SOLUZIONI

1) (i) $$\lim_{x\to 0} \frac{\sin(\log(1+2x)) - e^{2x} + 1}{tg(x^2)}$$

$$= \lim_{x\to 0} \frac{\sin\left(2x - \frac{(2x)^2}{2} + o(x^2)\right) - \left(1 + 2x + \frac{(2x)^2}{2} + o(x^2)\right) + 1}{tg(x^2)}$$

$$= \lim_{x\to 0} \frac{\sin(2x - 2x^2 + o(x^2)) - 2x - 2x^2 + o(x^2)}{x^2 + o(x^2)}$$

$$= \lim_{x\to 0} \frac{\cancel{2x} - 2x^2 + o(x^2) - \cancel{2x} - 2x^2 + o(x^2)}{x^2 + o(x^2)}$$

$$= \lim_{x\to 0} \frac{-4x^2 + o(x^2)}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{-4 + o(1)}{1 + o(1)} = \boxed{-4}$$

(ii) $$\lim_{x\to 0} \frac{\log(\cos x) + \log(e^x - x) - \frac{x^3}{6}}{x^3 \operatorname{arctg} x}$$

$$= \lim_{x\to 0} \frac{\log\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) + \log\left(1 + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4) - \cancel{x}\right) - \frac{x^3}{6}}{x^4 + o(x^4)}$$

$$= \lim_{x\to 0} \frac{\left(-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)\right) - \frac{\left(-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)\right)^2}{2} + o(x^4) + \left(\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+o(x^4)\right) - \frac{\left(\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+o(x^4)\right)^2}{2} + o(x^4) - \frac{x^3}{6}}{x^4+o(x^4)}$$

$$= \lim_{x\to 0} \frac{\cancel{-\frac{x^2}{2}}+\frac{x^4}{24}-\frac{x^4}{8}+o(x^4)+\cancel{\frac{x^2}{2}}+\cancel{\frac{x^3}{6}}+\frac{x^4}{24}-\frac{x^4}{8}+o(x^4)\cancel{-\frac{x^3}{6}}}{x^4+o(x^4)}$$

$$= \lim_{x\to 0} \frac{-\frac{4}{24}x^4+o(x^4)}{x^4+o(x^4)} = \lim_{x\to 0} \frac{-\frac{1}{6}+o(1)}{1+o(1)} = \boxed{-\frac{1}{6}}$$

(iii) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

$$= \lim_{x\to 0} \frac{x(1+x+o(x)) - \left(x-\frac{x^2}{2}+o(x^2)\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x}+x^2+o(x^2)\cancel{-x}+\frac{x^2}{2}+o(x^2)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{3}{2}x^2+o(x^2)}{x^2} = \lim_{x\to 0} \frac{3}{2}+o(1) = \boxed{\frac{3}{2}}$$

(iv) $$\lim_{x\to 0} \frac{\sin x\, e^{x} - 2\sqrt{1+x} + 2}{\log(1+x^2)}$$

$$= \lim_{x\to 0} \frac{(x+o(x^2))\left(1+x+\frac{x^2}{2}+o(x^2)\right) - 2\left(1+\frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)\right) + 2}{x^2+o(x^2)}$$

$$= \lim_{x\to 0} \frac{\cancel{x} + x^2 + o(x^2) \cancel{-2} \cancel{-x} + \frac{1}{4}x^2 + o(x^2) \cancel{+2}}{x^2+o(x^2)}$$

$$= \lim_{x\to 0} \frac{\frac{5}{4}x^2+o(x^2)}{x^2+o(x^2)} = \lim_{x\to 0} \frac{\frac{5}{4}+o(1)}{1+o(1)} = \boxed{\frac{5}{4}}$$

(v) $$\lim_{x\to 0} \frac{e^{x(1-x)} - \sin x + \log(1+x^2) - 1}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(1 + x(1-x) + \frac{(x(1-x))^2}{2} + o((x(1-x))^2)\right) - (x+o(x^2)) + (x^2+o(x^2)) - 1}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(1+x-x^2+\frac{x^2}{2}+o(x^2)\right) - x + x^2 + o(x^2) - 1}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2}+o(x^2)}{x^2} = \lim_{x\to 0} \frac{1}{2}+o(1) = \boxed{\frac{1}{2}}$$

2) (i) $\sum_{n=1}^{+\infty} \frac{1-e^{\sqrt{\frac{1}{n}}}}{n} \sim \sum_{n=1}^{+\infty} \frac{1-\left(1+\sqrt{\frac{1}{n}}\right)}{n}$

$= \sum_{n=1}^{+\infty} -\sqrt{\frac{1}{n}} \cdot \frac{1}{n}$

$= -\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$ converge perché $\frac{3}{2} > 1$

$$\boxed{\text{Ricordiamo che } \sum_{n} \frac{1}{n^{\alpha}} \text{ converge} \iff \alpha > 1}$$

(ii) $\sum_{n=1}^{+\infty} \frac{n+3}{n^3-n^2+4} \sim \sum_{n=1}^{+\infty} \frac{n}{n^3} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$

converge $(2>1)$

(iii) $\sum_{n=2}^{+\infty} \frac{1}{\sqrt{n^3-n^2}} \sim \sum_{n=2}^{+\infty} \frac{1}{(n^3)^{1/2}} = \sum_{n=2}^{+\infty} \frac{1}{n^{3/2}}$

converge

(iv) $\sum_{n=1}^{+\infty} \frac{(n+2)^n}{n^{n+2}} = \sum_{n=1}^{+\infty} \frac{(n+2)^n}{n^n} \cdot \frac{1}{n^2}$

$= \sum_{n=1}^{+\infty} \left(\frac{n+2}{n}\right)^n \cdot \frac{1}{n^2}$

$= \sum_{n=1}^{+\infty} \left(1+\frac{2}{n}\right)^n \cdot \frac{1}{n^2} \sim \sum_{n=1}^{+\infty} \frac{e^2}{n^2} = e^2 \sum_{n=1}^{+\infty} \frac{1}{n^2}$

converge

(v) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{+\infty} \frac{n+1-n}{n(n+1)} = \sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$

$$\sim \sum_{n=1}^{+\infty} \frac{1}{n^2} \ \underline{\text{converge}}$$

Osserviamo che questa serie é particolare, infatti

$$\sum_{n=1}^{+\infty} \frac{1}{n} - \frac{1}{n+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots$$

$$= 1 + \underbrace{\left(-\frac{1}{2} + \frac{1}{2}\right)}_{=0} + \underbrace{\left(-\frac{1}{3} + \frac{1}{3}\right)}_{=0} + \underbrace{\left(-\frac{1}{4} + \frac{1}{4}\right)}_{=0} + \dots$$

$$= 1$$

Questa serie é detta telescopica, perché rimane solo il primo termine (1) e l'ultimo (che tende a 0)

(vi) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \left(\sin\left(\frac{1}{n}\right)\right)^2$

Qui vogliamo applicare il criterio di Leibniz
Dobbiamo allora verificare che

- $\left(\sin \frac{1}{n}\right)^2$ é definitivamente decrescente
- $\displaystyle\lim_{n\to+\infty} \left(\sin \frac{1}{n}\right)^2 = 0$

Chiaramente quest'ultimo limite é zero.
Inoltre $\sin^2\left(\frac{1}{n}\right)$ é decrescente, infatti

$$\sin^2\left(\frac{1}{n}\right) > \sin^2\left(\frac{1}{n+1}\right) \Longrightarrow \frac{1}{n} > \frac{1}{n+1} \Longrightarrow 1 > 0 \ \text{vero}$$

Dunque per Leibniz converge

(vii) $\displaystyle\sum_{n=2}^{+\infty} (-1)^n \frac{1}{n\log^2 n}$

- $\displaystyle\lim_{n\to+\infty} \frac{1}{n\log^2 n} = 0$
- Mostriamo che $\dfrac{1}{n\log^2 n}$ è decrescente

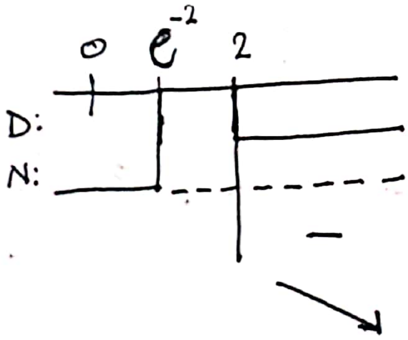
$$\Rightarrow f(x) = \frac{1}{x\log^2 x} \qquad f'(x) = \frac{-\left(\log^2 x + x\cdot 2\log x\cdot \frac{1}{x}\right)}{x^2\log^4 x}$$

$$\Rightarrow f'(x) = -\frac{\log x + 2}{x^2\log^3 x}$$

Consideriamo la funzione solo per $x \geq 2$

- $f'(x) \geq 0$ N: $-\log x - 2 \geq 0 \rightsquigarrow \log x \leq -2 \rightsquigarrow x \leq e^{-2}$

  D: $x^2\log^3 x \geq 0 \rightsquigarrow$ è positiva per le x che stiamo considerando noi



Per $x \geq 2$ $f'(x) < 0$ quindi f è decrescente

Allora $\dfrac{1}{n\log^2 n}$ è decrescente

Quindi per il criterio di Leibniz la serie converge

(viii) $\sum_{n=1}^{+\infty} \frac{1}{n!}$

La serie chiaramente converge per confronto,

infatti $n! > n^2$ per $n \geq 4$

Quindi $\frac{1}{n!} < \frac{1}{n^2}$ per $n \geq 4$

$$\Rightarrow \sum_{n=4}^{+\infty} \frac{1}{n!} < \sum_{n=4}^{+\infty} \frac{1}{n^2} < +\infty$$ quindi <u>converge</u>

Inoltre osserviamo che lo sviluppo di MacLaurin di e^x é

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$ (con la convenzione che $0! = 1$)

Allora per $x = 1 \Rightarrow e = \sum_{n=0}^{+\infty} \frac{1}{n!}$

$$\Rightarrow 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} = e$$

$$\Rightarrow \boxed{\sum_{n=1}^{+\infty} \frac{1}{n!} = e - 1}$$