Problem Session 4 (Pumping Lemma for RLs)

Problem 1

Prove that $L = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

Proof:
Assume that L is regular. Let p be the pumping length guaranteed by the Pumping Lemma. Take $w = 0^p10^p1$. Clearly, $w \in L$, and $|w| \geq p$. We need to show that for any partition $w = xyz$, with $|xy| \leq p$, and $y \neq \varepsilon$, there exists $i \geq 0$, such that $xy^iz \notin L$.

Let $k = |y|$. Note that $0 < k \leq p$. Then, $xy^0z = 0^{p-k}10^p1 \notin L$, because if it were in L, then for some string u, we must have $0^{p-k}10^p1 = uu$, which implies that u ends with 1, and so $p - k = p$, which is impossible because $k \neq 0$.

□

Problem 2

Prove that $L = \{w \mid w = w^R, w \in \{0,1\}^*\}$ is not regular (This is the language of binary palindromes).

Proof:
Assume that L is regular. Let p be the pumping length guaranteed by the Pumping Lemma. Take $w = 0^p10^p$. Clearly, $w \in L$, and $|w| \geq p$. We need to show that for any partition $w = xyz$, with $|xy| \leq p$, and $y \neq \varepsilon$, there exists $i \geq 0$, such that $xy^iz \notin L$.

Let $k = |y|$. Note that $0 < k \leq p$. Then, $xy^0z = 0^{p-k}10^p \notin L$, because if it were in L, then $0^{p-k}10^p = (0^{p-k}10^p)^R = 0^p10^{p-k}$, and so $p - k = p$, which is impossible because $k \neq 0$.

□

Problem 3

Prove that $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular.

Proof:
Assume that L is regular. Let p be the pumping length guaranteed by the Pumping Lemma. Take $w = 0^p110^p$. Clearly, $w \in L$, and $|w| \geq p$. We need to show that for any partition $w = xyz$, with $|xy| \leq p$, and $y \neq \varepsilon$, there exists $i \geq 0$, such that $xy^iz \notin L$.

Let $k = |y|$. Note that $0 < k \leq p$. Then, $xy^0z = 0^{p-k}110^p \notin L$, because either it has an odd length, or if it has an even length, then k is at least 2, and so the first half will contain two 1's, while the second half will contain none.

□

Problem 4

Prove that $L = \{w \mid \#_0(w) \neq \#_1(w), win\{0,1\}^*\}$ is not regular. So, this is the language of all strings with non-equal number of 0's and 1's.

Proof 1:
Assume that L is regular. Let p be the pumping length guaranteed by the Pumping Lemma. Ok, now we need to figure out the string w. Remember that we need to be able to find i for any partition of w into xyz, such that $y \neq \varepsilon$, and $|xy| \leq p$

- Lets try $w = 0^p1^{p+1}$. If $|y| = 1$, then we're good: take $i = 2$, then $xy^iz = 0^{p+1}0^{p+1} \notin L$. However, if $|y| = 2$, then $xy^2z = 0^{p+2}0^{p+1}$, so we "skipped" $p+1$, and now we have more 0s than 1s. Ok, let fix it ...

- Take $w = 0^p1^{p+2}$. Then if $|y| = 1$, $xy^3z \notin L$, and if $|y| = 2$, $xy^2z \notin L$. Great. But what if $|y| = 3$? Then even for $i = 2$ we will have more 0's than 1's. Ok, maybe we should take 0^p1^{p+3} ? This will work for $|y| = 1$, and $|y| = 3$ (take $i = 2$). But now it won't work for $|y| = 2$, because we can only get $p+2$ or $p+4$ 0's, again "skipping" $p+3$.

  So, looks like we need to take something that will be divisible both by 3 and by 2. This way it will work for $|y| = 3$, and still work $|y| = 2$. Ok, take 6, so we have $w = 0^p1^{p+6}$. Let's check: if $|y| = 1$, $xy^7z \notin L$, if $|y| = 2$, $xy^4z \notin L$, and if $|y| = 3$, $xy^3z \notin L$.

- But, now $|y| = 4$ won't work. So we need to do the same thing: take 0^p1^{p+k}, where k is divisible by 1,2,3,and 4. That number is 4! (well, 12 will work too, but the factorial will give us a general solution).

- Continuing this reasoning, we realize that to cover all possible values of $|y|$, which are $1, 2, 3, \ldots, p$, we have to take the string $w = 0^p1^{p+p!}$.

Now, let's finish the proof. Take $w = 0^p1^{p+p!}$. Clearly, $w \in L$, and $|w| \geq p$. We need to show that for any partition $w = xyz$, with $|xy| \leq p$, and $y \neq \varepsilon$, there exists $i \geq 0$, such that $xy^iz \notin L$.

Let $k = |y|$. Note that $0 < k \leq p$. Then, $xy^iz = 0^{p+(i-1)k}1^{p+p!}$. We need to show that for any $0 < k \leq p$, we can find i such that $0^{p+(i-1)k}1^{p+p!} \notin L$, i.e. such that $p + (i-1)k = p + p!$. Solving this equality for i gives us $i = \frac{p!}{k} + 1$. i has to be an integer, and it is indeed an integer because $p!$ is divisible by any number between 1 and p.

$\square$

Proof 2 (much easier):

- $L_1 = \{0^n1^n \mid n \geq 0\}$ is not regular (saw this in the class).

- $L_2 = \{w \mid \#_0(w) = \#_1(w)\}$ is not regular because $L_1 = L_2 \cup 0^*1^*$, and so if L_2 were regular, then L_1 had to be regular too (because regular languages are closed under union).
- $L = \overline{L_2}$, and so L is not regular, as otherwise L_2 would be regular too (because regular languages are closed under complement).