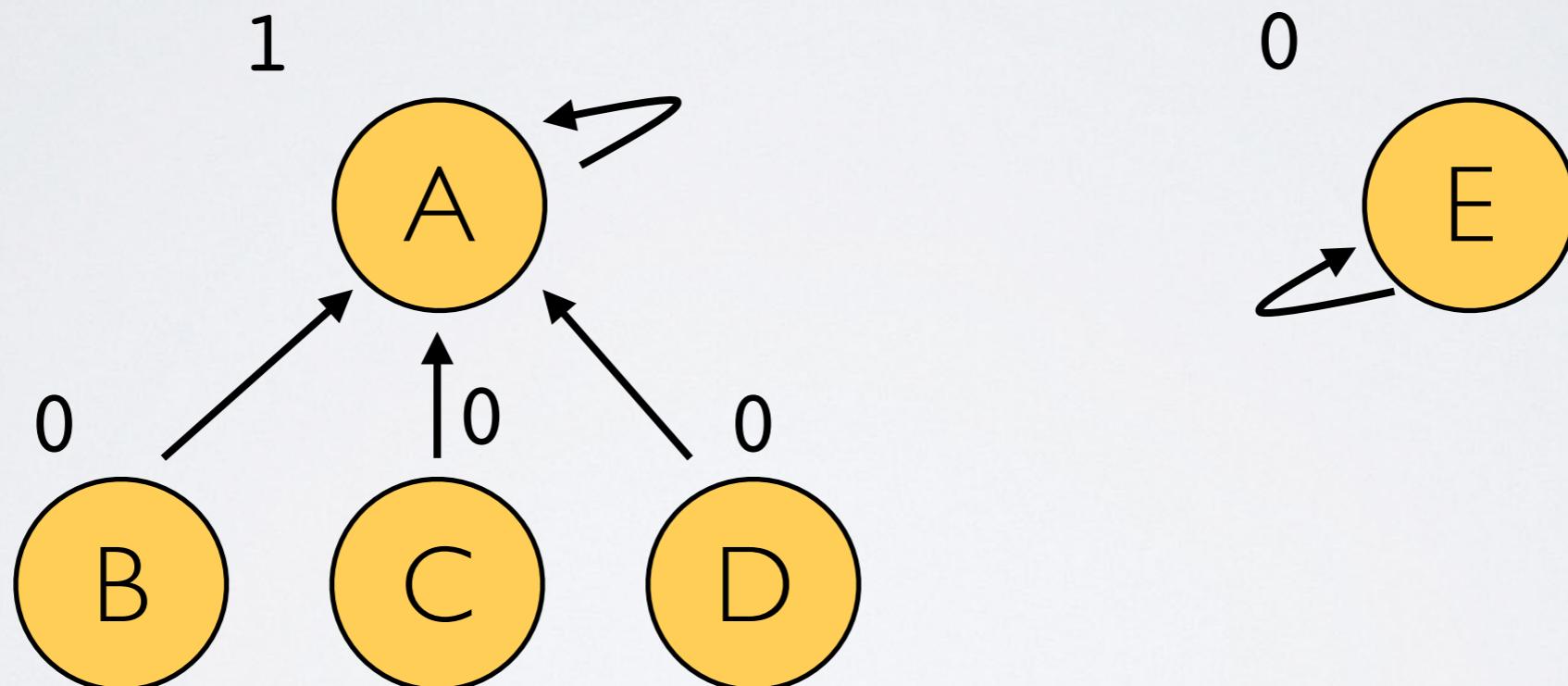
Implementing Union-Find

- ▶ Merging trees with same rank



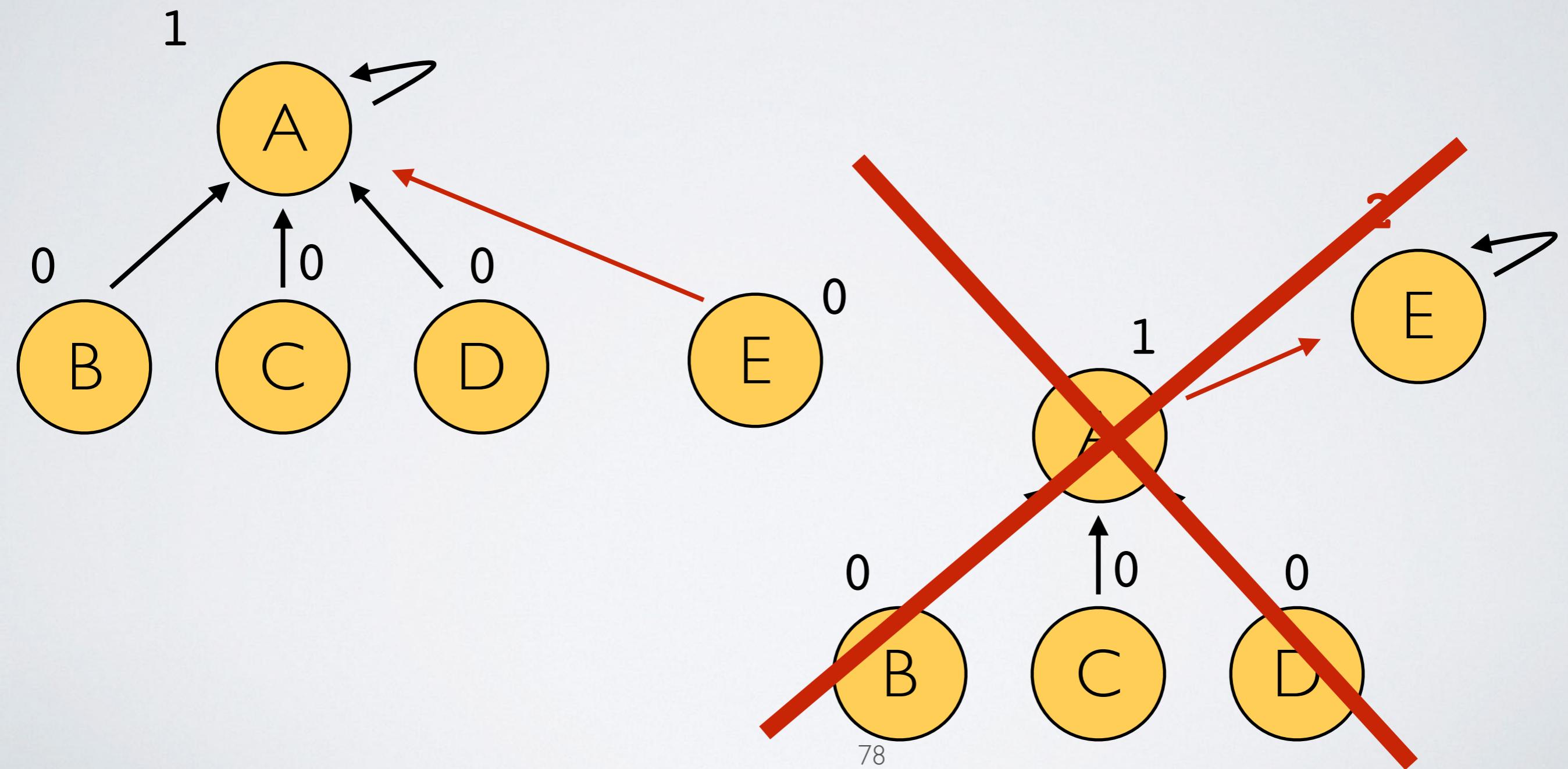
Implementing Union-Find

- ▶ Merging trees with different ranks



Implementing Union-Find

- ▶ Merging trees with different ranks



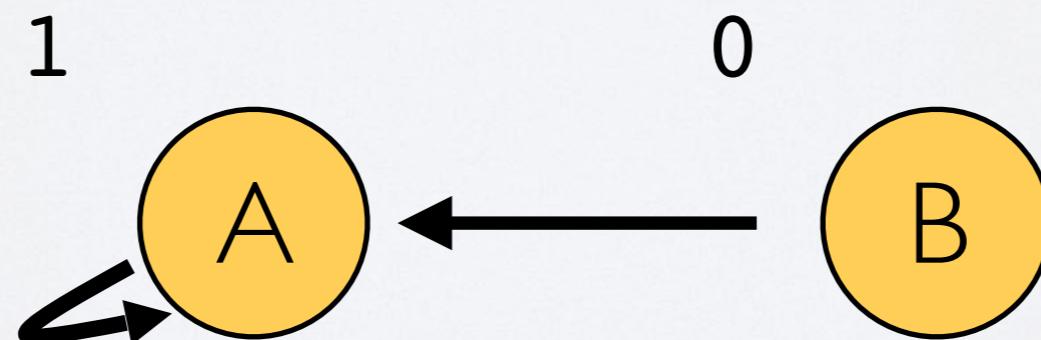
Implementing Union-Find

```
// Merges two clouds, given the root of each cloud
function union(root1, root2):
    if root1.rank > root2.rank:
        root2.parent = root1
    elif root1.rank < root2.rank:
        root1.parent = root2
    else:
        root2.parent = root1
        root1.rank++
```

Implementing Union-Find

- ▶ To find the cloud of B
 - ▶ follow B's parent pointer all the way up to root

```
// Finds the cloud of a given vertex
function find_root(x):
    while x.parent != x:
        x = x.parent
    return x
```

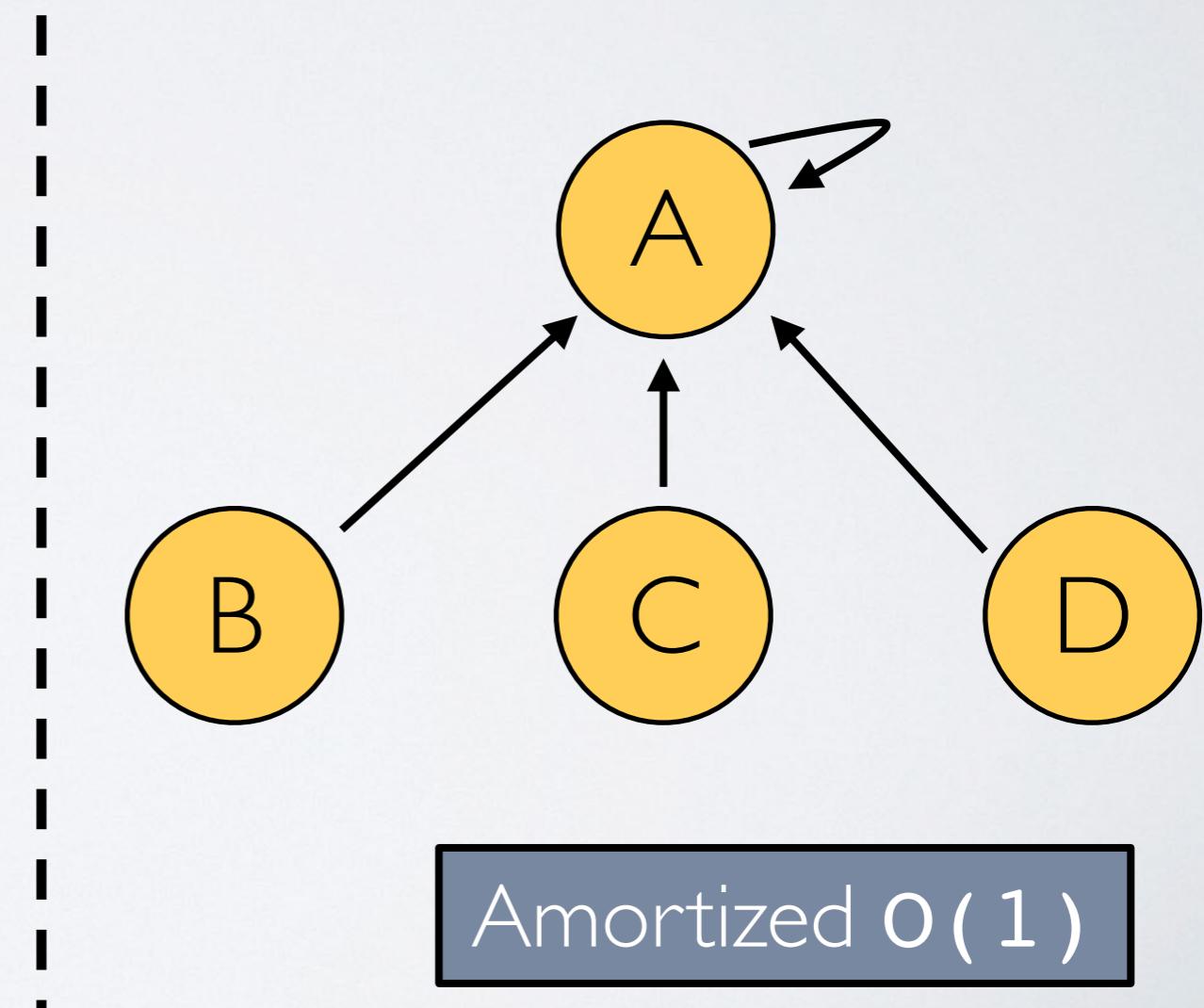
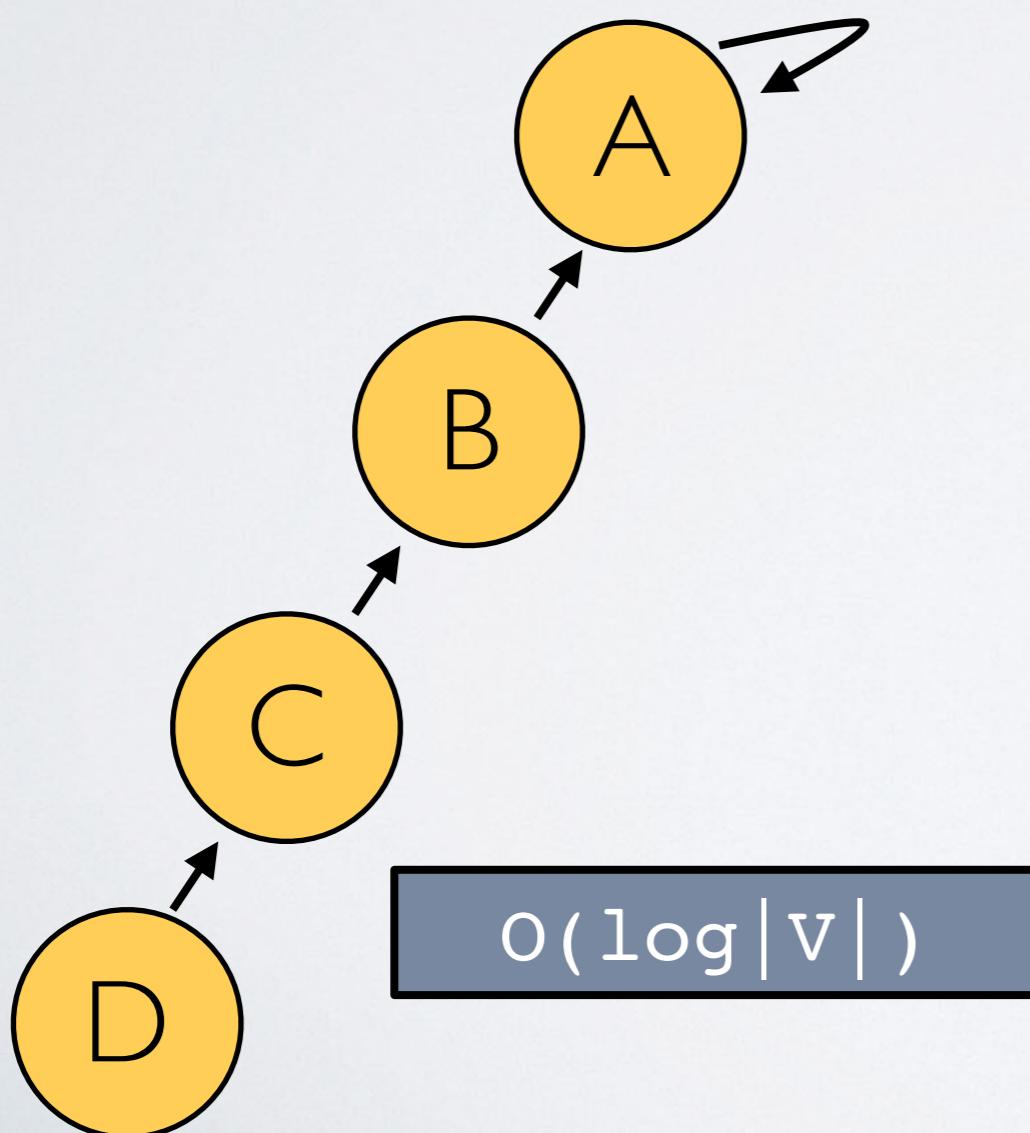


Path Compression

- ▶ This approach to implementing **find** runs in
 - ▶ $O(\log |v|)$
 - ▶ not obvious to see why and proof beyond CS16
- ▶ We can bring this down to amortized $O(1)$
 - ▶ with path compression...
 - ▶ ...a way of flattening the structure of the tree...
 - ▶ ...whenever **find()** is used on it

Path Compression

- ▶ Instead of traversing up tree every time **D**'s cloud is asked for
 - ▶ We only search for **D**'s root once
 - ▶ As we follow chain of parents to **A** we set parents of **D** & **C** to **A**



Path Compression Pseudo-code

```
function find_root(x):
    if x.parent != x:
        x.parent = find_root(x.parent)
    return x.parent
```

Runtime of Kruskal w/ Path Compression

1 min

Activity #5

Runtime of Kruskal w/ Path Compression

1 min

Activity #5

Runtime of Kruskal w/ Path Compression

O min

Activity #5

Runtime of Kruskal w/ Path Compression

```
function kruskal(G):
    // Input: undirected, weighted graph G
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        makeCloud(v)
    MST = []
    Sort all edges ←  $O(|E| \log |E|)$ 
    for all edges (u,v) in G sorted by weight: ←  $O(|E|)$ 
        if u and v are not in same cloud:
            add (u,v) to MST
            merge clouds containing u and v ←  $O(1)$ 
    return MST
    amortized
```

Kruskal Runtime

- $O(|V|)$ for iterating through vertices
- $O(|E| \log |E|)$ for sorting edges
- $O(|E| \times 1)$ for iterating through edges and merging clouds with path compression
- $O(|V| + |E| \log |E| + |E| \times 1)$
 - = $O(|V| + |E| \log |E|)$
- $O(|V| + |E| \log |E|)$ better than $O(|V|^3)$

Readings

- ▶ Dasgupta Section 5.1
- ▶ Explanations of MSTs
- ▶ and both algorithms discussed in this lecture