THEORETICAL COMPUTER SCIENCE TUTORING (3)

Maurizio Fiusco

TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Problem 1 from the exam held on July 4, 2019

Remember how Turing machines can be encoded as integers. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a function defined as follows:

$$\boldsymbol{f(i)=\begin{cases} 0 \text{ if } i \text{ is the encoding of the Turing machine} \\ 1 \text{ if } i \text{ isn't the encoding of the Turing machine} \end{cases}}$$

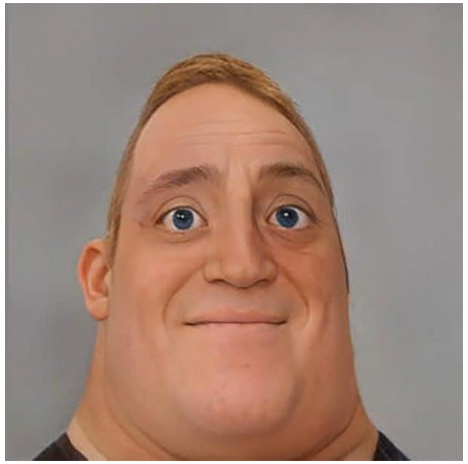
After **defining the concept of computability** of a function, **discuss the computability of $f(n)$** by demonstrating your claims.

Problem 1 from the exam held on July 4, 2019

Definition

A function $f\colon \Sigma_1^* \longrightarrow \Sigma_2^*$ is **computable** if there exists a Turing transducer T such that for every $x \in \Sigma_1^*$ for which x is defined, $T(x) = f(x)$

discuss the computability of $\boldsymbol{f(n)}$



Problem 1 from the exam held on July 4, 2019

🤔 f is the characteristic function associated with the language L_T of Turing machine encoded as integers

Claim

L_T is decidable

Prove it by yourselves (build a Turing machine)

💡

Theorem

A language $\boldsymbol{L}$ **is decidable** if and only if the associated characteristic function $\boldsymbol{f}$ **is computable**

Let's see the proof

Problem 1 from the exam held on July 4, 2019

Proof

L is decidable $\Rightarrow$ f is computable

There exists a recognizer T such that $\forall\ x \in \Sigma^*$:

$$o_T(x) = \begin{cases} q_A \text{ if } x \in L \\ q_R \text{ if } x \notin L \end{cases}$$

Suppose that T has only one tape

$\boldsymbol{T}$

$x \in \Sigma^*$

Let's build a transducer T' that will compute $f(x)$ on two tapes

Problem 1 from the exam held on July 4, 2019

Proof

L is decidable $\Rightarrow$ f is computable

T'

$x \in \Sigma^*$

output tape

1. On the first tape, which contains the input x, it performs the computation $T(x)$
2. If $T(x)$ terminates in q_a, it writes the value 0 on the output tape; **otherwise**, it writes the value 1

here ends the exercise, let's finish the proof anyway

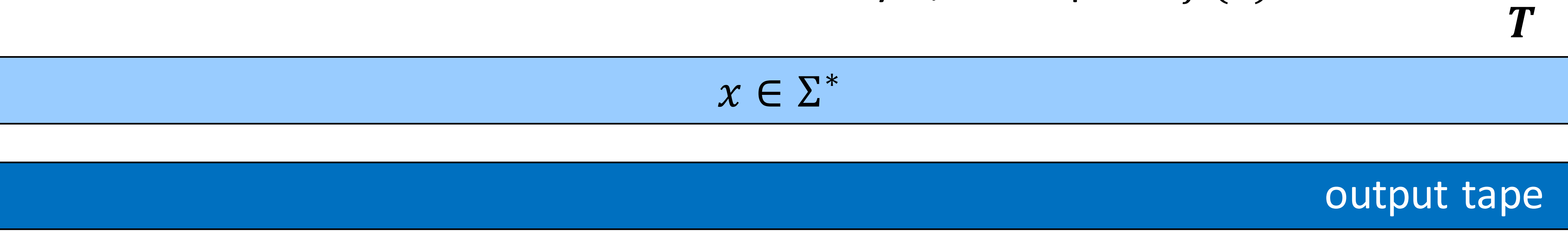
Problem 1 from the exam held on July 4, 2019

Proof

f is computable $\Rightarrow$ L is decidable

f is a total function by definition

There exists a transducer T such that for every x, it computes $f(x)$



Let’s build a recognizer T' that will decide L on two tapes

Problem 1 from the exam held on July 4, 2019

Proof

L is decidable $\Rightarrow$ f is computable

T'

$x \in \Sigma^*$

1/0

1. On the first tape, which contains the input x, it performs the computation $T(x)$, writing the result on the second tape
2. If **0** has been written on the second tape, then the computation of T' terminates in the accepting state; **otherwise**, it terminates in the rejecting state

Problem 3.1 from the exam held on July 9, 2018

Let $\boldsymbol{L_1} \subseteq \boldsymbol{\Sigma}^*$ be a **decidable** language decided by machine T_1, and let $\boldsymbol{L_2} \subseteq \boldsymbol{\Sigma}^*$ be an **acceptable** but undecidable language accepted by machine T_2. Consider the following language

$$L = \{(x, k) : \boldsymbol{x} \in \boldsymbol{\Sigma}^* \wedge \boldsymbol{k} \in \mathbb{N} \wedge [\boldsymbol{x} \notin \boldsymbol{L_1} \vee (\boldsymbol{x} \notin \boldsymbol{L_2} \wedge \boldsymbol{T_2(x)} \textbf{ rejects in } \boldsymbol{k} \textbf{ steps})]\}$$

Show whether L is an acceptable or decidable language

$$\boldsymbol{L_{blue}} = \{(\boldsymbol{x}, \boldsymbol{k}) : \boldsymbol{x} \in \boldsymbol{\Sigma}^* \wedge \boldsymbol{k} \in \mathbb{N} \wedge \boldsymbol{x} \notin \boldsymbol{L_1}\}$$

$$\boldsymbol{L_{orange}} = \{(\boldsymbol{x}, \boldsymbol{k}) : \boldsymbol{x} \notin \boldsymbol{L_2} \wedge \boldsymbol{T_2(x)} \textbf{ rejects in } \boldsymbol{k} \textbf{ steps}\}$$

$$\boldsymbol{L} = \boldsymbol{L_{blue}} \cup \boldsymbol{L_{orange}}$$

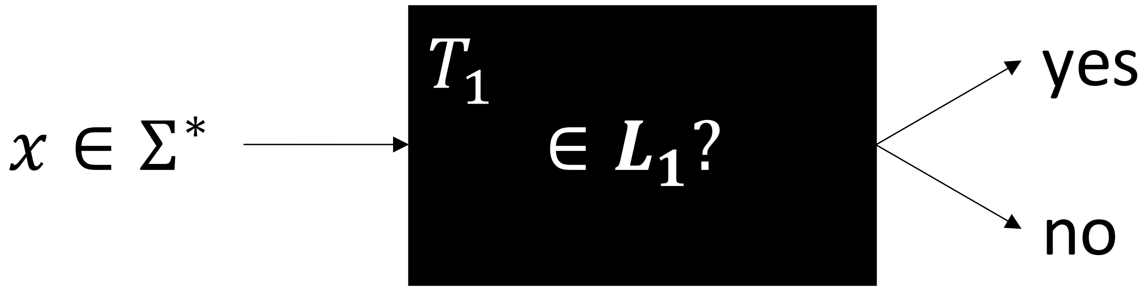
Problem 3.1 from the exam held on July 9, 2018

Claim

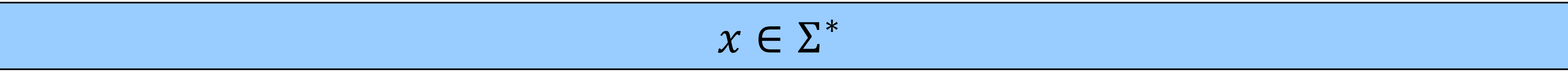
$\boldsymbol{L_{blue} = \{(x, k): x \in \Sigma^* \wedge k \in \mathbb{N} \wedge x \notin L_1\}}$ **is decidable**

Proof

$\boldsymbol{L_1}$ is decidable



Suppose that T_1 has only one tape



Problem 3.1 from the exam held on July 9, 2018

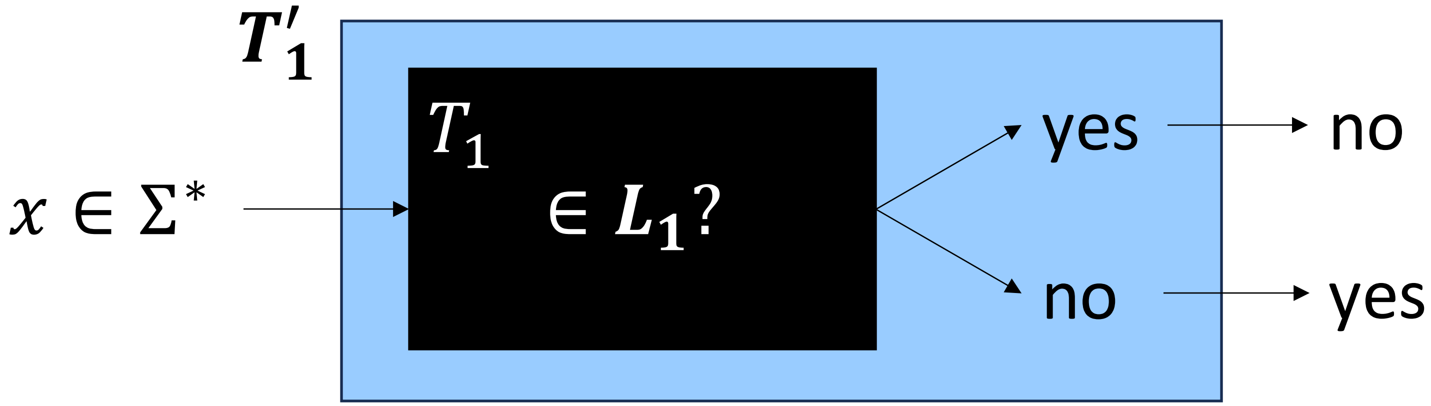
Let's build a recognizer $\boldsymbol{T'_1}$ that will decide $\boldsymbol{L_{blue}}$

$\boldsymbol{T'_1}$

$x \in \Sigma^*$

Simulate $T_1(x)$

- $T_1(x)$ ends in the accepting state — $\boldsymbol{T'_1}$ rejects
- $T_1(x)$ ends in the rejecting state — $\boldsymbol{T'_1}$ accepts



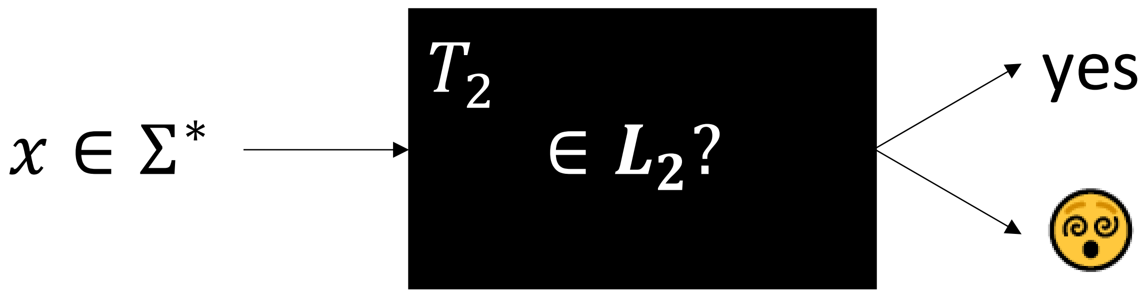
Problem 3.1 from the exam held on July 9, 2018

Claim

$L_{orange} = \{(x, k) : x \notin L_2 \wedge T_2(x)$ **rejects in** k **steps**$\}$ **is decidable**

Proof

L_2 is accepatable but not decidable



L_2^c is not acceptable but…

Suppose that T_2 has only one tape

$x \in \Sigma^*$

Problem 3.1 from the exam held on January 21, 2019

Let's build a recognizer $\boldsymbol{T'_2}$ that will decide $\boldsymbol{L_{orange}}$

$\boldsymbol{T'_2}$

$x \in \Sigma^*$

$\square \cdots \square \underbrace{11 \ldots 1}_{\#1\,=\,k} \square \cdots \square$

1. Simulate one instruction of $T_2(x)$ on the first tape
2. Move the head on the second tape to the right

if $T_2(x)$ ends in the rejecting state ✔ ❌
else if $T_2(x)$ ends in the accepting state or on the second tape the head reads $\square$
else 🤷 **1**

Problem 3.1 from the exam held on January 21, 2019

Claim

$L = L_{blue} \cup L_{orange}$ **decidable**

Proof

L_{blue} and L_{orange} are decidable, we proved it in the last lesson

Problem 3.1 from the exam held on June 18, 2018

Let $\boldsymbol{L_1} \subseteq \boldsymbol{\Sigma}^*$ be an **acceptable** but undecidable language and let $\boldsymbol{L_2} \subseteq \boldsymbol{\Sigma}^*$ be a decidable language. Consider the following function $\boldsymbol{f}: \sigma^* \to \mathbb{N} : \ \forall x \in \Sigma^*$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } x \in L_1 \\ |x| & \text{if } x \notin L_1 \wedge x \in L_2 \\ 0 & \text{otherwise} \end{cases}$$

Show whether $\boldsymbol{f}$ is a computable function