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Numerical Methods (ENUME)
Assignment A: Accuracy of computation
Spring Semester 2024

Task #1

Determine the function $T(x)$ characterising the propagation of the relative error corrupting the values of $x \in [10^{-2}, 10^{2}]$ to the result of computing the following expression:

$$y = f(x) \equiv \frac{\arctan(x)}{x^2} - x^3$$

Compare the formulae of the function $T(x)$, determined using the method of *epsilon calculus* and the method of symbolic differentiation.

Task #2

Determine the relative errors corrupting the values of y computed on the basis of the values $\{x_1, ..., x_N\}$ of x stored using the single-precision floating-point representation, where:

$$10^{-2} = x_1 \leq x_2 \leq ... \leq x_N = 10^{2}, \quad N \in \mathbb{N}$$

Proceed as follows:

a) compute the values $\{y_1, ..., y_N\}$:

$$y_n = f(x_n) \quad \text{for} \quad n = 1, ..., N$$

using the double-precision representation (default in MATLAB);

b) compute the values $\{\tilde{y}_1, ..., \tilde{y}_N\}$:

$$\tilde{y}_n = f(\tilde{x}_n) \quad \text{for} \quad n = 1, ..., N$$

where $\tilde{x}_n$ denotes the result of storing x_n using the single-precision representation (`single`); perform the computation using the double-precision representation (`double`);

c) compute the relative errors according to the formula:

$$\delta[\tilde{y}_n)] = \frac{\tilde{y}_n - y_n}{y_n} \quad \text{for} \quad n = 1, ..., N$$

Compare the computed errors with the worst-case estimate $|T(x)|\ eps_{\text{single}}$, where eps_{single} denotes the upper bound of the relative errors corrupting numeric values stored using the single-precision representation.

Task #3

Using the method of *epsilon calculus*, determine the functions $K_{\mathrm{A1}}(x)$ and $K_{\mathrm{A2}}(x)$, characterising the propagation of the relative errors caused by rounding the intermediate results of computing $\{y_1, ..., y_N\}$ according to the following algorithms:

$$\mathrm{A1}: \quad [x] \rightarrow \begin{bmatrix} v_1 = \arctan(x) \\ v_2 = x^2 \end{bmatrix} \rightarrow \begin{bmatrix} v_3 = v_1 / v_2 \\ v_4 = x^3 \end{bmatrix} \rightarrow [v_5 = v_3 - v_4] = [y]$$

$$\mathrm{A2}: \quad [x] \rightarrow \begin{bmatrix} v_1 = \arctan(x) \\ v_2 = x^5 \end{bmatrix} \rightarrow \begin{bmatrix} v_3 = v_1 - v_2 \\ v_4 = x^2 \end{bmatrix} \rightarrow [v_5 = v_3 / v_4] = [y]$$

Task #4

Determine the relative errors which corrupt the values $\{y_1, ..., y_N\}$ when the results of all intermediate operations are stored using the single-precision representation. Proceed as follows:

a) compute $\{y_1, ..., y_N\}$ using the double-precision representation (default in MATLAB);

b) by means of the algorithms A1 and A2, compute the approximate values $\left\{\tilde{y}_1^{\mathrm{A1}}, ..., \tilde{y}_N^{\mathrm{A1}}\right\}$ and $\left\{\tilde{y}_1^{\mathrm{A2}}, ..., \tilde{y}_N^{\mathrm{A2}}\right\}$ by performing the intermediate operations one by one and storing their results using single-precision representation (`single`);

c) compute the relative errors according to the formulae:

$$\delta\left[\tilde{y}_n^{\mathrm{A1}}\right] = \frac{\tilde{y}_n^{\mathrm{A1}} - y_n}{y_n} \quad \text{and} \quad \delta\left[\tilde{y}_n^{\mathrm{A2}}\right] = \frac{\tilde{y}_n^{\mathrm{A2}} - y_n}{y_n} \quad \text{for} \quad n = 1, ..., N$$

Compare the computed errors with the worst-case estimates: $\left|K_{\mathrm{A1}}(x)\right| \, eps_{\mathrm{single}}$, $\left|K_{\mathrm{A2}}(x)\right| \, eps_{\mathrm{single}}$.

Task #5

Compare the functions $T(x)$, $K_{\mathrm{A1}}(x)$ and $K_{\mathrm{A2}}(x)$ for $x \in [10^{-2}, 10^{2}]$.