Holographic Complexity in Non-Commutative Gauge Theory

Work with Stefan Eccles, Willy Fischler, and Ming-Lei Xiao
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Introduction

In this talk, we will consider the growth rate of holographic complexity in a geometry dual to non-commutative super Yang-Mills (NCSYM), according to 'complexity = action'. A brief outline is as follows:

- First, we will recall the holographic complexity conjectures, in particular 'complexity = action', or CA.
- Next, we will briefly discuss the geometry dual to ($\mathcal{N} = 4$) NCSYM.
- We will then consider a qualitative argument that we should expect the complexity of the dual non-local field theory to increase with the Moyal scale.
- We will review results for the D3-brane system, first looking at the finite time behavior ...
- ... then focusing in on the late time asymptotic behavior.
- Briefly we will look at the late time results generalized to a Dp-brane system for $2 \leq p \leq 5$
- And finally we will discuss the implications of these results, and possible future directions.

Holographic Complexity Recap

Recap of a few points from the last talk:

- Holographic complexity is motivated by the growth of the behind the horizon geometry [1]
- Originally it was conjectured by Susskind and others [1, 2] that the volume of a maximal spatial slice is dual to the circuit complexity of the boundary state (relative to some reference state).
- Relative circuit complexity of a state $|\psi\rangle$ is defined as the minimum number of unitary gates g_i from some gate set G needed to build a quantum circuit which when applied to a reference state $|\psi_0\rangle$ will produce ψ to within a tolerance ϵ
- The 'complexity = volume' conjecture is supported by the switchback effect, where we reproduce the expected behavior holographically [3, 4]
- It was later proposed in Brown et al. that the complexity is dual to the action of a Wheeler-DeWitt patch, rather than to a maximal spatial slice [5, 6]. It is this proposal with which we will primarily be concerned.

Complexity = Action

Complexity = volume has a few unpleasant features

- For example, in order to reproduce the correct boundary behavior, the volume must be multiplied by a non-universal length scale

One might seek an alternative proposal, which still captures something about the behind the horizon geometry, but which does not have these features. In fact, such an alternative has been proposed by Susskind et al., and it goes by 'complexity = action.'

- According to complexity = action, the complexity is dual to the action of a so-called 'Wheeler-DeWitt' (WDW) patch.
- Because this is an action, it can be nondimensionalized with some multiple of $\hbar$.
- A universal choice for this coefficient is consistent with the expected large temperature behavior.
- The action of a WDW patch also behaves in the appropriate way in the presence of shockwaves, so it still reproduces the switchback effect.

The WDW patch

- The WDW patch is defined by a spatial slice of the boundary.

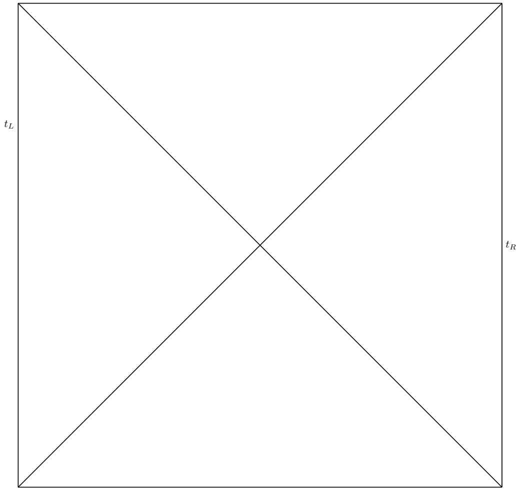

Figure : Boundary times t_L and t_R

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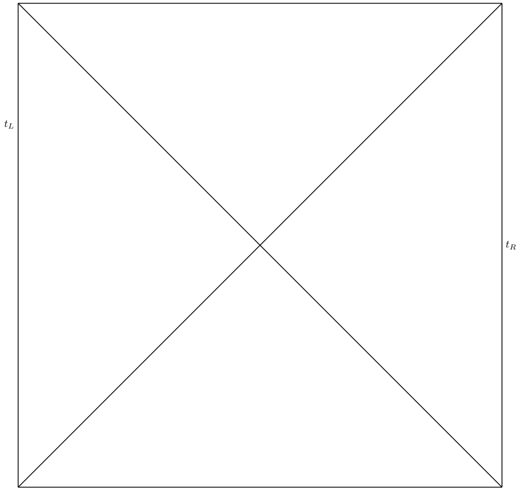

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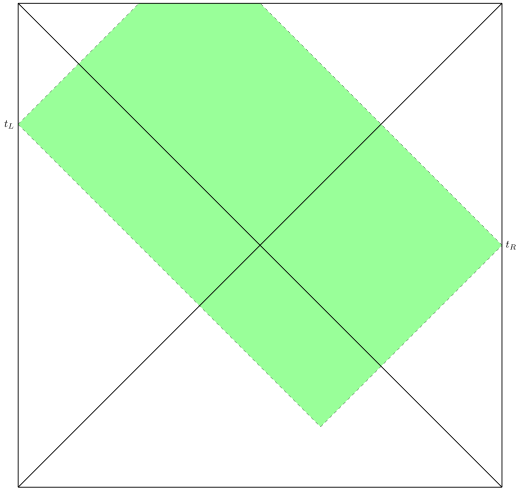

Figure : The WDW patch defined by boundry times t_L and t_R

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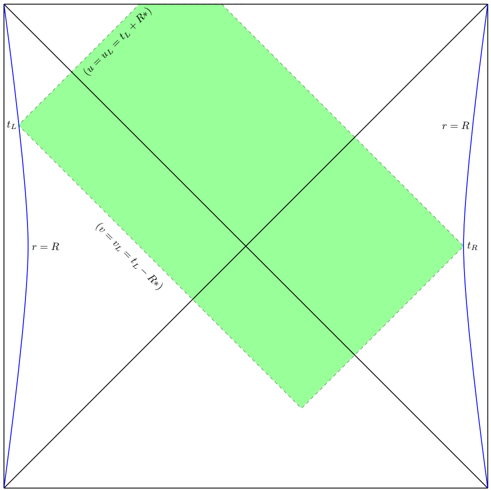

Figure : The WDW patch defined by boundry times t_L and t_R and a cutoff R

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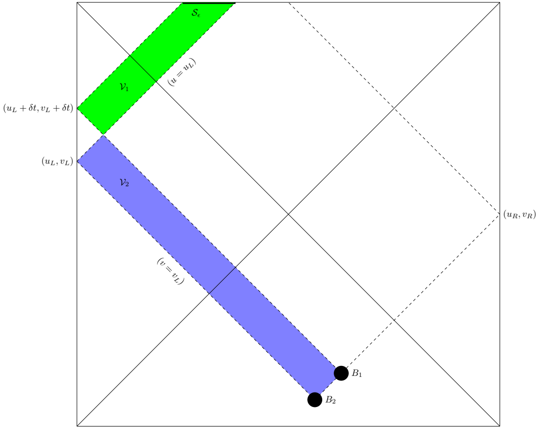

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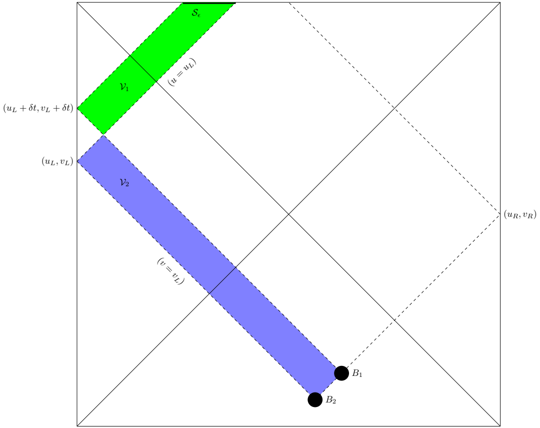
- This may be computed by subtracting the actions of two WDW patches, whose left time is separated by δt, and then taking the limit where $\delta t \to 0$.



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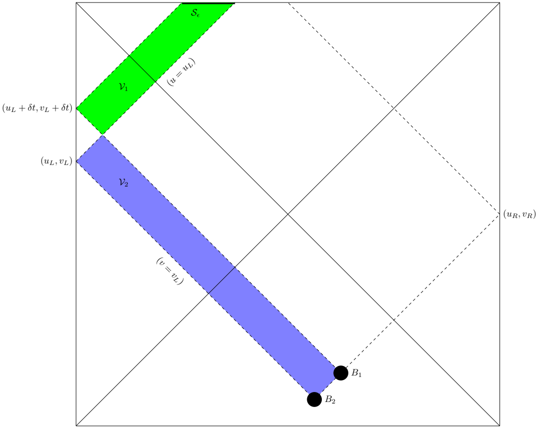
- This difference of actions decomposes to the difference of two bulk pieces, a piece from the spacelike boundary of a near singularity cutoff, and two codimension two contributions from the past corners of the patches.



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Why study NCSYM?

- We would like to test complexity = action in a new context.
- Tests from strongly coupled field theory are hard, so we would like field theory with a known gravity dual
- A gravity solution dual to NCSYM was derived in the late 90's by a number of authors [7, 8].
- And this solution has been well studied in other contexts (see e.g. Cai et al. [9], Edlati et al. [10], Fischler et al. [11], Karczmarek et al. [12]).
- Also, generalizations of this system to other numbers of dimensions have also been considered in, e.g., Alishahiha et al. [13] and Berman et al. [14].
- These theories come with a tunable parameter in the Moyal scale, so that the behavior of complexity with this parameter may be studied

The gravity dual to NCSYM

The gravity dual to NCSYM is obtained in a manner similar to that of standard $\mathcal{N} = 4$ SYM.

- Consider a stack of D3-branes
- We turn on a 2-form B parallel to the two spatial dimensions of the branes
- This can be achieved by applying a combination of T-dualities and Gauge transformations
- The B-field has the effect of introducing non-commutativity in the worldbrane theory
- The near horizon geometry in the usual limit gives the gravity dual, which is a IIB SUGRA solution.

The resulting solution, at finite teperature, is given in Einstein frame as follows:

$$ds^2 = \alpha' \left[\left(\frac{r}{L}\right)^2 \left(-f(r)dt^2 + dx_1^2 + h(r)(dx_2^2 + dx_3^2)\right) + \left(\frac{L}{r}\right)^2 \left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right) \right], \tag{1}$$

$$e^{2\Phi} = \hat{g}_s^2 h(r), \quad B_{23} = B_\infty(1 - h(r)), \quad C_{01} = -\frac{\alpha' a^2 r^4}{\hat{g}_s R^2}, \quad F_{0123r} = \frac{4\alpha'^2 r^3}{\hat{g}_s R^4} h(r) \tag{2}$$

$$f(r) = 1 - \left(\frac{r_+}{r}\right)^4, \quad h(r) = \frac{1}{1 + a^4 r^4}, \quad B_\infty = -\frac{\alpha'}{a^2 L^2}. \tag{3}$$

Here L is the AdS length scale, r_+ is the bulk coordinate of the horizon, $\hat{g}_s$ is the and closed string coupling, and a is the Moyal scale (i.e. $[x_2, x_3] = ia^2$ on the boundary).

The gravity dual to NCSYM

A few additional notes about the gravity dual:

- The bulk coordinate r has units of inverse length.
- Though the boundary field theory lives on a non-commutative manifold, the bulk geometry is commutative.
- The metric degenerates at the boundary. This should be fine, however, provided we always work with a finite cutoff.
- The dimension of the Hilbert space in the dual theory was found to be independent of the Moyal scale by Maldacena and Russo in [8].

We will now consider an intuition based heuristic argument that we should expect the complexity at a given (late) time should be higher in NCSYM than in it's commutative counterpart.

Non-Commutativity and Complexity: A heuristic argument

- Consider the unitary operator U which translates our state in time by a small time δt.

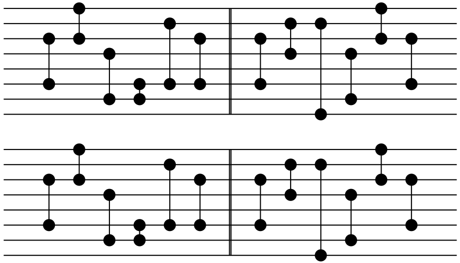

Figure : A small piece of a circuit implementing time translations on a local (top) and non-local (bottom) theory. This cartoon is inspired by another which appears in certain talks by Adam Brown.

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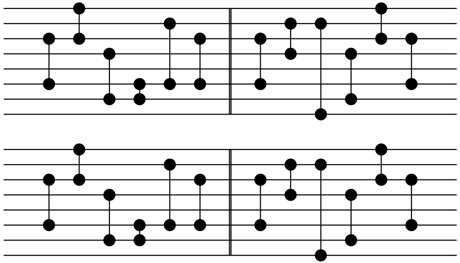

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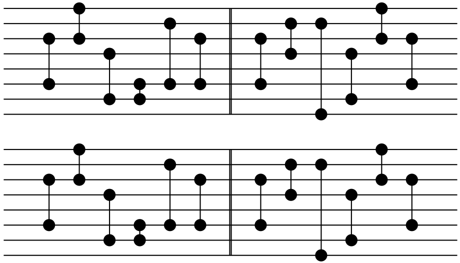

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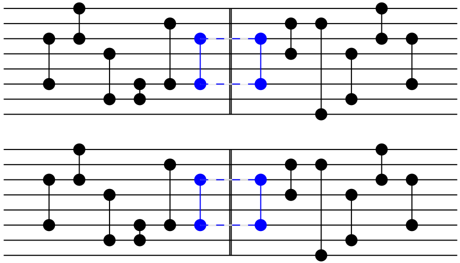

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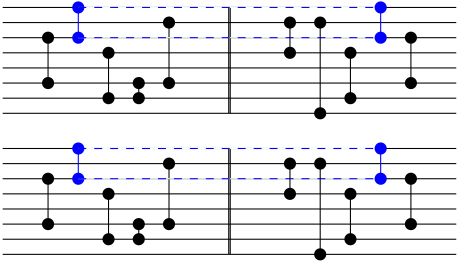

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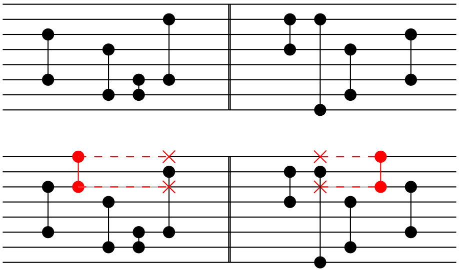
- These cancelations lead to a circuit for the same operator of lower complexity
- In a non-local theory (such as a non-commutative theory), fewer operators commute past one another, and so there will be more obstruction to such cancelations, leading to a higher final complexity.



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Results

Now that we have this naïve expectation, we will compare with the complexity in NCSYM according to CA. We will find it convenient to define the parameters $b = ar_+ = \pi a T$, $\rho = \frac{r_B}{r_+}$, and $\gamma = \frac{c\bar{c}\sqrt{\hat{g}_s}L^2}{\alpha' \pi^2 T^2}$.

- $T = \pi r_+$ is the temperature
- c and $\bar{c}$ are the normalizations of the null generators

With these conventions, the complexification rate after the critical time is given by

$$\dot{C} = \frac{\Omega_5 V_3 r_+^4}{(2\pi)^7 \hat{g}_s^2} \left(\frac{-2\log(1 + b^4\rho^4)}{b^4} + 4\rho^4 + 6 + 3(1-\rho^4)\log\left|\frac{\gamma\rho^2}{(1+b^4\rho^4)^{1/4}(1-\rho^4)}\right| \right). \tag{4}$$

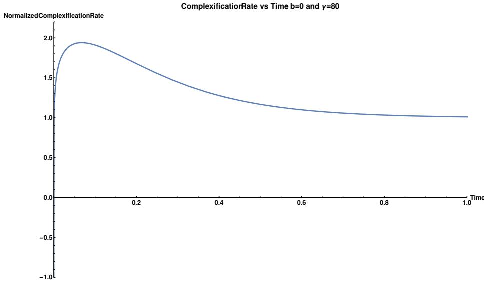
Here V_3 is the volume of a spatial slice of the boundary theory (which is infinite) and Ω_3 is the volume of a unit 5-sphere. In what follows, we will normalize the rates by the $b = 0$ result.

D3-brane results: Finite time behavior

In the commutative limit, we see similar behavior to that reported by Carmi et al. [15].

- Logarithmic divergence at the critical time.
- Reaches true global maximum in order one time (in thermal units).
- approaches asymptotic value from above.

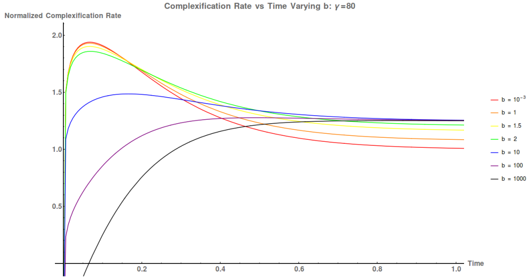
This behavior persists for all values of the Moyal scale but as the Moyal scale becomes very large, the true maximum shrinks and the asymptotic value increases, so that the gap between them decreases

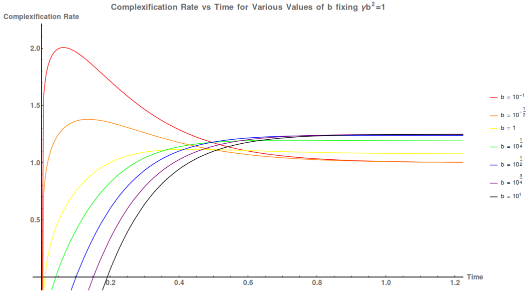


This behavior calls into question the normalization to complexity = action as set by Brown et al. [5, 6], though the logic that to that normalization would seem already to be contradicted by Cottrell and Montero [16].

Finite time behavior

Here we see how the finite time behavior changes as we vary the parameters.





D3-brane results: Late time limit

We take the late time limit by sending r_b to r_+ (i.e. we send ρ to 1). In this limit, the normalized complexification rate becomes

$$\dot{C}_{\text{normalized}}\Big|_{t\to\infty} = \frac{5}{4} - \frac{\log(1+b^4)}{4b^4}. \tag{5}$$

Recalling that $b = \pi a T$, we see that at a fixed temperature, as we send the Moyal scale to infinity, we get $5/4$, which given the normalization scheme tells us that we get exactly a 25% enhancement in of the complexification rate in the large Moyal scale regime.

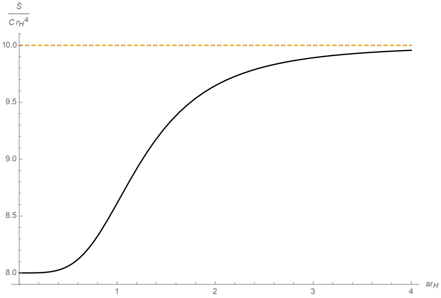


Figure : Late time action growth rate normalized by $C = \frac{\alpha^4 \Omega_5 V_3}{\hat{g}_s^2}$ and extra r_H dependence, versus ar_+, which is the Moyal scale measured in units of thermal length.

Results for other values of p

As discussed before, the geometry we have considered so far can be generalized by starting with stacks of $Dp - branes$ with for other values of p. For $p \neq 3$, the resulting geometry is not asymptotically AdS, even in the commutative limit. We considered $2 \leq p \leq 5$. For $p \geq 4$ there is the possibility of introducing non-commutativity between multiple pairs of coordinates. The table below summarizes our results for the late time rate of change of the complexity density, with a common normalization for all results. Here m indicates the number of pairs of non-commuting coordinates on the boundary.

p	$m = 0$	$m = 1$	$m = 2$
2	12	12	-
3	8	10	-
4	5	5	8
5	4	5	6

Conclusions

- For $p = 3, 5$ we do see an increase at late time, as expected.
- Though we did not see an increase for $p = 2$ or for $p = 4$ with a single non-trivial commutator, at least we did not see a decrease either.
- Overall, the results are consistent with the heuristic argument above.
- This result is in tension with the idea that commutative black holes are the fastest possible computers
- In future work, we plan to repeat our calculations for complexity $=$ volume.

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