Purification Complexity of Gaussian States

arxiv:181x.xxxxx, work in progress with Elena Cáceres, Shira Chapman, Juan Pablo Hernandez, Rob Myers, and Shan-Ming Ruan

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Introduction

- We are interested in studying the *purification complexity* of mixed states of free scalar field theories in 1+1 dimensions.
- In particular, we will be interested in thermal states, and in the states which arise as the reduced state on a small interval.
- We will approach this by studying mixed states of a small number of harmonic oscillators.
- We will follow previous work by Jefferson and Myers (2017) and by Chapman et al. (2017).
- We are motivated by the *holographic complexity* conjectures of Susskind and collaborators.
- These conjectures state that in the *AdS/CFT correspondence*, either the volume of a maximal spatial slice or the action of a Wheeler-DeWitt patch in bulk is dual to the circuit complexity of the corresponding CFT state.
- The ultimate goal is to compare to results in holographic complexity as a test of those conjectures.

The AdS/CFT correspondence

- 'bulk' gravity theory in asymptotically $d+1$ dimensional AdS spacetime $\leftrightarrow$ 'boundary' d dimensional conformal field theory

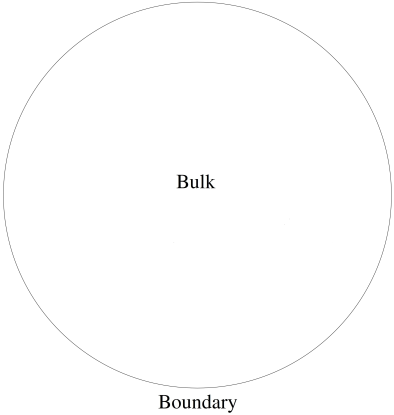

Figure : The AdS/CFT Correspondence

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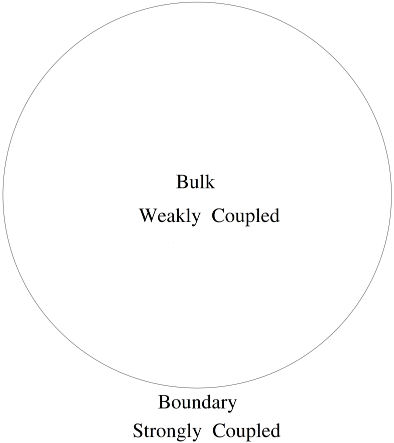

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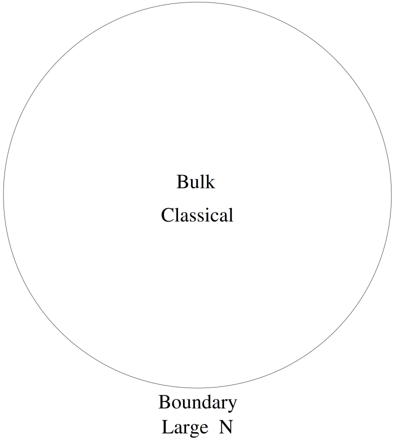

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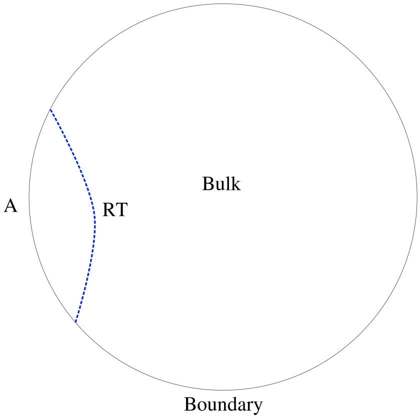
- boundary entanglement entropy $\leftrightarrow$ minimal bulk surface area (RT)



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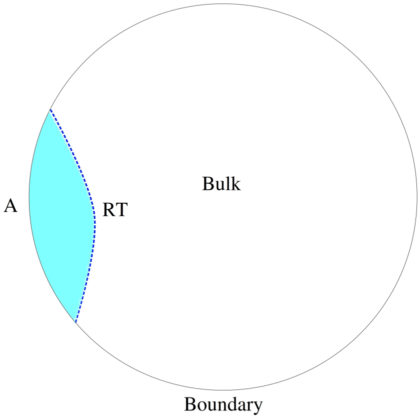
- boundary subregion $\leftrightarrow$ entanglement wedge



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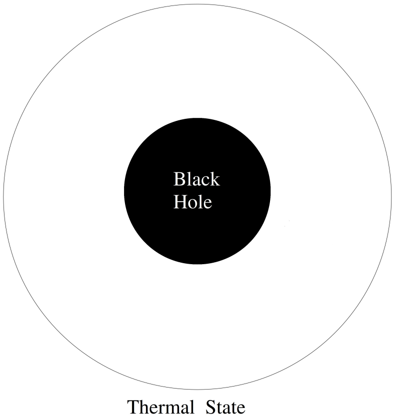
- boundary thermal state $\leftrightarrow$ black hole (above Hawking-Page transition)



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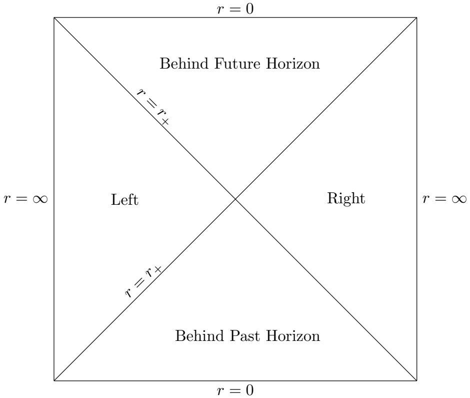
- boundary subregion $\leftrightarrow$ entanglement wedge
- boundary thermal state $\leftrightarrow$ black hole (above Hawking-Page transition)
- boundary therofield double state $\leftrightarrow$ two-sided eternal black hole

$$|TFD\rangle = \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R$$



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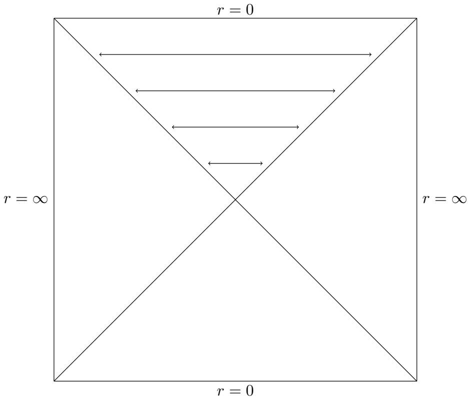
Holographic Complexity

- Black holes are dual to thermal states.
- Thermal state are in equilibrium, so observables don't generally evolve.
- Yet, volume behind the horizon keeps growing. What could it be dual to?
- Susskind suggested *quantum circuit complexity*
- Complexity = Volume: the volume of a maximal spatial slice is dual to complexity.
- Complexity = Action: The action on the Wheeler-DeWitt patch is dual to complexity
- What evidence is there for this? Can we test it?
- Check field theory!



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Circuit Complexity

What is quantum circuit complexity?

- Consider a Hilbert space $\mathcal{H}$, e.g., the Hilbert space for N quantum bits.
- A universal gate set $\{g_i\}$ for $\mathcal{H}$ is a set of unitary operators on the Hilbert space such that any unitary U acting on $\mathcal{H}$ can be approximated by some product $\prod_i g_{\alpha_i}$ to within a small tolerance ϵ.
- Such a product of gates is referred to as a quantum circuit.
- The quantum circuit complexity of a unitary U is then the minimum number of gates needed to approximate U to within the tolerance.
- In the example of qubits, one typically considers gates that act on a single qubit or pairs of qubits at a time.
- Given some reference state $|\psi_R\rangle$, one may define the complexity of a state $|\psi\rangle$ as the minimum of complexity $C(U)$ over all unitaries U such that $|\psi\rangle = U\,|\psi_R\rangle$.

Complexity in field theory?

- Would like to compute the circuit complexity in the a CFT, and look for agreement.
- Complexity in quantum field theories is not well understood, so just try to understand this.
- Jefferson and Myers (2017) and Chapmap et al. (2017): Start with free scalar field theory in 1+1.
- Actually, start with just lattice of harmonic oscillators.
- consider Gaussian reference state, $|R\rangle \propto e^{-\frac{1}{2}\omega_0|\vec{x}|^2}$, and gates which only take Gaussian states to other Guassian states.
- Reduce problem to finding geodesics by going to 'complexity geometry'
- Jefferson and Myers found that for a state with normal modes ω_i,

$$C = \sum_{i=1}^{N} \log \left| \frac{\omega_i}{\omega_0} \right|$$

Subregion Complexity and Purification Complexity

Subregion Complexity:

- Apply holographic complexity inside of the entanglement wedge (EW)
- Since EW is dual to subregion, perhaps this 'subregion complexity' is dual to complexity of reduced state?
- But what does complexity mean for a mixed state?
- One definition (among many possible), suggested as promising by Agón et al. (2018) is *purification complexity*

Purification Complexity:

- Given a mixed state ρ, and the set $\mathcal{P}$ of all purifications of ρ, the purification complexity of ρ is

$$C^P(\rho) = \min_{|\psi\rangle \in \mathcal{P}} C(|\psi\rangle)$$

- Actually, we should restrict to only consider purifications $|psi\rangle$ such that all auxiliary systems are entangled with original system.

Purification complexity in Field Theory

Can we compute purification complexity in FT?

- Well, we can do small numbers of harmonic oscillators. Start with one.
- Consider arbitrary Gaussian mixed state

$$\rho(x, x') := \langle x| \, \rho \, |x'\rangle \propto e^{-\frac{1}{2}[a(x^2+x'^2)-2bxx']}$$

- An arbitrary purification to two oscillator state looks like

$$\psi(x) \propto e^{-\frac{1}{2}(\omega_1 x_1^2+\omega_2 x_2^2-2\beta x_1 x_2)}$$

- To be a purification of the mixed state above, we must require

$$\omega_1 = a - b; \; \beta = \sqrt{b\omega_2}$$

- ω_2 may be freely chosen, we will vary it to minimize the complexity of this purification.
- Normal modes:

$$\omega_\pm = \frac{1}{2}\left[a - b + \omega_2 \pm \sqrt{(\omega_2 + b - a)^2 - 4b\omega_2}\right]$$

Purification Complexity in FT (continued)

- We can now minimize $\log\left|\frac{\omega_+}{\omega_0}\right| + \log\left|\frac{\omega_-}{\omega_0}\right|$ over ω_2.
- But is this enough? Do we need to consider all purifications to 3 particle states? 4 particles?
- Numerical studies seem to indicate that we get no smaller complexity from 3-particle purifications. Hopefully this result extrapolates to N particles.
- Can we do this for a whole lattice?
  - Can write down arbitrary purification, find normal modes, and try minimization.
  - But we are minimizing over a high dimensional space, so computationally hard
  - Can 'cheat' by distilling entangled d.o.fs and purifying them pairwise.
  - But in general, the cheat does not yield the global minimum. It is still an upper bound though.
- Ultimately, we aim to compute the complexity of a (regulated) field theory subregion, and compare the result to holographic subregion complexity.
  - Consider lattice of harmonic oscillators in ground state, and trace out all sites not in a given interval.
  - Study dependence on cutoff (lattice spacing)
  - Compare to subregion complexity of an interval of the boudary of AdS_3.
  - This is work in progress.