SOLUZIONI

(i) $$\lim_{x\to 0} \frac{x^3-3x^2+4x}{x^5-x} = \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{x^2-3x+4}{x^4-1} = \boxed{-4}$$

(ii) $$\lim_{n\to +\infty} \frac{\log(n^3)}{\log(n^3+3n^2)} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{n\to +\infty} \frac{\log(n^3)}{\log\left(n^3\left(1+\frac{3}{n}\right)\right)} = \lim_{n\to +\infty} \frac{\log(n^3)}{\log(n^3)+\log\left(1+\frac{3}{n}\right)}$$

$$= \lim_{n\to +\infty} \frac{1}{1+\frac{\log\left(1+\frac{3}{n}\right)}{\log(n^3)}} = \boxed{1}$$

(iii) $$\lim_{n\to +\infty} \frac{n^2+n\sin(n)}{1+n^2+n}$$

$$= \lim_{n\to +\infty} \frac{\cancel{n^2}\left(1+\frac{\sin(n)}{n}\right)}{\cancel{n^2}\left(\frac{1}{n^2}+1+\frac{1}{n}\right)} = \boxed{1}$$

perché $\frac{\sin(n)}{n} \xrightarrow{n\to+\infty} 0$ per teorema carabinieri

(iv) $\lim_{x \to +\infty} \frac{\log(3+\sin x)}{x^3}$

Teorema carabinieri

$$\lim_{x \to +\infty} \frac{\log(2)}{x^3} \leq \lim_{x \to +\infty} \frac{\log(3+\sin x)}{x^3} \leq \lim_{x \to +\infty} \frac{\log(4)}{x^3}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad\qquad 0$$

$$\Rightarrow \lim_{x \to +\infty} \frac{\log(3+\sin x)}{x^3} = \boxed{0}$$

(v) $\lim_{x \to 5} \frac{x-5}{\sqrt{x}-\sqrt{5}} = \left[\frac{0}{0}\right]$

$$\lim_{x \to 5} \frac{(x-5)(\sqrt{x}+\sqrt{5})}{(\sqrt{x}-\sqrt{5})(\sqrt{x}+\sqrt{5})} = \lim_{x \to 5} \frac{\cancel{(x-5)}(\sqrt{x}+\sqrt{5})}{\cancel{x-5}}$$

$$\lim_{x \to 5} \sqrt{x}+\sqrt{5} = \boxed{2\sqrt{5}}$$

(vi) $$\lim_{n\to+\infty} \frac{\log\left(\sqrt[3]{1+\frac{9}{n^2}}\right)}{\log\left(\cos\left(\frac{6}{n}\right)\right)}$$

Cambio variabile $\boxed{\frac{1}{n}=x}$

$$\Rightarrow \lim_{x\to 0} \frac{\log\left(\left(1+9x^2\right)^{1/3}\right)}{\log(\cos 6x)}$$

$$= \lim_{x\to 0} \underbrace{\frac{\frac{1}{3}\log(1+9x^2)}{9x^2}}_{\downarrow\, 1} \cdot \frac{9x^2}{\log(1+(\cos 6x-1))}$$

$$= \lim_{x\to 0} \frac{9x^2}{3} \cdot \underbrace{\frac{(\cos 6x-1)}{\log(1+(\cos 6x-1))}}_{\downarrow\, 1} \cdot \underbrace{\frac{(6x)^2}{(\cos 6x-1)}}_{\downarrow\, -2} \cdot \frac{1}{(6x)^2}$$

Per il coseno ho usato il limite notevole $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$, però con segno cambiato e fatto il reciproco, cioè

$$\lim_{x\to 0} \frac{x^2}{\cos x-1} = -2$$

$$\Rightarrow \lim_{x\to 0} \frac{9\cancel{x^2}}{3} \cdot (-2) \cdot \frac{1}{36\cancel{x^2}} = \frac{-18}{3 \cdot 36} = \boxed{-\frac{1}{6}}$$

(vii) $\lim_{x\to 0} \frac{\log(2-\cos x)}{\sin^2 x} = \left[\frac{0}{0}\right]$

$$\lim_{x\to 0} \underbrace{\frac{\log(1+(1-\cos x))}{(1-\cos x)}}_{\downarrow \atop 1} \cdot \underbrace{\frac{(1-\cos x)}{x^2}}_{\downarrow \atop 1/2} \cdot \underbrace{\frac{x^2}{\sin^2 x}}_{\downarrow \atop 1}$$

$$= \boxed{\frac{1}{2}}$$

(viii) $\lim_{x\to +\infty} \sqrt{x} - 1 + \cos x$

teorema carabinieri

$$\underbrace{\lim_{x\to+\infty} \sqrt{x} - 2}_{\downarrow \atop +\infty} \leq \lim_{x\to+\infty} \sqrt{x} - 1 + \cos x \leq \underbrace{\lim_{x\to+\infty} \sqrt{x}}_{\downarrow \atop +\infty}$$

$$\Rightarrow \lim_{x\to+\infty} \sqrt{x} - 1 + \cos x = +\infty$$

(ix) $\lim_{n\to+\infty} n^2 \cos\left(\frac{1}{n}\right) = \boxed{+\infty}$

(x) $\lim_{x\to-\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \left[\frac{\infty}{\infty}\right]$

Cambio variabile per non confondermi con i segni $\boxed{x = -y}$

$$\Rightarrow \lim_{y\to+\infty} \frac{3^{-y} - 3^{y}}{3^{-y} + 3^{y}}$$

$$= \lim_{y\to+\infty} \frac{\cancel{3^y}(3^{-2y} - 1)}{\cancel{3^y}(3^{-2y} + 1)} = \boxed{-1}$$

perché $\lim_{y\to+\infty} 3^{-2y} = 0$

(xi) $\lim_{x\to0} \frac{1-\cos 2x}{\sin^2 3x}$

$$= \lim_{x\to0} \underbrace{\frac{1-\cos 2x}{(2x)^2}}_{\downarrow\ 1/2} \cdot \frac{(2x)^2}{(3x)^2} \cdot \underbrace{\frac{(3x)^2}{\sin^2(3x)}}_{\downarrow\ 1}$$

$$= \lim_{x\to0} \frac{1}{2} \cdot \frac{4\cancel{x^2}}{9\cancel{x^2}} = \boxed{\frac{2}{9}}$$

(xii) $\lim_{x\to 0} \dfrac{x^3 + x^2 \sin x + \sin^2 x}{x^4 + x^3 + x \sin x}$

$$= \lim_{x\to 0} \frac{\cancel{x^2}\left(x + \sin x + \dfrac{\sin^2 x}{x^2}\right)}{\cancel{x^2}\left(x^2 + x + \dfrac{\sin x}{x}\right)} = \boxed{1}$$

(xiii) $\lim_{x\to 0} \left(\dfrac{1}{\cos x}\right)^{1/x^2}$

$$= \lim_{x\to 0} e^{\log\left(\frac{1}{\cos x}\right)^{1/x^2}}$$

$$= \lim_{x\to 0} e^{-\frac{\log(\cos x)}{x^2}}$$

Studiamoci solo l'esponente

$$\lim_{x\to 0} -\frac{\log(\cos x)}{x^2} = \lim_{x\to 0} \underbrace{-\frac{\log(1+(\cos x-1))}{(\cos x - 1)}}_{\to -1} \cdot \underbrace{\frac{(\cos x - 1)}{x^2}}_{\to -\frac{1}{2}}$$

$$= -1 \cdot \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} e^{-\frac{\log(\cos x)}{x^2}} = \boxed{e^{1/2}}$$

(xiv) $\lim_{n\to+\infty} n^2 2^{-\sqrt{n}}$

$$= \lim_{n\to+\infty} \frac{n^2}{2^{\sqrt{n}}} = \boxed{0}$$ per confronto tra infiniti

Altrimenti possiamo farlo così:

$$= \lim_{n\to+\infty} e^{\log n^2} \cdot e^{\log 2^{-\sqrt{n}}}$$

$$= \lim_{n\to+\infty} e^{2\log n - \sqrt{n}\log 2}$$

$$= \lim_{n\to+\infty} e^{\underbrace{-\sqrt{n}}_{\downarrow -\infty}\underbrace{\left(\log 2 - \frac{2\log n}{\sqrt{n}}\right)}_{\downarrow \log 2}} = 0$$

(sempre per confronto tra infiniti)

(xv) $\lim_{x\to 0} \frac{\sin(\sqrt{1+x^2}-1)}{x}$

$$= \lim_{x\to 0} \underbrace{\frac{\sin(\sqrt{1+x^2}-1)}{(\sqrt{1+x^2}-1)}}_{\to 1} \cdot \underbrace{\frac{\sqrt{1+x^2}-1}{x^2}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{x^2}{x}}_{\to 0}$$

$= \boxed{0}$

Qui abbiamo usato il limite notevole

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$$

(xvi) $\lim_{x\to 0} \frac{\sin(\pi \cos x)}{x \sin x}$

Qui vorremmo applicare il limite notevole $\lim_{x\to 0} \frac{\sin x}{x} = 1$, il problema peró é che l'argomento del seno non tende a 0 ma a π. Quindi facciamo cosí:

$$\lim_{x\to 0} \frac{\sin(\pi + (\pi\cos x - \pi))}{x \sin x}$$

E sul seno del numeratore applichiamo le formule di addizione - sottrazione del seno

$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha \\ \sin(\alpha-\beta) = \sin\alpha \cos\beta - \sin\beta \cos\alpha \end{cases}$$

$$= \lim_{x\to 0} \frac{\overset{=0}{\sin(\pi)} \cos(\pi\cos x - \pi) + \sin(\pi\cos x - \pi) \overset{-1}{\overset{\shortparallel}{\cos(\pi)}}}{x \sin x}$$

$$= \lim_{x\to 0} - \frac{\sin(\pi\cos x - \pi)}{x \sin x}$$

Ora possiamo applicare il limite notevole del seno

$$\Rightarrow \lim_{x\to 0} - \underbrace{\frac{\sin(\pi\cos x - \pi)}{(\pi\cos x - \pi)}}_{\to 1} \cdot \frac{\pi\cos x - \pi}{x\sin x}$$

$$= \lim_{x\to 0} \frac{\pi - \pi\cos x}{x\sin x}$$

$$= \lim_{x\to 0} \pi \underbrace{\frac{(1-\cos x)}{x^2}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{x}{\sin x}}_{\to 1} = \boxed{\frac{\pi}{2}}$$

(xvii) $\lim_{n\to+\infty} \log^n\left(e + \frac{1}{n}\right)$

$$= \lim_{n\to+\infty} \left[\log\left(e\left(1+\frac{1}{en}\right)\right)\right]^n$$

$$= \lim_{n\to+\infty} \left[\log e + \log\left(1+\frac{1}{en}\right)\right]^n$$

Cambio di variabile $\boxed{\frac{1}{n} = x}$

$$= \lim_{x\to 0} \left[1 + \log(1+ex)\right]^{1/x}$$

$$= \lim_{x\to 0} e^{\log\left[1+\log(1+ex)\right]^{1/x}}$$

$$= \lim_{x\to 0} e^{\frac{1}{x}\cdot \underbrace{\frac{\log(1+\log(1+ex))}{\log(1+ex)}}_{\downarrow\, 1}\cdot \log(1+ex)}$$

$$= \lim_{x\to 0} e^{\frac{1}{x}\cdot \log(1+ex)}$$

$$= \lim_{x\to 0} e^{\underbrace{\frac{\log(1+ex)}{ex}}_{\downarrow\, 1}\cdot e} = \boxed{e^e}$$

(XVIII) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

$$= \lim_{x\to 0} \underbrace{\frac{\sqrt{1+x+x^2}-1}{x+x^2}}_{\downarrow\, \frac{1}{2}} \cdot \frac{x+x^2}{x}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\cancel{x}(1+x)}{\cancel{x}} = \boxed{\frac{1}{2}}$$

Scansionato con CamScanner

(xix) $$\lim_{n\to+\infty} \frac{2^{\sqrt{\log^2 n + \log(n^2)}}}{n^2+1}$$

$$= \lim_{n\to+\infty} \frac{2^{\sqrt{\log^2 n \left(1+\frac{2}{\log n}\right)}}}{n^2+1}$$

$$= \lim_{n\to+\infty} \frac{2^{\log n \underbrace{\sqrt{1+\frac{2}{\log n}}}_{\to 1}}}{\underbrace{n^2\left(1+\frac{1}{n^2}\right)}_{\to 1}}$$

$$= \lim_{n\to+\infty} \frac{2^{\log n}}{n^2}$$

Chi ha la crescita maggiore?

$$\left[\begin{aligned} 2^{\log n} = e^{\log 2^{\log n}} &= e^{\log n \cdot \log 2} \\ &= \left(e^{\log n}\right)^{\log 2} \\ &= n^{\log 2} \end{aligned}\right]$$

$$\lim_{n\to+\infty} \frac{n^{\log 2}}{n^2}$$

Ora è più chiaro?

Siccome $\log 2 < 1 < 2$

$$\Rightarrow \lim_{n\to+\infty} \frac{n^{\log 2}}{n^2} = \boxed{0}$$

(XX) $\displaystyle\lim_{x\to 0} \frac{\sin(\pi+4x)}{x}$

Qui come prima vogliamo applicare il limite notevole del seno ma l'argomento del seno non tende a 0. Allora usiamo le formule di addizione del seno

$$\Rightarrow \lim_{x\to 0} \frac{\overbrace{\sin(\pi)}^{=0}\cos(4x) + \overbrace{\cos(\pi)}^{=-1}\sin(4x)}{x}$$

$$= \lim_{x\to 0} -\frac{\sin(4x)}{4x}\cdot 4 = \boxed{-4}$$

(XXi) $\displaystyle\lim_{x\to 0} \frac{(1-\cos x)^2}{\log(1+\sin^4 x)}$

$$= \lim_{x\to 0} \underbrace{\frac{(1-\cos x)^2}{(x^2)^2}}_{\to \frac{1}{4}} \cdot \underbrace{\frac{(x^2)^2}{\sin^4 x}}_{\to 1} \cdot \underbrace{\frac{\sin^4 x}{\log(1+\sin^4 x)}}_{\to 1}$$

perché $\lim_{x\to 0} \underbrace{\frac{(1-\cos x)}{x^2}}_{\downarrow 1/2} \cdot \underbrace{\frac{(1-\cos x)}{x^2}}_{\downarrow 1/2} = \frac{1}{4}$

$$= \boxed{\frac{1}{4}}$$

(xxii) $\lim_{n\to+\infty} \left(1+\sin\left(\frac{1}{n}\right)\right)^{n+\sqrt{n}}$

$$= \lim_{n\to+\infty} e^{(n+\sqrt{n})\cdot\log\left(1+\sin\left(\frac{1}{n}\right)\right)}$$

$$= \lim_{n\to+\infty} e^{(n+\sqrt{n})\cdot \underbrace{\frac{\log\left(1+\sin\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)}}_{\to 1}\cdot\sin\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to+\infty} e^{\underbrace{n\sin\left(\frac{1}{n}\right)}_{\to 1}} \cdot e^{\sqrt{n}\sin\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to+\infty} e \cdot e^{\underbrace{n\sin\left(\frac{1}{n}\right)}_{\to 1}\cdot\frac{1}{\sqrt{n}}}$$

Volendo si può fare il cambio di variabile $\frac{1}{n} = x$

$$= \lim_{n\to+\infty} e \cdot e^{\frac{1}{\sqrt{n}}} = \boxed{e}$$

(xxiii) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{1-\sin x}$

Cambio di variabile $\boxed{x - \frac{\pi}{2} = t}$

$$\Rightarrow \lim_{t \to 0} \frac{\cos(t+\pi/2)}{1-\sin\left(t+\frac{\pi}{2}\right)}$$

Qui usiamo le formule di addizione di seno e coseno

$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{cases}$$

$$= \lim_{t \to 0} \frac{\cos(t)\cos(\pi/2) - \sin(t)\sin(\pi/2)}{1-(\sin(t)\cos(\pi/2) + \cos(t)\sin(\pi/2))}$$

$\sin\left(\frac{\pi}{2}\right) = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$

$$\Rightarrow \lim_{t \to 0} \frac{-\sin t}{1-\cos t}$$

$$= \lim_{t \to 0} \underbrace{-\frac{\sin t}{t}}_{\to -1} \cdot \frac{t}{t^2} \cdot \underbrace{\frac{t^2}{1-\cos t}}_{\to 2}$$

$$= \lim_{t\to 0} \frac{-2t}{t^2} = \lim_{t\to 0} \frac{-2}{t} \Rightarrow \boxed{\text{Non esiste il limite}}$$

Perché se si avviciniamo da destra

$$\lim_{t\to 0^+} \frac{-2}{t} = -\infty$$

Invece da sinistra $\rightarrow \lim_{t\to 0^-} \frac{-2}{t} = +\infty$

Dunque il limite non esiste

(XXIV) $\lim_{x\to 0^+} \frac{3^{\cos\frac{1}{x}} - 5}{x \log x}$

Cambio di variabile $\boxed{x = \frac{1}{t}}$

$$\lim_{t\to +\infty} \frac{3^{\cos(t)} - 5}{\frac{1}{t} \log\left(\frac{1}{t}\right)}$$

$$= \lim_{t\to +\infty} -\frac{t}{\log(t)} \cdot \left(3^{\cos(t)} - 5\right)$$

Usiamo ora il teorema dei carabinieri

$$\underbrace{\lim_{t\to+\infty} -\frac{t}{\log t}\left(3^{-1}-5\right)}_{\downarrow \atop +\infty} \leq \lim_{t\to+\infty} -\frac{t}{\log t}\left(3^{\cos t}-5\right) \leq \underbrace{\lim_{t\to+\infty} \frac{2t}{\log t}}_{\downarrow \atop +\infty}$$

perché $3^{-1}-5 = \frac{1}{3}-5 < 0$

$$\Rightarrow \lim_{t\to+\infty} -\frac{t}{\log t}\left(3^{\cos t}-5\right) = \boxed{+\infty}$$

Scansionato con CamScanner