Purification Complexity of Gaussian States

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Introduction

- We are interested in studying the *purification complexity* of mixed states of free scalar field theories in 1+1 dimensions.
- In particular, we will be interested in thermal states, and in the states which arise as the reduced state on a small interval.
- We will approach this by studying mixed states of a small number of harmonic oscillators.
- We will follow previous work by Jefferson and Myers (2017) and by Chapman et al. (2017).
- We are motivated by the *holographic complexity* conjectures of Susskind and collaborators.
- These conjectures state that in the *AdS/CFT correspondence*, either the volume of a maximal spatial slice or the action of a Wheeler-DeWitt patch in bulk is dual to the circuit complexity of the corresponding CFT state.
- The ultimate goal is to compare to results in holographic complexity as a test of those conjectures.

The AdS/CFT correspondence

- 'bulk' gravity theory in asymptotically $d+1$ dimensional AdS spacetime $\leftrightarrow$ 'boundary' d dimensional conformal field theory
- boundary strong coupling $\leftrightarrow$ bulk weak coupling
- large N boundary $\leftrightarrow$ classical bulk
- boundary entanglement entropy $\leftrightarrow$ minimal bulk surface area (RT)
- boundary subregion $\leftrightarrow$ entanglement wedge
- boundary thermal state $\leftrightarrow$ black hole (above Hawking-Page transition)
- boundary therofield double state $\leftrightarrow$ two-sided eternal black hole

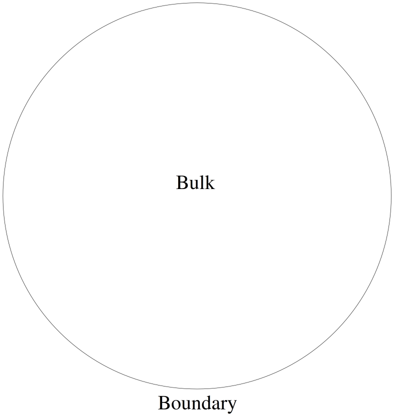

Figure : The AdS/CFT Correspondence

Holographic Complexity

- Black holes are dual to thermal states.
- Thermal state are in equilibrium, so observables don't generally evolve.
- Yet, volume behind the horizon keeps growing. What could it be dual to?
- Susskind suggested *quantum circuit complexity*
- Complexity = Volume: the volume of a maximal spatial slice is dual to complexity.
- Complexity = Action: The action on the Wheeler-DeWitt patch is dual to complexity
- What evidence is there for this? Can we test it?
- Check field theory!

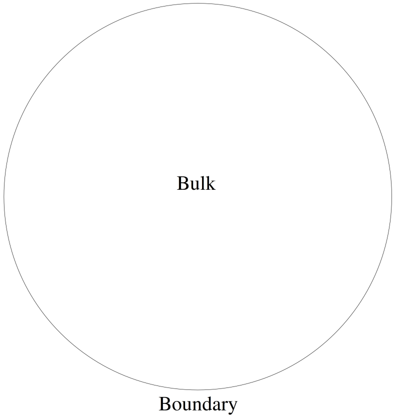

Figure : The AdS/CFT Correspondence

Circuit Complexity

What is quantum circuit complexity?

- Consider a Hilbert space $\mathcal{H}$, e.g., the Hilbert space for N quantum bits.
- A universal gate set $\{g_i\}$ for $\mathcal{H}$ is a set of unitary operators on the Hilbert space such that any unitary U acting on $\mathcal{H}$ can be approximated by some product $\prod_i g_{\alpha_i}$ to within a small tolerance ϵ.
- Such a product of gates is referred to as a quantum circuit.
- The quantum circuit complexity of a unitary U is then the minimum number of gates needed to approximate U to within the tolerance.
- In the example of qubits, one typically considers gates that act on a single qubit or pairs of qubits at a time.
- Given some reference state $|\psi_R\rangle$, one may define the complexity of a state $|\psi\rangle$ as the minimum of complexity $C(U)$ over all unitaries U such that $|\psi\rangle = U\,|\psi_R\rangle$.

Complexity in field theory?

- Would like to compute the circuit complexity in the a CFT, and look for agreement.
- Complexity in quantum field theories is not well understood, so just try to understand this.
- Jefferson and Myers (2017) and Chapmap et al. (2017): Start with free scalar field theory in 1+1.
- Actually, start with just lattice of harmonic oscillators.
- consider Gaussian reference state, $|R\rangle \propto e^{-\frac{1}{2}\omega_0|\vec{x}|^2}$, and gates which only take Gaussian states to other Guassian states.
- Reduce problem to finding geodesics by going to 'complexity geometry'
- Jefferson and Myers found that for a state with normal modes ω_i,

$$C = \sum_{i=1}^{N} \log\left|\frac{\omega_i}{\omega_0}\right|$$

Subregion Complexity and Purification Complexity

Subregion Complexity:

- Apply holographic complexity inside of the entanglement wedge (EW)
- Since EW is dual to subregion, perhaps this 'subregion complexity' is dual to complexity of reduced state?
- But what does complexity mean for a mixed state?
- One definition (among many possible), suggested as promising by Agón et al. (2018) is *purification complexity*

Purification Complexity:

- Given a mixed state ρ, and the set $\mathcal{P}$ of all purifications of ρ, the purification complexity of ρ is

$$C^P(\rho) = \min_{|\psi\rangle \in \mathcal{P}} C(|\psi\rangle)$$

- Actually, we should restrict to only consider purifications $|psi\rangle$ such that all auxiliary systems are entangled with original system.

Purification complexity in Field Theory

Can we compute purification complexity in FT?

- Well, we can do small numbers of harmonic oscillators. Start with one.
- Consider arbitrary Gaussian mixed state

$$\rho(x, x') := \langle x| \rho |x'\rangle \propto e^{-\frac{1}{2}[a(x^2+x'^2)-2bxx']}$$

- An arbitrary purification to two oscillator state looks like

$$\psi(x) \propto e^{-\frac{1}{2}(\omega_1 x_1^2+\omega_2 x_2^2-2\beta x_1 x_2)}$$

- To be a purification of the mixed state above, we must require

$$\omega_1 = a - b;\ \beta = \sqrt{b\omega_2}$$

- ω_2 may be freely chosen, we will vary it to minimize the complexity of this purification.
- Normal modes:

$$\omega_\pm = \frac{1}{2}\left[a - b + \omega_2 \pm \sqrt{(\omega_2 + b - a)^2 - 4b\omega_2}\right]$$

Purification Complexity in FT (continued)

- We can now minimize $\log\left|\frac{\omega_+}{\omega_0}\right| + \log\left|\frac{\omega_-}{\omega_0}\right|$ over ω_2.
- But is this enough? Do we need to consider all purifications to 3 particle states? 4 particles?
- Numerical studies seem to indicate that we get no smaller complexity from 3-particle purifications. Hopefully this result extrapolates to N particles.
- Can we do this for a whole lattice?
  - Can write down arbitrary purification, find normal modes, and try minimization.
  - But we are minimizing over a high dimensional space, so computationally hard
  - Can 'cheat' by distilling entangled d.o.fs and purifying them pairwise.
  - But in general, the cheat does not yield the global minimum. It is still an upper bound though.
- Ultimately, we aim to compute the complexity of a (regulated) field theory subregion, and compare the result to holographic subregion complexity.
  - Consider lattice of harmonic oscillators in ground state, and trace out all sites not in a given interval.
  - Study dependence on cutoff (lattice spacing)
  - Compare to subregion complexity of an interval of the boudary of AdS_3.
  - This is work in progress.

Introduction

Around 2014, Leonard Susskind and collaborators proposed a new entry in the AdS/CFT dictionary, namely that the volume of a maximal spatial slice of of asymptotically AdS spacetime is dual quantum circuit complexity in the dual CFT.

The AdS/CFT correspondence:

- Duallity between quantum gravity in $d+1$ dimensions and a conformal field theory in d dimensions.
- The gravity theory lives on an Asymptotically *AdS* (constant negative curvature) spacetime.
- Relates weakly coupled gravity to strongly coupled CFT, and classical limit of gravity to limit of CFT with infinite d.o.fs.
- Quantities/observables from the CFT are related to those in the gravity theory and vice versa through the *'dictionary'*.

Quantum Circuit Complexity:

- Consider a Hilbert space $\mathcal{H}$, e.g., the Hilbert space for N quantum bits, and a (computationally) universal set of unitary *gates* $\{g_i\}$ on $\mathcal{H}$
- Consider also a reference state $|R\rangle$
- A product of gates $Q = \prod_i g_i$ is a quantum circuit, whose complexity is the number of gates in the product.
- Then the circuit complexity of a state $|\psi\rangle$ is the minimum complexity over all circuits Q such that $|\psi\rangle = Q\,|R\rangle$

Holographic Complexity

So why would circuit complexity have anything to do with volume, anyway? Susskind was motivated by certain facts about the behind the horizon geometry of AdS black holes:

- (Large) black holes are dual to thermal states on the boundry. Two sided black holes are dual to a purification of the thermal state, the thermofield double state: $|\mathrm{TFD}\rangle = \sum_n e^{-\beta E_n/2} |n\rangle \otimes |n\rangle$
- The volume behind the black hole horizon increases in time, yet ordinary field theory obervables are not normally time dependent in a thermal state.
- However, we expect the *circuit complexity* of the TFD state to increase under time evolution, like the volume
- The volume also also reproduces the so-called *switchback effect*, an effect complexity is expected to exhibit.
- With a proper choice of normalization constant, the 'holographic complexity' computed by volume grows like TS at late time. This matches the behavior expected of complexity based on circuit arguments.

Susskind later updated his conjecture to 'complexity = action', in which complexity is instead thought to be dual to the causal development of such a maximal slice, termed the Wheeler-DeWitt (WDW) patch.

Complexity in field theory?

While we have some indication holographic complexity *might* be correct, stronger evidence is needed before it is accepted as a valid entry in the AdS/CFT dictionary.

- Ideally we would compute the circuit complexity in the a CFT, and look for agreement.
- However, complexity in quantum field theories (as opposed to systems of qubits) is not well understood, so as a first step, we should understand circuit complexity in field theories.
- In the past two years or so, there has been progress towards this, by e.g. Hashimoto et al. (2017) in for lattice gauge theory, and by Jefferson et al. (2017) and Chapmap et al. (2017) respectivley for lattice regularized scalar FT.
- In particular, Jefferson et al. considered Guassian states of harmonic oscillators, and a gate set which, though not universal, is at least universal on Gaussian states, and a reference state

$$|R\rangle \propto e^{-\frac{1}{2}\omega_0|\vec{x}|^2}$$

- They found that the complexity of a Gaussian state on N oscillators, with normal mode frequencies ω_i, is given by

$$C = \sum_{i=1}^{N} \log\left|\frac{\omega_i}{\omega_0}\right|$$

Subregion Complexity and Purification Complexity

Subregion Complexity:

- It is widely believed that the reduced state on a subregion of the CFT is dual to the bulk *entanglement wedge* associated to that region. This is called subregion duality.
- We may apply the holographic complexity conjectures to the entanglement wedge just as easily as to the whole geometry.
- The resulting quantity is termed *subregion complexity*. It is conjectured that it is dual to some notion of complexity on the reduced state.
- However, there is not a unique way to extend the definition of complexity to mixed states.

Purification Complexity:

- Recently, Agón et al. studied different definitions of mixed state complexity, and compared expectations about them to holographic subregion complexity.
- They suggested that the closest match to holographic complexity (in its complexity = action form, anyway) was the *purification complexity*.
- The purification complexity of a state ρ is defined as the minimum complexity $C(|\psi\rangle)$ over all purifcations $|\psi\rangle$ of ρ, which don't have any seperable factor which is also a purification of ρ.

Purification complexity in field theory

Can we compute purification complexity in FT?

- Well, we can do small numbers of harmonic oscillators.
- Hard to minimize over all purifcations $\rightarrow$ try just those of on to copies of original Hilbert space. Is this enough?
- Parameterize all purifications to the Guassian state, and numerically minimize the complexity as found by Jefferson et al.

$$C = \sum_i \log \left| \frac{\omega_i}{\omega_0} \right|$$

- Consider mixed states that come by tracing out all but a few sites on a lattice, as well as thermal states
- Compare result to holographic computation.

Results

- It seems one doesn't need extra factors of the Hilbert space, e.g. purifying one oscillator onto three doesn't seem to improve on purifying to two (No proof though).