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Numerical Methods (ENUME)
Assignment C: Ordinary differential equations
Spring Semester 2024

The so-called *planar three-body problem* is a set of nonlinear second-order ordinary differential equations modelling the two-dimensional movement trajectories of three objects which attract each other:

$$\begin{cases} \dfrac{d^2x_1(t)}{dt^2} = -Gm_2\dfrac{x_1(t)-x_2(t)}{r_{12}^3(t)} - Gm_3\dfrac{x_1(t)-x_3(t)}{r_{31}^3(t)} \\ \dfrac{d^2y_1(t)}{dt^2} = -Gm_2\dfrac{y_1(t)-y_2(t)}{r_{12}^3(t)} - Gm_3\dfrac{y_1(t)-y_3(t)}{r_{31}^3(t)} \\ \dfrac{d^2x_2(t)}{dt^2} = -Gm_3\dfrac{x_2(t)-x_3(t)}{r_{23}^3(t)} - Gm_1\dfrac{x_2(t)-x_1(t)}{r_{12}^3(t)} \\ \dfrac{d^2y_2(t)}{dt^2} = -Gm_3\dfrac{y_2(t)-y_3(t)}{r_{23}^3(t)} - Gm_1\dfrac{y_2(t)-y_1(t)}{r_{12}^3(t)} \\ \dfrac{d^2x_3(t)}{dt^2} = -Gm_1\dfrac{x_3(t)-x_1(t)}{r_{31}^3(t)} - Gm_2\dfrac{x_3(t)-x_2(t)}{r_{23}^3(t)} \\ \dfrac{d^2y_3(t)}{dt^2} = -Gm_1\dfrac{y_3(t)-y_1(t)}{r_{31}^3(t)} - Gm_2\dfrac{y_3(t)-y_2(t)}{r_{23}^3(t)} \end{cases} \tag{1}$$

where:

- t is the time,
- $x_k(t)$ and $y_k(t)$ are the coordinates of the kth object's position ($k = 1, 2, 3$),
- m_k is the kth object's mass ($k = 1, 2, 3$),
- G is the gravitational constant,
- $r_{jk}(t) \equiv \sqrt{\left[x_k(t) - x_j(t)\right]^2 + \left[y_k(t) - y_j(t)\right]^2}$ for $j, k = 1, 2, 3$.

Task #1

Solve the set of equations (1) using:

- the MATLAB function `ode45` with the parameters `AbsTol` and `RelTol` set to 10^{-12},
- the Adams-Bashforth method of order 2,
- the Kutta method,
- the "classical" Runge-Kutta method

for $m_1 = m_2 = m_3 = G = 1$, $t \in [0, 5.226525]$ and initial conditions specified in Table 1.

For the methods other than the MATLAB function `ode45`, pick any value of the differentiation step smaller or equal to 0.01.

Table 1. Exemplary initial conditions for the planar three-body problem.

k	1	2	3
$x_k(0)$	0.8083106230	−0.4954148566	−0.3128957664
$y_k(0)$	0.0000000000	0.0000000000	0.0000000000
$\frac{dx_k}{dt}(0)$	0.0000000000	0.0000000000	0.0000000000
$\frac{dy_k}{dt}(0)$	0.9901979166	−2.7171431768	1.7269452602

Task #2

Find the value of r_0 for which the solution of the set of equations (1) has the form:

$$\left.\begin{aligned} x_1(t) &= r_0\ \sin\left(t - \tfrac{\pi}{2}\right) \\ x_2(t) &= r_0\ \sin\left(t + \tfrac{\pi}{6}\right) \\ x_3(t) &= r_0\ \sin\left(t + \tfrac{5}{6}\pi\right) \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} y_1(t) &= r_0\ \cos\left(t - \tfrac{\pi}{2}\right) \\ y_2(t) &= r_0\ \cos\left(t + \tfrac{\pi}{6}\right) \\ y_3(t) &= r_0\ \cos\left(t + \tfrac{5}{6}\pi\right) \end{aligned}\right. \tag{2}$$

if $G = 1$ and $m_1 = m_2 = m_3 = \sqrt{3}$.

Task #3

Solve the set of equations (1) with the initial conditions corresponding to the solution described in Task #2 for $t \in [0, 2\pi]$ using the methods listed in Task #1. Compute the mean-square error Δ_{x_2}:

$$\Delta_{x_2} = \frac{1}{N} \sum_{n=1}^{N} \left[\hat{x}_{2,n} - x_2(t_n)\right]^2 \tag{3}$$

where:

- $t_1, ..., t_N$ are the values of t for which the solution has been obtained numerically (with N denoting their total number),
- $\hat{x}_{2,1}, ..., \hat{x}_{2,N}$ are the estimates of $x_2(t_1), ..., x_2(t_N)$, obtained numerically,
- $x_2(t)$ is the reference solution, defined by one of the equations in the set (2).

Determine the dependence of Δ_{x_2} on the step of integration $h \in [10^{-3}, 10^{0}]$ for the Adams-Bashforth method of order 2, the Kutta method and the "classical" Runge-Kutta method.

Task #4

The data stored in the file *data_13.csv* represent the results of measurement of the two-dimensional movement trajectories of three objects which satisfy the planar three-body problem for $G = 1$. Use these data to estimate the masses m_1, m_2 and m_3 of those objects.