Θέμα ②

Σχεδιάστε ένα ταξινομητή LDA (Linear Discriminant Analysis)

ω_1, ω_2, $\mu_1 = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$, $\mu_2 = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$\Sigma_1 = \begin{bmatrix} 11 & 9 \\ 9 & 11 \end{bmatrix}$, $\Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Λύση.

$$\Sigma_\omega = P(\omega_1)\Sigma_1 + P(\omega_2)\Sigma_2 = \frac{1}{2}\begin{pmatrix} 11 & 9 \\ 9 & 11 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 13/2 & 9/2 \\ 9/2 & 13/2 \end{pmatrix}$$

$$\det(\Sigma_\omega) = \left(\frac{13}{2}\right)^2 - \left(\frac{9}{2}\right)^2 = \frac{169}{4} - \frac{81}{4} = \frac{88}{4} = 22$$

$$\Sigma_\omega^{-1} = \frac{1}{\det(\Sigma_\omega)} \begin{pmatrix} \frac{13}{2} & -\frac{9}{2} \\ -\frac{9}{2} & \frac{13}{2} \end{pmatrix} = \begin{pmatrix} \frac{13}{44} & -\frac{9}{44} \\ -\frac{9}{44} & \frac{13}{44} \end{pmatrix}$$

$$w = \Sigma_\omega^{-1}(\mu_1 - \mu_2) = \begin{pmatrix} \frac{13}{44} & -\frac{9}{44} \\ -\frac{9}{44} & \frac{13}{44} \end{pmatrix} \begin{pmatrix} -5 & -10 \\ 5 & -15 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{13}{44} & -\frac{9}{44} \\ -\frac{9}{44} & \frac{13}{44} \end{pmatrix} \begin{pmatrix} -15 \\ -10 \end{pmatrix} = \begin{pmatrix} -\frac{105}{44} \\ \frac{5}{44} \end{pmatrix}$$

$$\Rightarrow \boxed{w = \frac{1}{44}\begin{pmatrix} -105 \\ 5 \end{pmatrix}}$$

Θέμα 4

Bayes.

ω_1, ω_2, $P(\omega_1), P(\omega_2)$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$P(x|\omega_1) = N(\mu_1, \Sigma_1)$
$P(x|\omega_2) = N(\mu_2, \Sigma_2)$, με $\mu_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $\Sigma_1 = \begin{pmatrix} 1{,}2 & -0{,}4 \\ -0{,}4 & 1{,}2 \end{pmatrix}$
$\mu_2 = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, $\Sigma_2 = \begin{pmatrix} 1{,}2 & 0{,}4 \\ 0{,}4 & 1{,}2 \end{pmatrix}$

(α) Το σύνορο απόφασης: όταν:

$$P(\omega_1) \cdot P(x|\omega_1) = P(\omega_2) \cdot P(x|\omega_2)$$

$$P(x|\omega_1) = \frac{1}{\sqrt{(2\pi)^2 |\Sigma_1|}} \cdot e^{-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)}$$

→ για ω_1: $\det(\Sigma_1) = 1{,}2^2 - 0{,}4^2 = 1{,}28$

$$\Sigma_1^{-1} = \frac{1}{1{,}28} \begin{pmatrix} 1{,}2 & 0{,}4 \\ 0{,}4 & 1{,}2 \end{pmatrix} = \begin{pmatrix} 0{,}93 & 0{,}31 \\ 0{,}31 & 0{,}93 \end{pmatrix}$$

$$(x-\mu_1)^T \cdot \Sigma_1^{-1} (x-\mu_1) = (x_1 - 3 \quad x_2 - 3) \begin{pmatrix} 0{,}93 & 0{,}31 \\ 0{,}31 & 0{,}93 \end{pmatrix} \begin{pmatrix} x_1 - 3 \\ x_2 - 3 \end{pmatrix}$$

$$= 0{,}93 \cdot x_1^2 + 0{,}62 x_1 \cdot x_2 - 7{,}44 \cdot x_1 + 0{,}93 x_2^2 + 22{,}32 - 7{,}44 \cdot x_2$$

→ για ω_2: $\det(\Sigma_2) = 1{,}2^2 - 0{,}4^2 = 1{,}28$

$$\Sigma_2^{-1} = \frac{1}{1{,}28} \begin{pmatrix} 1{,}2 & -0{,}4 \\ -0{,}4 & 1{,}2 \end{pmatrix} = \begin{pmatrix} 0{,}93 & -0{,}31 \\ -0{,}31 & 0{,}93 \end{pmatrix}$$

$$(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2) = 0{,}93 x_1^2 - 0{,}62 \cdot x_1 \cdot x_2 - 7{,}44 \cdot x_1 + 0{,}93 x_2^2 + 44{,}64 - 7{,}44 \cdot x_2$$

$$P(\omega_1)\cdot \frac{1}{2\pi|\Sigma_1|^{\frac{1}{2}}}\cdot e^{-\frac{1}{2}(x-\mu_1)^T\cdot \Sigma_1^{-1}(x-\mu_1)}$$

$$= P(\omega_2)\cdot \frac{1}{2\pi|\Sigma_2|^{\frac{1}{2}}}\cdot e^{-\frac{1}{2}(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)}$$

$$\Rightarrow P(\omega_1)\cdot e^{-1/2(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)} = P(\omega_2)\cdot e^{-1/2\cdot(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)}$$

$$\Rightarrow \ln(P(\omega_1)) - \frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) = \ln(P(\omega_2)) - \frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)$$

$$\Rightarrow \ln(P(\omega_1)) - \cancel{\frac{0{,}93}{2}x_1^2} - \frac{0{,}62}{2}x_1\cdot x_2 + \cancel{\frac{7{,}44}{2}x_1} - \cancel{\frac{0{,}93}{2}x_2^2}$$

$$-\frac{22{,}32}{2} + \cancel{\frac{7{,}44}{2}x_2} =$$

$$\ln(P(\omega_2)) - \cancel{\frac{0{,}93}{2}x_1^2} + \frac{0{,}62}{2}x_1x_2 + \cancel{\frac{7{,}44}{2}x_1} - \cancel{\frac{0{,}93}{2}x_2^2}$$

$$-\frac{44{,}64}{2} + \cancel{\frac{7{,}44}{2}x_2}$$

$$\Rightarrow \ln(P(\omega_1)) - 0{,}31\cdot x_1x_2 - 11{,}16 = \ln(P(\omega_2)) + 0{,}31\cdot x_1\cdot x_2 - 22{,}32$$

$$\Rightarrow \ln(P(\omega_1)) - \ln(P(\omega_2)) + 11{,}16 = 0{,}62\cdot x_1\cdot x_2$$

$$\Rightarrow \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 11{,}16 = 0{,}62\cdot x_1\cdot x_2$$

$$\Rightarrow \boxed{x_1 = \frac{\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 11{,}16}{0{,}62\cdot x_2},\quad x_2 \neq 0}$$

$$\Rightarrow \boxed{x_1\cdot x_2 = 1{,}61\cdot \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 18}$$

(β) $\Sigma = \Sigma_1 = \Sigma_2 = \begin{pmatrix} 1,2 & 0,4 \\ 0,4 & 1,2 \end{pmatrix}$

$\det(\Sigma) = \det(\Sigma_1) = 1,28$

$\underbrace{\text{και } \Sigma^{-1} = \Sigma_1^{-1} = \begin{pmatrix} 0,93 & 0,31 \\ 0,31 & 0,93 \end{pmatrix}}_{\text{από ερώτημα (α)}}$

$\rightarrow$ για ω_1: $(x-\mu_1)^T \cdot \Sigma^{-1}(x-\mu_1)$

$= 0,93x_1^2 + 0,62x_1 \cdot x_2 - 7,44x_1 + 0,93x_2^2 + 22,32 - 7,44 \cdot x_2$

ερώτημα (α).

$\rightarrow$ για ω_2: $(x-\mu_2)^T \Sigma^{-1}(x-\mu_2)$

$= (x_1-6 \;\; x_2-6) \begin{pmatrix} 0,93 & 0,31 \\ 0,31 & 0,93 \end{pmatrix} \begin{pmatrix} x_1-6 \\ x_2-6 \end{pmatrix}$

$= 0,93x_1^2 + 0,62 \cdot x_1 \cdot x_2 - 14,88x_1 + 0,93 \cdot x_2^2 + 89,28 - 14,88 \cdot x_2$.

Άρα $\Rightarrow P(\omega_1) \cdot P(x|\omega_1) = P(\omega_2) \cdot P(x|\omega_2)$

$\Rightarrow \ln(P(\omega_1)) - \frac{1}{2}(x-\mu_1)^T \cdot \Sigma^{-1}(x-\mu_1)$

$= \ln(P(\omega_2)) - \frac{1}{2}(x-\mu_2)^T \cdot \Sigma^{-1}(x-\mu_2)$.

$\Rightarrow \ln(P(\omega_1)) - \cancel{\frac{0,93}{2}x_1^2} - \cancel{\frac{0,62}{2}x_1 x_2} + \frac{7,44}{2} \cdot x_1$

$- \cancel{\frac{0,93}{2}x_2^2} - \frac{22,32}{2} + \frac{7,44}{2}x_2$.

$= \ln(P(\omega_2)) - \cancel{\frac{0,93}{2} \cdot x_1^2} - \cancel{\frac{0,62}{2}x_1 \cdot x_2} + \frac{14,88}{2}x_1$.

$- \cancel{\frac{0,93}{2}x_2^2} - \frac{89,28}{2} + \frac{14,88}{2} \cdot x_2$.

$$\Rightarrow \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 3{,}72 \cdot x_1 - 11{,}16 + 3{,}72 \cdot x_2$$

$$= 7{,}44 x_1 - 44{,}64 + 7{,}44 x_2$$

$$\Rightarrow \boxed{x_1 = \frac{\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) - 3{,}72 \cdot x_2 + 33{,}48}{3{,}72}}$$

Θέμα 6

Minimum Risk

ω_1, ω_2 , $P(\omega_1) = P(\omega_2)$

$$p(x|\omega_i) = \begin{cases} \frac{x}{\sigma_i^2} \cdot e^{-\frac{x^2}{2\sigma_i^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \sigma_1 = 1, \ \sigma_2 = 2$$

$$L = \begin{pmatrix} 0 & 0{,}5 \\ 1{,}0 & 0 \end{pmatrix}$$

$$R_1 = \lambda_{11} \cdot P(\omega_1) \cdot P(x|\omega_1) + \lambda_{21} P(\omega_2) \cdot P(x|\omega_2).$$

$$\Rightarrow R_1 = P(\omega_2) \cdot P(x|\omega_2)$$

$$R_2 = \lambda_{12} \cdot P(\omega_1) \cdot P(x|\omega_1) + \lambda_{22} \cdot P(\omega_2) \cdot P(x|\omega_2)$$

$$\Rightarrow R_2 = \frac{1}{2} P(\omega_1) \cdot P(x|\omega_1)$$

με $L = \begin{pmatrix} 0 & 0{,}5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$

Αν $R_1 < R_2 \Rightarrow$ αποφασίζω ω_1
Αν $R_1 > R_2 \Rightarrow$ αποφασίζω ω_2

θέλω το x_0, θα είναι όταν $R_1 = R_2$.

$P_1 = P_2$.

$$\Rightarrow P(\omega_2)\cdot P(x|\omega_2) = \frac{1}{2} P(\omega_1)\cdot P(x|\omega_1)$$

$$\Rightarrow \frac{x_0}{2}\cdot e^{-\frac{x_0^2}{2\cdot 4}} = \frac{1}{2} x_0 e^{-\frac{x_0^2}{2}}$$

$$\overset{x_0 \neq 0}{\Rightarrow} \ln\left(\frac{1}{2} e^{-\frac{x_0^2}{8}}\right) = \ln\left(e^{-\frac{x_0^2}{2}}\right)$$

$$\Rightarrow -\frac{x_0^2}{8} - \ln 2 = -\frac{x_0^2}{2}$$

$$\Rightarrow -x_0^2 - 8\ln 2 = -4x_0^2$$

$$\Rightarrow 3x_0^2 = 8\ln 2 \Rightarrow \boxed{x_0 = \sqrt{\frac{8\ln 2}{3}}}$$

Θέμα 7:

Singular Value Decomposition (SVD)

$$X = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$$ υπολογίστε rank-1 προσέγγιση κατά την.

1. $$X^T \cdot X = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 7 & 14 \end{pmatrix}$$

$\Rightarrow$ ιδιοτιμές $X^T \cdot X$:

$$|X^T X - \lambda I_2| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & 7 \\ 7 & 14-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)(14-\lambda) - 49 = 0 \Rightarrow \lambda^2 - 20\lambda + 35 = 0$$

$$\Delta = \beta^2 - 4\alpha\gamma = 260$$

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\Delta}}{2\alpha} = \frac{20 \pm 2\sqrt{65}}{2} \quad \begin{array}{l} \nearrow \lambda_1 = 10 + \sqrt{65} \\ \searrow \lambda_2 = 10 - \sqrt{65} \end{array}$$

→ Ιδιοδιανύσματα $X^T X$:

$(X^T X - \lambda_i I_2) \cdot V_i = 0$, με $V_i = \begin{pmatrix} V_{i1} \\ V_{i2} \end{pmatrix}$.

→ για $\lambda_1 = 10 + \sqrt{65}$:

$$(X^T X - \lambda_1 I_2) V_1 = 0$$

⇒ Επαυξημένος πίνακας με τα στοιχεία του V_1

$$\left(\begin{array}{cc|c} -4-\sqrt{65} & 7 & V_{11}=0 \\ 7 & 4-\sqrt{65} & V_{i2}=0 \end{array}\right)$$

Χρησιμοποιώ απαλοιφή Gauss.

$$\left(\begin{array}{cc|c} -4-\sqrt{65} & 7 & 0 \\ 7 & 4-\sqrt{65} & 0 \end{array}\right) \xrightarrow{R_1 \leftarrow \frac{R_1}{-4-\sqrt{65}}} \left(\begin{array}{cc|c} 1 & \frac{4-\sqrt{65}}{7} & 0 \\ 7 & 4-\sqrt{65} & 0 \end{array}\right)$$

$$R_2 \leftarrow R_2 - 7R_1 \rightarrow \left(\begin{array}{cc|c} 1 & \frac{4-\sqrt{65}}{7} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Άρα $V_{11} + \frac{4-\sqrt{65}}{7} \cdot V_{12} = 0$
και $V_{12} = V_{12}$. $\Big\}$ ⇒

για $v_{12}=1$: $v_{11}=\frac{-4+\sqrt{65}}{7}$

Επομένως $v_1=\begin{pmatrix}\frac{-4+\sqrt{65}}{7}\\ 1\end{pmatrix}$

→ για $\lambda_2=10-\sqrt{65}$

Επαυξημένος πίνακας με στοιχεία τα v_2.

$$\left(\begin{array}{cc|c} -4+\sqrt{65} & 7 & v_{21}=0 \\ 7 & 4+\sqrt{65} & v_{22}=0 \end{array}\right)$$

Χρησιμοποιώ την απαλοιφή Gauss

$$\left(\begin{array}{cc|c} -4+\sqrt{65} & 7 & 0 \\ 7 & 4+\sqrt{65} & 0 \end{array}\right) R_1 \leftarrow \frac{R_1}{-4+\sqrt{65}} \rightarrow \left(\begin{array}{cc|c} 1 & \frac{4+\sqrt{65}}{7} & 0 \\ 7 & 4+\sqrt{65} & 0 \end{array}\right)$$

$$R_2 \leftarrow R_2 - 7R_1 \rightarrow \left(\begin{array}{cc|c} 1 & \frac{4+\sqrt{65}}{7} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Άρα $\left.\begin{array}{l} v_{12}+\frac{4+\sqrt{65}}{7}\cdot v_{22}=0 \\ \text{και } v_{22}=v_{22} \end{array}\right\} \Rightarrow$

για $v_{22}=1$: $v_{21}=\frac{-4-\sqrt{65}}{7}$

Επομένως $v_2=\begin{pmatrix}\frac{-4-\sqrt{65}}{7}\\ 1\end{pmatrix}$

Πρέπει να γίνει κανονικοποίηση

των v_1, v_2 έτσι ώστε να ισχύει $|v_1|=|v_2|=1$, για να δημιουργηθεί ο ~~τελικός~~ τελικός πίνακας V.

$$|v_1| = \sqrt{\left(\frac{-4+\sqrt{65}}{7}\right)^2+1} = 1,15$$

$$|v_2| = \sqrt{\left(\frac{-4-\sqrt{65}}{7}\right)^2+1} = 1,99$$

$\Rightarrow$ ~~Άρα~~ $v_{1,norm} = \frac{v_1}{|v_1|} = \frac{1}{1,15}\begin{pmatrix} \frac{-4+\sqrt{65}}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} 0,5 \\ 0,86 \end{pmatrix}$

$\Rightarrow$ $v_{2,norm} = \frac{v_2}{|v_2|} = \frac{1}{1,99}\begin{pmatrix} \frac{-4-\sqrt{65}}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} -0,86 \\ 0,5 \end{pmatrix}$

Ο Τελικός πίνακας είναι

$$\boxed{V = \begin{pmatrix} 0,5 & -0,86 \\ 0,86 & 0,5 \end{pmatrix}}$$

② Υπολογισμός singular values.

$$s_1 = \sqrt{\lambda_1} = \sqrt{10+\sqrt{65}} = 4,24$$

$$s_2 = \sqrt{\lambda_2} = \sqrt{10-\sqrt{65}} = 1,39$$

$$\Rightarrow \Sigma = \begin{pmatrix} 4,24 & 0 \\ 0 & 1,39 \end{pmatrix}$$

③ Υπολογισμός ιδιοδιανυσμάτων XX^T, δηλαδή τον πίνακα U.

$$u_1 = \frac{1}{s_1} \cdot X \cdot v_{1,norm} = \frac{1}{4{,}24} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0{,}5 \\ 0{,}86 \end{pmatrix}$$

$$= \begin{pmatrix} 0{,}52 \\ 0{,}43 \\ 0{,}72 \end{pmatrix}$$

$$u_2 = \frac{1}{s_2} \cdot X \cdot v_{2,norm} = \frac{1}{1{,}39} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -0{,}86 \\ 0{,}5 \end{pmatrix}$$

$$= \begin{pmatrix} 0{,}1 \\ -0{,}87 \\ 0{,}46 \end{pmatrix}$$

$$X = U \cdot \Sigma V^T = \begin{pmatrix} 0{,}52 & 0{,}10 \\ 0{,}43 & -0{,}87 \\ 0{,}72 & 0{,}46 \end{pmatrix} \begin{pmatrix} 4{,}24 & 0 \\ 0 & 1{,}39 \end{pmatrix} \begin{pmatrix} 0{,}5 & 0{,}86 \\ -0{,}86 & 0{,}5 \end{pmatrix}$$

↳ λόγω προσεγγίσεων σε δεκαδικά θα υπάρξουν μικρές διαφορές.

④ $$\hat{X} = s_1 \cdot u_1 \cdot v_{1,norm}^T = 4{,}24 \begin{pmatrix} 0{,}52 \\ 0{,}43 \\ 0{,}72 \end{pmatrix} \begin{pmatrix} 0{,}5 & 0{,}86 \end{pmatrix}$$

$$= \begin{pmatrix} 1{,}10 & 1{,}89 \\ 0{,}91 & 1{,}56 \\ 1{,}52 & 2{,}62 \end{pmatrix}$$