

# Recursive Rows in Rome

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## 1 IX: THE INDEX CALCULUS

### 1.1 Syntax

Sorts	$\sigma ::= \star \mid \mathcal{U}$
Terms	$M, N, T ::= \star \mid x \mid$ $\mathbb{N} \mid Z \mid S M \mid$ $\text{case}_{\mathbb{N}} M N T \mid$ $\text{Ix } M \mid \text{I}_0 \mid \text{I}_S M \mid$ $\text{case}_{\text{Ix}} M N \mid \text{case}_{\text{Ix}} M N T \mid$ $\{\!\{M_1, \dots, M_n\}\!\} \mid \lambda() \mid$ $\tau \mid \text{tt} \mid$ $\forall \alpha : T. N \mid \lambda x : T. N \mid M N \mid$ $\exists \alpha : T. M \mid \langle\langle \alpha : T, M \rangle\rangle \mid \text{case}_{\exists} M N \mid$ $M + N \mid \text{left } M \mid \text{right } M \mid$ $\text{case}_+ M N T \mid$ $M \equiv N \mid \text{refl } T M N \mid$
Environments	$\Gamma ::= \varepsilon \mid \Gamma, \alpha : T$

Fig. 1. Syntax

#### 1.1.1 Meta-syntax & syntactic sugar. Let

- (1)  $\tau \rightarrow v$  denote the non-dependent universal quantification  $\forall(\_ : \tau).v$ ;
- (2)  $\tau \times v$  denote the non-dependent existential quantification  $\exists(\_ : \tau).v$ ;
- (3)  $0, 1, 2, \dots$  denote object-level natural numbers in the intuitive fashion;
- (4)  $i_n$  denote the index obtained by  $n$  applications of  $\text{I}_S$  to  $\text{I}_0$ ; and

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(5) the syntax

$$\{\!\{M_1, \dots, M_n\}\!\}$$

denote the large elimination of a known, finite quantity of indices to types  $M_1, \dots, M_n$ , elaborated by the equations:

$$\begin{aligned}\{\!\{M_1\}\!\} &:= \lambda(i : \text{Ix } 1).\text{case}_{\text{Ix}} i M_1 \\ \{\!\{M_1, \dots, M_n\}\!\} &:= \lambda(i : \text{Ix } n).\text{case}_{\text{Ix}} i M_1 \{\!\{M_2, \dots, M_n\}\!\}\end{aligned}$$

## 1.2 Typing

Many of these are fucked or in need of repair; refer to the translation as the SSOT.

$$\begin{array}{c} \boxed{\vdash \Gamma} \\ \text{(EMP)} \frac{}{\vdash \varepsilon} \quad \text{(VAR)} \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M} \\ \boxed{\Gamma \vdash M : \sigma} \\ \begin{array}{llll} (\star) \frac{\vdash \Gamma}{\Gamma \vdash \star : \mathcal{U}} & (\top) \frac{\vdash \Gamma}{\Gamma \vdash \top : \sigma} & (\text{NAT}) \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{N} : \star} & (\text{IX}) \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \text{Ix } n : \star} \\ (\forall) \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \forall \alpha : M.N : \sigma_2} & (\exists) \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \exists \alpha : M.N : \sigma_2} & & \\ (+) \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} & (\equiv) \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \equiv N : \sigma} & & \end{array} \end{array}$$

Fig. 2. Context and type well-formedness

$$\boxed{\Gamma \vdash M : N}$$

$$\begin{array}{c}
(\text{VAR}) \frac{\vdash \Gamma \quad x : M \in \Gamma}{\Gamma \vdash x : M} \quad (\text{tt}) \frac{\vdash \Gamma}{\Gamma \vdash \text{tt} : \top} \\
(Z) \frac{\vdash \Gamma}{\Gamma \vdash Z : \mathbb{N}} \quad (S) \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash S n : \mathbb{N}} \quad (\mathbb{N}E) \frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \mathbb{N} \quad \Gamma \vdash N : A \quad \Gamma \vdash P : \mathbb{N} \rightarrow A}{\Gamma \vdash \text{case}_{\mathbb{N}} M N P : A} \\
(l_0) \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash l_0 : \text{Ix}(S n)} \quad (l_S) \frac{\Gamma \vdash n : \mathbb{N} \quad \Gamma \vdash i : \text{Ix } n}{\Gamma \vdash l_S i : \text{Ix}(S n)} \quad (\lambda()) \frac{\Gamma \vdash A : \sigma}{\Gamma \vdash \lambda() : \text{Ix } 0 \rightarrow A} \\
(l_S E) \frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \text{Ix}(S n) \quad \Gamma \vdash N : A \quad \Gamma \vdash P : \text{Ix } n \rightarrow A}{\Gamma \vdash \text{case}_{\text{Ix}} M N P : A} \\
(\forall I) \frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T. M : \forall(x : T). N} \quad (\forall E) \frac{\Gamma \vdash M : \forall(x : T_1). T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M N : T_2[N/x]} \\
(\exists I) \frac{\Gamma \vdash T_1 : \sigma \quad \Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash \langle\langle M : T_1, N \rangle\rangle : \exists(x : T_1). T_2} \\
(\exists E) \frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \exists(x : T_1). T_2 \quad \Gamma \vdash N : \forall(x : T_1). T_2 \rightarrow A}{\Gamma \vdash \text{case}_{\exists} M N : A} \\
(+_1 I) \frac{\Gamma \vdash M : A}{\Gamma \vdash \text{left } M : A + B} \quad (+_2 I) \frac{\Gamma \vdash N : B}{\Gamma \vdash \text{right } N : A + B} \\
(+E) \frac{\Gamma \vdash C : \sigma \quad \Gamma \vdash M : A + B \quad \Gamma \vdash N : A \rightarrow C \quad \Gamma \vdash P : B \rightarrow C}{\Gamma \vdash \text{case}_+ M N P : C} \\
(\equiv I) \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \text{refl} : M \equiv M} \quad (\text{CONV}) \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma}{\Gamma \vdash M : T_2} \\
\begin{array}{c}
\Gamma \vdash P : T_1 \equiv T_2 \\
\Gamma \vdash M : T_1 \\
\Gamma \vdash N : T_1 \\
\Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\
\Gamma, z : T_1 \vdash H : T[z/x, z/y, \text{refl}/p] \\
(\equiv E) \frac{}{\Gamma \vdash \mathcal{J} H M N P : T[M/x, N/y, P/p]}
\end{array}
\end{array}$$

Fig. 3. Typing lx terms

$$\boxed{\Gamma \vdash M = N : \sigma}$$

$$\begin{array}{c}
(\text{E-REFL}) \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \quad (\text{E-SYM}) \frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \quad (\text{E-TRANS}) \frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma} \\
\boxed{\Gamma \vdash M = N : T} \\
(\text{C-REFL}) \frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \quad (\text{C-SYM}) \frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \quad (\text{C-TRANS}) \frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}
\end{array}$$

Fig. 4. Definitional equality &amp; computational laws

### 1.3 Properties

**THEOREM 1 (WELL-SORTEDNESS).** *if  $\Gamma \vdash M : N$  then  $\vdash \Gamma$  and there exists  $\sigma$  such that  $\Gamma \vdash N : \sigma$ .*

## 2 TRANSLATION FROM $R\omega$

### 2.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of  $R\omega$  types. Figure 5 describe the untyped translation, which is used to show translational soundness of the typed translation (Figure 5).

$$\boxed{\llbracket \kappa \rrbracket}$$

$$\llbracket \star \rrbracket = \star$$

$$\llbracket \mathbf{L} \rrbracket = \top$$

$$\llbracket \kappa_1 \rightarrow \kappa_2 \rrbracket = \llbracket \kappa_1 \rrbracket \rightarrow \llbracket \kappa_2 \rrbracket$$

$$\llbracket \mathbf{R}^\kappa \rrbracket = \exists (n : \mathbb{N}). \text{Ix } n \rightarrow \llbracket \kappa \rrbracket$$

$$\boxed{\llbracket \Gamma \vdash \tau : \kappa \rrbracket}$$

$$\llbracket \Gamma \vdash \alpha : \kappa \rrbracket = \alpha$$

$$\llbracket \Gamma \vdash \tau_1 \rightarrow \tau_2 : \star \rrbracket = \llbracket \tau_1 \rrbracket \rightarrow \llbracket \tau_2 \rrbracket$$

$$\llbracket \Gamma \vdash \forall \alpha : \kappa. \tau : \star \rrbracket = \forall (\alpha : \llbracket \kappa \rrbracket). \llbracket \tau \rrbracket$$

$$\llbracket \Gamma \vdash \lambda \alpha : \kappa. \tau : \kappa \rightarrow \kappa' \rrbracket = \forall (\alpha : \llbracket \kappa \rrbracket). \llbracket \tau \rrbracket$$

$$\llbracket \Gamma \vdash \pi \Rightarrow \tau : \kappa \rrbracket = \llbracket \pi \rrbracket \rightarrow \llbracket \tau \rrbracket$$

$$\llbracket \Gamma \vdash \tau v : \kappa \rrbracket = \llbracket \tau \rrbracket \llbracket v \rrbracket$$

$$\llbracket \Gamma \vdash \ell : \mathbf{L} \rrbracket = \top$$

$$\llbracket \Gamma \vdash \lfloor \xi \rfloor : \star \rrbracket = \top$$

$$\llbracket \Gamma \vdash (\xi \triangleright \tau) : \kappa \rrbracket = \llbracket \tau \rrbracket$$

$$\llbracket \Gamma \vdash \Pi \rho : \star \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda P : \text{Ix } n \rightarrow \star. \forall (i : \text{Ix } n). P i)$$

$$\llbracket \Gamma \vdash \Sigma \rho : \star \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda P : \text{Ix } n \rightarrow \star. \exists (i : \text{Ix } n). P i)$$

$$\llbracket \Gamma \vdash \epsilon : \mathbf{R}^\kappa \rrbracket = \langle \langle 0 : \mathbb{N}, \lambda () \rangle \rangle$$

$$\llbracket \Gamma \vdash \rho [v] : \mathbf{R}^{\kappa_2} \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa_1 \rrbracket \rightarrow \llbracket \kappa_2 \rrbracket). \langle \langle n : \mathbb{N}, \lambda (j : \text{Ix } n). (P j) \llbracket \tau \rrbracket \rangle \rangle)$$

$$\llbracket \Gamma \vdash \lceil \tau \rceil \rho : \mathbf{R}^{\kappa_2} \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa_1 \rrbracket). \langle \langle n : \mathbb{N}, \lambda (j : \text{Ix } n). \llbracket \tau \rrbracket (P j) \rangle \rangle)$$

$$\llbracket \Gamma \vdash (\xi \triangleright_{\mathbf{R}} \tau) : \mathbf{R}^\kappa \rrbracket = \langle \langle 1 : \mathbb{N}, \llbracket \tau \rrbracket \rangle \rangle$$

$$\boxed{\llbracket \Gamma \vdash \pi : \kappa \rrbracket \text{ (Predicate well-formedness)}}$$

$$\text{case } \llbracket \rho_1 \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa \rrbracket).$$

$$\llbracket \Gamma \vdash \rho_1 \lesssim \rho_2 : \kappa \rrbracket = \text{case } \llbracket \rho_2 \rrbracket (\lambda m : \mathbb{N}. \lambda (Q : \text{Ix } m \rightarrow \llbracket \kappa \rrbracket). \forall (i : \text{Ix } n). \exists (j : \text{Ix } m). P i \equiv Q j))$$

$$\text{case } \llbracket \rho_1 \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa \rrbracket).$$

$$\text{case } \llbracket \rho_2 \rrbracket (\lambda m : \mathbb{N}. \lambda (Q : \text{Ix } m \rightarrow \llbracket \kappa \rrbracket).$$

$$\text{case } \llbracket \rho_3 \rrbracket (\lambda l : \mathbb{N}. \lambda (R : \text{Ix } l \rightarrow \llbracket \kappa \rrbracket).$$

$$(\forall (i : \text{Ix } n). \exists (k : \text{Ix } l). P i \equiv R k)$$

$$\llbracket \Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3 : \kappa \rrbracket = \times (\forall (j : \text{Ix } m). \exists (k : \text{Ix } l). Q j \equiv R k)$$

$$\times (\forall (k : \text{Ix } l).$$

$$(\exists (i : \text{Ix } n). P i \equiv R k)$$

$$+ (\exists (j : \text{Ix } m). Q j \equiv R k))))$$

Fig. 5. A compositional translation of typed  $\mathbf{R}\omega$  kinds and predicates to untyped  $\text{Ix}$  terms

## 2.2 Typed translation

$$\begin{array}{c}
 \boxed{\Gamma \vdash \tau \rightsquigarrow v : \kappa} \\
 \\
 \text{(C-FOO)} \frac{A}{B} \\
 \boxed{\Gamma \vdash M \rightsquigarrow N : \tau} \\
 \\
 \text{(C-FOO)} \frac{A}{B} \\
 \boxed{\Gamma \Vdash \pi \rightsquigarrow N} \\
 \\
 \text{(C-FOO)} \frac{A}{B} \\
 \boxed{\tau \equiv v \rightsquigarrow P} \\
 \\
 \text{(C-FOO)} \frac{A}{B}
 \end{array}$$

Fig. 6. Translation of  $R\omega$  derivations to  $lx$  derivations

## 2.3 Properties of Translation

Presume an  $R\omega$  instantiation of the simple row theory. A lot of this is likely bullshit.

**THEOREM 2 (TRANSLATIONAL SOUNDNESS (TYPES)).** *if  $\Gamma \vdash \tau : \kappa$  such that  $\Gamma \vdash \tau \rightsquigarrow v : \kappa$  then  $\llbracket \Gamma \rrbracket \vdash v : \llbracket \kappa \rrbracket$ .*

**THEOREM 3 (TRANSLATIONAL SOUNDNESS (TYPE EQUIVALENCE)).** *if*

- (1)  $\Gamma \vdash \tau_1 \rightsquigarrow v_1 : \kappa_1$ ;
- (2)  $\Gamma \vdash \tau_2 \rightsquigarrow v_2 : \kappa_2$ ; and
- (3)  $\tau_1 \equiv \tau_2 \rightsquigarrow P$ ,

*then  $\llbracket \Gamma \rrbracket \vdash P : v_1 \equiv v_2$ .*

**THEOREM 4 (TRANSLATIONAL SOUNDNESS (PREDICATES)).** *if  $\Gamma \Vdash \pi$  such that  $\Gamma \Vdash \pi \rightsquigarrow N$  then  $\llbracket \Gamma \rrbracket \vdash N : \llbracket \pi \rrbracket$ .*

Finally,

**THEOREM 5 (TRANSLATIONAL SOUNDNESS).** *if  $\Gamma \vdash M : \tau$  such that  $\Gamma \vdash M \rightsquigarrow N : \tau$  then  $\llbracket \Gamma \rrbracket \vdash N : \llbracket \tau \rrbracket$ .*

## 3 OPERATIONAL SEMANTICS

### REFERENCES

- J. Garrett Morris and James McKinna. 2019. Abstracting extensible data types: or, rows by any other name. *Proc. ACM Program. Lang.* 3, POPL (2019), 12:1–12:28. <https://doi.org/10.1145/3290325>