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Peano Arithmetic

Peano Arithmetic is the formalisation of arithmetic operations over the natural numbers.

Peano Axioms:

1) $0 \in \mathbb{N}$. Stating that at least a single natural number exists (originally 1).
2) $\forall n \in \mathbb{N}(n = n)$. I.e. equality is reflexive.
3) $\forall x, y \in \mathbb{N}(x = y \Leftrightarrow y = x)$. I.e. equality is symmetric.
4) $\forall x, y, z \in \mathbb{N}(x = y \land y = z \Rightarrow x = z)$. I.e. equality is transitive.
5) $\forall a \forall b(a \in \mathbb{N} \land a = b \Rightarrow b \in \mathbb{N})$. I.e. the natural numbers are closed under equality.
6) $\forall n \in \mathbb{N}(s(n) \in \mathbb{N})$. I.e. the natural numbers are closed under the successor function s.
7) $\forall n, m \in \mathbb{N}(s(n) = s(m) \Rightarrow n = m)$. I.e. s is an injection.
8) $\forall n \in \mathbb{N}(s(n) \neq 0)$. I.e. there is no natural number whose successor is 0.
9) Axiom of induction: for a set K, $\big(0 \in K \land \forall n \in \mathbb{N}(n \in K \Rightarrow s(n) \in K)\big) \Rightarrow \forall n \in \mathbb{N}(n \in K)$
   Note: $s(0) = 1, s(1) = s, \dots, s(n) = n + 1$.

A version of Peano Arithmetic is defined by the first-order language $\Sigma_{PA} = (\{0, s, +, \times\}, \{=, <\})$.

Symbol	Arity	Type	Meaning (semantics)
$\mathbf{0}$	0	Constant	Zero
$\boldsymbol{s}$	1	Function	Successor (addition of 1)
$+$	2	Function	Addition
$\times$	2	Function	Multiplication
$=$	2	Predicate	Equality
$<$	2	Predicate	Inequality

A first-order structure M for Σ_{PA} is $M = (\mathbb{N}, \{0^M, s^M, +^M, \times^M\}, \{=^M, <^M\})$, where:

- $\mathbb{N}$ is the domain of natural numbers.
- 0^M is the constant symbol for the natural number 0.
- s^M is the unary successor function on $\mathbb{N}$.
- $+^M$ is the binary addition function $+^M: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
- $\times^M$ is the binary multiplication function $\times^M: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
- $=^M$ is the binary relation of equality on $\mathbb{N}$, i.e. $=^M \coloneqq \{(n, n) \mid n \in \mathbb{N}\}$.
- $<^M$ is the binary relation of strict inequality on $\mathbb{N}$, i.e. $<^M \coloneqq \{(n, m) \mid n, m \in \mathbb{N}, n \textit{ is less than } m\}$.

Facts about the natural numbers:

- Linearly ordered, i.e. every natural number is strictly greater than, equal to, or strictly less than any other natural number.
- 0 is less than or equal to any natural number.