Reading:The Crossing Tverberg Theorem

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1 Introduction

The paper focuses on a new extension of the classical *Tverberg Theorem*, which is fundamental in discrete geometry. The original Tverberg theorem asserts that for any set X of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$, one can partition X into r disjoint sets such that the convex hulls of these sets intersect at a common point, called a *Tverberg point*. This result has spurred numerous developments in geometry and combinatorics since it was first proved by Helge Tverberg.

In this work, the authors present a *Crossing Tverberg Theorem*, an enhancement of the original result. It guarantees that not only do the convex hulls of the partitioned sets share a common point but also that the boundaries of the convex hulls have non-empty pairwise intersections, referred to as *crossing convex hulls*. The significance of this result is demonstrated through applications in planar geometry, particularly with pairwise crossing triangles.

2 The Tverberg Theorem and Its Extension

The original Tverberg theorem applies to any set X in $\mathbb{R}^d$, provided the set has sufficient points to be partitioned into r subsets. The convex hulls of these subsets must contain a shared Tverberg point. In this paper, the authors extend this theorem to show that for a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$, it is possible to partition the points into r sets such that the boundaries of the convex hulls of these sets intersect pairwise.

The extension is referred to as the *Crossing Tverberg Theorem*, and it has applications in both low and high dimensions. Specifically, the authors prove that any set of points in general position in $\mathbb{R}^d$ can be partitioned into r sets such that their convex hulls cross. For example, in the plane, one can find at least $\lfloor n/3 \rfloor$ pairwise crossing triangles, a result that generalizes to higher dimensions.

3 Results in Planar Geometry

A notable application of the Crossing Tverberg Theorem is in planar geometry. The authors build on previous results by Álvarez-Rebollar et al., who showed that any point set in general position can be partitioned into $\lfloor n/6 \rfloor$ pairwise crossing triangles. The Crossing Tverberg Theorem improves this by showing that $\lfloor n/3 \rfloor$ triangles can be formed, which is the maximum number possible given that at most $\lfloor n/3 \rfloor$ disjoint triangles exist in a planar point set.

The significance of this improvement lies in the concept of *crossing triangles*. In previous results, the triangles were only required to have intersecting edges. In the new result, the entire boundaries of the triangles must intersect, ensuring a stronger form of intersection.

4 Generalizations to Higher Dimensions

The authors extend their results to simplices in higher-dimensional spaces. For a point set in general position in $\mathbb{R}^d$, the Crossing Tverberg Theorem guarantees that one can find $\lfloor n/(d+1) \rfloor$ disjoint and pairwise crossing simplices. This generalization opens the door for future applications of the Crossing Tverberg Theorem in higher-dimensional discrete geometry and topology.

One of the corollaries presented by the authors is that for any set of $(d+1)r$ points in $\mathbb{R}^d$, there is a partition into r subsets whose convex hulls cross pairwise, and this is guaranteed for sets of points exactly divisible by $d+1$.

5 Proof Strategy

The authors use several combinatorial and geometric tools to prove the Crossing Tverberg Theorem. One key result is a lemma stating that for any two disjoint sets of $d+1$ points, whose convex hulls intersect, it is always possible to partition these points into two new sets such that their convex hulls cross. This lemma plays a crucial role in constructing the crossing partition in the proof.

The proof itself relies on iteratively applying a *fixing operation* to eliminate nested simplices in the partition. The fixing operation modifies the sets involved in a pair of nested convex hulls, ensuring that the resulting simplices are not nested and instead cross. This process is guaranteed to terminate after a finite number of steps due to a lexicographical ordering of the volumes of the simplices.

6 Applications and Computational Complexity

The authors discuss the algorithmic aspects of finding a Crossing Tverberg partition. Although the existence of such a partition is guaranteed by their theorem, the computational complexity of finding the partition remains an open problem. While approximate Tverberg partitions can be computed efficiently, the exact computation of a Crossing Tverberg partition may require exponentially many steps in some cases.

The paper raises interesting questions about whether there are polynomial-time algorithms for finding a Crossing Tverberg partition. While the problem of determining whether a given point is a Tverberg point is known to be NP-complete, it remains to be seen whether the general problem of finding the crossing partition is also computationally hard.

7 Conclusion

The Crossing Tverberg Theorem strengthens the classical Tverberg theorem by guaranteeing not only that the convex hulls of the partitioned sets have a common point, but also that their boundaries intersect pairwise. This result has important implications in planar and higher-dimensional geometry, particularly in the study of crossing triangles and simplices.

The authors present a robust proof using geometric and combinatorial techniques and open several avenues for future research, especially in the algorithmic and computational aspects of finding such partitions. The results provide an optimal bound for crossing triangles in the plane and extend naturally to higher-dimensional spaces, enriching the understanding of geometric partitions in discrete geometry.