Recursive Rows in Rome

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1 IX: THE INDEX CALCULUS

1.1 Syntax

Let $0, 1, 2, \ldots$ denote object-level natural numbers in the intuitive fashion and let i_n be the finite natural obtained by n applications of FSuc to FZero.

$$
\begin{array}{llcl}
\text{Sorts} & \sigma & ::= & \star \mid \mathcal{U} \\
\text{Terms} & A, B, M, N, T & ::= & \star \mid x \mid \\
 & & & \mathrm{Nat} \mid \mathrm{Zero} \mid \mathrm{Suc}\, M \mid \\
 & & & \mathrm{case}_{\mathbb{N}}\, M \,\mathrm{of}\, \{\mathrm{Zero} \mapsto N_1; \mathrm{Suc}\, x \mapsto N_2\} \mid \\
 & & & \mathrm{Ix}\, M \mid \mathrm{FZero} \mid \mathrm{FSuc}\, M \mid \\
 & & & \mathrm{case}_{\mathrm{Fin}}\, M \,\mathrm{of}\, \{\mathrm{FZero} \mapsto N_1; \mathrm{FSuc}\, x \mapsto N_2\} \mid \\
 & & & \{\!\!\{ M_1, \ldots, M_n \}\!\!\} \mid \\
 & & & \top \mid \mathtt{tt} \mid \\
 & & & \forall \alpha : T.N \mid \lambda x : T.N \mid M\, N \mid \\
 & & & \exists \alpha : T.M \mid (\alpha : T, M) \mid \\
 & & & M + N \mid \mathrm{left}\, M \mid \mathrm{right}\, M \mid \\
 & & & \mathrm{case}_{+}\, M \,\mathrm{of}\, \{\mathrm{left}\, x \mapsto N_1; \mathrm{right}\, y \mapsto N_2\} \mid \\
 & & & M \equiv N \mid \mathrm{refl}\, T\, M\, N \mid \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : T
\end{array}
$$

Fig. 1. Syntax

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1.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(\textsc{emp})\,\frac{}{\vdash \varepsilon} \qquad (\textsc{var})\,\frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\star)\,\frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad (\top)\,\frac{}{\Gamma \vdash \top : \sigma} \qquad (\textsc{Nat})\,\frac{}{\Gamma \vdash \mathrm{Nat} : \star} \qquad (\mathrm{Ix})\,\frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\, n : \star}$$

$$(\forall)\,\frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \forall \alpha : M.N : \sigma_2} \qquad (\exists)\,\frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \exists \alpha : M.N : \sigma_2}$$

$$(+)\,\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} \qquad (\equiv)\,\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \equiv N : \sigma}$$

Fig. 2. Context and type formation rules

$$\boxed{\Gamma \vdash M : N}$$

$$(\textsc{Var})\,\frac{x : M \in \Gamma}{\Gamma \vdash x : M} \qquad (\mathtt{tt})\,\frac{}{\Gamma \vdash \mathtt{tt} : \top}$$

$$(\mathrm{Zero})\,\frac{}{\Gamma \vdash \mathrm{Zero} : \mathrm{Nat}} \qquad (\mathrm{Suc})\,\frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Suc}\, n : \mathrm{Nat}}$$

$$(\mathrm{FZero})\,\frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{FZero} : \mathrm{Ix}\,(\mathrm{Suc}\, n)} \qquad (\mathrm{FSuc})\,\frac{\Gamma \vdash n : \mathrm{Nat} \quad \Gamma \vdash i : \mathrm{Ix}\, n}{\Gamma \vdash \mathrm{FSuc}\, i : \mathrm{Ix}\,(\mathrm{Suc}\, n)}$$

$$(\forall I)\,\frac{\Gamma \vdash T : \star \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \forall (x : T).N} \qquad (\forall E)\,\frac{\Gamma \vdash M : \forall (x : T_1).T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\, N : T_2[N/x]}$$

$$(\exists I)\,\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \exists (x : T_1).T_2} \qquad (\exists E_1)\,\frac{\Gamma \vdash M : \Sigma (x : T_1).T_2}{\Gamma \vdash \mathrm{fst}\, M : T_1} \qquad (\exists E_2)\,\frac{\Gamma \vdash M : \Sigma (x : T_1).T_2}{\Gamma \vdash \mathrm{snd}\, M : T_1[\mathrm{fst}\, M/x]}$$

$$(\equiv I)\,\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathrm{refl} : M \equiv M} \qquad (\textsc{conv})\,\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma \quad \sigma\ \mathrm{Safe}}{\Gamma \vdash M : T_2}$$

$$(\equiv E)\,\frac{\begin{array}{c}\Gamma \vdash P : T_1 \equiv T_2 \\ \Gamma \vdash M : T_1 \\ \Gamma \vdash N : T_1 \\ \Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\ \Gamma, z : T_1 \vdash H : T[z/x, z/y, \mathrm{refl}/p]\end{array}}{\Gamma \vdash \mathcal{J}\, H\, M\, N\, P : T[M/x, N/y, P/p]}$$

Fig. 3. Typing rules

$$\boxed{\Gamma \vdash M = N : \sigma}$$

$$(\text{E-REFL})\ \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \qquad (\text{E-SYM})\ \frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \qquad (\text{E-TRANS})\ \frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma}$$

$$\boxed{\Gamma \vdash M = N : T}$$

$$(\text{C-REFL})\ \frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \qquad (\text{C-SYM})\ \frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \qquad (\text{C-TRANS})\ \frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}$$

Fig. 4. Definitional equality & computational laws

1.3 A Comparison to $\lambda^{\Pi\mathcal{U}\mathbb{N}}$ [Abel et al. 2018]

2 TRANSLATION FROM Rω

2.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of Rω types. Figure 5 describe the untyped translation, which is used to show translational soundness of the typed translation (Figure 5).

$$\boxed{(\kappa)^\bullet}$$

$$\begin{aligned}
(\star)^\bullet &= \star \\
(\mathsf{L})^\bullet &= \top \\
(\kappa_1 \longrightarrow \kappa_2)^\bullet &= \forall(\alpha : (\kappa_1)^\bullet).(\kappa_2)^\bullet \\
(\mathsf{R}^\kappa)^\bullet &= \exists(n : \mathrm{Nat}).\forall(j : \mathrm{Ix}\, n).(\kappa)^\bullet
\end{aligned}$$

$$\boxed{(\Gamma \vdash \tau : \kappa)^\bullet}$$

$$\begin{aligned}
(\alpha)^\bullet &= \alpha \\
(\tau_1 \longrightarrow \tau_2)^\bullet &= \forall(\alpha : (\tau_1)^\bullet).(\tau_2)^\bullet \\
(\forall \alpha : \kappa.\tau)^\bullet &= \forall(\alpha : (\kappa)^\bullet).(\tau)^\bullet \\
(\lambda \alpha : \kappa.\tau)^\bullet &= \forall(\alpha : (\kappa)^\bullet).(\tau)^\bullet \\
(\pi \Longrightarrow \tau)^\bullet &= \forall(\alpha : (\pi)^\bullet).(\tau)^\bullet \\
(\tau\, \upsilon)^\bullet &= (\tau)^\bullet\, (\upsilon)^\bullet \\
(\rho\, \lceil \upsilon \rceil)^\bullet &= (\mathrm{fst}\, (\rho)^\bullet : \mathrm{Nat}, \lambda(j : \mathrm{Ix}\, (\mathrm{fst}\, (\rho)^\bullet)).(\mathrm{snd}\, (\rho)^\bullet)(\upsilon)^\bullet) \\
(\lceil \tau \rceil\, \rho)^\bullet &= (\mathrm{fst}\, (\rho)^\bullet : \mathrm{Nat}, \lambda(j : \mathrm{Ix}\, (\mathrm{fst}\, (\rho)^\bullet)).(\tau)^\bullet\, (\mathrm{snd}\, (\rho)^\bullet)) \\
(\ell)^\bullet &= \top \\
(\lfloor \xi \rfloor)^\bullet &= \top \\
(\xi \triangleright \tau)^\bullet &= (1,\ \lambda(i : \mathrm{Ix}\, 1).(\tau)^\bullet) \\
(\Pi \rho)^\bullet &= \forall(i : \mathrm{Ix}\, (\mathrm{fst}\, (\rho)^\bullet)).(\mathrm{snd}\, (\rho)^\bullet)\ i \\
(\Sigma \rho)^\bullet &= \exists(i : \mathrm{Ix}\, (\mathrm{fst}\, (\rho)^\bullet)).(\mathrm{snd}\, (\rho)^\bullet)\ i
\end{aligned}$$

$$\boxed{(\Gamma \vdash \pi : \kappa)^\bullet}$$

$$\ldots$$

$$\boxed{(\Gamma \Vdash \pi)^\bullet}$$

$$\ldots$$

$$\boxed{(\Gamma \vdash M : \tau)^\bullet}$$

$$\ldots$$

Fig. 5. A compositional translation of Rω judgments to (untyped) Ix terms

2.2 Typed translation

$$\boxed{\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \vdash M \rightsquigarrow N : \tau}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \Vdash \pi \rightsquigarrow N}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\tau \equiv \upsilon \rightsquigarrow P}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Fig. 6. Translation of Rω derivations to Ix derivations

2.3 Properties of Translation

Presume an Rω instantiation of the simple row theory. A lot of this is likely bullshit.

THEOREM 1 (TRANSLATIONAL SOUNDNESS (TYPES)). *if* $\Gamma \vdash \tau : \kappa$ *such that* $\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa$ *then* $(\Gamma)^\bullet \vdash \upsilon : (\kappa)^\bullet$.

THEOREM 2 (TRANSLATIONAL SOUNDNESS (TYPE EQUIVALENCE)). *if*

(1) $\Gamma \vdash \tau_1 \rightsquigarrow \upsilon_1 : \kappa_1$;
(2) $\Gamma \vdash \tau_2 \rightsquigarrow \upsilon_2 : \kappa_2$; *and*
(3) $\tau_1 \equiv \tau_2 \rightsquigarrow P$,

then $(\Gamma)^\bullet \vdash P : \upsilon_1 \equiv \upsilon_2$.

THEOREM 3 (TRANSLATIONAL SOUNDNESS (OF PREDICATES)). *if* $\Gamma \Vdash \pi$ *such that* $\Gamma \Vdash \pi \rightsquigarrow N$ *then* $(\Gamma)^\bullet \vdash N : (\pi)^\bullet$.

Finally,

THEOREM 4 (TRANSLATIONAL SOUNDNESS). *if* $\Gamma \vdash M : \tau$ *such that* $\Gamma \vdash M \rightsquigarrow N : \tau$ *then* $(\Gamma)^\bullet \vdash N : (\tau)^\bullet$.

3 OPERATIONAL SEMANTICS

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