SOLUZIONI

(i)

$$f(x) = \log\left(\frac{x^2}{x+1}\right)$$

- Dominio

$$\frac{x^2}{x+1} > 0 \quad \rightsquigarrow \quad \begin{matrix} x \neq 0 \\ x > -1 \end{matrix} \quad \Rightarrow (-1,0) \cup (0,+\infty)$$

- Comportamento ai bordi del dominio

$$\lim_{x \to -1^+} \log\left(\frac{x^2}{x+1}\right) = +\infty$$

$$\lim_{x \to 0^-} \log\left(\frac{x^2}{x+1}\right) = -\infty$$

$$\lim_{x \to 0^+} \log\left(\frac{x^2}{x+1}\right) = -\infty$$

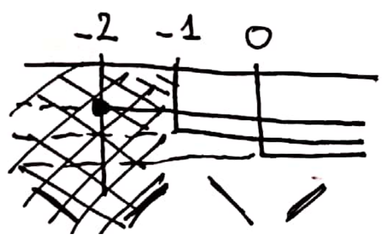
$$\lim_{x \to +\infty} \log\left(\frac{x^2}{x+1}\right) = +\infty$$

- Non ci sono asintoti obliqui
- Derivata

$$f'(x) = \frac{1}{\frac{x^2}{x+1}} \cdot \frac{2x(x+1)-x^2}{(x+1)^2}$$

$$= \frac{x+1}{x^2} \cdot \frac{x^2+2x}{(x+1)^2} = \frac{x+2}{x(x+1)}$$

$$f'(x) \geq 0 \quad \rightsquigarrow \quad \frac{x+2}{x(x+1)} \geq 0 \qquad \begin{matrix} x \geq -2 \\ x > 0 \\ x > -1 \end{matrix}$$



- Derivata seconda

$$f''(x) = \frac{x^2+x-(x+2)(2x+1)}{x^2(x+1)^2}$$

$$= \frac{x^2 \cancel{+x} - 2x^2 \cancel{-x} - 4x - 2}{x^2(x+1)^2}$$

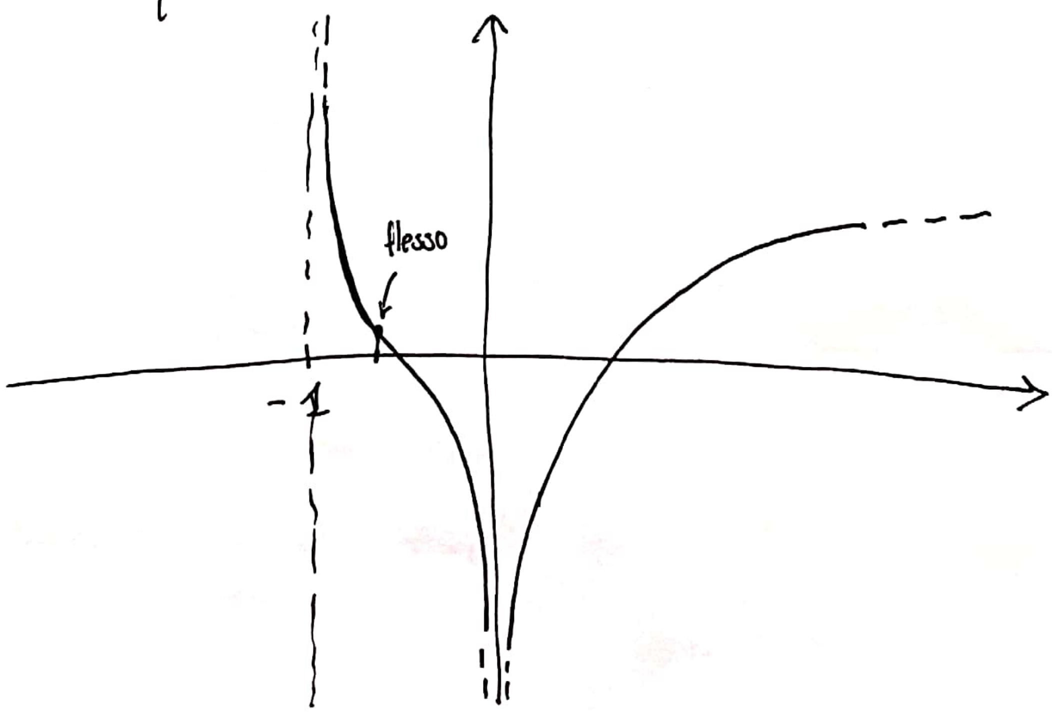
$$= \frac{-x^2-4x-2}{x^2(x+1)^2} \geq 0$$

$$x^2+4x+2 \leq 0 \rightsquigarrow -2-\sqrt{2} \leq x \leq -2+\sqrt{2}$$

$$\frac{-4 \pm \sqrt{16-8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

Dunque per $x \in (-1, -2+\sqrt{2})$ la funzione é convessa mentre per $x \in (-2+\sqrt{2}, +\infty)$ la funzione é concava.

- Grafico qualitativo



(ii) $f(x) = 2x + \sqrt{x^2-1}$

• Dominio

$x^2 - 1 \geq 0 \longrightarrow x \leq -1$ e $x \geq 1$

$\longrightarrow (-\infty, -1] \cup [1, +\infty)$

• Comportamento ai bordi del dominio

$$\lim_{x \to -\infty} 2x + \sqrt{x^2-1}$$

$$= \lim_{x \to -\infty} \frac{(2x+\sqrt{x^2-1})(2x-\sqrt{x^2-1})}{2x-\sqrt{x^2-1}}$$

$$= \lim_{x \to -\infty} \frac{4x^2 - x^2 + 1}{2x - \sqrt{x^2-1}}$$

$$= \lim_{x \to -\infty} \frac{3x^2+1}{x\left(2 - \frac{|x|}{x}\sqrt{1-\frac{1}{x^2}}\right)} = \lim_{x \to -\infty} \frac{3x + \frac{1}{x}}{2 - \frac{|x|}{x}\sqrt{1-\frac{1}{x^2}}}$$

$$= -\infty$$

$$\lim_{x \to +\infty} 2x + \sqrt{x^2-1} = +\infty$$

$$\lim_{x \to -1} 2x + \sqrt{x^2-1} = -2$$

$$\lim_{x \to 1} 2x + \sqrt{x^2-1} = 2$$

• Non ci sono asintoti verticali e orizzontali. Andiamo a vedere quelli obliqui

$$m_1: \lim_{x\to+\infty} \frac{2x+\sqrt{x^2-1}}{x} = \lim_{x\to+\infty} \frac{2x+|x|\sqrt{1-\frac{1}{x^2}}}{x}$$

$$= \lim_{x\to+\infty} 2+\sqrt{1-\frac{1}{x^2}} = 3$$

$$q_1: \lim_{x\to+\infty} 2x+\sqrt{x^2-1}-3x = \lim_{x\to+\infty} \sqrt{x^2-1}-x$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2}-1-\cancel{x^2}}{\sqrt{x^2-1}+x} = \lim_{x\to+\infty} \frac{-1}{\sqrt{x^2-1}+x} = 0$$

Dunque a $+\infty$ la retta di equazione $y=3x$ é un asintoto obliquo.

$$m_2: \lim_{x\to-\infty} \frac{2x+\sqrt{x^2-1}}{x} = \lim_{x\to-\infty} \frac{4x^2-x^2+1}{x(2x-\sqrt{x^2-1})}$$

$$= \lim_{x\to-\infty} \frac{3x^2+1}{2x^2-x|x|\sqrt{1-\frac{1}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{3+\frac{1}{x^2}}{2-\frac{|x|}{x}\sqrt{1-\frac{1}{x^2}}} = \frac{3}{2-(-1)} = 1$$

q_2: $\lim_{x \to -\infty} 2x + \sqrt{x^2-1} - x$

$$= \lim_{x \to -\infty} \frac{\left(x + \sqrt{x^2-1}\right)\left(x - \sqrt{x^2-1}\right)}{x - \sqrt{x^2-1}}$$

$$= \lim_{x \to -\infty} \frac{x^2 - x^2 + 1}{x - |x|\sqrt{1 - \frac{1}{x^2}}} = \lim_{x \to -\infty} \frac{1}{x\left(1 - \frac{|x|}{x}\sqrt{1 - \frac{1}{x^2}}\right)}$$

$$= 0$$

Dunque a $-\infty$ la retta di equazione $y = x$ è un asintoto obliquo per la funzione

- Derivata

$$f'(x) = 2 + \frac{1}{2} \cdot \frac{1}{\sqrt{x^2-1}} \cdot 2x$$

$$= 2 + \frac{x}{\sqrt{x^2-1}} = \frac{2\sqrt{x^2-1} + x}{\sqrt{x^2-1}}$$

$$f'(x) \geq 0 \longrightarrow \frac{2\sqrt{x^2-1} + x}{\sqrt{x^2-1}} \geq 0$$

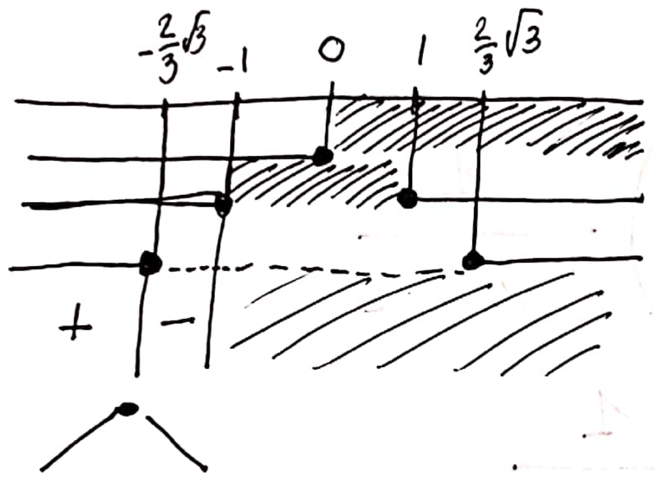
$N: 2\sqrt{x^2-1} + x \geq 0$

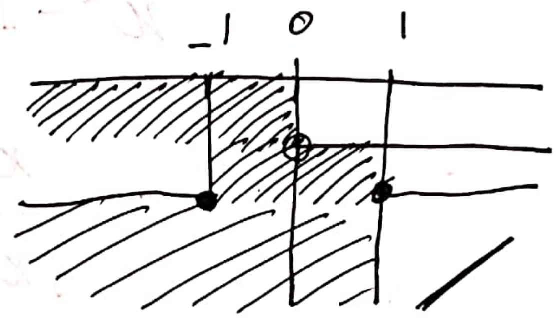
$$\sqrt{x^2-1} \geq -\frac{x}{2} \longrightarrow \begin{cases} -\frac{x}{2} \geq 0 \\ x^2 - 1 \geq \frac{x^2}{4} \end{cases} \cup \begin{cases} -\frac{x}{2} < 0 \\ x^2 - 1 \geq 0 \end{cases}$$

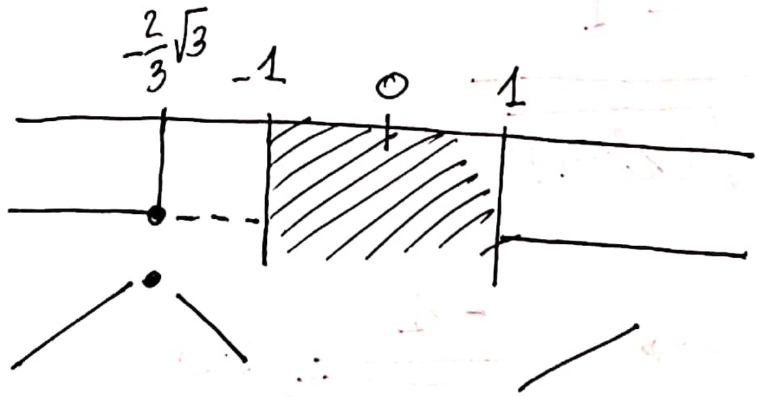
Scansionato con CamScanner

$$\begin{cases} x \leq 0 \\ \frac{3}{4}x^2 - 1 \geq 0 \end{cases} \cup \begin{cases} x > 0 \\ x \leq -1 \cup x \geq 1 \end{cases}$$

$$\begin{cases} x \leq 0 \\ x \leq -\frac{2}{3}\sqrt{3} \cup x \geq \frac{2}{3}\sqrt{3} \end{cases} \cup \begin{cases} x > 0 \\ x \leq -1 \cup x \geq 1 \end{cases}$$







Dunque c'è un massimo in corrispondenza di $-\frac{2}{3}\sqrt{3}$

$D: \sqrt{x^2-1} > 0 \longrightarrow x > 1 \cup x < -1$

Osserviamo che $\lim_{x \to 1^+} f'(x) = +\infty$

$\lim_{x \to -1^-} f'(x) = -\infty$

Perciò la funzione in -1 e 1 non é derivabile.

Potremmo già disegnare un grafico qualitativo ma per completezza:

- Derivata seconda:

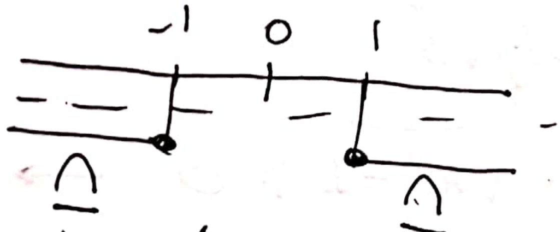
$$f''(x) = \frac{\sqrt{x^2-1} - x \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2-1}}}{x^2-1}$$

$$= \left(\sqrt{x^2-1} - \frac{x^2}{\sqrt{x^2-1}}\right) \cdot \frac{1}{x^2-1}$$

$$= \frac{|x^2-1| - x^2}{\sqrt{x^2-1}} \cdot \frac{1}{x^2-1}$$

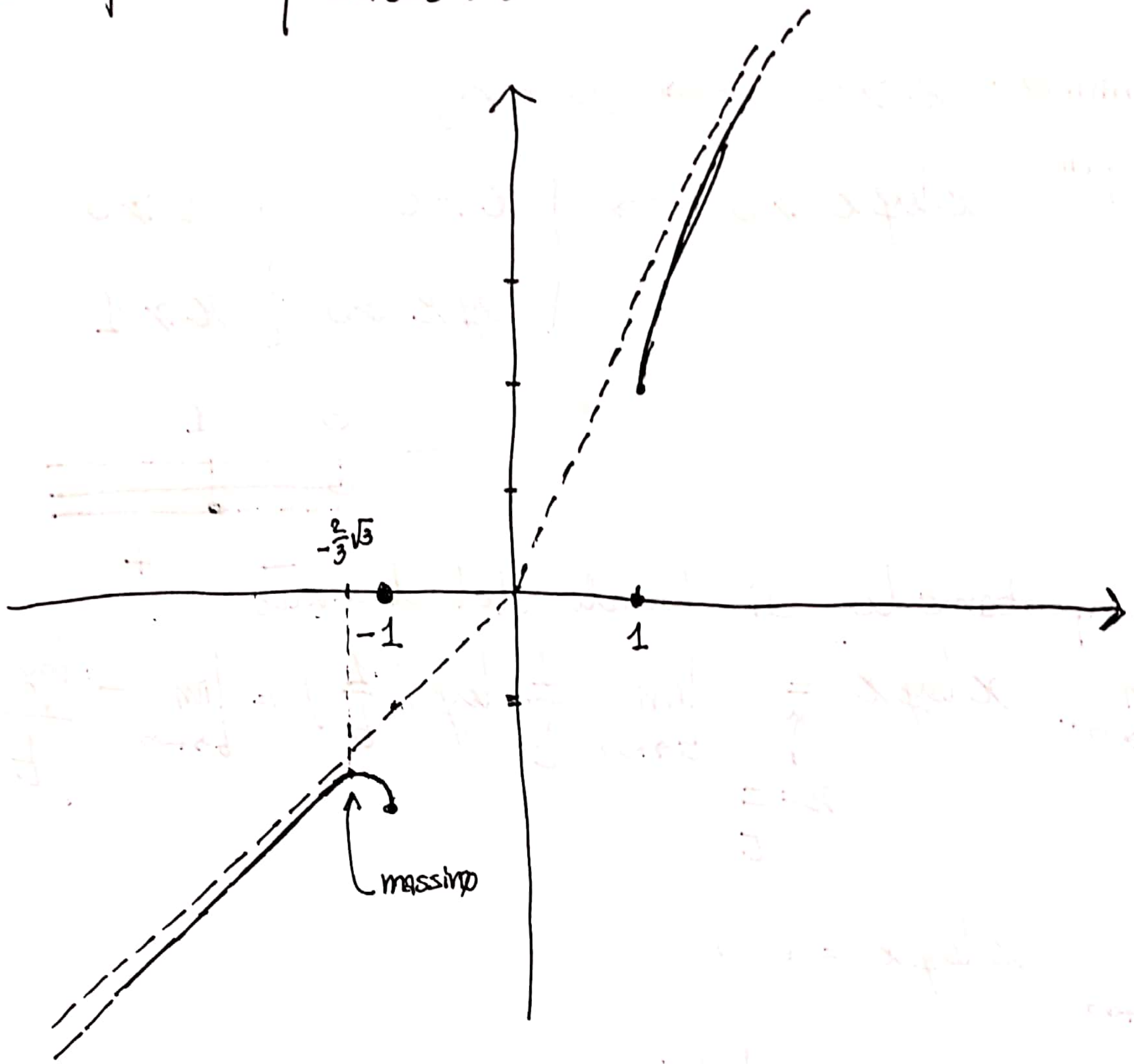
$$= \frac{-1}{\sqrt{(x^2-1)^3}}$$

$$f''(x) \geq 0 \quad \rightsquigarrow \quad \frac{-1}{\sqrt{(x^2-1)^3}} \geq 0$$



La funzione non ha flessi ed é sempre concava.

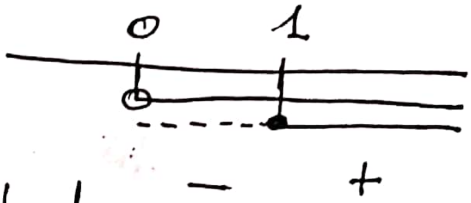
• Grafico qualitativo



Scansionato con CamScanner

(iii) $f(x) = x \log x$

- Dominio: $x > 0 \rightarrow (0, +\infty)$
- Segno: $x \log x \geq 0 \rightarrow \begin{cases} x \geq 0 \\ \log x \geq 0 \end{cases} \begin{cases} x \geq 0 \\ x \geq 1 \end{cases}$



- Comportamento ai bordi del dominio

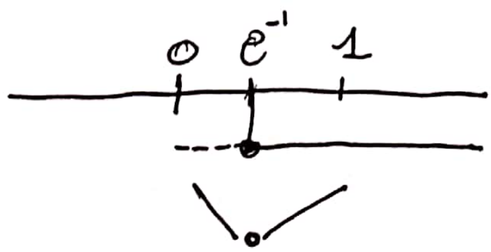
$$\lim_{x \to 0^+} x \log x \underset{x = \frac{1}{t}}{=} \lim_{t \to +\infty} \frac{1}{t} \log\left(\frac{1}{t}\right) = \lim_{t \to +\infty} -\frac{\log(t)}{t} = 0^-$$

$$\lim_{x \to +\infty} x \log x = +\infty$$

- Non ci sono asintoti
- Derivata

$$f'(x) = \log x + x \cdot \frac{1}{x} = \log x + 1$$

$$f'(x) \geq 0 \rightarrow \log x \geq -1 \rightarrow x \geq e^{-1}$$



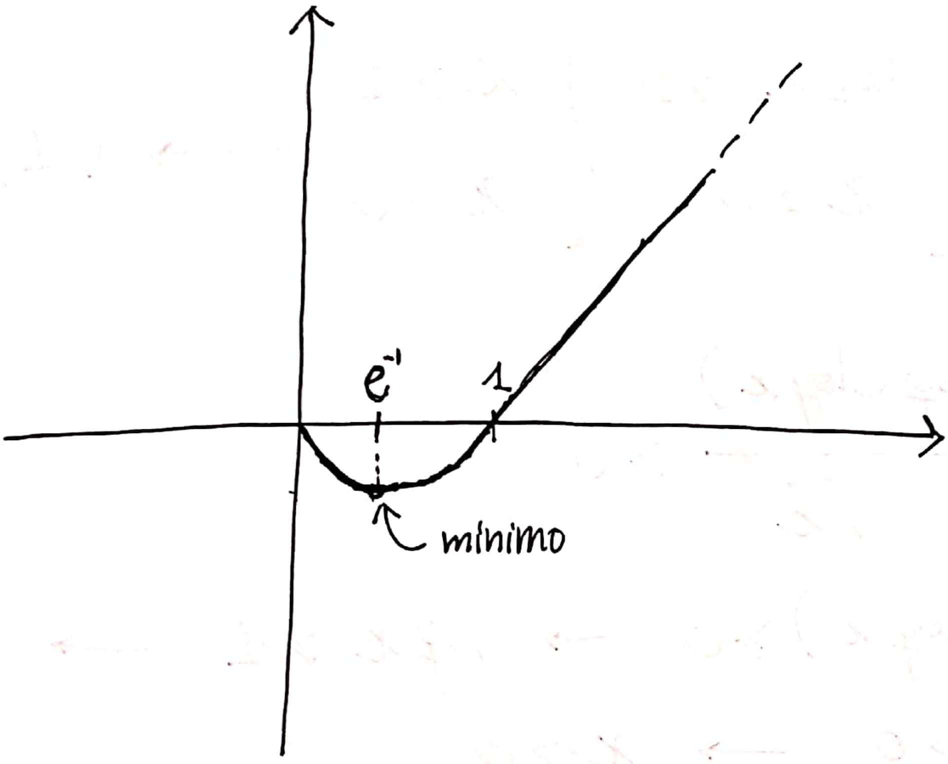
e^{-1} è un minimo

- Derivata seconda

$$f''(x) = \frac{1}{x} \rightarrow f''(x) \geq 0 \quad \frac{1}{x} \geq 0 \rightarrow x > 0$$

La funzione è sempre convessa

- Grafico



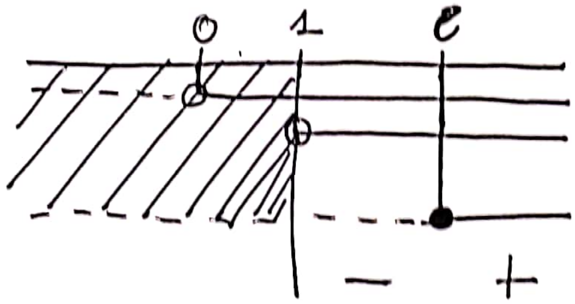
(iv) $f(x) = \dfrac{\log(\log x)}{\sqrt{x}}$

- Dominio: $\begin{cases} \log x > 0 \\ x > 0 \end{cases} \quad \begin{cases} x > 1 \\ x > 0 \end{cases} \longrightarrow (1, +\infty)$

- Segno: $\dfrac{\log(\log x)}{\sqrt{x}} \geq 0$

N: $\log(\log x) \geq 0 \rightarrow \log x \geq 1 \longrightarrow x \geq e$

D: $\sqrt{x} > 0 \longrightarrow x > 0$



- Comportamento ai bordi del dominio

$$\lim_{x \to 1^+} \frac{\log(\log x)}{\sqrt{x}} = -\infty$$

$$\lim_{x \to +\infty} \frac{\log(\log x)}{\sqrt{x}} = 0^+$$

- Derivata

$$f'(x) = \left(\frac{1}{\log x} \cdot \frac{1}{x} \cdot \cancel{\sqrt{x}} - \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot \log(\log x) \right) \cdot \frac{1}{x}$$

$$f'(x) = \frac{\left(\frac{1}{\sqrt{x}\cdot\log x} - \frac{\log(\log x)}{2\sqrt{x}}\right)}{x}$$

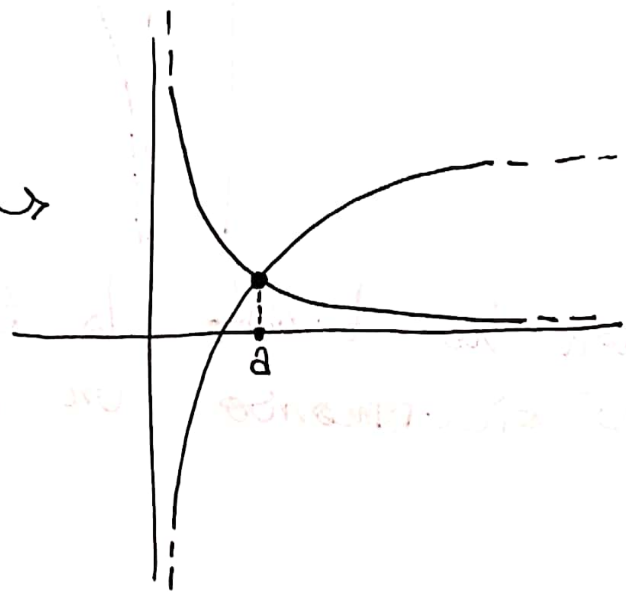
$$= \frac{2 - \log x \cdot \log(\log x)}{2\sqrt{x}\log x \cdot x}$$

$$f'(x) \geq 0 \rightarrow \frac{2-\log x\cdot \log(\log x)}{2\sqrt{x^3}\log x} \geq 0$$

$N: \; 2-\log x \cdot \log(\log x) \geq 0$

$\log x = t \quad \rightarrow 2 - t\cdot\log(t) \geq 0$

$$\log t \leq \frac{2}{t}$$

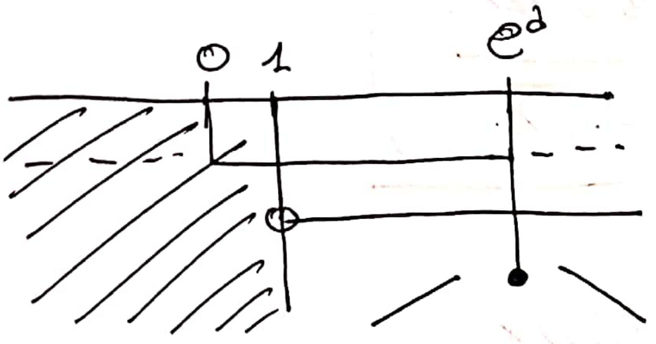


Detto a l'ascissa del punto d'intersezione tra $\log t$ e $\frac{2}{t}$, abbiamo che $\log t \leq \frac{2}{t}$ quando $0 < t \leq a$

$\Rightarrow 0 < \log x \leq a$

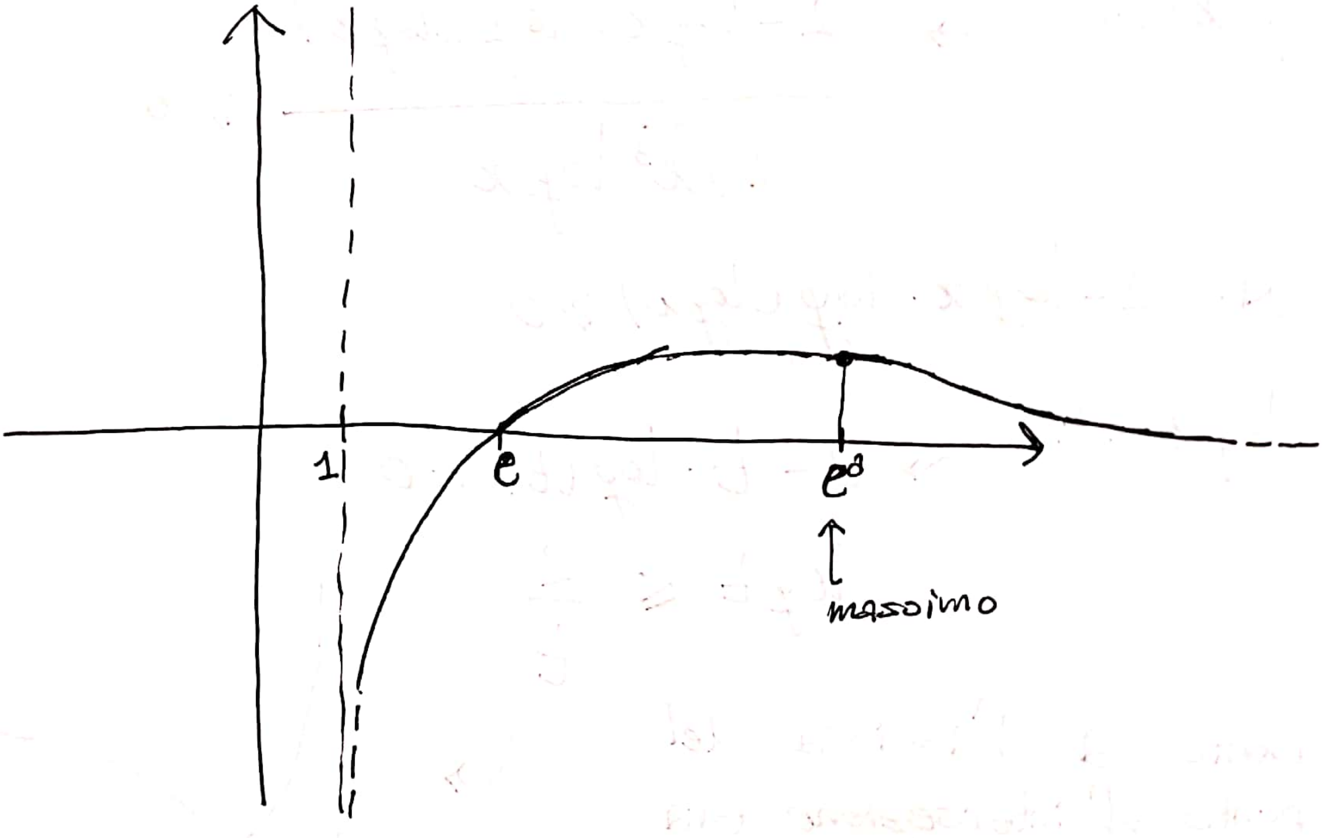
$\quad 0 < x \leq e^a$

$D: \; 2\sqrt{x^3}\log x > 0 \quad \longrightarrow \quad x > 1$



In corrispondenza di e^2 abbiamo un massimo

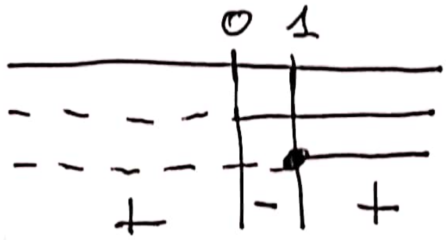
• Grafico



Non ho studiato la derivata seconda ma c'è sicuramente un flesso.

(v) $f(x) = \dfrac{x-1}{x^3}$

- Dominio: $x \neq 0 \rightarrow (-\infty;0) \cup (0,+\infty)$
- Segno: $\dfrac{x-1}{x^3} \geq 0$ N: $x \geq 1$ D: $x > 0$



- Comportamento ai bordi del dominio

$$\lim_{x \to -\infty} \frac{x-1}{x^3} = 0$$

$$\lim_{x \to 0^-} \frac{x-1}{x^3} = +\infty$$

$$\lim_{x \to 0^+} \frac{x-1}{x^3} = -\infty$$

$$\lim_{x \to +\infty} \frac{x-1}{x^3} = 0$$

- Derivata

$$f'(x) = \frac{x^3 - (x-1)3x^2}{x^6} = \frac{x^3 - 3x^3 + 3x^2}{x^6}$$

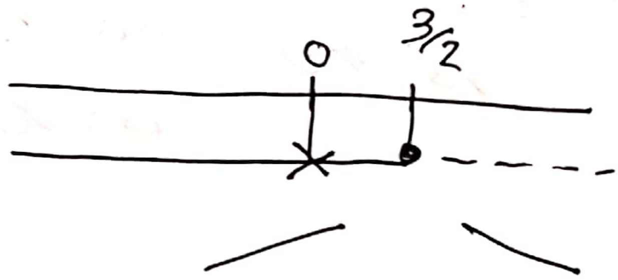
$$= \frac{3x^2 - 2x^3}{x^6} = \frac{3-2x}{x^4}$$

$$f'(x) \geq 0 \longrightarrow \frac{3-2x}{x^4} \geq 0$$

N: $3-2x \geq 0$

$x \leq \frac{3}{2}$

D: $x^4 > 0 \rightarrow x \neq 0$



- Derivata seconda

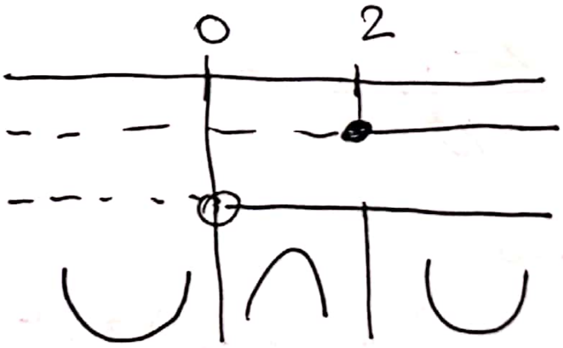
$$f''(x) = \frac{-2x^4 - (3-2x)4x^3}{x^8} = \frac{-2x^4 - 12x^3 + 8x^4}{x^8}$$

$$= \frac{6x-12}{x^5}$$

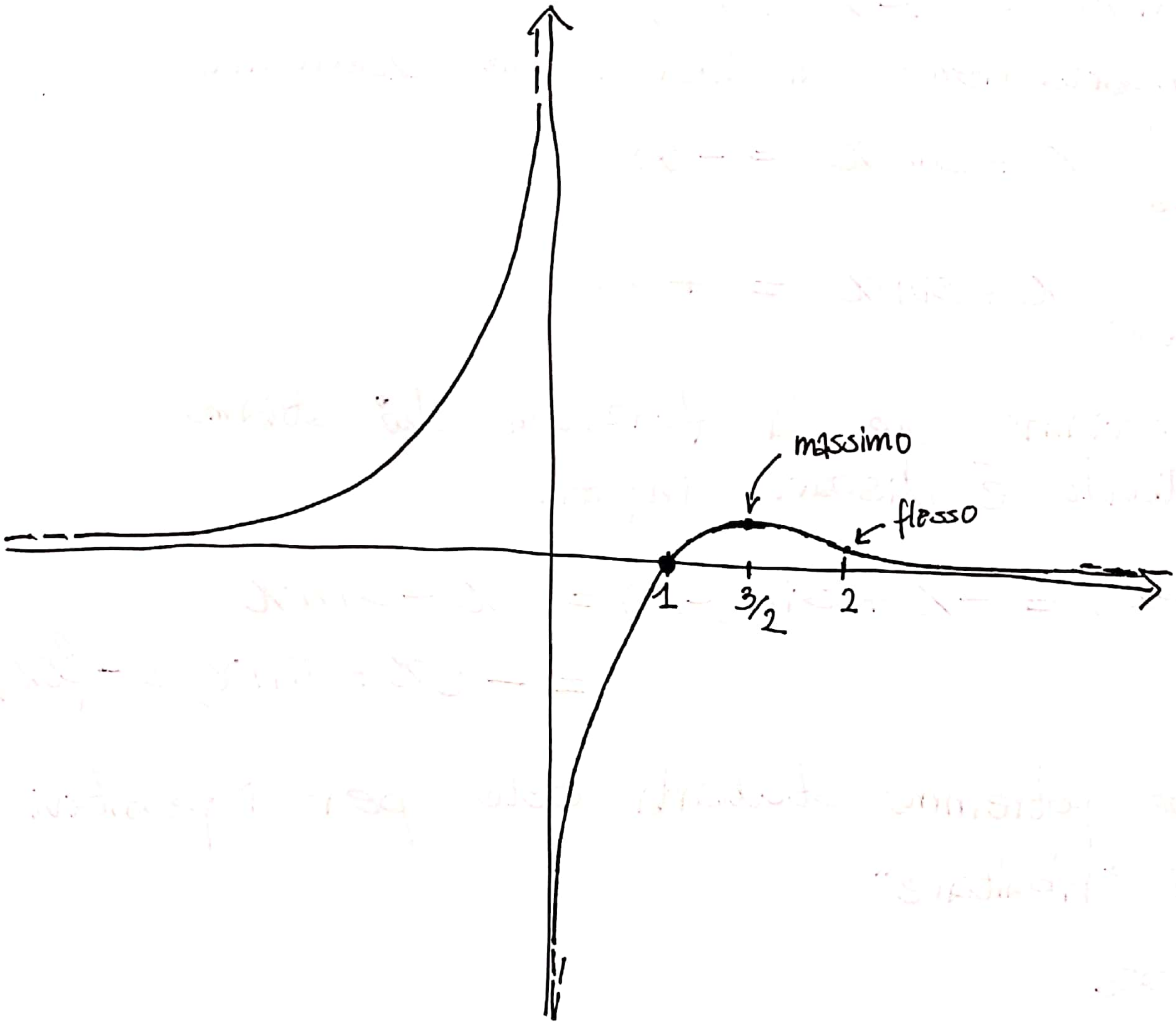
$f''(x) \geq 0$

N: $6x - 12 \geq 0 \rightarrow x \geq 2$

D: $x^5 > 0 \rightarrow x > 0$



- Grafico qualitativo



(vi) $f(x) = x + \sin x$

- Dominio : $(-\infty, +\infty)$
- Comportamento ai bordi del dominio

$$\lim_{x\to-\infty} x + \sin x = -\infty$$

$$\lim_{x\to+\infty} x + \sin x = +\infty$$

- Osserviamo che la funzione che stiamo studiando è dispari, infatti:

$$f(-x) = -x + \sin(-x) = -x - \sin x$$
$$= -(x + \sin x) = -f(x)$$

Dunque potremmo studiarla solo per i positivi e poi "ribaltare"

- Derivata

$$f'(x) = 1 + \cos x$$

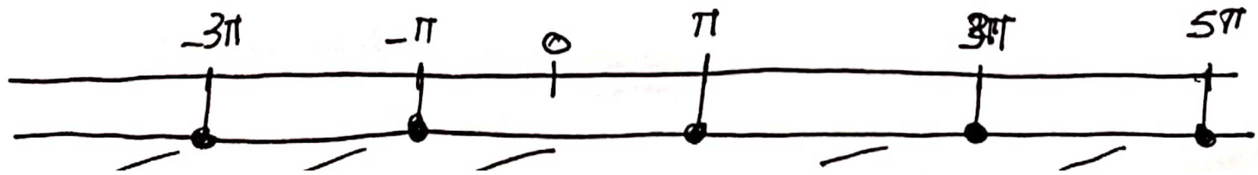
$$f'(x) \geq 0 \rightarrow 1 + \cos x \geq 0$$

$$\forall x \in \mathbb{R} \quad 1 + \cos x \geq 0$$

Dunque la funzione è sempre crescente, tuttavia ci sono dei punti un po' speciali, ovvero

$$1 + \cos x = 0 \rightarrow x = \pi + 2k\pi, \ k \in \mathbb{Z}$$

Che sono flessi a tangente orizzontale



• Grafico qualitativo