

# The Index Calculus and its translation from $R\omega$

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## 1 Ix: The Index Calculus

### 1.1 Syntax

	Term variables $x \ \alpha$
Sorts	$\sigma ::= \star \mid \mathcal{U}$
Terms	$M, N, T ::= x \mid \star \mid \text{Nat} \mid \text{Zero} \mid \text{Suc } M \mid$ $\text{Ix } M \mid \text{FZero} \mid \text{FSuc } M \mid$ $\top \mid \mathbf{tt} \mid$ $\Pi\alpha : T.N \mid \lambda x : T.N \mid M \ N \mid$ $\Sigma\alpha : T.M \mid (\alpha : T, M) \mid M.1 \mid M.2$ $M + N \mid \text{left } M \mid \text{right } M \mid \text{case } M \text{ of } \{\text{left} \mapsto M; \text{right} \mapsto M\}$ $M \equiv N \mid \text{refl} \mid \dots$
Environments	$\Gamma ::= \varepsilon \mid \Gamma, \alpha : T$

Figure 1: Syntax

## 1.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(C\text{-EMP}) \frac{}{\vdash \varepsilon} \quad (C\text{-VAR}) \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$\begin{array}{c} (T\text{-}\star) \frac{}{\Gamma \vdash \star : \mathcal{U}} \quad \frac{}{\Gamma \vdash \top : \star} \quad \frac{}{\Gamma \vdash \text{Nat} : \star} \quad \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Ix } n : \star} \\[10pt] \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Pi \alpha : M.N : \star} \quad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \sigma}{\Gamma \vdash \Sigma \alpha : M.N : \star} \\[10pt] \frac{\Gamma \vdash M : \star \quad \Gamma \vdash N : \star}{\Gamma \vdash M + N : \star} \quad \frac{\Gamma \vdash M_1 : N_1 \quad \Gamma \vdash N_1 : \sigma \quad \Gamma \vdash M_2 : N_2 \quad \Gamma \vdash N_2 : \sigma}{\Gamma \vdash M_1 \equiv M_2 : \star} \end{array}$$

$$\boxed{\Gamma \vdash M : N}$$

$$\begin{array}{c} \frac{x : M \in \Gamma}{\Gamma \vdash x : M} \quad \frac{}{\Gamma \vdash \mathbf{tt} : \top} \\[10pt] \frac{}{\Gamma \vdash \text{Zero} : \text{Nat}} \quad \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Suc } n : \text{Nat}} \quad \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{FZero} : \text{Ix}(\text{Suc } n)} \quad \frac{\Gamma \vdash n : \text{Nat} \quad \Gamma \vdash i : \text{Ix } n}{\Gamma \vdash \text{FSuc } i : \text{Ix}(\text{Suc } n)} \\[10pt] \frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \Pi(x : T).N} \quad \frac{\Gamma \vdash M : \Pi(x : T_1).T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M N : T_2[N/x]} \\[10pt] \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1).T_2} \quad \frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash M.1 : T_1} \quad \frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash M.2 : T_2[M.1/x]} \\[10pt] \dots \end{array}$$

## 2 Translation From $R\omega$

### 2.1 Untyped Translation

$$\begin{aligned}
 & \llbracket \kappa \rrbracket \\
 & \llbracket \star \rrbracket = \star \\
 & \llbracket L \rrbracket = \star \\
 & \llbracket \kappa_1 \rightarrow \kappa_2 \rrbracket = \dots \\
 & \llbracket R^\kappa \rrbracket = \dots
 \end{aligned}$$

$$\llbracket \tau \rrbracket$$

...

$$\llbracket M \rrbracket$$

...

Figure 2: A compositional translation of  $R\omega$  to  $\mu\text{Ix}$

## 2.2 Typed translation

$$\begin{array}{c}
\boxed{\Delta \vdash \tau : \kappa \rightsquigarrow \llbracket \Gamma \rrbracket \vdash v : \sigma} \\
\\
(\text{C-FOO}) \frac{A}{B} \\
\boxed{\Delta \vdash M : \tau \rightsquigarrow \Gamma \vdash N : v} \\
\\
(\text{C-FOO}) \frac{A}{B} \\
\boxed{\Delta \Vdash \pi \rightsquigarrow \Gamma \vdash N : v} \\
\\
(\text{C-FOO}) \frac{A}{B} \\
\boxed{\tau \equiv v \rightsquigarrow \llbracket \Delta \rrbracket \vdash P : \llbracket \tau \rrbracket \sim \llbracket v \rrbracket} \\
\\
(\text{C-FOO}) \frac{A}{B}
\end{array}$$

## 2.3 Example translations of $\mathbf{R}\omega$ terms and types

**Record selection.** In  $\mathbf{R}\omega$ ,

$$\forall \rho : \mathbf{R}^*, \ell : \mathbf{L}, \tau : \star. \{\ell \triangleright \tau\} \lesssim \rho \Rightarrow \lfloor \ell \rfloor \rightarrow \Pi \rho \rightarrow \tau$$

translates to

$$\Pi(\rho : \text{Row } \star). \Pi(\ell : \top). \Pi(\tau : \star). \llbracket \{\ell \triangleright \tau\} \lesssim \rho \rrbracket. \Pi(- : \top). \Pi(i : \text{Ix } \rho.1). \rho.2 \ i$$

where

$$\begin{aligned}
\text{Row } \kappa &:= \Sigma(n : \text{Nat}). \Pi(i : \text{Ix } n). \kappa \\
\llbracket \{\ell \triangleright \tau\} \lesssim \rho \rrbracket &= \Pi(i : \text{Ix } \llbracket \{\ell \triangleright \tau\} \rrbracket.1). \Sigma(j : \text{Ix } \rho.1). \llbracket \{\ell \triangleright \tau\} \rrbracket.1 \ i \equiv \rho.2 \ j \\
\llbracket \{\ell \triangleright \tau\} \rrbracket &= (\text{Suc Zero} : \text{Nat}, \lambda(i : \text{Ix } (\text{Suc Zero})). \llbracket \tau \rrbracket)
\end{aligned}$$

Putting this all together:

$$\begin{aligned}
& \Pi(\rho : (\Sigma(n : \text{Nat}).\Pi(i : \text{Ix } n).\star)). \\
& \Pi(\ell : \top). \\
& \Pi(\tau : \star). \\
& \Pi(P : \\
& \quad \Pi(i : \text{Ix } (\text{Suc Zero} : \text{Nat}, \lambda(i : \text{Ix } (\text{Suc Zero})).\llbracket \tau \rrbracket).1). \\
& \quad \Sigma(j : \text{Ix } \rho.1). \\
& \quad (\text{Suc Zero} : \text{Nat}, \lambda(i : \text{Ix } (\text{Suc Zero})).\llbracket \tau \rrbracket).2 \ i \equiv \rho.2 \ j) \\
& \Pi(- : \top). \\
& \Pi(i : \text{Ix } \rho.1). \rho.2 \ i
\end{aligned}$$

which should normalize to

$$\begin{aligned}
& \Pi(\rho : (\Sigma(n : \text{Nat}).\Pi(i : \text{Ix } n).\star)). \\
& \Pi(\ell : \top). \\
& \Pi(\tau : \star). \\
& \Pi(P : \\
& \quad \Pi(i : \text{Ix } 1). \\
& \quad \Sigma(j : \text{Ix } \rho.1). \\
& \quad \llbracket \tau \rrbracket \equiv \rho.2 \ j). \\
& \Pi(- : \top). \\
& \Pi(i : \text{Ix } \rho.1). \rho.2 \ i
\end{aligned}$$

## A The static semantics of $\mathbf{R}\omega$

### A.1 Syntax

The syntax of  $\mathbf{R}\omega(\mathcal{T})$  is given in Figure 3.

Term variables $x$	Type variables $\alpha$	Labels $\ell$	Directions $d \in \{\mathbf{L}, \mathbf{R}\}$
Kinds	$\kappa ::= \star \mid \mathbf{L} \mid \mathbf{R}^\kappa \mid \kappa \rightarrow \kappa$		
Predicates	$\pi, \psi ::= \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho$		
Types	$\phi, \tau, v, \rho, \xi ::= \alpha \mid (\rightarrow) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa. \tau \mid \lambda \alpha : \kappa. \tau \mid \tau \tau$ $\mid \ell \mid [\xi] \mid \xi \triangleright \tau \mid \{\tau_1, \dots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho$		
Terms	$M, N ::= x \mid \lambda x : \tau. M \mid M N \mid \Lambda \alpha : \kappa. M \mid M [\tau]$ $\mid \ell \mid M \triangleright M \mid M / M \mid \mathbf{prj}_d M \mid M ++ M \mid \mathbf{inj}_d M \mid M \nabla M$ $\mid \mathbf{syn}_\phi M \mid \mathbf{ana}_\phi M \mid \mathbf{fold} M M M M$		
Environments	$\Gamma ::= \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi$		

Figure 3: Syntax

## A.2 Types and Kinds

Figure 4 gives rules for context formation ( $\vdash \Gamma$ ), kinding ( $\Gamma \vdash \tau : \kappa$ ), and predicate formation ( $\Gamma \vdash \pi$ ), parameterized by row theory  $\mathcal{T}$ .

$$\begin{array}{c}
\boxed{\vdash \Gamma} \\
\\
\text{(C-EMP)} \frac{}{\vdash \varepsilon} \quad \text{(C-TVAR)} \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \quad \text{(C-VAR)} \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau} \quad \text{(C-PRED)} \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi} \\
\\
\boxed{\Gamma \vdash \tau : \kappa} \quad \boxed{\Gamma \vdash \pi} \\
\\
\text{(K-VAR)} \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \quad \text{(K-}(\rightarrow)\text{)} \frac{\vdash \Gamma}{\Gamma \vdash (\rightarrow) : \star \rightarrow \star \rightarrow \star} \quad \text{(K-}\Rightarrow\text{)} \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star} \\
\\
\text{(K-}\forall\text{)} \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa. \tau : \star} \quad \text{(K-}\rightarrow I\text{)} \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \quad \text{(K-}\rightarrow E\text{)} \frac{\Gamma \vdash \tau_1 : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1 \tau_2 : \kappa_2} \\
\\
\text{(K-LAB)} \frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathbf{L}} \quad \text{(K-SING)} \frac{\Gamma \vdash \xi : \mathbf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star} \quad \text{(K-LTY)} \frac{\Gamma \vdash \xi : \mathbf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa} \quad \text{(K-ROW)} \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathbf{R}^\kappa}{\Gamma \vdash \{\xi \triangleright \tau\} : \mathbf{R}^\kappa} \\
\\
\text{(K-II)} \frac{\Gamma \vdash \rho : \mathbf{R}^\kappa}{\Gamma \vdash \Pi \rho : \kappa} \quad \text{(K-}\Sigma\text{)} \frac{\Gamma \vdash \rho : \mathbf{R}^\kappa}{\Gamma \vdash \Sigma \rho : \kappa} \quad \text{(K-LIFT}_1\text{)} \frac{\Gamma \vdash \rho : \mathbf{R}^{\kappa_1 \rightarrow \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho \tau : \mathbf{R}^{\kappa_2}} \\
\\
\text{(K-LIFT}_2\text{)} \frac{\Gamma \vdash \phi : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \rho : \mathbf{R}^{\kappa_1}}{\Gamma \vdash \phi \rho : \mathbf{R}^{\kappa_2}} \quad \text{(K-}\lesssim_d\text{)} \frac{\Gamma \vdash \rho_i : \mathbf{R}^\kappa}{\Gamma \vdash \rho_1 \lesssim_d \rho_2} \quad \text{(K-}\odot\text{)} \frac{\Gamma \vdash \rho_i : \mathbf{R}^\kappa}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}
\end{array}$$

Figure 4: Contexts and kinding.

$$\begin{array}{c}
\boxed{\tau \equiv \tau} \quad \boxed{\pi \equiv \pi} \\
\\
(\text{E-REFL}) \frac{}{\tau \equiv \tau} \quad (\text{E-SYM}) \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \quad (\text{E-TRANS}) \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \quad (\text{E-}\beta) \frac{}{(\lambda \alpha : \kappa. \tau) v \equiv \tau[v/\alpha]} \\
\\
(\text{E-}\xi_{\Rightarrow}) \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \quad (\text{E-}\xi_{\forall}) \frac{\tau[\gamma/\alpha] \equiv v[\gamma/\beta]}{\forall \alpha : \kappa. \tau \equiv \forall \beta : \kappa. v} (\gamma \notin fv(\tau, v)) \quad (\text{E-}\xi_{\text{APP}}) \frac{\tau_i \equiv v_i}{\tau_1 \tau_2 \equiv v_1 v_2} \\
\\
(\text{E-}\xi_{\triangleright}) \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \quad (\text{E-ROW}) \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\xi_i \triangleright \tau_i\} \equiv \{\xi'_j \triangleright \tau'_j\}} \quad (\text{E-}\xi_{[\cdot]}) \frac{\xi_1 \equiv \xi_2}{[\xi_1] \equiv [\xi_2]} \\
\\
(\text{E-LIFT}_1) \frac{}{\{\xi \triangleright \phi\} \tau \equiv \{\xi \triangleright \phi \tau\}} \quad (\text{E-LIFT}_2) \frac{}{\phi \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi \tau\}} \\
\\
(\text{E-}\xi_{\Pi\Sigma}) \frac{\rho_1 \equiv \rho_2}{K \rho_1 \equiv K \rho_2} \quad (\text{E-LIFT}_3) \frac{}{(K \rho) \tau \equiv K(\rho \tau)} \quad (\text{E-SING}) \frac{}{K \{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \quad (K \in \{\Pi, \Sigma\}) \\
\\
(\text{E-}\xi_{\lesssim_d}) \frac{\tau_i \equiv v_i}{\tau_1 \lesssim_d \tau_2 \equiv v_1 \lesssim_d v_2} \quad (\text{E-}\xi_{\odot}) \frac{\tau_i \equiv v_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv v_1 \odot v_2 \sim v_3}
\end{array}$$

Figure 5: Type and predicate equivalence



### A.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$\begin{array}{c}
(\text{T-VAR}) \frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \quad (\text{T-}\rightarrow I) \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1. M : \tau_1 \rightarrow \tau_2} \quad (\text{T-}\rightarrow E) \frac{\Gamma \vdash M_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash M_2}{\Gamma \vdash M_1 M_2 : \tau_2} \\
(\text{T-}\equiv) \frac{\Gamma \vdash M : \tau \quad \tau \equiv v}{\Gamma \vdash M : v} \quad (\text{T-}\Rightarrow I) \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \quad (\text{T-}\Rightarrow E) \frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau} \\
(\text{T-}\forall I) \frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa. M : \forall \alpha : \kappa. \tau} \quad (\text{T-}\forall E) \frac{\Gamma \vdash M : \forall \alpha : \kappa. \tau \quad \Gamma \vdash v : \kappa}{\Gamma \vdash M[v] : \tau[v/\alpha]} \\
(\text{T-SING}) \frac{\vdash \Gamma}{\Gamma \vdash \ell : [\ell]} \quad (\text{T-}\triangleright I) \frac{\Gamma \vdash M_1 : [\ell] \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \quad (\text{T-}\triangleright E) \frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : [\ell]}{\Gamma \vdash M_1 / M_2 : \tau} \\
(\text{T-II}E) \frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \text{prj}_d M : \Pi \rho_2} \quad (\text{T-II}I) \frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 ++ M_2 : \Pi \rho_3} \\
(\text{T-}\Sigma I) \frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \text{inj} M : \Sigma \rho_2} \quad (\text{T-}\Sigma E) \frac{\Gamma \vdash M_1 : \Sigma \rho_1 \rightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim}{\Gamma \vdash M_1 \nabla M_2 : \Sigma \rho_3 \rightarrow \tau} \\
(\text{T-ana}) \frac{\Gamma \vdash \rho : \mathbf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : \mathbf{L}, u : \kappa, y_1, z, y_2 : \mathbf{R}^\kappa. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u \rightarrow \tau}{\Gamma \vdash \text{ana}_\phi M : \Sigma(\phi \rho) \rightarrow \tau} \\
(\text{T-syn}) \frac{\Gamma \vdash \rho : \mathbf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : \mathbf{L}, u : \kappa, y_1, z, y_2 : \mathbf{R}^\kappa. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u}{\Gamma \vdash \text{syn}_\phi M : \Pi(\phi \rho)} \\
(\text{T-fold}) \frac{M_1 : \forall l : \mathbf{L}, t : \star, y_1, z, y_2 : \mathbf{R}^\kappa. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow t \rightarrow v \quad \Gamma \vdash M_2 : v \rightarrow v \rightarrow v \quad \Gamma \vdash M_3 : v \quad \Gamma \vdash N : \Pi \rho}{\Gamma \vdash \text{fold} M_1 M_2 M_3 N : v}
\end{array}$$

Figure 6: Typing

Minimal Rows

Figure 7 gives the minimal row theory  $\mathcal{M}$ .

$$\begin{array}{c}
\boxed{\Gamma \vdash_{\mathbf{m}} \rho : \kappa} \quad \boxed{\rho \equiv_{\mathbf{m}} \rho} \\
\\
(\text{K-MROW}) \frac{\Gamma \vdash \xi : \mathbf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathbf{m}} \{\xi \triangleright \tau\} : \mathbf{R}^\kappa} \quad (\text{E-MROW}) \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathbf{m}} \{\xi' \triangleright \tau'\}} \\
\\
\boxed{\Gamma \Vdash_{\mathbf{m}} \pi} \\
\\
(\text{N-AX}) \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathbf{m}} \pi} \quad (\text{N-REFL}) \frac{}{\Gamma \Vdash_{\mathbf{m}} \rho \lesssim_d \rho} \quad (\text{N-TRANS}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathbf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathbf{m}} \rho_1 \lesssim_d \rho_3} \\
\\
(\text{N-}\equiv) \frac{\Gamma \Vdash_{\mathbf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathbf{m}} \pi_2} \quad (\text{N-}\lesssim_{\text{LIFT1}}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathbf{m}} \phi \rho_1 \lesssim_d \phi \rho_2} \quad (\text{N-}\lesssim_{\text{LIFT2}}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathbf{m}} \rho_1 \tau \lesssim_d \rho_2 \tau} \\
\\
(\text{N-}\odot_{\text{LIFT1}}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathbf{m}} \rho_1 \tau \odot \rho_2 \tau \sim \rho_3 \tau} \quad (\text{N-}\odot_{\text{LIFT2}}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathbf{m}} \phi \rho_1 \odot \phi \rho_2 \sim \phi \rho_3} \\
\\
(\text{N-}\odot_{\lesssim_{\mathbf{L}}}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathbf{m}} \rho_1 \lesssim_{\mathbf{L}} \rho_3} \quad (\text{N-}\odot_{\lesssim_{\mathbf{R}}}) \frac{\Gamma \Vdash_{\mathbf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathbf{m}} \rho_2 \lesssim_{\mathbf{R}} \rho_3}
\end{array}$$

Figure 7: Minimal row theory  $\mathcal{M} = \langle \vdash_{\mathbf{m}}, \equiv_{\mathbf{m}}, \Vdash_{\mathbf{m}} \rangle$