

THEORETICAL COMPUTER SCIENCE TUTORING (5)

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TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Problem 3 from the exam held on June 19, 2019

Given an undirected graph $G = (V, E)$ and an integer h , we say that a subset V' of the nodes of G is α -colorable with h colors if there exists a function $c : V' \rightarrow \{1, \dots, h\}$ such that, for every $u, v \in V'$, if $c(u) = c(v)$ then $(u, v) \in E$.

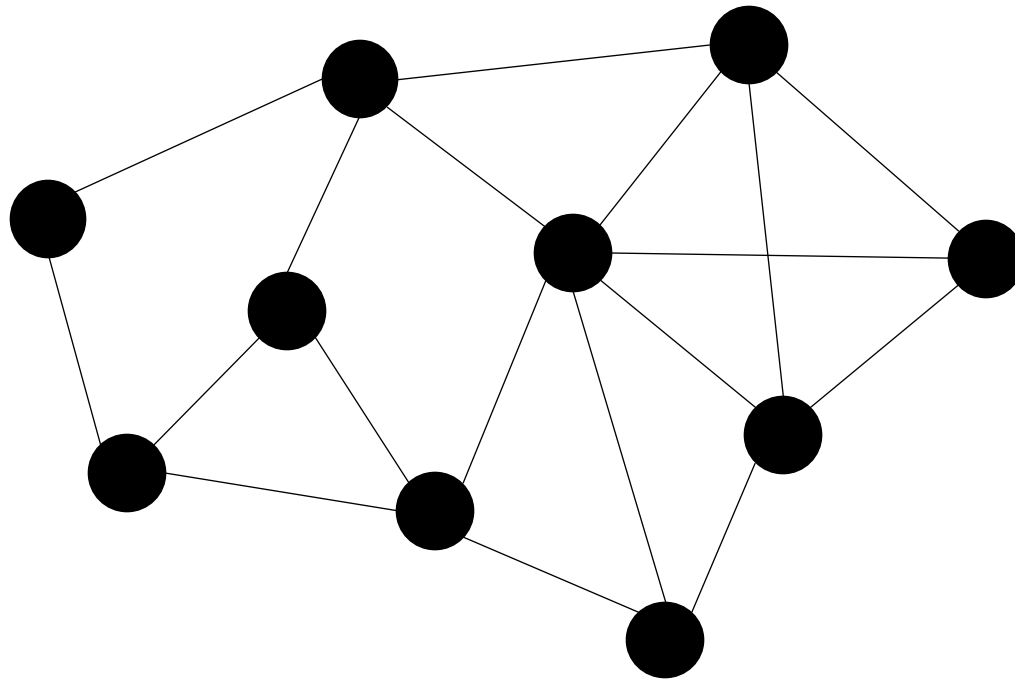
Consider the following decision problem: given an undirected graph $G = (V, E)$ and an integer k , decide whether there does not exist a subset $V' \subseteq V$ of cardinality at least k that is α -colorable with only one color.

After formalizing the problem using the triple $\langle I, S, \pi \rangle$, answer the following questions (in the order deemed appropriate), providing justification for each response.

- Is the problem in P?
- Is the problem in NP?
- Is the problem in coNP?

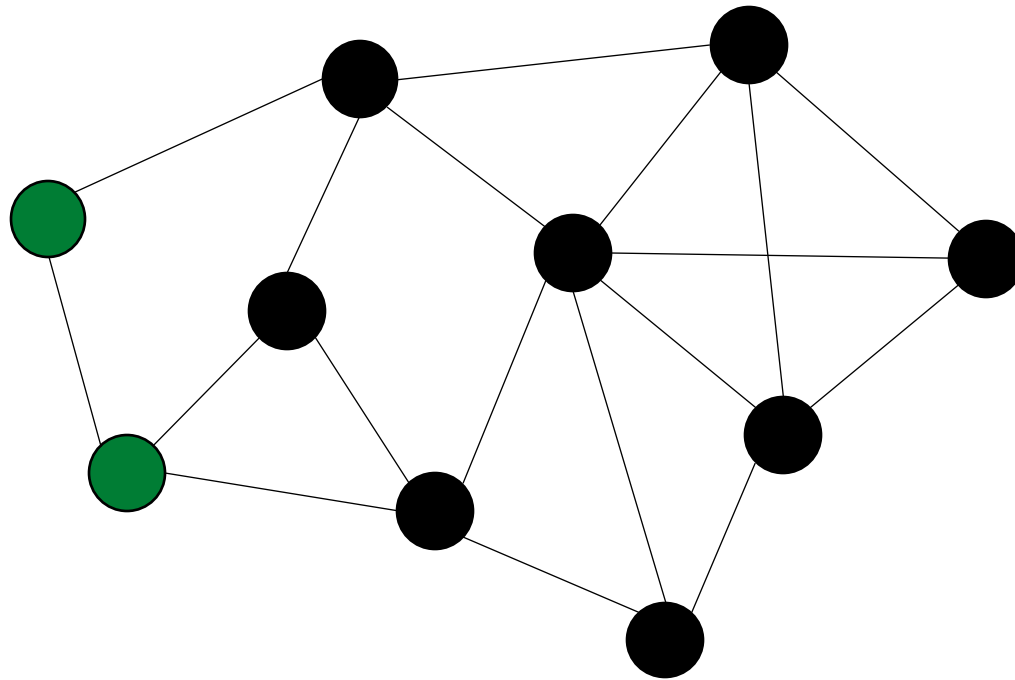
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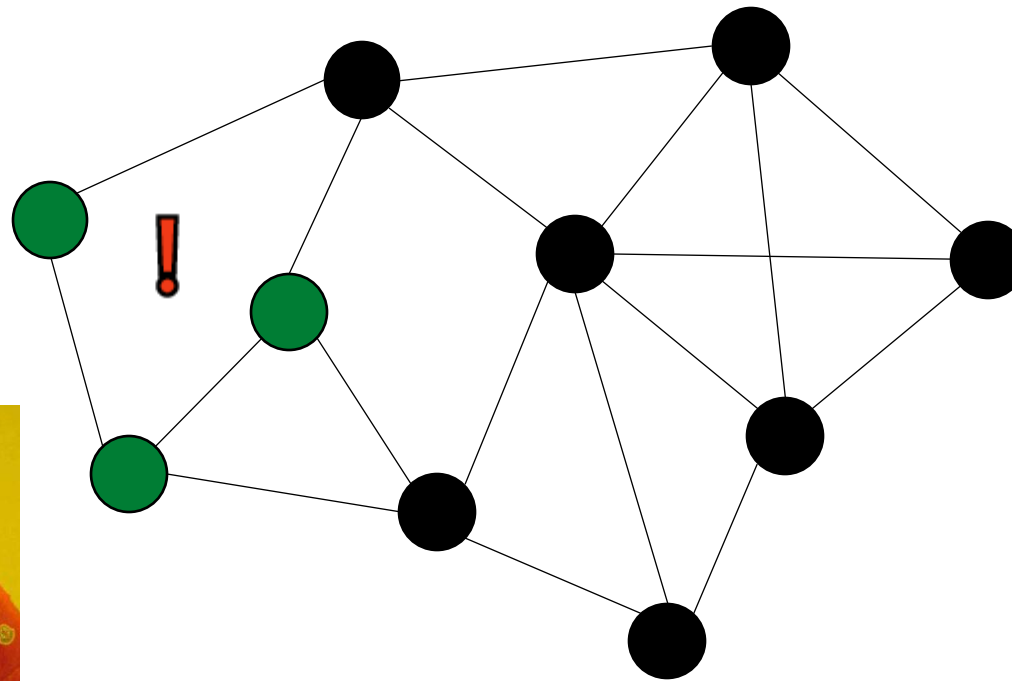
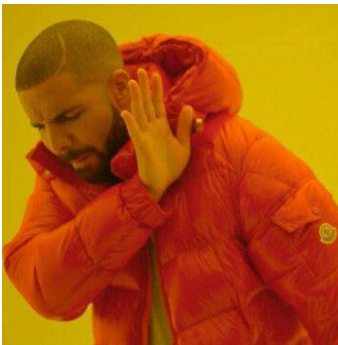
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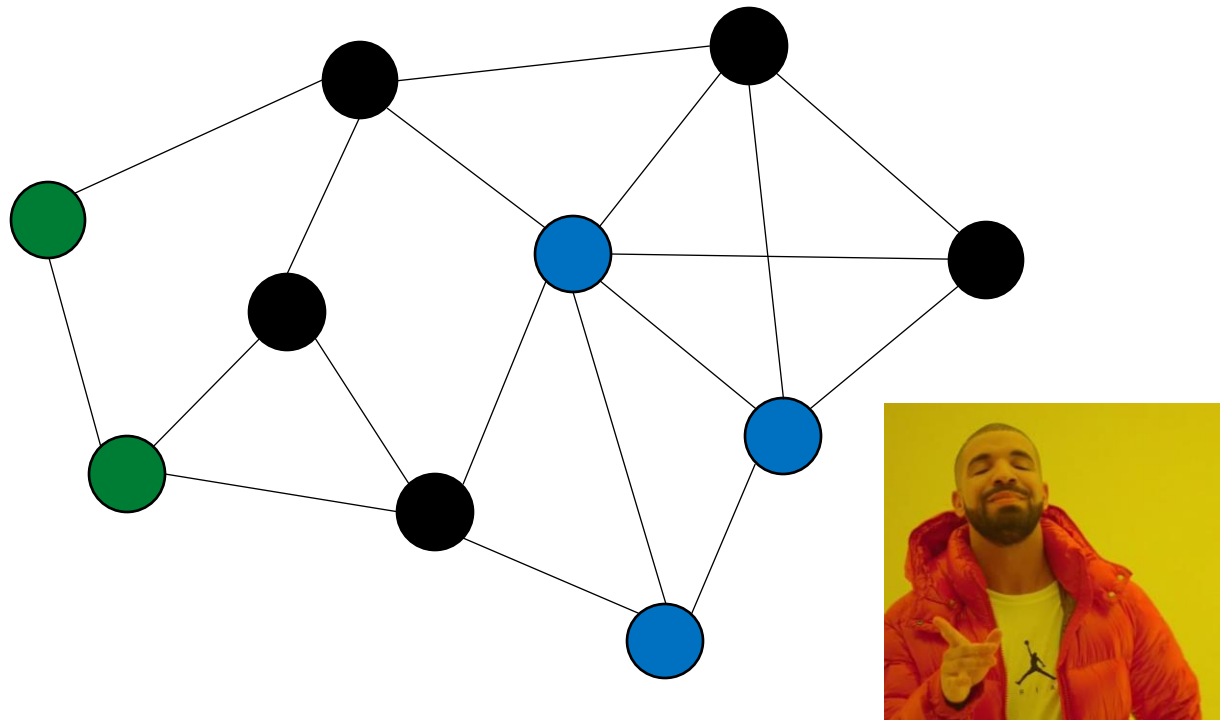
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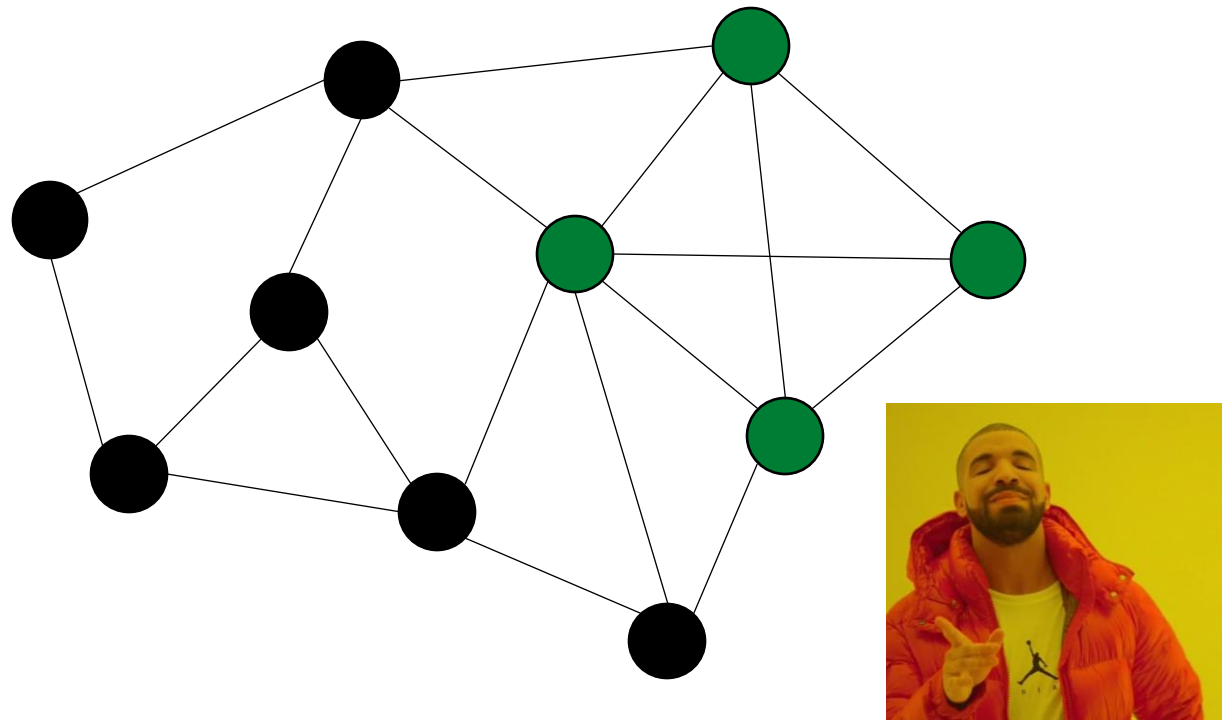
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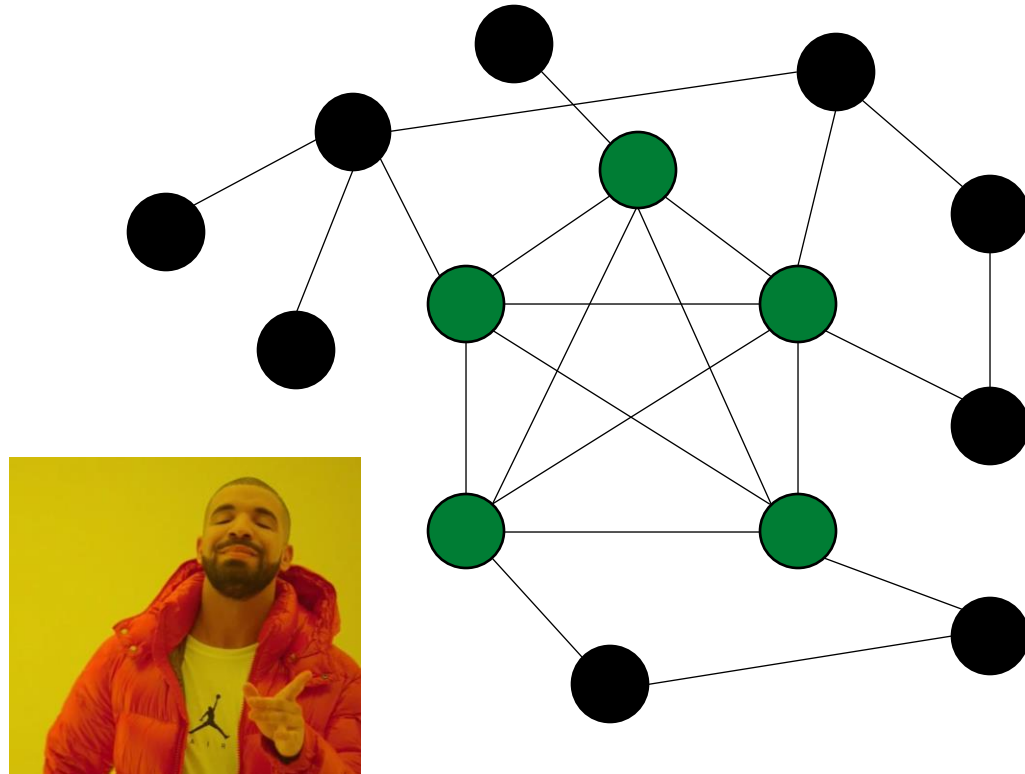
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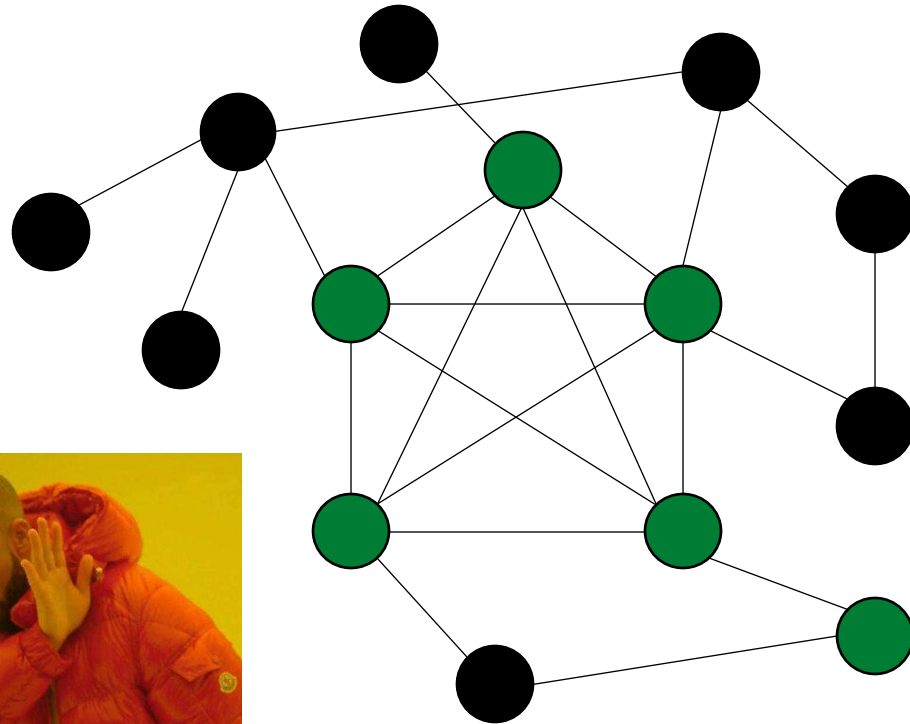


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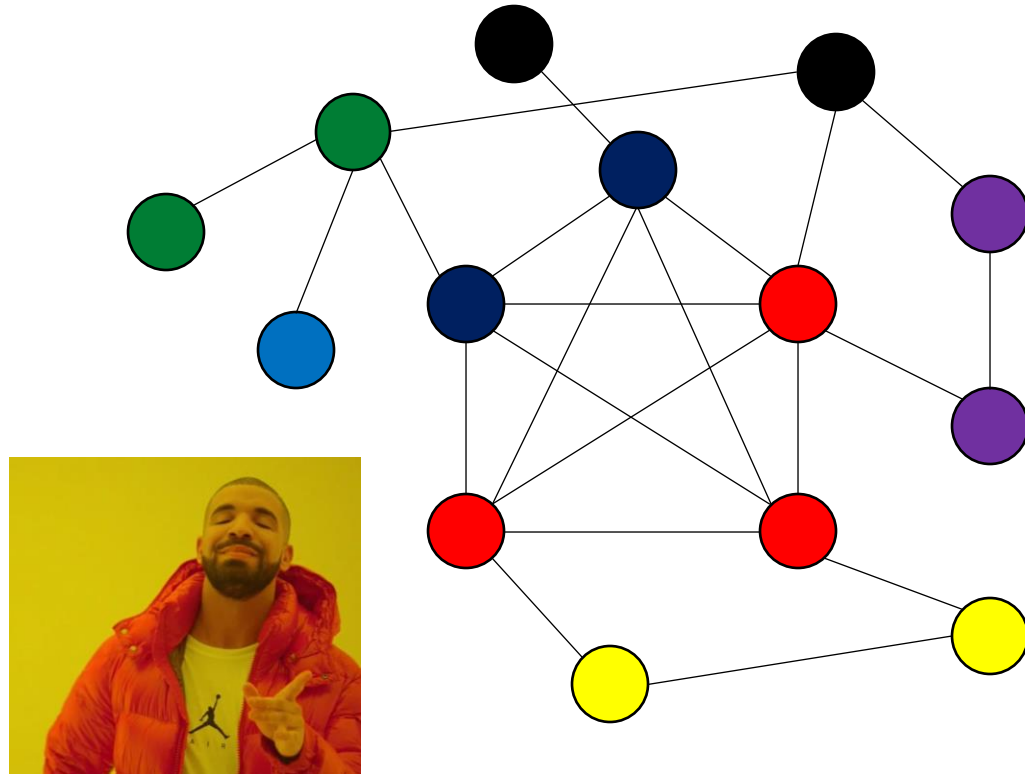


I can only color
the cliques



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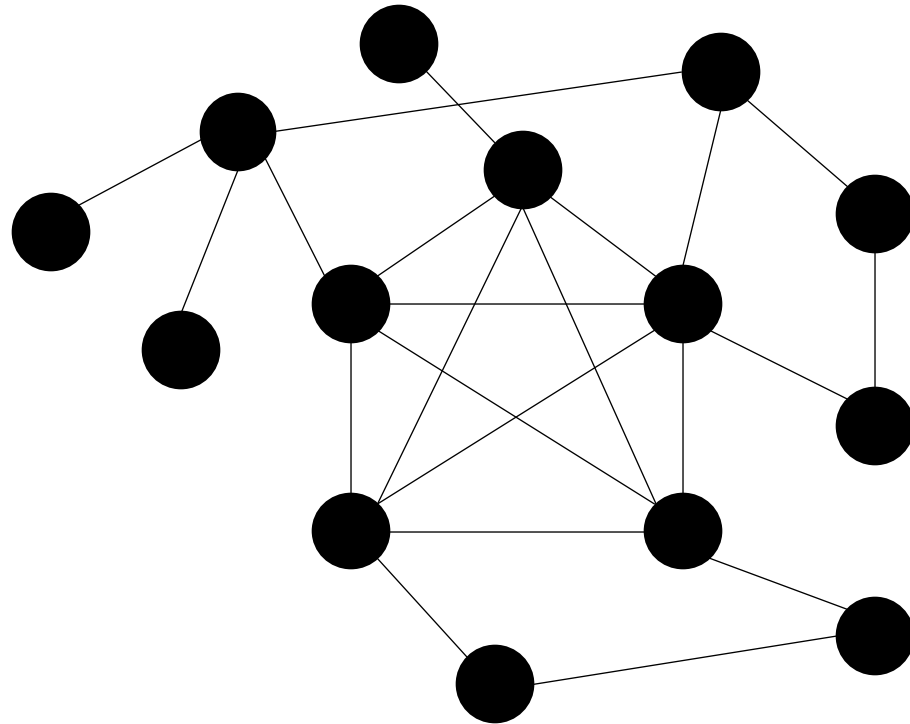
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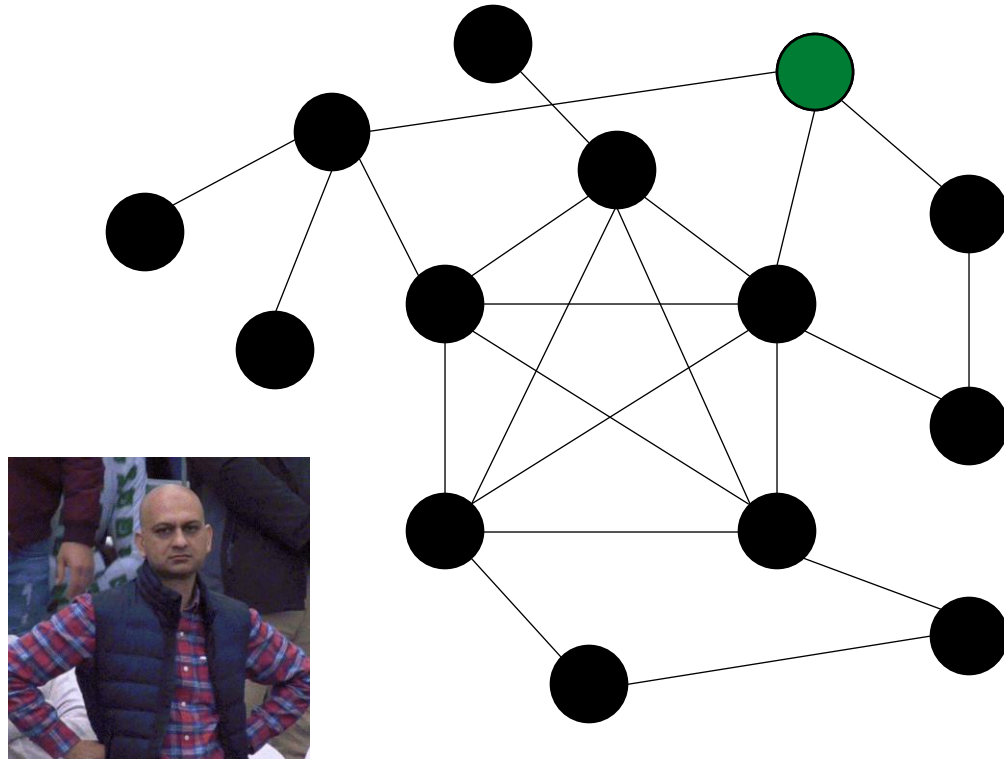
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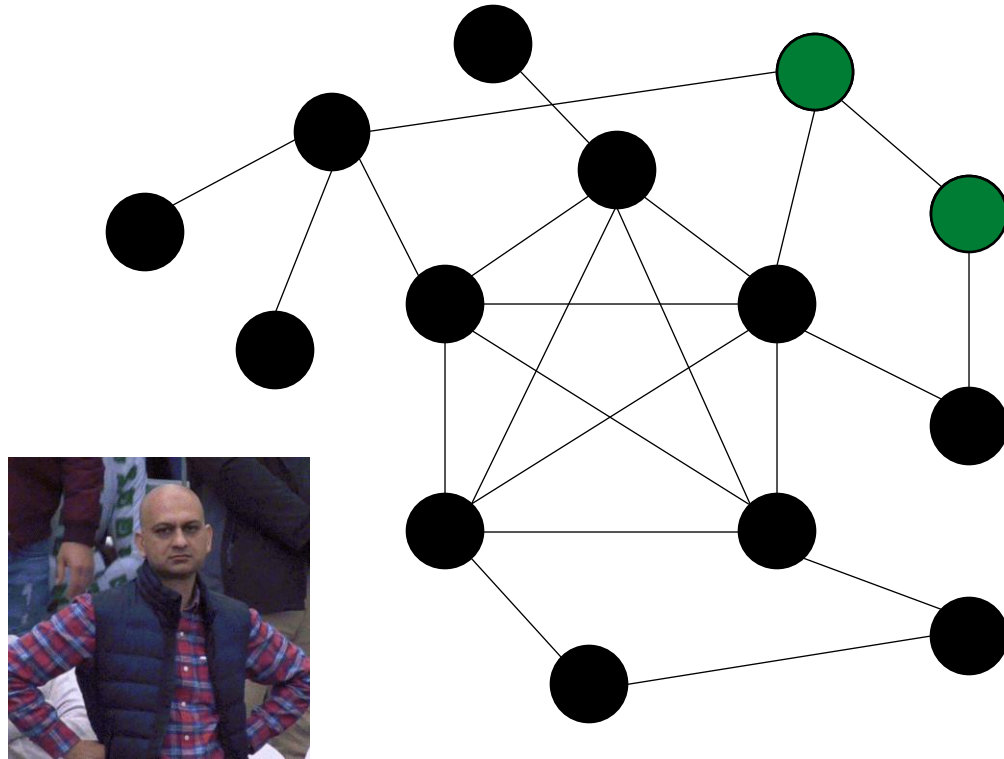
$$k = 1$$



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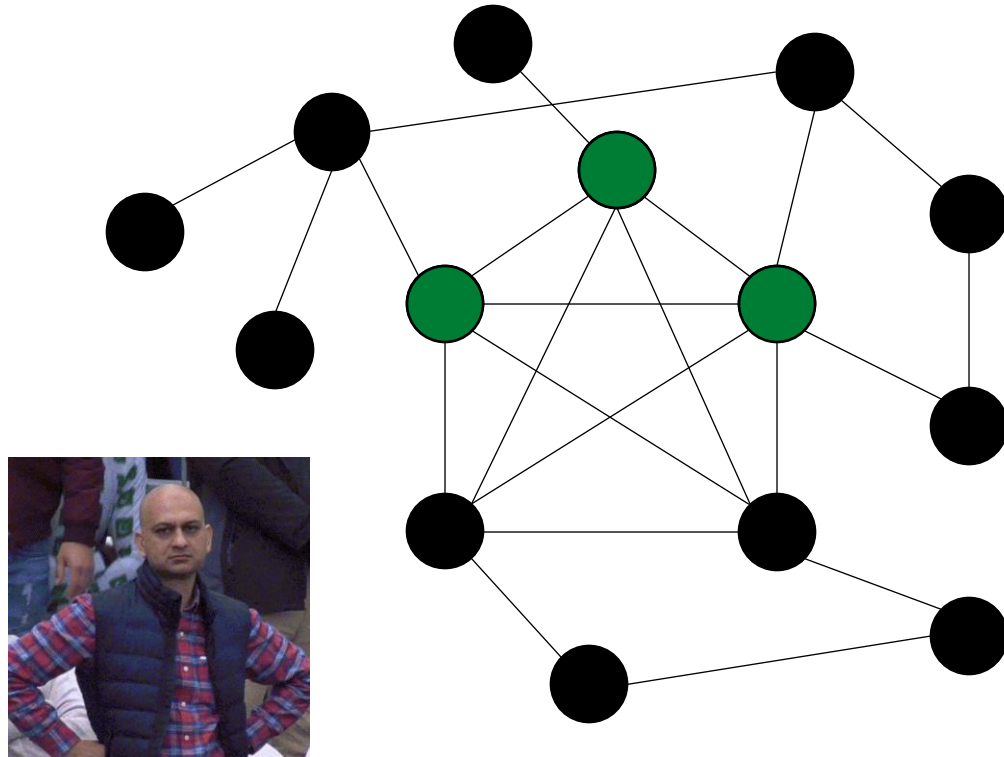
$$k = 2$$



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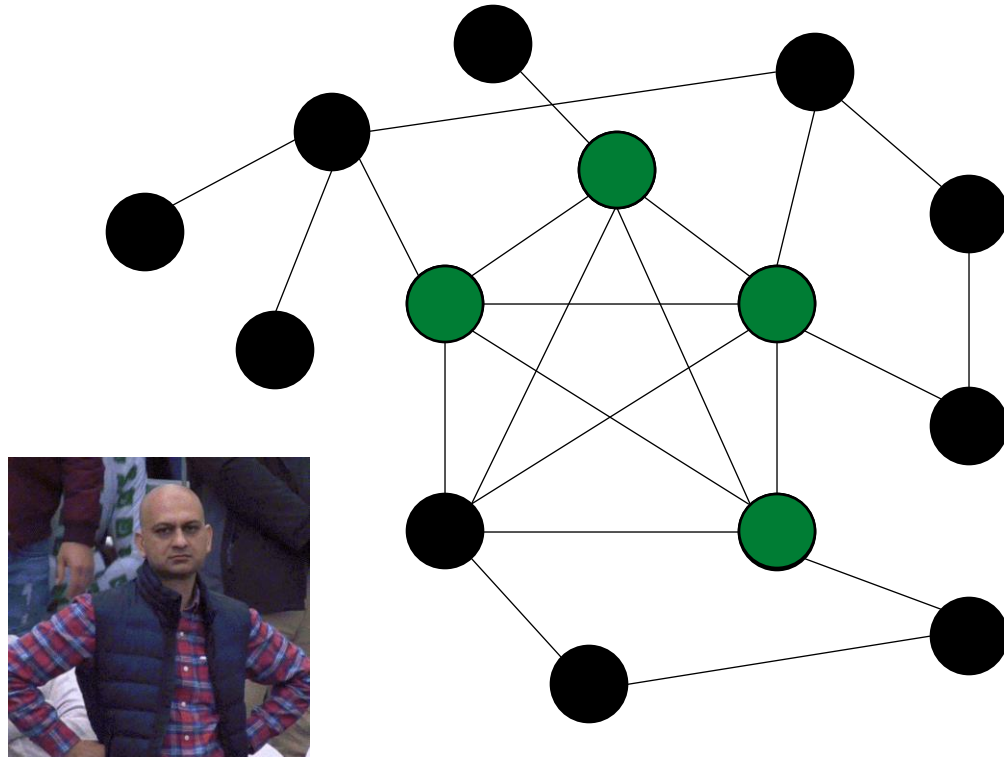
$$k = 3$$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $G = (V, E)$ and an integer k , decide whether there **does not exist** a subset $V' \subseteq V$ of cardinality at least k that is **α -colorable** with only **one color**.

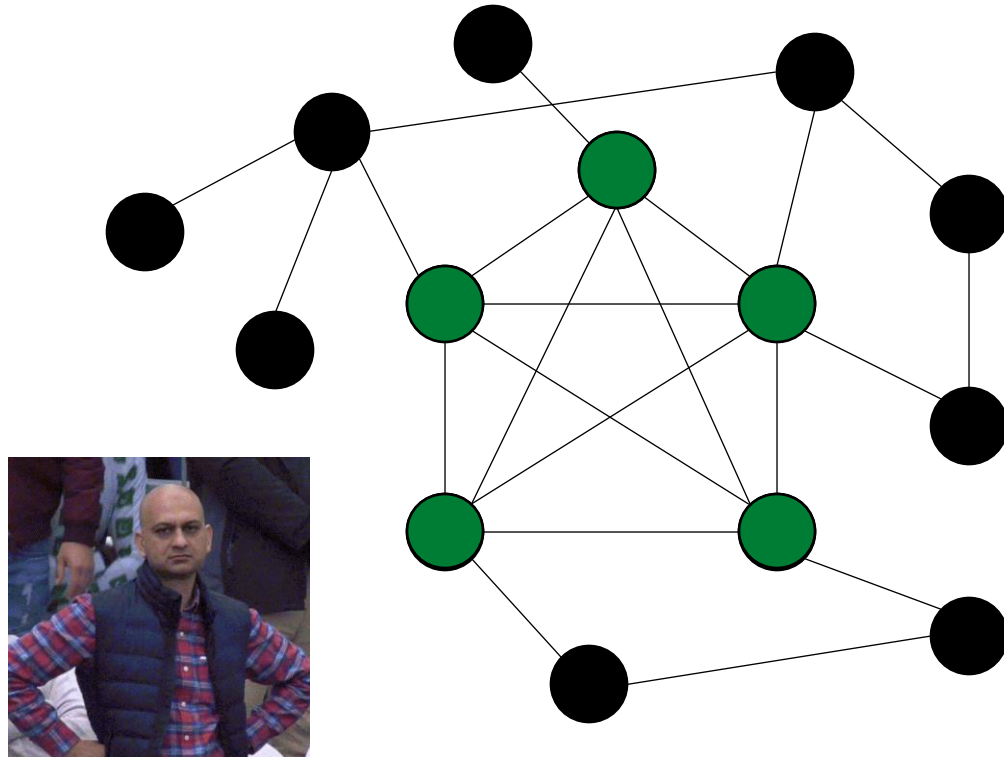
$$k = 4$$



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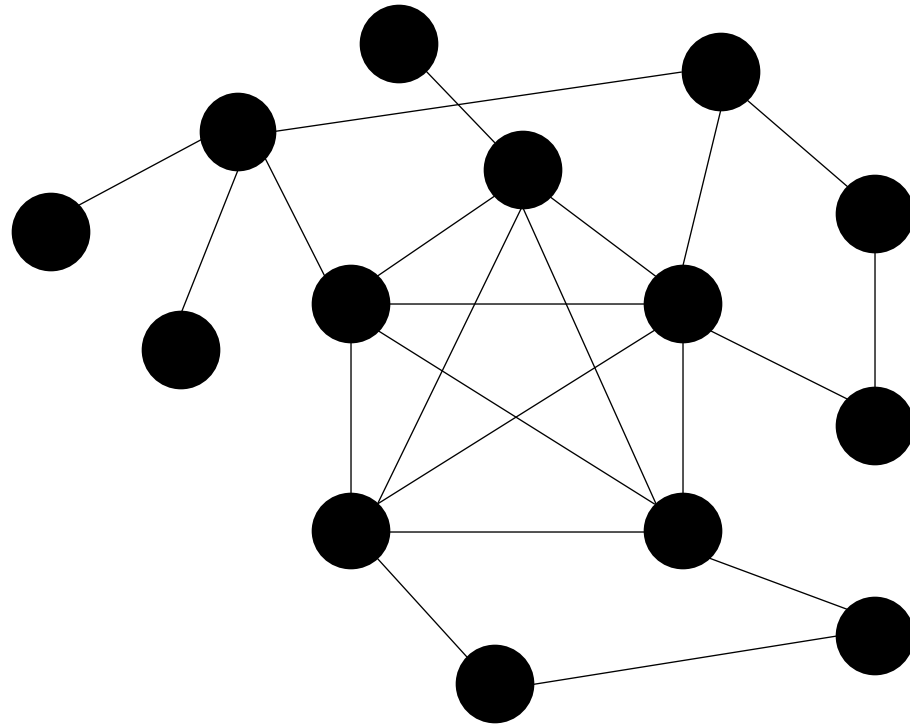
$$k = 5$$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $G = (V, E)$ and an integer k , decide whether there **does not exist** a subset $V' \subseteq V$ of cardinality at least k that is α -colorable with only **one color**.

$k \geq 6$?



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- $I_{exam} = \{\langle G = (V, E), k \rangle : G \text{ is an undirected graph} \wedge k \in \mathbb{N}\}$
- $S_{exam}(G, k) = \{V' \subseteq V\}$
- $\pi_{exam}(G, k, S_{exam}(G, k)) = \forall V' \in S_{exam}(G, k), c: V' \rightarrow \{1\} [|V'| < k \vee \exists u, v \in V' [c(u) = c(v) \wedge (u, v) \notin E]]$

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Problem 3 from the exam held on June 19, 2019

- $I_{\neg exam} = \{\langle G = (V, E), k \rangle : G \text{ is an undirected graph} \wedge k \in \mathbb{N}\}$
- $S_{\neg exam}(G, k) = \{V' \subseteq V\}$
- $\pi_{\neg exam}(G, k, S_{\neg exam}(G, k)) = \exists V' \in S_{\neg exam}(G, k) [|V'| \geq k \wedge \forall u, v \in V' [(u, v) \in E]]$

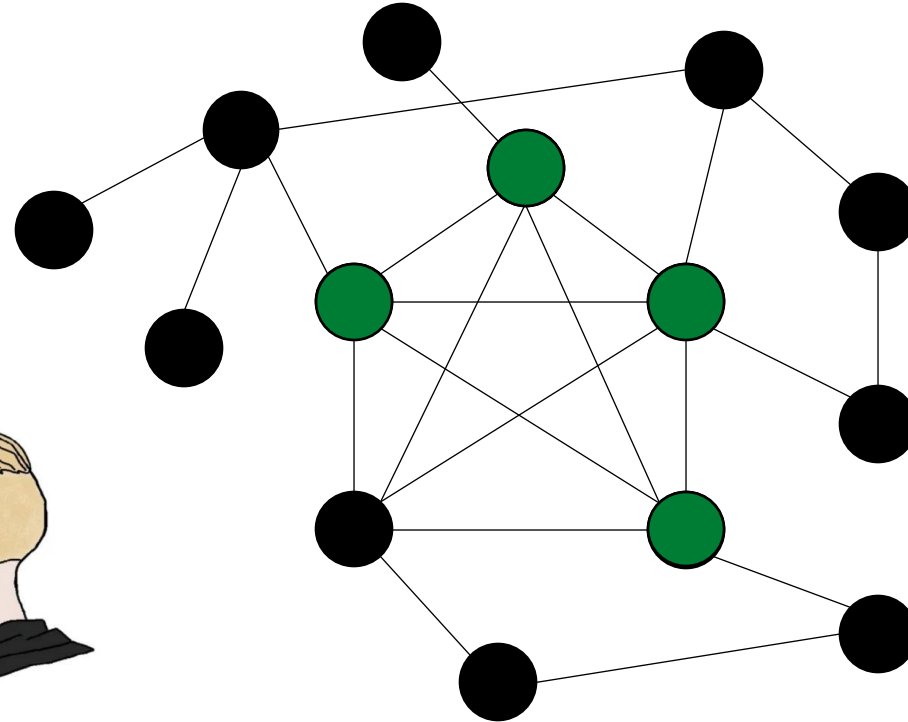
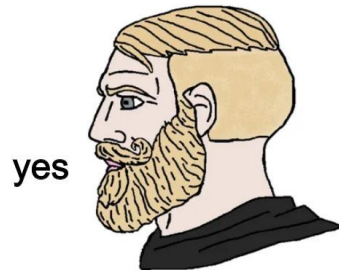


- $I_{clique} = \{\langle G = (V, E), k \rangle : G \text{ is an undirected graph} \wedge k \in \mathbb{N}\}$
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$k = 4$



Problem 3 from the exam held on June 19, 2019

Clique is **NP-complete** \Rightarrow \neg **exam** is **NP-complete** \Rightarrow **exam** is **coNP-complete**



To be more formal, you should formally specify the reduction between **Clique** and \neg **exam**, but the problems are practically identical

Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $G = (V, E, w)$ with $w: E \rightarrow \mathbb{N}$ and two integers h and k , decide whether G has a clique of k nodes such that the sum of the weights of its edges is at most h .

After formalizing the problem using the triple $\langle I, S, \pi \rangle$, answer the following questions (in the order deemed appropriate), providing justification for each response.

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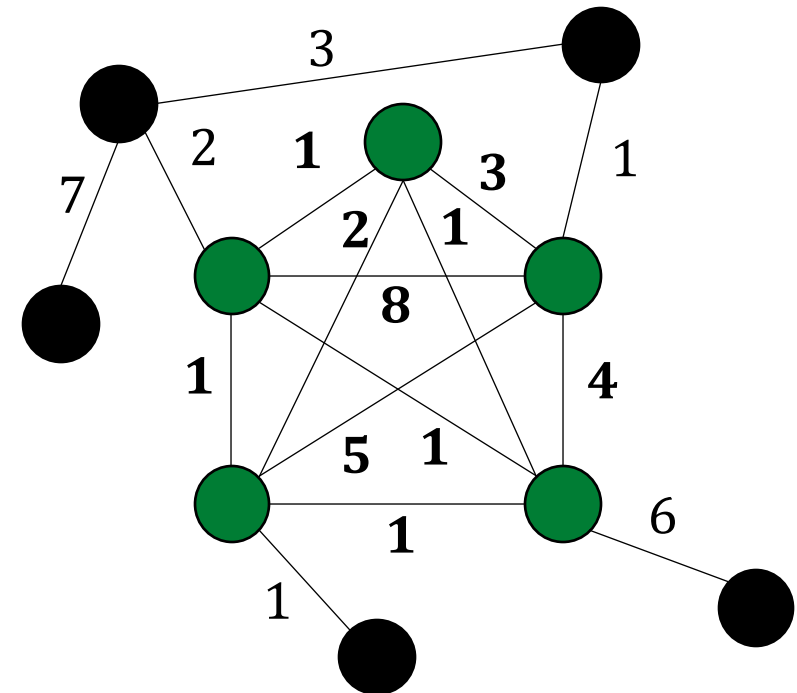
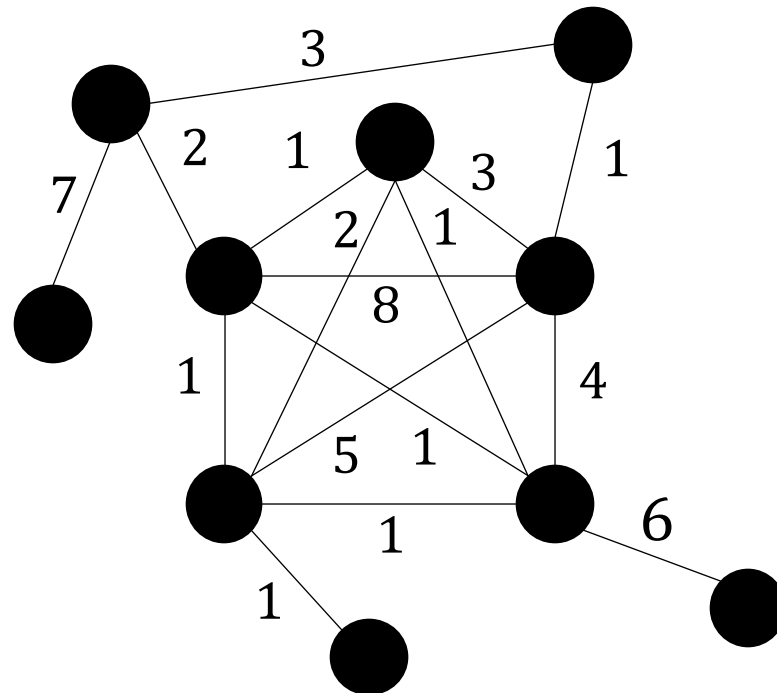
Note well: Deciding whether a graph contains a clique of exactly k nodes is an NP-complete problem.

Suggestion: Remember the reduction that demonstrates that TSP is NP-complete.

Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $G = (V, E, w)$ with $w: E \rightarrow \mathbb{N}$ and two integers h and k , decide whether G has a **clique** of k nodes such that the **sum of the weights** of its edges is at most h .

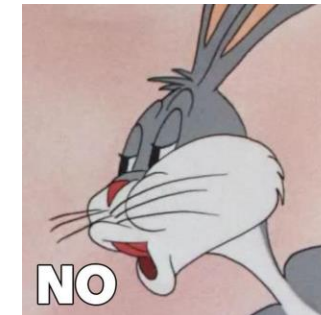
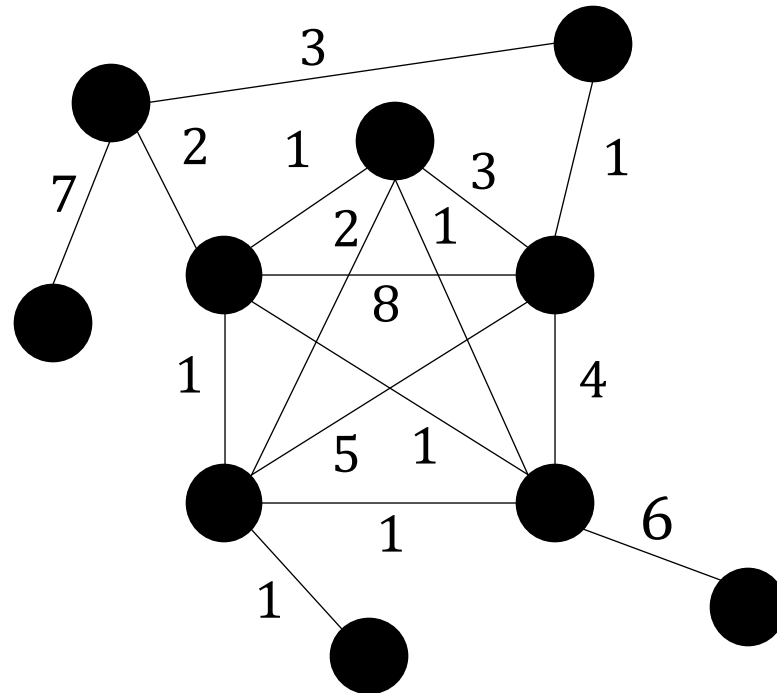
$$k = 5$$
$$h = 100$$



Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $G = (V, E, w)$ with $w: E \rightarrow \mathbb{N}$ and two integers h and k , decide whether G has a **clique** of k nodes such that the **sum of the weights** of its edges is at most h .

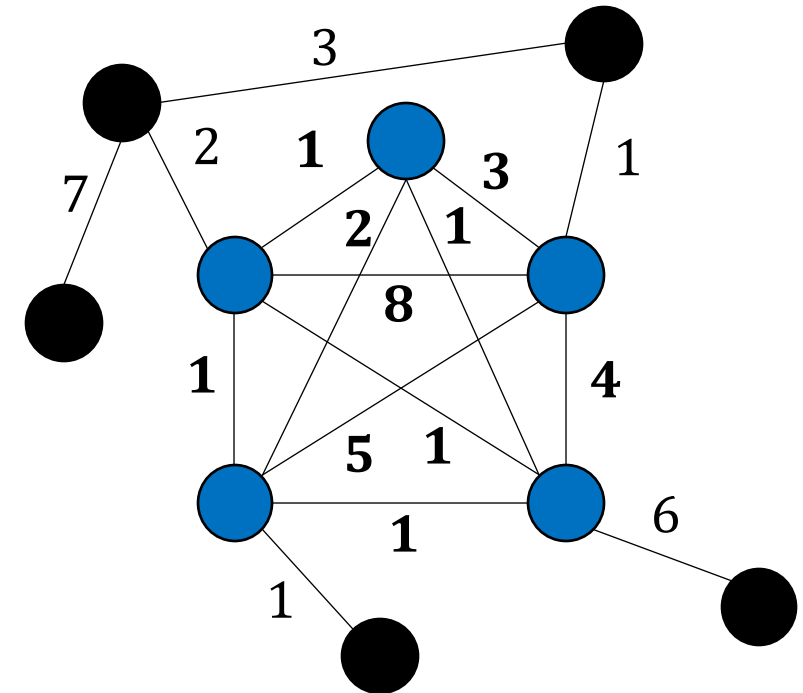
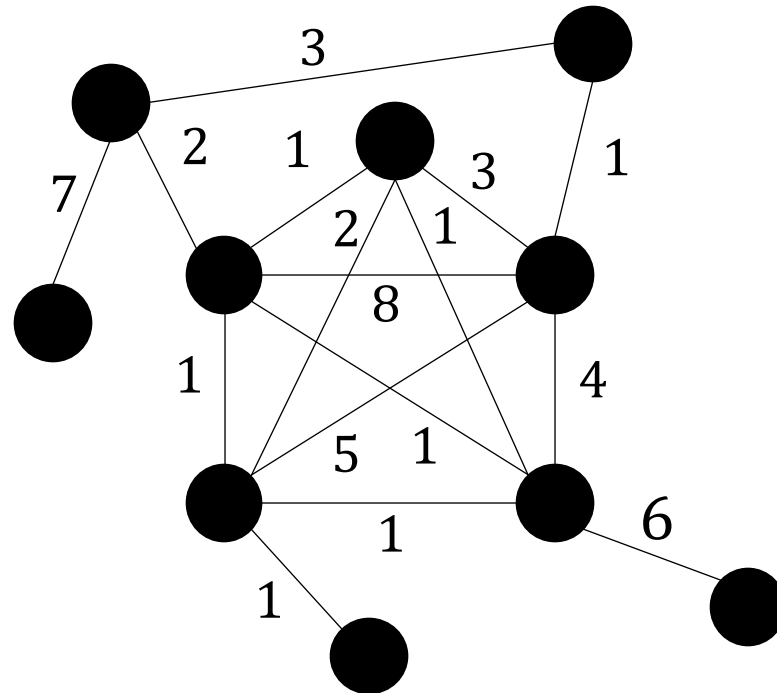
$$k = 6$$
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Consider the following decision problem: given an undirected and weighted graph $G = (V, E, w)$ with $w: E \rightarrow \mathbb{N}$ and two integers h and k , decide whether G has a **clique** of k nodes such that the **sum of the weights** of its edges is at most h .

$$k = 5$$
$$h = 15 \quad \text{X}$$



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- $I_{ex} = \{\langle G = (V, E, w), k, h \rangle : G \text{ is an undirected and weighted graph } \wedge w: E \rightarrow \mathbb{N} \wedge k, h \in \mathbb{N}\}$
- $S_{ex}(G, k, h) = \{V' \subseteq V\}$
- $\pi_{ex}(G, k, S_{ex}(G, k)) = \exists V' \in S_{ex}(G, k) [|V'| = k \wedge \forall u, v \in V' [(u, v) \in E] \wedge \sum_{u, v \in V'} w(\{u, v\}) \leq h]$

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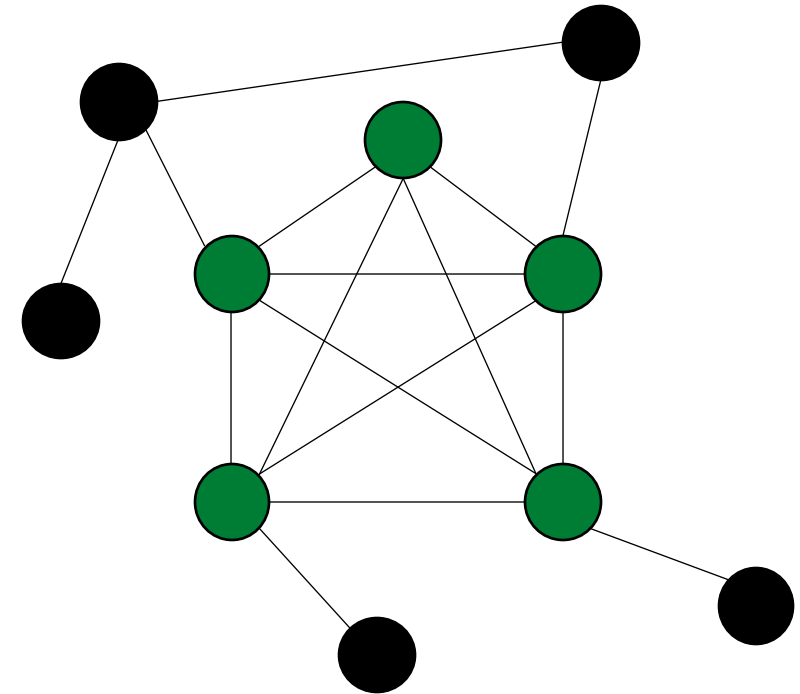
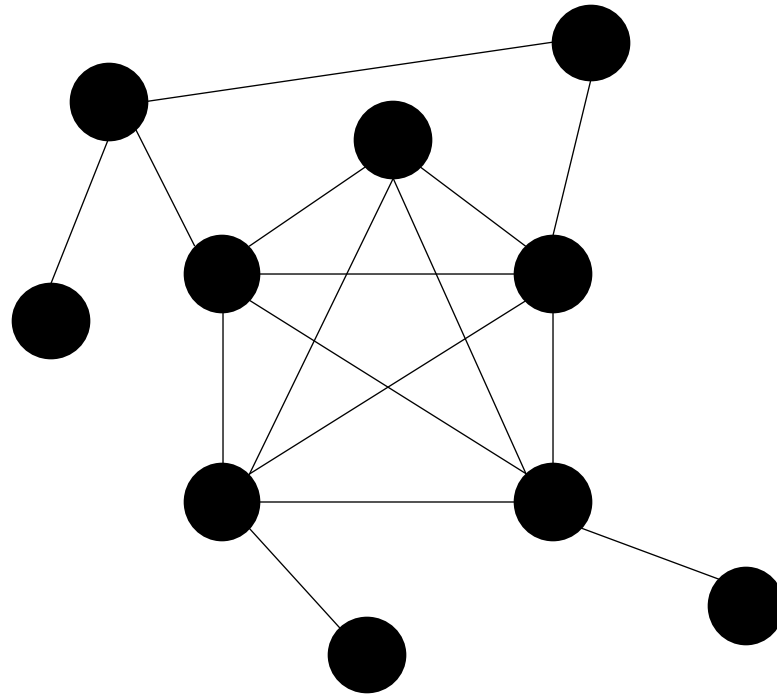
- $I_{ex} = \{\langle G = (V, E, \text{X}), k, \text{X} \rangle : G \text{ is an undirected and weighted graph } \wedge w: E \rightarrow \mathbb{N} \wedge k, h \in \mathbb{N}\}$
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Note well: Deciding whether a graph contains a clique of exactly k nodes is an NP-complete problem.

Problem 3 from the exam held on June 16, 2022

- $I_{nw} = \{\langle G = (V, E, w), k \rangle : G \text{ is an undirected graph} \wedge w: E \rightarrow \mathbb{N} \wedge k \in \mathbb{N}\}$
- $S_{nw}(G, k) = \{V' \subseteq V\}$
- $\pi_{nw}(G, k, S_{nw}(G, k)) = \exists V' \in S_{nw}(G, k) [|V'| = \mathbf{k} \wedge \forall \mathbf{u}, \mathbf{v} \in V' [(u, v) \in E]]$
 \geq (it's **Clique**)

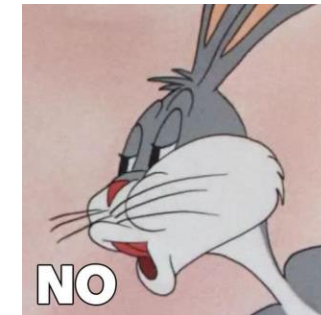
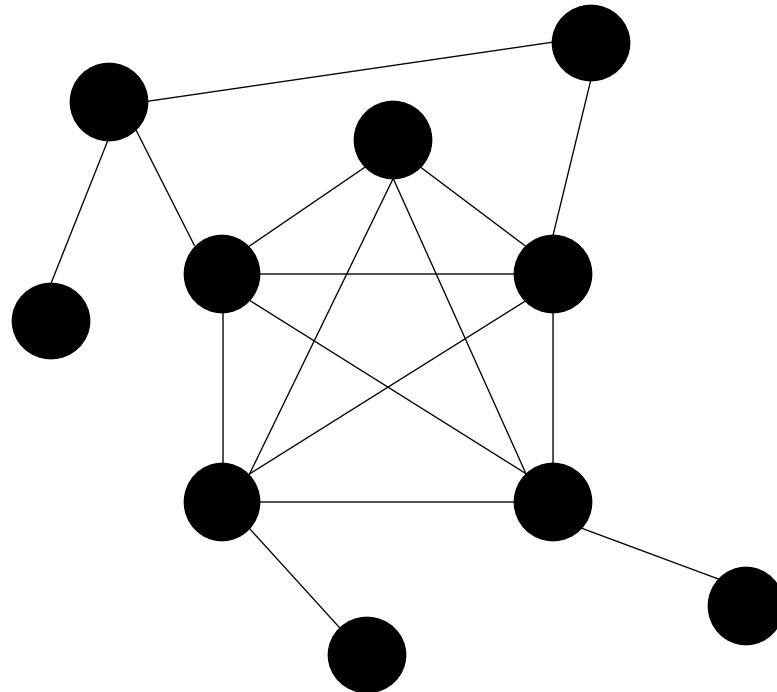
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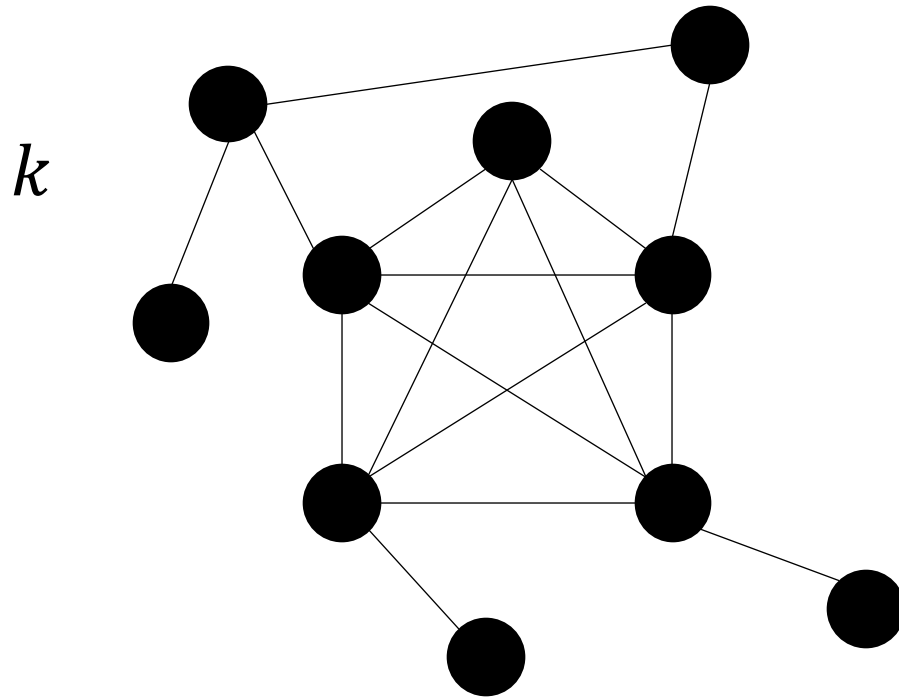
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Let's reduce from **Clique**

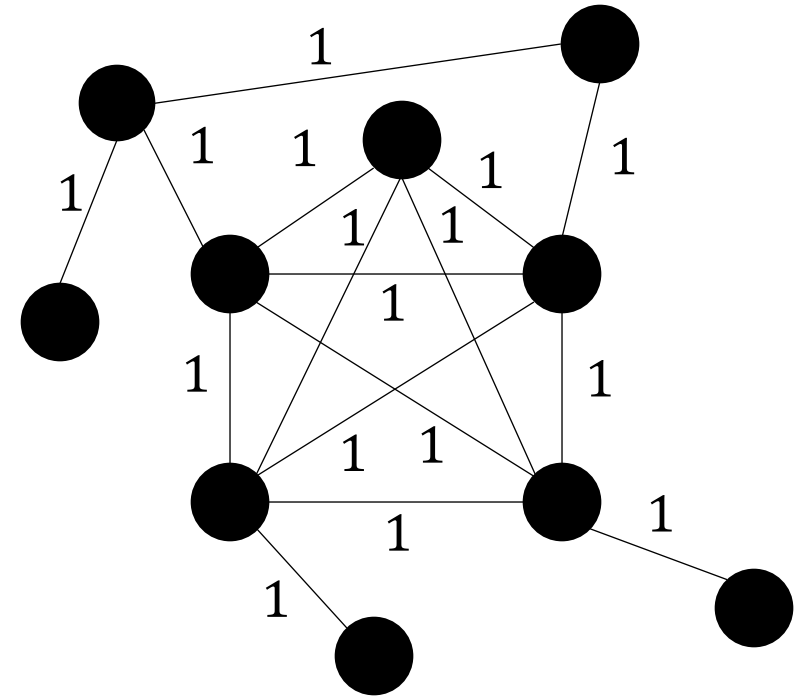
- $I_{Clique} = \{\langle G = (V, E, w), k \rangle : G \text{ is an undirected graph } \wedge w: E \rightarrow \mathbb{N} \wedge k \in \mathbb{N}\}$
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Instance of **Clique**

$$k, h = |E|$$



Instance of **our problem**

Problem 3 from the exam held on June 16, 2022

Instance of **Clique**

$$I_{Clique} = \{\langle G = (V, E), k \rangle : G \text{ is an undirected graph} \wedge k \in \mathbb{N}\}$$

Instance of **our problem**

$$I_{ex} = \{\langle G_{ex} = (V, E, w), k, h = |E| \rangle$$

: G_{ex} is an undirected and weighted graph $\wedge w: E \rightarrow \mathbb{N} \wedge k \in \mathbb{N}\}$

If G has a **clique** of k vertices, G_{ex} has a **clique** of k vertices, such that the **sum of the weights** of its edges is at most h , I_{ex} is a “yes” instance

If G doesn't have a **clique** of k vertices, G_{ex} doesn't have a **clique** of k vertices, I_{ex} is a “no” instance