Assignment2

> Computer Vision 2023 Spring

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1

(**Math**) In the augmented Euclidean plane, there is a line $x - 3y + 4 = 0$, what is the homogeneous coordinate of the infinity point of this line?

The homogeneous coordinate of the line is $x - 3y + 4z = 0$. In the projective plane, a point at infinity is represented by setting $z = 0$. We plug $z = 0$ into the equation to find the homogeneous coordinates of the infinity point

$$x - 3y = 0$$

let $x = 3$, we have $y = 1$

Therefore, the homogeneous coordinate of the infinity point is $(3, 1, 0)$.

2

(**Math**) On the normalized retinal plane, suppose that $\mathbf{p}_n$ is an ideal point of projection without considering distortion. If distortion is considered, $\mathbf{p}_n=(x, y)^T$ is mapped to $\mathbf{p}_d=(x_d, y_d)^T$ which is also on the normalized retinal plane. Their relationship is,

$$\begin{cases} x_d = x\left(1+k_1r^2+k_2r^4\right)+2\rho_1 xy+\rho_2\left(r^2+2x^2\right)+xk_3r^6 \\ y_d = y\left(1+k_1r^2+k_2r^4\right)+2\rho_2 xy+\rho_1\left(r^2+2y^2\right)+yk_3r^6 \end{cases}$$

where $r^2 = x^2 + y^2$

For performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian matrix of $\mathbf{p}_d$ w.r.t $\mathbf{p}_n$, i.e.,

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$$

It should be noted that in this question $\mathbf{p}_d$ is the function of $\mathbf{p}_n$ and all the other parameters can be regarded as constants.

We plug $r^2 = x^2 + y^2$

$$\begin{cases} x_d = x[1 + k_1(x^2+y^2) + k_2(x^4+2x^2y^2+y^4)] + 2\rho_1 xy + \rho_2(3x^2+y^2) + xk_3(x^6+3x^4y^2+3x^2y^4+y^6) \\ y_d = y[1 + k_1(x^2+y^2) + k_2(x^4+2x^2y^2+y^4)] + 2\rho_2 xy + \rho_1(x^2+3y^2) + yk_3(x^6+3x^4y^2+3x^2y^4+y^6) \end{cases}$$

$\mathbf{p}_n = (x, y)^T$ is mapped to $\mathbf{p}_d = (x_d, y_d)^T$, we have $F: \mathbb{R}_2 \to \mathbb{R}_2$

$$\frac{\mathrm{d}\mathbf{p}_d}{\mathrm{d}\mathbf{p}_n^T} = J_F(x,y) = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} 1 + k_1(3x^2+y^2) + k_2(5x^4+6x^2y^2+y^4)+ \\ 2\rho_1 y + 6\rho_2 x + k_3(7x^6+15x^4y^2+9x^2y^4+y^6) \end{matrix} & \begin{matrix} 2k_1xy + k_2(4x^3y+4xy^3)+ \\ 2\rho_1 x + 2\rho_2 y + k_3(6x^5y+12x^3y^3+6xy^5) \end{matrix} \\ \begin{matrix} 2k_1xy + k_2(4x^3y+4xy^3)+ \\ 2\rho_1 x + 2\rho_2 y + k_3(6x^5y+12x^3y^3+6xy^5) \end{matrix} & \begin{matrix} 1 + k_1(x^2+3y^2) + k_2(x^4+6x^2y^2+5y^4)+ \\ 2\rho_2 x + 6\rho_1 y + k_3(x^6+9x^4y^2+15x^2y^4+7y^6) \end{matrix} \end{bmatrix}$$

3

(**Math**) In our lecture, we mentioned that for performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian of the rotation matrix (represented in a vector) w.r.t its axis-angle representation. In this question, your task is to derive the concrete formula of this Jacobian matrix.
Suppose that

$\mathbf{d} = \theta\mathbf{n} \in \mathbb{R}^{3\times 1}$, where $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a 3D unit vector and θ is a real number denoting the rotation angle.

With Rodrigues formula, $\mathbf{d}$ can be converted to its rotation matrix form,

$$\mathbf{R} = \cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta\mathbf{n}^\wedge$$

and obviously $\mathbf{R} \triangleq \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ is a 3×3 matrix.

Denote $\mathbf{r}$ by the vectorized form of $\mathbf{R}$, i.e.,

$$\mathbf{r} \triangleq (r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{31}, r_{32}, r_{33})^T$$

Please give the concrete form of Jacobian matrix of $\mathbf{r}$ w.r.t $\mathbf{d}$, i.e., $\frac{d\mathbf{r}}{d\mathbf{d}^T} \in \mathbb{R}^{9\times 3}$.

In order to make it easy to check your result, please follow the following notation requirements,

$$\alpha \triangleq \sin\theta, \beta \triangleq \cos\theta, \gamma \triangleq 1-\cos\theta$$

In other words, the ingredients appearing in your formula are restricted to $\alpha, \beta, \gamma, \theta, n_1, n_2, n_3$.

We have

$$\mathbf{n}\mathbf{n}^T = \begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ n_1n_2 & n_2^2 & n_2n_3 \\ n_1n_3 & n_2n_3 & n_3^2 \end{bmatrix} \quad \mathbf{n}^\wedge = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

We plug them into $\mathbf{R}$

$$\mathbf{R} = \begin{bmatrix} \cos\theta + n_1^2(1-\cos\theta) & n_1n_2(1-\cos\theta) - n_3\sin\theta & n_1n_3(1-\cos\theta) + n_2\sin\theta \\ n_1n_2(1-\cos\theta) + n_3\sin\theta & \cos\theta + n_2^2(1-\cos\theta) & n_2n_3(1-\cos\theta) - n_1\sin\theta \\ n_1n_3(1-\cos\theta) - n_2\sin\theta & n_2n_3(1-\cos\theta) + n_1\sin\theta & \cos\theta + n_3^2(1-\cos\theta) \end{bmatrix}$$

let $\mathbf{d} = (d_1, d_2, d_3)^T$, we have

$$\mathbf{d} = \theta\mathbf{n} = \begin{bmatrix} \theta n_1 \\ \theta n_2 \\ \theta n_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$d_i = \theta n_i \quad (i = 1, 2, 3)$$

$\mathbf{n}$ is a 3D unit vector, therefore

$$d_1^2 + d_2^2 + d_3^2 = \theta^2(n_1^2 + n_2^2 + n_3^2) = \theta^2$$

We plug the relationship above, have

$$\frac{\partial\alpha}{\partial d_i} = \frac{\partial\sin\theta}{\partial d_i} = \cos\theta\frac{\partial\theta}{\partial d_i} = \cos\theta\frac{\partial\sqrt{d_1^2+d_2^2+d_3^2}}{\partial d_i} = \cos\theta\frac{d_i}{\theta} = n_i\cos\theta$$

$$\frac{\partial\beta}{\partial d_i} = \frac{\partial\cos\theta}{\partial d_i} = -\sin\theta\frac{\partial\theta}{\partial d_i} = -n_i\sin\theta$$

$$\frac{\partial\gamma}{\partial d_i} = \frac{\partial 1-\cos\theta}{\partial d_i} = -\frac{\partial\cos\theta}{\partial d_i} = n_i\sin\theta$$

$$\frac{\partial n_i}{\partial d_i} = \frac{\partial\frac{d_i}{\theta}}{\partial d_i} = \frac{\theta - d_i\frac{\partial\theta}{\partial d_i}}{\theta^2} = \frac{\theta - \frac{d_i^2}{\theta}}{\theta^2} = \frac{1-n_i^2}{\theta}$$

$$\frac{\partial n_j}{\partial d_i} = \frac{\partial\frac{d_j}{\theta}}{\partial d_i} = \frac{-d_j\frac{\partial\theta}{\partial d_i}}{\theta^2} = \frac{-\frac{d_id_j}{\theta}}{\theta^2} = -\frac{n_in_j}{\theta}$$

$\mathbf{d} = (d_1, d_2, d_3)^T$ is mapped to $\mathbf{r} = (r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{31}, r_{32}, r_{33})^T$, we have $F: \mathbb{R}_3 \to \mathbb{R}_9$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\mathbf{d}^T} &= J_F(r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{31}, r_{32}, r_{33}) \\
&= \begin{bmatrix} \frac{\partial r_{11}}{\partial d_1} & \frac{\partial r_{11}}{\partial d_2} & \frac{\partial r_{11}}{\partial d_3} \\ \frac{\partial r_{12}}{\partial d_1} & \frac{\partial r_{12}}{\partial d_2} & \frac{\partial r_{12}}{\partial d_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial r_{33}}{\partial d_1} & \frac{\partial r_{33}}{\partial d_2} & \frac{\partial r_{33}}{\partial d_3} \end{bmatrix} \\
= \begin{bmatrix}
n_1(1-n_1^2)(\frac{2\gamma}{\theta}-\alpha) & n_1^2 n_2(\alpha-\frac{2\gamma}{\theta})-n_2\alpha & n_1^2 n_3(\alpha-\frac{2\gamma}{\theta})-n_3\alpha \\
n_1^2 n_2(\alpha-\frac{2\gamma}{\theta})+n_1 n_3(\frac{\alpha}{\theta}-\beta)+n_2\frac{\gamma}{\theta} & n_1 n_2^2(\alpha-\frac{2\gamma}{\theta})+n_2 n_3(\frac{\alpha}{\theta}-\beta)+n_1\frac{\gamma}{\theta} & n_1 n_2 n_3(\alpha-\frac{2\gamma}{\theta})-(1-n^3)\frac{\alpha}{\theta}-n_3^2\beta \\
n_1^2 n_3(\alpha-\frac{2\gamma}{\theta})-n_1 n_2(\frac{\alpha}{\theta}-\beta)+n_3\frac{\gamma}{\theta} & n_1 n_2 n_3(\alpha-\frac{2\gamma}{\theta})+(1-n_2^2)\frac{\alpha}{\theta}+n_2^2\beta & n_1 n_3^2(\alpha-\frac{2\gamma}{\theta})-n_2 n_3(\frac{\alpha}{\theta}-\beta)+n_1\frac{\gamma}{\theta} \\
n_1^2 n_2(\alpha-\frac{2\gamma}{\theta})-n_1 n_3(\frac{\alpha}{\theta}-\beta)+n_2\frac{\gamma}{\theta} & n_1 n_2^2(\alpha-\frac{2\gamma}{\theta})-n_2 n_3(\frac{\alpha}{\theta}-\beta)+n_1\frac{\gamma}{\theta} & n_1 n_2 n_3(\alpha-\frac{2\gamma}{\theta})+(1-n^3)\frac{\alpha}{\theta}+n_3^2\beta \\
n_1 n_2^2(\alpha-\frac{2\gamma}{\theta})-n_1\alpha & n_2(1-n_2^2)(\frac{2\gamma}{\theta}-\alpha) & n_2^2 n_3(\alpha-\frac{2\gamma}{\theta})-n_3\alpha \\
n_1 n_2 n_3(\alpha-\frac{2\gamma}{\theta})-(1-n_1^2)\frac{\alpha}{\theta}-n_1^2\beta & n_2^2 n_3(\alpha-\frac{2\gamma}{\theta})+n_1 n_2(\frac{\alpha}{\theta}-\beta)+n_3\frac{\gamma}{\theta} & n_2 n_3^2(\alpha-\frac{2\gamma}{\theta})+n_1 n_3(\frac{\alpha}{\theta}-\beta)+n_2\frac{\gamma}{\theta} \\
n_1^2 n_3(\alpha-\frac{2\gamma}{\theta})+n_1 n_2(\frac{\alpha}{\theta}-\beta)+n_3\frac{\gamma}{\theta} & n_1 n_2 n_3(\alpha-\frac{2\gamma}{\theta})-(1-n_2^2)\frac{\alpha}{\theta}-n_2^2\beta & n_1 n_3^2(\alpha-\frac{2\gamma}{\theta})+n_2 n_3(\frac{\alpha}{\theta}-\beta)-n_1\frac{\gamma}{\theta} \\
n_1 n_2 n_3(\alpha-\frac{2\gamma}{\theta})+(1-n_1^2)\frac{\alpha}{\theta}+n_1^2\beta & n_2^2 n_3(\alpha-\frac{2\gamma}{\theta})-n_1 n_2(\frac{\alpha}{\theta}-\beta)+n_3\frac{\gamma}{\theta} & n_2 n_3^2(\alpha-\frac{2\gamma}{\theta})-n_1 n_3(\frac{\alpha}{\theta}-\beta)+n_2\frac{\gamma}{\theta} \\
n_1 n_3^2(\alpha-\frac{2\gamma}{\theta})-n_1\alpha & n_2 n_3^2(\alpha-\frac{2\gamma}{\theta})-n_2\alpha & n_3(1-n_3^2)(\alpha-\frac{2\gamma}{\theta})
\end{bmatrix}
\end{aligned}
$$