Universal Property of a Set – a property which is true for every element of the set.

Well-Ordered Set – a set where every nonempty subset of the set has a least element (or smallest), i.e. $\forall S' \subseteq S\left(S' \neq \emptyset \Rightarrow \exists s \in S'\big(\forall s' \in S'(s \leq s')\big)\right)$, where $a \leq b$ means that an element a is less than or equal to an element b.

Inductively/Recursively Defined Structures/Sets:

- A set of objects is inductively defined when:
    - At least one base case/atomic element (or 'case') has been defined as a member of the set. This is the smallest structure allowed.
    - At least one inductive (or 'step') case has been defined for an accepted structure of a member of the set, which has at least one of it's contained elements restricted to being part of the inductively defined set. Such an element may be termed the successor of it's contained element, if it contains only one element. The inductive member is often formed by applying a **constructor** to elements.
    - Nothing else is a part of the set.
- This method of defining an inductive structure can be related for how terms are defined in a first-order language.
- It's very common that functions applied to inductively defined structures are defined inductively themselves, notably for the base case pattern as an argument, and the inductive case pattern as an argument. Inductive/recursive definitions for functions are very useful in proofs by induction, as they match the base cases and inductive cases for a structure.

Proving Conditionals:

- A lemma in the form $P(x) \Rightarrow Q(y)$ is notably true in all valuations bar when the antecedent $P(x)$ is true and the consequent $Q(y)$ is false.
- From this fact, the lemma in the form of a conditional statement can be proved to be valid by showing that assuming the antecedent true, it follows that the consequent is always true. I.e. assume some property holds on some variable, and that from that it can be shown that a property holds on some other variable.

Proof by (Mathematical) Induction

A proof by induction establishes that a property (a first-order predicate here) holds for all base cases of an inductively defined structure, and then that by assuming it holds for an arbitrary member(s) of the set of inductively defined objects it holds for all constructions which contain that arbitrary member(s). This inducts that the property holds for all members of the set of objects.

Each inductively defined structure has it's own principle of (structural) induction/induction principle, which is a formalisation of the above description specific to an inductively defined structure.

Induction on the Natural Numbers:

- The natural numbers are clearly defined inductively, which corresponds to the Peano Axioms for a base and recursive case $0 \in \mathbb{N}$ and $\forall n. n \in \mathbb{N} \Rightarrow s(n) \in \mathbb{N}$ respectively:

  - 0 is a natural number.
  - If n is a natural number, then so is its successor $s(n)$.
  - Nothing else is a natural number.
- The recursive grammar for the natural numbers is $n ::= 0 \mid s(n)$.
- Proving a universal property P holds for all natural numbers:
  - Proving that $\forall n. n \in \mathbb{N} \Rightarrow P(n)$ is valid in the domain of natural numbers and the Peano axioms.
  - Base Case: show that P holds on the base case 0, i.e. show that $P(0)$ is true.
  - Induction Hypothesis (IH): assume that P holds for some arbitrary k, $k \in \mathbb{N}$.
  - Step Case: using the IH to determine the cases for k where $P(k)$ holds, show that $P\big(s(k)\big)$ is true.
  - As an inference rule, the principle of induction for natural numbers:

$$\frac{\begin{array}{l} P(0) \\ P(k) \Rightarrow P\big(s(k)\big) \end{array}}{\forall n. n \in \mathbb{N} \Rightarrow P(n)} \qquad (k\ is\ arbitrary\ from\ \mathbb{N})$$

    The inference rule clearly shows the two premises, that P holds for 0 and that if P holds for an arbitrary k then P holds for $s(k)$, and the conclusion that P holds for all $n \in \mathbb{N}$.
- This proof method makes sense, as it has been shown the property holds on the smallest element of the natural numbers, and that when for any arbitrary natural number the property holds it holds for the successor of that number. Since all natural numbers are successively found from the smallest, it makes sense that k can be initially taken as 0, and then the property holds for $s(0)$, and this will then continue for all natural numbers.
- **Domino Visualisation:**
  - Due to this nature, proof my mathematical induction is often likened to dominoes.
  - By pushing over the first, and knowing that any will fall when their predecessor falls, clearly all dominoes which are connected to the original will fall.

Structural Induction:

- A form of induction which generalises induction used on the natural numbers to all recursively defined mathematical structures to prove a universal property holds for that structure.
- Fundamentally, the same concept as with natural numbers is used. First show that all base cases for such a structure hold for the universal property, and then show that the property is conserved (still holds) for all the constructor operations which can be applied to members on which the property holds.

Structural Induction on Lists:

- The list structure can be defined inductively with the grammar $\ell ::= [\,] \mid x :: \ell$, where [ ] is nil (the empty list) and the base case, x is an arbitrary element of some other set, and the binary constructor :: is used for the inductive definition of lists. Nothing else is a list.
- Common terminology for lists:
  - **Head** – the 'first' or 'front' element of a list, i.e. for the list given by $x :: \ell$, x is the head of the list. The empty list has no head.

- **Tail** – the remaining list from removing the head, i.e. for the list given by $x :: \ell$, ℓ is the tail of the list. The empty list has no tail, as it has no head.
- **Singleton** – a singleton list is given by the form of $x :: [\,]$, i.e. it contains exactly one element, x, and is also represented $[x]$.
- **Cons** – a common abbreviation for the list constructor $::$.
- It is common to write variables for lists as some variable symbol followed by 's', to indicate plurality of elements.

- Proving a universal property P holds for all lists (the set ℓ):
  - Proving $\forall l.\, l \in \ell \Rightarrow P(l)$ is valid.
  - Base Case: show $P([\,])$ is true.
  - Induction Hypothesis: assume $P(xs)$ is true for some arbitrary list xs.
  - Step Case: using the IH, show that $P(x :: xs)$ is true for any element x.
  - Induction principle for lists:

$$\frac{\begin{array}{l} P([\,]) \\ P(xs) \Rightarrow P(x :: xs) \end{array}}{\forall l.\, l \in \ell \Rightarrow P(l)} \qquad (x\ and\ xs \in \ell\ are\ arbitrary)$$

Structural Induction on Binary Trees:

- A binary tree can be defined inductively with the grammar $t ::= Lf \mid Br(x, t, t)$, where Lf is a 'leaf', and Br is the 'branch' constructor from an arbitrary element x, from another set, to two other subtrees. The 'root' of a tree of the form $Br(x, t_1, t_2)$ is x.
- Proving a universal property P holds for all binary trees (the set T):
  - Proving $\forall t.\, t \in T \Rightarrow P(t)$ is valid.
  - Base Case: show $P(Lf)$ is true.
  - Induction Hypothesis: assume $P(t_1)$ and $P(t_2)$ are true for arbitrary trees t_1 and t_2.
  - Step Case: using the IH, show that $P\big(Br(x, t_1, t_2)\big)$ is true for any element x.
  - Principle of induction for binary trees:

$$\frac{\begin{array}{l} P(Lf) \\ P(t_1) \wedge P(t_2) \Rightarrow P\big(Br(x, t_1, t_2)\big) \end{array}}{\forall t.\, t \in T \Rightarrow P(t)} \qquad (x, t_1 \in T, and\ t_2 \in T\ are\ arbitrary)$$

Structural Induction on (Propositional) Formulas:

- For formula defined with the inductive grammar $\phi ::= A \mid \neg\phi \mid \phi \vee \phi$, which notably can be used to express any propositional formula, where A is any propositional symbol.
- Proving a universal property P holds for all formulas (the set ϕ):
  - Proving $\forall \varphi.\, \varphi \in \phi \Rightarrow P(\varphi)$ is valid.
  - Base Case: show $P(A)$ is true for any propositional symbol.
  - Induction Hypothesis 1: assume $P(\varphi_1)$ is true for an arbitrary formula φ_1.
  - Induction Hypothesis 2: assume $P(\varphi_1)$ and $P(\varphi_2)$ are true for arbitrary formulas φ_1 and φ_2.
  - Step Case 1: using IH 1, show that $P(\neg\varphi_1)$ is true.
  - Step Case 2: using IH 2, show that $P(\varphi_1 \vee \varphi_2)$ is true.
  - Principle of induction for formulas:

$$\begin{array}{ll} P(A) & \\ P(\varphi_1) \Rightarrow P(\neg\varphi_1) & (for\ arbitrary\ \varphi_1 \in \phi) \\ \underline{P(\varphi_1) \wedge P(\varphi_2) \Rightarrow P(\varphi_1 \vee \varphi_2)} & (for\ arbitrary\ \varphi_1 \in \phi\ and\ \varphi_2 \in \phi) \end{array}$$

$$\forall\varphi.\varphi \in \phi \Rightarrow P(\varphi)$$

Two-Step Induction on the Natural Numbers:

- An alternative method to prove a universal property P of $\mathbb{N}$:
    - Base Case 1 (even): show $P(0)$ is true.
    - Base Case 2 (odd): show $P(s(0))$ is true.
    - Induction Hypothesis: assume $P(k)$ and $P(s(k))$ hold for an arbitrary $k, k \in \mathbb{N}$. I.e. P holds for a pair of successive natural numbers.
    - Step Case: using the IH, show that $P\left(s(s(k))\right)$ is true.
    - Principle of two-step induction on the natural numbers:

$$\frac{\begin{array}{l} P(0) \\ P(s(0)) \\ P(k) \land P(s(k)) \Rightarrow P(s(s(k)) \end{array}}{\forall n. n \in \mathbb{N} \Rightarrow P(n)} \quad (k\ is\ arbitrary\ from\ \mathbb{N})$$

- **Domino Visualisation:**
    - This can be likened to the domino visualisation previously used if the dominoes are considered to be split into two distinct categories (here even and odd), i.e. they might be coloured red or blue.
    - If the dominoes are then arranged alternately, they can be split into pairs, i.e. the first red and first blue domino.
    - By showing that the first pair is pushed, and that for some arbitrary pair being pushed it pushes the next domino, this is an equivalent method of inductive proof.
- Proof that ordinary induction is the same as two-step induction on the natural numbers:
    - Showing ordinary induction implies two-step induction:
        - By assuming the premises are true for the normal induction inference rule, show using two-step induction that the consequent is true. **This shows the equivalence of the premises.**
        - The inference rule for normal induction: $P(0) \land \left(\forall k \in \mathbb{N} \,.\, P(k) \rightarrow P(s(k))\right) \rightarrow \forall n \in \mathbb{N}. P(n)$.
        - Assume the premises $P(0)$ and $\forall k \in \mathbb{N} \,.\, P(k) \rightarrow P(s(k))$ hold.
        - Two-step induction:
            - Base Case 1: $P(0)$, which is true by assuming the premises.
            - Base Case 2: $P(s(0))$ which is true by assuming the premises, i.e. that $P(0) \land \left(P(k) \rightarrow P(s(k))\right) \rightarrow P(s(0))$ (modus ponens).
            - Induction Hypothesis: assume $P(x)$ and $P(s(x))$ hold for an arbitrary $x, x \in \mathbb{N}$.
            - Step Case: show that $P\left(s(s(x))\right)$ holds, by the assumptions and the IH it can be seen that $P(s(x)) \land \left(P(k) \rightarrow P(s(k))\right)$, hence by modus ponens it follows that $P\left(s(s(x))\right)$.
            - Hence, by two-step induction, $\forall n \in \mathbb{N}. P(n)$ follows when the premises of ordinary induction are true.
    - Showing two-step induction implies ordinary induction:

        - By assuming the premises are true for the two-step induction inference rule, show using ordinary induction that the consequent is true.
        - The inference rule for two-step induction: $P(0) \wedge P(s(0)) \wedge \left(\forall k \in \mathbb{N} . P(k) \wedge P(s(k)) \rightarrow P\left(s(s(k))\right)\right) \rightarrow \forall n \in \mathbb{N} . P(n)$.
        - Assume the premises $P(0)$ and $P(s(0))$ and $\forall k \in \mathbb{N} . P(k) \wedge P(s(k)) \rightarrow P\left(s(s(k))\right)$ hold.
        - Ordinary induction:
            - Base Case: $P(0) \wedge P(s(0))$, which is true by assuming the premises. (this reflects the form of ordinary induction being able to be seen exactly as two-step induction, considering associativity of conjunctions)
            - Induction Hypothesis: assume $P(x) \wedge P(s(x))$ holds for an arbitrary $x, x \in \mathbb{N}$.
            - Step Case: show that $P(s(x)) \wedge P\left(s(s(x))\right)$ holds. It is clear from the IH that $P(s(x))$ is true, and by the premise and assumptions $P(x) \wedge P(s(x)) \wedge \forall k \in \mathbb{N} . P(k) \wedge P(s(k)) \rightarrow P\left(s(s(k))\right)$, hence by modus ponens it follows that $P\left(s(s(x))\right)$ is true, and hence $P(s(x)) \wedge P\left(s(s(x))\right)$ is true.
            - Hence, by ordinary induction, $\forall n \in \mathbb{N}. P(n)$ follows when the premises of two-step induction are true.
    - Since it has been shown that ordinary induction implies two-step induction, and conversely that two-step induction implies ordinary induction, the two methods are clearly equivalent.

Infinite Descent:

- Another alternative proof method for a universal property P of $\mathbb{N}$. This method of proof was known to the Ancient Greeks, however, Pierre de Fermat brought it into modern mathematics.
- Overview of infinite descent to prove P is a universal property of $\mathbb{N}$:
    - By showing the property P is not true for some natural number k, i.e. a counterexample.
    - And showing that for any arbitrary natural number which is a counterexample (for which P does not hold), there is a smaller counterexample (for which P does not hold).
    - Since $\mathbb{N}$ is infinite in only the increasing (successor) 'direction', i.e. it will always have an atomic case which cannot be reduced, and by showing that a counterexample to P always has a smaller counterexample, there is an infinite number of smaller counterexamples, which does not in fact follow. Since there will always be more counterexamples than the natural numbers from an arbitrary counterexample k to 0, as this range is finite.
- This description is essentially the opposite or negation of proof by induction, since it involves assuming the property is not universal, and then shows by that assumption that a contradiction is reached, and hence that the property is universal as the assumption cannot be true.

- This is proof relies on the property which the set of natural numbers is well-ordered, i.e. there is a smallest element to the set.
- There are no infinitely descending chains of natural numbers from any natural number, $\forall n \in \mathbb{N}. \neg \bigwedge_{i\in\mathbb{N}} n_i \in \mathbb{N} \wedge n > n_i \wedge n_i > n_{i+1}$. I.e. no chain $n_0 > n_1 > n_2 > \ldots$, ad infinitum.
- Principle of infinite descent on the natural numbers:
$$\frac{\neg P(k) \Rightarrow \exists k'\big(k' < k \wedge \neg P(k')\big)}{\forall n. n \in \mathbb{N} \Rightarrow P(n)} \quad (k\ is\ arbitrary\ from\ \mathbb{N})$$
I.e. there is a counterexample $k \in \mathbb{N}$ to $\forall n. n \in \mathbb{N} \Rightarrow P(n)$, then it must be there exists a strictly smaller counterexample, and hence an infinitely descending chain of natural number. This is a contradiction, so there exists no counterexamples to $\forall n. n \in \mathbb{N} \Rightarrow P(n)$.

Equivalence of Infinite Descent and Complete Induction:

- Principle for infinite descent:
$$\frac{\neg P(k) \Rightarrow \exists k'\big(k' < k \wedge \neg P(k')\big)}{\forall n. n \in \mathbb{N} \Rightarrow P(n)} \quad (k\ is\ arbitrary\ from\ \mathbb{N})$$
- Equivalent principle, by using equivalences on the premise:
  - Premise:
$$\begin{aligned}&\neg P(k) \Rightarrow \exists k'\big(k' < k \wedge \neg P(k')\big)\\ &\equiv \neg\exists k'\big(k' < k \wedge \neg P(k')\big) \Rightarrow \neg\neg P(k)\\ &\equiv \forall k'\neg\big(k' < k \wedge \neg P(k')\big) \Rightarrow P(k)\\ &\equiv \forall k'\neg\big(k' < k \wedge \neg P(k')\big) \Rightarrow P(k)\\ &\equiv \forall k'\big(\neg(k' < k) \vee \neg\neg P(k')\big) \Rightarrow P(k)\\ &\equiv \forall k'\big(\neg(k' < k) \vee P(k')\big) \Rightarrow P(k)\\ &\equiv \forall k'\big(k' < k \Rightarrow P(k')\big) \Rightarrow P(k)\end{aligned}$$
  - Principle:
$$\frac{\forall k'\big(k' < k \Rightarrow P(k')\big) \Rightarrow P(k)}{\forall n. n \in \mathbb{N} \Rightarrow P(n)} \quad (k\ is\ arbitrary\ from\ \mathbb{N})$$

Complete (or Strong) Induction:

- Principle of complete induction for the universality of a property P of $\mathbb{N}$:
$$\frac{\forall k'\big(k' < k \Rightarrow P(k')\big) \Rightarrow P(k)}{\forall n. n \in \mathbb{N} \Rightarrow P(n)} \quad (k\ is\ arbitrary\ from\ \mathbb{N})$$
- It is called 'strong' induction as it uses a stronger induction hypothesis than ordinary induction, notably that for all smaller elements the property holds, rather than only the predecessor element.
- To show the premise $\forall k'\big(k' < k \Rightarrow P(k')\big) \Rightarrow P(k)$ is valid:
  - Pick an arbitrary $k, k \in \mathbb{N}$.
  - Induction Hypothesis: assume the antecedent holds, i.e. $\forall k'\big(k' < k \Rightarrow P(k')\big)$.
  - Step Case: show, using the induction hypothesis, $P(k)$ follows.
  - There is no explicit base case, in the case of $k = 0$ the assumption is vacuously true, as there are no $k' \in \mathbb{N}$ such that $k' < 0$. The base case is implicitly included in the structure of complete induction. So the base case must still be shown, although it will be part of the step case, since there is no information provided by the induction hypothesis which can help to show it.

- Complete induction is equivalent to ordinary induction (can be shown similarly as how ordinary and two-step were shown, by showing both imply each other, i.e. the sets of premises are equivalent), but sometimes provides more concise proofs.