SOLUZIONI

1) (i) $\displaystyle\int \frac{x}{x^3-3x+2}\,dx$

Fattorizziamo x^3-3x+2

Osserviamo che $x=1$ è una radice del polinomio, quindi per Ruffini

$$\Rightarrow \begin{array}{c|ccc|c} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & // \end{array}$$

$$x^3-3x+2=\underbrace{(x^2+x-2)}_{(x+2)(x-1)}(x-1) = (x+2)(x-1)^2$$

$$\Rightarrow \frac{x}{x^3-3x+2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$x = A(x-1)^2+B(x+2)(x-1)+C(x+2)$$

$$x=(A+B)x^2+(B+C-2A)x+(A-2B+2C)$$

$$\begin{cases} A+B=0 \\ B+C-2A=1 \\ A-2B+2C=0 \end{cases} \qquad \begin{cases} B=-A \\ C-3A=1 \\ 3A+2C=0 \end{cases}$$

$$\begin{cases} B = -A \\ 3A = C - 1 \\ 3C = 1 \end{cases} \qquad \begin{cases} B = 2/9 \\ A = -2/9 \\ C = 1/3 \end{cases}$$

$$\Rightarrow \int -\frac{2}{9(x+2)} + \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2}\, dx$$

$$= -\frac{2}{9} \log|x+2| + \frac{2}{9} \log|x-1| - \frac{1}{3} \cdot \frac{1}{(x-1)} + c$$

$$\boxed{= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C}$$

(ii) $$\int \frac{1}{x^4 - 2x^3}\, dx = \int \frac{1}{x^3(x-2)}\, dx$$

$$\frac{1}{x^3(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2}$$

$$1 = A\left(x^2(x-2)\right) + B\left(x(x-2)\right) + C(x-2) + Dx^3$$

$$1 = A(x^3 - 2x^2) + B(x^2 - 2x) + Cx - 2C + Dx^3$$

$$1 = (A+D)x^3 + (B-2A)x^2 + (C-2B)x - 2C$$

$$\begin{cases} A+D=0 \\ B-2A=0 \\ C-2B=0 \\ -2C=1 \end{cases} \qquad \begin{cases} D=-A=1/8 \\ A=B/2=-1/8 \\ B=C/2=-1/4 \\ C=-1/2 \end{cases}$$

$$\Rightarrow \int -\frac{1}{8x}-\frac{1}{4x^2}-\frac{1}{2x^3}+\frac{1}{8(x-2)}\,dx$$

$$-\frac{1}{8}\log|x|+\frac{1}{4x}+\frac{1}{4x^2}+\frac{1}{8}\log|x-2|+c$$

$$=\boxed{\frac{1}{8}\log\left|\frac{x-2}{x}\right|+\frac{x+1}{4x^2}+c}$$

(iii) $$\int \frac{1}{2\sin x-\cos x+5}\,dx$$

Usiamo la sostituzione $tg\,\frac{x}{2}=t$

$$dx=\frac{2}{1+t^2}\,dt$$

$$\sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$\cos x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$\int \frac{1}{\frac{4t}{1+t^2}-\frac{1-t^2}{1+t^2}+5}\cdot\frac{2}{1+t^2}\,dt$$

$$=\int \frac{\cancel{1+t^2}}{4t-1+t^2+5+5t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt$$

$$=\int \frac{1}{3t^2+2t+2}\,dt=\int \frac{1}{\left(\sqrt{3}t+\frac{\sqrt{3}}{3}\right)^2+\frac{5}{3}}\,dt$$

$$=\int \frac{1}{\frac{5}{3}\left(1+\frac{3}{5}\left(\sqrt{3}t+\frac{\sqrt{3}}{3}\right)^2\right)}\,dt$$

$$=\frac{3}{5}\int \frac{1}{1+\left(\frac{\sqrt{3}}{\sqrt{5}}\cdot\left(\sqrt{3}t+\frac{\sqrt{3}}{3}\right)\right)^2}\,dt$$

$$=\frac{3}{5}\int \frac{1}{1+\left(\frac{3}{\sqrt{5}}t+\frac{1}{\sqrt{5}}\right)^2}\,dt \qquad \frac{3}{\sqrt{5}}t+\frac{1}{\sqrt{5}}=y$$

$$dt=\frac{\sqrt{5}}{3}\,dy$$

$$=\frac{\sqrt{5}}{5}\int \frac{1}{1+y^2}\,dy=\frac{\sqrt{5}}{5}\operatorname{arctg}(y)+c$$

$$=\boxed{\frac{\sqrt{5}}{5}\operatorname{arctg}\left(\frac{3}{\sqrt{5}}\operatorname{tg}\frac{x}{2}+\frac{1}{\sqrt{5}}\right)+c}$$

(iv) $\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x^2-1)(x^2+1)}\,dx$

$$= \int \underbrace{\frac{1}{(x-1)(x+1)(x^2+1)}}\,dx$$

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1) = 1$$

$$A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + (Cx^3+Dx^2-Cx-D) = 1$$

$$\begin{cases} A+B+C=0 \\ A-B+D=0 \\ A+B-C=0 \\ A-B-D=1 \end{cases} \qquad \begin{cases} C=0 \\ B=-A=-1/4 \\ D=-1/2 \\ A=1/4 \end{cases}$$

$$= \int \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}\,dx$$

$$= \boxed{\frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\operatorname{arctg} x + c}$$

(v) $\displaystyle\int \frac{1}{4x^2+12x+12}\,dx$

$4x^2+12x+12 \rightsquigarrow \Delta = 12^2-4\cdot 4\cdot 12 < 0$

$$\Rightarrow \int \frac{1}{(2x+3)^2+3}\,dx = \frac{1}{3}\int \frac{1}{1+\left(\frac{2x+3}{\sqrt{3}}\right)^2}\,dx$$

$$\left[\begin{array}{l} \frac{2}{\sqrt{3}}x+\frac{3}{\sqrt{3}}=t \\ \frac{2}{\sqrt{3}}dx = dt \end{array}\right]$$

$$\Rightarrow \frac{\sqrt{3}}{6}\int \frac{1}{1+t^2}\,dt = \frac{\sqrt{3}}{6}\operatorname{arctg}(t)+c$$

$$= \frac{\sqrt{3}}{6}\operatorname{arctg}\left(\frac{2}{3}\sqrt{3}\,x+\sqrt{3}\right)+c$$

(vi) $\displaystyle\int \frac{1}{x^4+1}\,dx$

Dobbiamo scomporre x^4+1 come prodotto di due polinomi di secondo grado.

$(ax^2+bx+c)(dx^2+ex+f) = x^4+1$

Qui possiamo supporre che $a=d=1$

Supponiamo per semplicità che $c=f=1$

Proviamo a trovare allora b ed e

$$(x^2+bx+1)(x^2+ex+1) = x^4+1 =$$
$$x^4+(b+e)x^3+(be+2)x^2+(b+e)x+1$$

$$\Rightarrow \begin{cases} b+e=0 \\ be+2=0 \\ b+e=0 \end{cases} \Rightarrow \begin{cases} e=-b \\ b^2=2 \\ \end{cases} \quad \begin{cases} e=-\sqrt{2} \\ b=\sqrt{2} \end{cases}$$

$$\Rightarrow (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1) = x^4+1$$

$$\frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} = \frac{1}{x^4+1}$$

Troviamo A, B, C e D

$$(Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) = 1$$
$$(A+C)x^3+(D+B-\sqrt{2}A+\sqrt{2}C)x^2$$
$$+(A-\sqrt{2}B+C+\sqrt{2}D)x+(B+D) = 1$$

$$\begin{cases} A+C=0 \\ D+B-\sqrt{2}A+\sqrt{2}C=0 \\ A-\sqrt{2}B+C+\sqrt{2}D=0 \\ B+D=1 \end{cases} \quad \begin{cases} C=-A \\ 1-2\sqrt{2}A=0 \\ D-B=0 \\ B+D=1 \end{cases}$$

$$\begin{cases} C=-A=-\sqrt{2}/4 \\ A=\sqrt{2}/4 \\ B=1/2 \\ D=B=1/2 \end{cases}$$

$$\Rightarrow$$

$$\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

$$\Rightarrow \frac{1}{4}\int \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\,dx$$

$$=\frac{1}{4\sqrt{2}}\int \frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx$$

$$=\frac{1}{4\sqrt{2}}\left[\log(x^2+\sqrt{2}x+1)-\log(x^2+\sqrt{2}x+1)\right]$$

$$+\frac{1}{4}\underbrace{\int \frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{x^2-\sqrt{2}x+1}\,dx}$$

$$\int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}+\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx$$

$$=\int \frac{2}{1+(\sqrt{2}x+1)^2}+\frac{2}{1+(\sqrt{2}x-1)^2}\,dx$$

$$=\frac{1}{2\sqrt{2}}\left(\operatorname{arctg}(\sqrt{2}x+1)+\operatorname{arctg}(\sqrt{2}x-1)\right)$$

$$\Rightarrow \boxed{\frac{1}{4\sqrt{2}}\log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{1}{2\sqrt{2}}\left(\operatorname{arctg}(\sqrt{2}x+1)+\operatorname{arctg}(\sqrt{2}x-1)\right)+c}$$

(vii) $\int \frac{2x+1}{x^2+3}\,dx$

$$= \int \frac{2x}{x^2+3}\,dx + \int \frac{1}{3+x^2}\,dx$$

$$\log(x^2+3) + \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$$= \boxed{\log(x^2+3) + \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{x}{\sqrt{3}}\right) + c}$$

(viii) $\int \frac{x^4}{x^4+5x^2+4}\,dx = \int 1 - \frac{5x^2+4}{x^4+5x^2+4}\,dx$

$$x^4+5x^2+4 = (x^2+1)(x^2+4)$$

$$\Rightarrow \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{5x^2+4}{(x^2+1)(x^2+4)}$$

$$(Ax+B)(x^2+4) + (Cx+D)(x^2+1) = 5x^2+4$$

$$(A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D) = 5x^2+4$$

$$\begin{cases} A+C=0 \\ B+D=5 \\ 4A+C=0 \\ 4B+D=4 \end{cases} \qquad \begin{cases} C=-A \\ D=5-B \\ 3A=0 \\ 4B+5-B=4 \end{cases} \qquad \begin{cases} A=C=0 \\ D=16/3 \\ B=-1/3 \end{cases}$$

$$\Rightarrow \int 1 - \left(-\frac{1}{3}\cdot\frac{1}{1+x^2} + \frac{16}{3}\cdot\frac{1}{x^2+4}\right) dx$$

$$= x + \frac{1}{3}\operatorname{arctg} x - \frac{4}{3}\int \frac{1}{1+\left(\frac{x}{2}\right)^2} dx$$

$$= \boxed{x + \frac{1}{3}\operatorname{arctg} x - \frac{8}{3}\operatorname{arctg}\left(\frac{x}{2}\right) + c}$$

(ix) $\int \log x \, dx$

Integrazione per parti

$$\int 1 \cdot \log x \, dx = \left[x \log x\right] - \int x \cdot \frac{1}{x} dx$$

$$= \boxed{x \log x - x + c}$$

(x) $$\int \frac{\sin^2 x}{1+\sin^2 x} dx = \int 1 - \frac{1}{1+\sin^2 x} dx$$

$$= x - \int \frac{1}{1+\sin^2 x} dx$$

Voglio usare la sostituzione $t = \operatorname{tg}\frac{x}{2}$ peró prima voglio abbassare il grado del seno. In che modo? Uso la formula

$$\cos 2x = 1 - 2\sin^2 x$$

In tal modo $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\Rightarrow \int \frac{1}{1+\sin^2 x}\,dx = \int \frac{1}{1+\frac{1-\cos 2x}{2}}\,dx$$

$$= 2\int \frac{1}{3-\cos 2x}\,dx$$

Dunque $\cos 2x = \frac{1-tg^2 x}{1+tg^2 x}$

Sostituzione $tg\,x = t$

$$dx = \frac{1}{1+t^2}\,dt$$

$$\Rightarrow 2\int \frac{1}{3-\frac{1-t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,dt$$

$$= 2\int \frac{1}{4t^2+2}\,dt = \int \frac{1}{1+2t^2}\,dt$$

$$= \int \frac{1}{1+(\sqrt{2}t)^2}\,dt$$

$$= \frac{1}{\sqrt{2}}\,\mathrm{arctg}(t\cdot\sqrt{2}) + c$$

$$= \frac{\sqrt{2}}{2}\,\mathrm{arctg}(\sqrt{2}\,tg\,x) + c$$

Allora il risultato finale é

$$\boxed{x - \frac{\sqrt{2}}{2} \operatorname{arctg}\left(\sqrt{2}\, tg\, x\right) + c}$$

2) (i) $\lim_{x\to 0} \frac{x^2 + e^x - 1 - x}{3x^2}$

$$= \lim_{x\to 0} \frac{x^2 + \left(1 + x + \frac{x^2}{2} + o(x^2)\right) - 1 - x}{3x^2}$$

$$= \lim_{x\to 0} \frac{\frac{3}{2}x^2 + o(x^2)}{3x^2} = \lim_{x\to 0} \frac{\frac{3}{2} + o(1)}{3} = \boxed{\frac{1}{2}}$$

(ii) $\lim_{x\to +\infty} x^3 \left[\log\left(1 + \frac{1}{x}\right) - \frac{1}{x} + \frac{1}{2x^2}\right]$

$x = \frac{1}{t}$

$$\lim_{t\to 0} \frac{1}{t^3}\left[\log(1+t) - t + \frac{t^2}{2}\right]$$

$$= \lim_{t\to 0} \frac{1}{t^3}\left[\cancel{t} - \cancel{\frac{t^2}{2}} + \frac{t^3}{3} + o(t^3) \cancel{-t} + \cancel{\frac{t^2}{2}}\right]$$

$$= \lim_{t\to 0} \frac{1}{t^3}\left[\frac{t^3}{3} + o(t^3)\right] = \lim_{t\to 0} \frac{1}{3} + o(1)$$

$$= \boxed{\frac{1}{3}}$$

(iii) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{e^{2x}-1-2x}$

$$= \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2}+o(x^2)\right)}{\left(1+2x+\frac{4x^2}{2}+o(x^2)\right)-1-2x}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2}+o(x^2)}{2x^2+o(x^2)} = \lim_{x\to 0} \frac{\frac{1}{2}+o(1)}{2+o(1)} = \boxed{\frac{1}{4}}$$

(iv) $\displaystyle\lim_{x\to 0} \frac{\sqrt[4]{1-4x^2+x^4}-1+x^2}{x^4}$

$$= \lim_{x\to 0} \frac{\left(1+(-4x^2+x^4)\right)^{1/4}-1+x^2}{x^4}$$

$$= \lim_{x\to 0} \frac{1+\frac{1}{4}(-4x^2+x^4)+\frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)}{2}\cdot(-4x^2+x^4)^2+o\left((-4x^2+x^4)^2\right)-1+x^2}{x^4}$$

<u>OSS</u>: $o\left((-4x^2+x^4)^2\right) = o(x^4)$

$$= \lim_{x\to 0} \frac{\cancel{1}\cancel{-x^2}+\frac{1}{4}x^4-\frac{3}{32}(16x^4)+o(x^4)\cancel{-1}\cancel{+x^2}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{1}{4}x^4 - \frac{3}{2}x^4 + o(x^4)}{x^4}$$

$$= \lim_{x \to 0} -\frac{5}{4} + o(1) = \boxed{-\frac{5}{4}}$$

(v) $$\lim_{x \to 0} \frac{1 - e^{-x^2} + x^3 \sin\left(\frac{1}{x}\right)}{x^2}$$

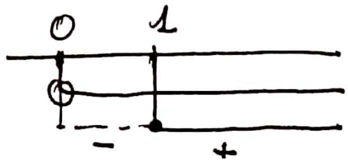
$$= \lim_{x \to 0} \frac{1 - (1 - x^2 + o(x^2)) + x^3 \sin\left(\frac{1}{x}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{x^2 + o(x^2) + x^3 \sin\left(\frac{1}{x}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{x^2 + o(x^2)}{x^2} = \lim_{x \to 0} 1 + o(1) = \boxed{1}$$

3) (i) $f(x) = \frac{\log x}{x}$

- Dominio : $x > 0 \rightsquigarrow (0, +\infty)$
- Segno : $\frac{\log x}{x} \geq 0$ $\begin{cases} \log x \geq 0 \rightsquigarrow x \geq 1 \\ x > 0 \end{cases}$



- Limiti ai bordi

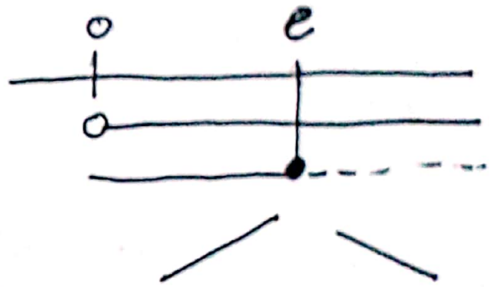
$$\lim_{x \to 0^+} \frac{\log x}{x} = -\infty \qquad \lim_{x \to +\infty} \frac{\log x}{x} = 0^+$$

- Derivata

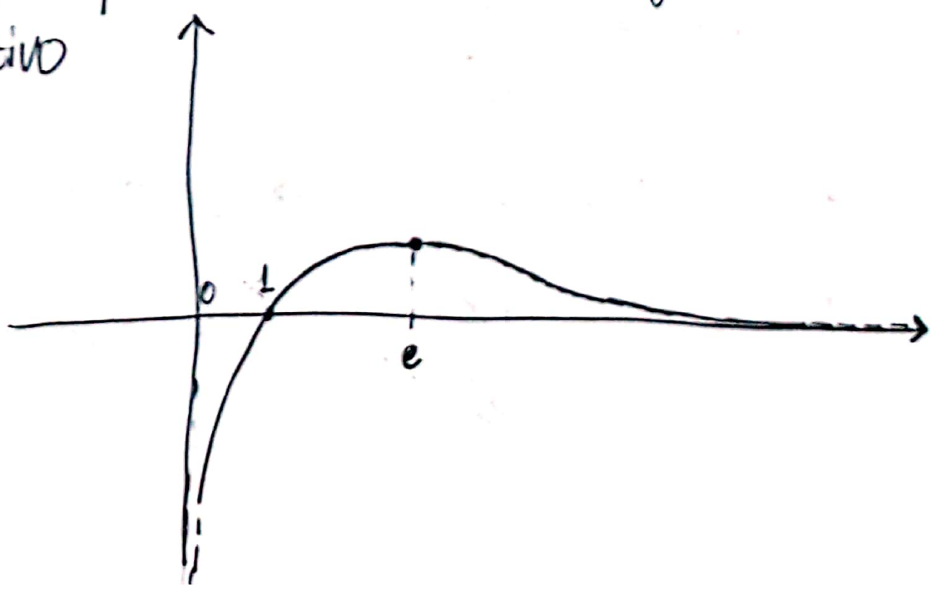
$$f'(x) = \frac{\frac{1}{x} \cdot x - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

$$f'(x) \geq 0 \rightarrow \frac{1 - \log x}{x^2} \geq 0$$

- $1 - \log x \geq 0 \rightarrow \log x \leq 1 \rightarrow x \leq e$
- $x^2 > 0 \rightarrow x \neq 0$



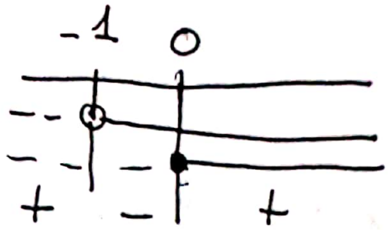
- Non studio la derivata seconda ma sicuramente c'è un flesso per considerazioni sugli asintoti
- Grafico qualitativo



(ii) $f(x) = \left| \dfrac{x}{x+1} \right|$

Per studiare una funzione col modulo si studiano prima di tutto gli intervalli di positività e negatività. Quelli in cui è positiva la funzione rimane uguale mentre in quelli negativi cambia di segno.

$$\Rightarrow \frac{x}{x+1} \geq 0 \qquad \begin{array}{l} x \geq 0 \\ x > -1 \end{array} \rightarrow$$



$$\Rightarrow f(x) = \begin{cases} \dfrac{x}{x+1} & x \in (-\infty, -1) \cup [0, +\infty) \\ -\dfrac{x}{x+1} & x \in (-1, 0) \end{cases}$$

- Dominio: $x \neq -1$
- Limiti ai bordi

$$\lim_{x \to +\infty} \frac{x}{x+1} = 1 \qquad \lim_{x \to -\infty} \frac{x}{x+1} = 1$$

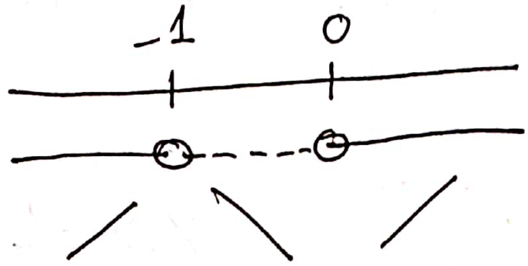
$$\lim_{x \to -1^+} -\frac{x}{x+1} = +\infty \qquad \lim_{x \to -1^-} \frac{x}{x+1} = +\infty$$

- Derivata

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$f'(x) = \begin{cases} \dfrac{1}{(x+1)^2} & x \in (-\infty, -1) \cup (0, +\infty) \\ -\dfrac{1}{(x+1)^2} & x \in (-1, 0) \end{cases}$$

$f'(x) \geq 0 \rightsquigarrow$



$x=0$ è un punto di non derivabilità
È un cosiddetto punto angoloso

- Grafico qualitativo