Recursive Rows in Rome

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1 IX: THE INDEX CALCULUS

1.1 Syntax

$$
\begin{array}{llcl}
\text{Sorts} & \sigma & ::= & \star \mid \mathcal{U} \\
\text{Terms} & M, N, T & ::= & \star \mid x \mid \\
 & & & \text{Nat} \mid Z \mid \text{S}\, M \mid \\
 & & & \text{case}_{\text{Nat}}\, M\, N\, T \mid \\
 & & & \text{Ix}\, M \mid \mathsf{I}_0 \mid \mathsf{I}_{\mathsf{S}}\, M \mid \\
 & & & \text{case}_{\text{Fin}}\, M\, N \mid \text{case}_{\text{Fin}}\, M\, N\, T \mid \\
 & & & \{\!\{M_1, ..., M_n\}\!\} \mid \\
 & & & \top \mid \texttt{tt} \mid \bot \mid \\
 & & & \forall \alpha : T.N \mid \lambda x : T.N \mid M\, N \mid \\
 & & & \exists \alpha : T.M \mid \langle\!\langle \alpha : T, M \rangle\!\rangle \mid \text{case}_{\exists}\, M\, N \mid \\
 & & & M + N \mid \text{left}\, M \mid \text{right}\, M \mid \\
 & & & \text{case}_{+}\, M\, N\, T \mid \\
 & & & M \equiv N \mid \text{refl}\, T\, M\, N \mid \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : T
\end{array}
$$

Fig. 1. Syntax

1.1.1 Meta-syntax & syntactic sugar. Let

(1) $\tau \longrightarrow \upsilon$ denote the unnamed quantification $\forall(\_ : \tau).\upsilon$;
(2) 0, 1, 2, ... denote object-level natural numbers in the intuitive fashion;
(3) i_n denote the index obtained by n applications of I_{S} to I_0; and

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(4) the syntax

$$\{\!\{M_1, \ldots, M_n\}\!\}$$

denote the large elimination of a known, finite quantity of indices to types $M_1, \ldots, M_n$, elaborated by the equations:

$$\begin{aligned}
\{\!\{M_1\}\!\} &:= \lambda(i : \mathrm{Ix}\ 1).\mathrm{case}_{\mathrm{Fin}}\ i\ M_1 \\
\{\!\{M_1, \ldots, M_n\}\!\} &:= \lambda(i : \mathrm{Ix}\ n).\mathrm{case}_{\mathrm{Fin}}\ i\ M_n\ \{\!\{M_1, \ldots, M_{n-1}\}\!\}
\end{aligned}$$

1.2 Typing

Many of these are fucked or in need of repair; refer to the translation as the SSOT.

$$\boxed{\vdash \Gamma}$$

$$(\textsc{emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{var})\ \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\star)\ \frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad (\top)\ \frac{}{\Gamma \vdash \top : \sigma} \qquad (\textsc{Nat})\ \frac{}{\Gamma \vdash \mathrm{Nat} : \star} \qquad (\textsc{Ix})\ \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\ n : \star}$$

$$(\forall)\ \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \forall \alpha : M.N : \sigma_2} \qquad (\exists)\ \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \exists \alpha : M.N : \sigma_2}$$

$$(+)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} \qquad (\equiv)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \equiv N : \sigma}$$

Fig. 2. Context and type formation rules

$$\boxed{\Gamma \vdash M : N}$$

$$(\textsc{Var})\,\frac{\vdash \Gamma \quad x : M \in \Gamma}{\Gamma \vdash x : M} \qquad (\mathtt{tt})\,\frac{\vdash \Gamma}{\Gamma \vdash \mathtt{tt} : \top}$$

$$(Z)\,\frac{\vdash \Gamma}{\Gamma \vdash Z : \mathrm{Nat}} \qquad (\mathrm{S})\,\frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{S}\, n : \mathrm{Nat}}$$

$$(\mathsf{I}_0)\,\frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathsf{I}_0 : \mathrm{Ix}\,(\mathrm{S}\, n)} \qquad (\mathsf{I}_\mathrm{S})\,\frac{\Gamma \vdash n : \mathrm{Nat} \quad \Gamma \vdash i : \mathrm{Ix}\, n}{\Gamma \vdash \mathsf{I}_\mathrm{S}\, i : \mathrm{Ix}\,(\mathrm{S}\, n)}$$

$$(\forall I)\,\frac{\Gamma \vdash T : \star \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \forall(x : T).N} \qquad (\forall E)\,\frac{\Gamma \vdash M : \forall(x : T_1).T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\, N : T_2[N/x]}$$

$$(\exists I)\,\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \exists(x : T_1).T_2} \qquad (\exists E_1)\,\frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash \mathrm{fst}\, M : T_1} \qquad (\exists E_2)\,\frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash \mathrm{snd}\, M : T_1[\mathrm{fst}\, M/x]}$$

$$(\equiv I)\,\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathrm{refl} : M \equiv M} \qquad (\textsc{conv})\,\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma \quad \sigma\ \mathrm{Safe}}{\Gamma \vdash M : T_2}$$

$$(\equiv E)\,\frac{\begin{array}{c}\Gamma \vdash P : T_1 \equiv T_2 \\ \Gamma \vdash M : T_1 \\ \Gamma \vdash N : T_1 \\ \Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\ \Gamma, z : T_1 \vdash H : T[z/x, z/y, \mathrm{refl}/p]\end{array}}{\Gamma \vdash \mathcal{J}\, H\, M\, N\, P : T[M/x, N/y, P/p]}$$

Fig. 3. Typing rules. Missing nat, fin, and sum elimination. Fin elimination should have special case for Ix 1.

$$\boxed{\Gamma \vdash M = N : \sigma}$$

$$(\textsc{e-refl})\,\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \qquad (\textsc{e-sym})\,\frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \qquad (\textsc{e-trans})\,\frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma}$$

$$\boxed{\Gamma \vdash M = N : T}$$

$$(\textsc{c-refl})\,\frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \qquad (\textsc{c-sym})\,\frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \qquad (\textsc{c-trans})\,\frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}$$

Fig. 4. Definitional equality & computational laws

1.3 A Comparison to $\lambda^{\Pi\mathcal{U}\mathbb{N}}$ [Abel et al. 2018]

2 TRANSLATION FROM Rω

2.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of Rω types. Figure 6 describe the untyped translation, which is used to show translational soundness of the typed translation (Figure 6).

$$\boxed{[\![\kappa]\!]}$$

$$\begin{aligned}
[\![\star]\!] &= \star \\
[\![\mathsf{L}]\!] &= \top \\
[\![\kappa_1 \to \kappa_2]\!] &= [\![\kappa_1]\!] \to [\![\kappa_2]\!] \\
[\![\mathsf{R}^{\kappa}]\!] &= \exists(n : \mathrm{Nat}).\mathrm{Ix}\; n \to [\![\kappa]\!]
\end{aligned}$$

$$\boxed{[\![\Gamma \vdash \tau : \kappa]\!]}$$

$$\begin{aligned}
[\![\Gamma \vdash \alpha : \kappa]\!] &= \alpha \\
[\![\Gamma \vdash \tau_1 \to \tau_2 : \star]\!] &= [\![\tau_1]\!] \to [\![\tau_2]\!] \\
[\![\Gamma \vdash \forall \alpha : \kappa.\tau : \star]\!] &= \forall(\alpha : [\![\kappa]\!]).[\![\tau]\!] \\
[\![\Gamma \vdash \lambda \alpha : \kappa.\tau : \kappa \to \kappa']\!] &= \forall(\alpha : [\![\kappa]\!]).[\![\tau]\!] \\
[\![\Gamma \vdash \pi \Rightarrow \tau : \kappa]\!] &= \forall(\alpha : [\![\pi]\!]).[\![\tau]\!] \\
[\![\Gamma \vdash \tau\, v : \kappa]\!] &= [\![\tau]\!]\; [\![v]\!] \\
[\![\Gamma \vdash \ell : \mathsf{L}]\!] &= \top \\
[\![\Gamma \vdash \lfloor \xi \rfloor : \star]\!] &= \top \\
[\![\Gamma \vdash \xi \triangleright \tau : \kappa]\!] &= [\![\tau]\!] \\
[\![\Gamma \vdash \Pi \rho : \star]\!] &= \mathrm{case}_{\exists}\; [\![\rho]\!]\; (\lambda n : \mathrm{Nat}.\lambda P : \mathrm{Ix}\; n \to \star.\forall(i : \mathrm{Ix}\; n).P\; i) \\
[\![\Gamma \vdash \Sigma \rho : \star]\!] &= \mathrm{case}_{\exists}\; [\![\rho]\!]\; (\lambda n : \mathrm{Nat}.\lambda P : \mathrm{Ix}\; n \to \star.\exists(i : \mathrm{Ix}\; n).P\; i) \\
[\![\Gamma \vdash \epsilon : \mathsf{R}^{\kappa}]\!] &= \langle\!\langle 0 : \mathrm{Nat}, \bot \rangle\!\rangle \\
[\![\Gamma \vdash \rho\, \lceil v \rceil : \mathsf{R}^{\kappa_2}]\!] &= \mathrm{case}_{\exists}\; [\![\rho]\!]\; (\lambda n : \mathrm{Nat}.\lambda(P : \mathrm{Ix}\; n \to [\![\kappa_1]\!] \to [\![\kappa_2]\!]).\langle\!\langle n : \mathrm{Nat}, \lambda(j : \mathrm{Ix}\; n).(P\; j)[\![\tau]\!]\rangle\!\rangle) \\
[\![\Gamma \vdash \lceil \tau \rceil\, \rho : \mathsf{R}^{\kappa_2}]\!] &= \mathrm{case}_{\exists}\; [\![\rho]\!]\; (\lambda n : \mathrm{Nat}.\lambda(P : \mathrm{Ix}\; n \to [\![\kappa_1]\!]).\langle\!\langle n : \mathrm{Nat}, \lambda(j : \mathrm{Ix}\; n).[\![\tau]\!](P\; j)\rangle\!\rangle) \\
[\![\Gamma \vdash \xi \triangleright_{\mathsf{R}} \tau : \mathsf{R}^{\kappa}]\!] &= \langle\!\langle 1 : \mathrm{Nat}, \{\!\{ [\![\tau]\!] \}\!\} \rangle\!\rangle
\end{aligned}$$

$$\boxed{[\![\Gamma \vdash \pi : \kappa]\!]}$$

$$\ldots$$

Fig. 5. A compositional translation of typed Rω kinds and predicates to untyped Ix terms

$$\boxed{[\![\Gamma \Vdash \pi]\!]}$$

...

$$\boxed{[\![\Gamma \vdash M : \tau]\!]}$$

...

Fig. 6. Translating predicates and terms

2.2 Typed translation

$$\boxed{\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \vdash M \rightsquigarrow N : \tau}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \Vdash \pi \rightsquigarrow N}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\tau \equiv \upsilon \rightsquigarrow P}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Fig. 7. Translation of Rω derivations to Ix derivations

2.3 Properties of Translation

Presume an Rω instantiation of the simple row theory. A lot of this is likely bullshit.

Theorem 1 (Translational Soundness (Types)). *if* $\Gamma \vdash \tau : \kappa$ *such that* $\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa$ *then* $[\![\Gamma]\!] \vdash \upsilon : [\![\kappa]\!]$.

Theorem 2 (Translational Soundness (Type Equivalence)). *if*

(1) $\Gamma \vdash \tau_1 \rightsquigarrow \upsilon_1 : \kappa_1$;
(2) $\Gamma \vdash \tau_2 \rightsquigarrow \upsilon_2 : \kappa_2$; *and*
(3) $\tau_1 \equiv \tau_2 \rightsquigarrow P$,

then $[\![\Gamma]\!] \vdash P : \upsilon_1 \equiv \upsilon_2$.

Theorem 3 (Translational Soundness (Of Predicates)). *if* $\Gamma \Vdash \pi$ *such that* $\Gamma \Vdash \pi \rightsquigarrow N$ *then* $[\![\Gamma]\!] \vdash N : [\![\pi]\!]$.

Finally,

Theorem 4 (Translational Soundness). *if* $\Gamma \vdash M : \tau$ *such that* $\Gamma \vdash M \rightsquigarrow N : \tau$ *then* $[\![\Gamma]\!] \vdash N : [\![\tau]\!]$.

3 OPERATIONAL SEMANTICS

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