Recursive Rows in Rome

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1 INTRODUCTION

1.1 The expression problem, in full

1.1.1 Seeking solutions sans encodings.

1.2 Recursion and rows

1.2.1 Row type systems with term- or type-level μ. There are none.

1.2.2 Structural typing of objects in recursive record calculi.

1.3 Challenges to practical extensibility

1.3.1 Polymorphic variants in OCaml.

1.3.2 Inheritance is not subtyping.

2 ROME: A ROW THEORETIC FOUNDATION FOR (CO)INDUCTIVE DATA TYPES

3 Rω—HIGHER ORDERED ROWS

Recursive types have a well known semantics as the least fixed-points of type-level operators. Rω is the only row calculus (to our knowledge) to include an (explicit) type-level λ operator. Like with $F\omega$, this necessarily sacrifices desirable metatheoretic properties, such as principality of types; hence, like $F\omega$, Rω may serve as a highly expressive intermediate or target language. Correspondingly, we perceive the addition of recursive terms and types to Rω to aid the adoption of recursion in surface languages.

We review the relevant syntax and typing of Rω now.

(Todo). It will be a challenge to trim this down, as [?] does with [?].

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3.1 Syntax

The syntax of $\mathrm{R}\omega(\mathcal{T})$ is given in Figure 6.

$$\text{Term variables } x \qquad \text{Type variables } \alpha \qquad \text{Labels } \ell \qquad \text{Directions } d \in \{\mathsf{L}, \mathsf{R}\}$$

$$
\begin{array}{llcl}
\text{Kinds} & \kappa & ::= & \star \mid \mathsf{L} \mid \mathsf{R}^{\kappa} \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi & ::= & \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi & ::= & \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa . \tau \mid \lambda \alpha : \kappa . \tau \mid \tau\, \tau \\
 & & \mid & \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho \\
\text{Terms} & M, N & ::= & x \mid \lambda x : \tau . M \mid M\, N \mid \Lambda \alpha : \kappa . M \mid M\, [\tau] \\
 & & \mid & \ell \mid M \triangleright M \mid M / M \mid \mathsf{prj}_d\, M \mid M \mathbin{+\!\!+} M \mid \mathsf{inj}_d\, M \mid M \triangledown M \\
 & & \mid & \mathsf{syn}_{\phi}\, M \mid \mathsf{ana}_{\phi}\, M \mid \mathsf{fold}\, M\, M\, M\, M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}
$$

Fig. 1. Syntax

3.2 Types and Kinds

Figure 2 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

$$\boxed{\vdash \Gamma}$$

$$
(\textsc{c-emp})\,\frac{}{\vdash \varepsilon}
\qquad
(\textsc{c-tvar})\,\frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa}
\qquad
(\textsc{c-var})\,\frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}
\qquad
(\textsc{c-pred})\,\frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}
$$

$$\boxed{\Gamma \vdash \tau : \kappa}\ \boxed{\Gamma \vdash \pi}$$

$$
(\textsc{k-var})\,\frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}
\qquad
(\textsc{k-}(\to))\,\frac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star}
\qquad
(\textsc{k-}\Rightarrow)\,\frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}
$$

$$
(\textsc{k-}\forall)\,\frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa . \tau : \star}
\qquad
(\textsc{k-}{\to} I)\,\frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1 . \tau : \kappa_1 \to \kappa_2}
\qquad
(\textsc{k-}{\to} E)\,\frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\, \tau_2 : \kappa_2}
$$

$$
(\textsc{k-lab})\,\frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}}
\qquad
(\textsc{k-sing})\,\frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star}
\qquad
(\textsc{k-lty})\,\frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}
\qquad
(\textsc{k-row})\,\frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^{\kappa}}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^{\kappa}}
$$

$$
(\textsc{k-}\Pi)\,\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa}}{\Gamma \vdash \Pi \rho : \kappa}
\qquad
(\textsc{k-}\Sigma)\,\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa}}{\Gamma \vdash \Sigma \rho : \kappa}
\qquad
(\textsc{k-lift}_1)\,\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\, \tau : \mathsf{R}^{\kappa_2}}
$$

$$
(\textsc{k-lift}_2)\,\frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\, \rho : \mathsf{R}^{\kappa_2}}
\qquad
(\textsc{k-}\lesssim_d)\,\frac{\Gamma \vdash \rho_i : \mathsf{R}^{\kappa}}{\Gamma \vdash \rho_1 \lesssim_d \rho_2}
\qquad
(\textsc{k-}\odot)\,\frac{\Gamma \vdash \rho_i : \mathsf{R}^{\kappa}}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}
$$

Fig. 2. Contexts and kinding.

$$\boxed{\tau \equiv \tau} \; \boxed{\pi \equiv \pi}$$

$$(\text{E-REFL})\ \frac{}{\tau \equiv \tau} \qquad (\text{E-SYM})\ \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad (\text{E-TRANS})\ \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad (\text{E-}\beta)\ \frac{}{(\lambda\alpha : \kappa.\tau)\ \upsilon \equiv \tau[\upsilon/\alpha]}$$

$$(\text{E-}\xi_{\Rightarrow})\ \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad (\text{E-}\xi_{\forall})\ \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha : \kappa.\tau \equiv \forall\beta : \kappa.\upsilon}\ (\gamma \notin fv(\tau, \upsilon)) \qquad (\text{E-}\xi_{\text{APP}})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1\ \tau_2 \equiv \upsilon_1\ \upsilon_2}$$

$$(\text{E-}\xi_{\triangleright})\ \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \qquad (\text{E-ROW})\ \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \qquad (\text{E-}\xi_{\lfloor\cdot\rfloor})\ \frac{\xi_1 \equiv \xi_2}{\lfloor\xi_1\rfloor \equiv \lfloor\xi_2\rfloor}$$

$$(\text{E-LIFT}_1)\ \frac{}{\{\xi \triangleright \phi\}\ \tau \equiv \{\xi \triangleright \phi\ \tau\}} \qquad (\text{E-LIFT}_2)\ \frac{}{\phi\ \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi\ \tau\}}$$

$$(\text{E-}\xi_{\Pi\Sigma})\ \frac{\rho_1 \equiv \rho_2}{K\rho_1 \equiv K\rho_2} \qquad (\text{E-LIFT}_3)\ \frac{}{(K\rho)\ \tau \equiv K(\rho\ \tau)} \qquad (\text{E-SING})\ \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \qquad (K \in \{\Pi, \Sigma\})$$

$$(\text{E-}\xi_{\lesssim_d})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad (\text{E-}\xi_{\odot})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Fig. 3. Type and predicate equivalence

3.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$(\text{T-VAR})\,\frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad (\text{T-}{\longrightarrow} I)\,\frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . M : \tau_1 \longrightarrow \tau_2} \qquad (\text{T-}{\longrightarrow} E)\,\frac{\Gamma \vdash M_1 : \tau_1 \longrightarrow \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1\, M_2 : \tau_2}$$

$$(\text{T-}{\equiv})\,\frac{\Gamma \vdash M : \tau \quad \tau \equiv v}{\Gamma \vdash M : v} \qquad (\text{T-}{\Longrightarrow} I)\,\frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \qquad (\text{T-}{\Longrightarrow} E)\,\frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$$

$$(\text{T-}\forall I)\,\frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa . M : \forall \alpha : \kappa . \tau} \qquad (\text{T-}\forall E)\,\frac{\Gamma \vdash M : \forall \alpha : \kappa . \tau \quad \Gamma \vdash v : \kappa}{\Gamma \vdash M\,[v] : \tau[v/\alpha]}$$

$$(\text{T-SING})\,\frac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor} \qquad (\text{T-}{\triangleright} I)\,\frac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \qquad (\text{T-}{\triangleright} E)\,\frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1 / M_2 : \tau}$$

$$(\text{T-}\Pi E)\,\frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\, M : \Pi \rho_2} \qquad (\text{T-}\Pi I)\,\frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \mathbin{+\!\!+} M_2 : \Pi \rho_3}$$

$$(\text{T-}\Sigma I)\,\frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \mathsf{inj}\, M : \Sigma \rho_2} \qquad (\text{T-}\Sigma E)\,\frac{\Gamma \vdash M_1 : \Sigma \rho_1 \longrightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \longrightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \mathbin{\triangledown} M_2 : \Sigma \rho_3 \longrightarrow \tau}$$

$$(\text{T-ana})\,\frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^{\kappa} \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^{\kappa}. \, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u \to \tau\end{array}}{\Gamma \vdash \mathsf{ana}_{\phi}\, M : \Sigma(\phi\, \rho) \to \tau}$$

$$(\text{T-syn})\,\frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^{\kappa} \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^{\kappa}. \, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u\end{array}}{\Gamma \vdash \mathsf{syn}_{\phi}\, M : \Pi(\phi\, \rho)}$$

$$(\text{T-fold})\,\frac{\begin{array}{c}M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^{\kappa}. \, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to t \to v \\ \Gamma \vdash M_2 : v \to v \to v \quad \Gamma \vdash M_3 : v \quad \Gamma \vdash N : \Pi \rho\end{array}}{\Gamma \vdash \mathsf{fold}\, M_1\, M_2\, M_3\, N : v}$$

Fig. 4. Typing

Minimal Rows

Figure 5 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathrm{m}} \rho : \kappa} \quad \boxed{\rho \equiv_{\mathrm{m}} \rho}$$

$$(\textsc{k-mrow})\ \frac{\Gamma \vdash \xi : \mathsf{L} \qquad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathrm{m}} \{\xi \triangleright \tau\} : \mathsf{R}^{\kappa}} \qquad (\textsc{e-mrow})\ \frac{\xi \equiv \xi' \qquad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathrm{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathrm{m}} \pi}$$

$$(\textsc{n-ax})\ \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathrm{m}} \pi} \qquad (\textsc{n-refl})\ \frac{}{\Gamma \Vdash_{\mathrm{m}} \rho \lesssim_d \rho} \qquad (\textsc{n-trans})\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \lesssim_d \rho_2 \qquad \Gamma \Vdash_{\mathrm{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathrm{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\textsc{n-}\equiv)\ \frac{\Gamma \Vdash_{\mathrm{m}} \pi_1 \qquad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathrm{m}} \pi_2} \qquad (\textsc{n-}\lesssim\textsc{lift}_1)\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathrm{m}} \phi\, \rho_1 \lesssim_d \phi\, \rho_2} \qquad (\textsc{n-}\lesssim\textsc{lift}_2)\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathrm{m}} \rho_1\, \tau \lesssim_d \rho_2\, \tau}$$

$$(\textsc{n-}\odot\textsc{lift}_1)\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathrm{m}} \rho_1\, \tau \odot \rho_2\, \tau \sim \rho_3\, \tau} \qquad (\textsc{n-}\odot\textsc{lift}_2)\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathrm{m}} \phi\, \rho_1 \odot \phi\, \rho_2 \sim \phi\, \rho_3}$$

$$(\textsc{n-}\odot\lesssim_{\mathsf{L}})\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathrm{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\textsc{n-}\odot\lesssim_{\mathsf{R}})\ \frac{\Gamma \Vdash_{\mathrm{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathrm{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Fig. 5. Minimal row theory $\mathcal{M} = \langle \vdash_{\mathrm{m}}, \equiv_{\mathrm{m}}, \Vdash_{\mathrm{m}} \rangle$

4 IX: THE INDEX CALCULUS

4.1 Syntax

Term variables $x\ \alpha$

$$\begin{array}{llcl}
\text{Sorts} & \sigma & ::= & \star \mid \mathcal{U} \\
\text{Terms} & M, N, T & ::= & x \mid \star \mid \mathrm{Nat} \mid \mathrm{Zero} \mid \mathrm{Suc}\, M \mid \\
& & & \mathrm{Ix}\, M \mid \mathrm{FZero} \mid \mathrm{FSuc}\, M \mid \\
& & & \top \mid \mathtt{tt} \mid \\
& & & \Pi\alpha : T.N \mid \lambda x : T.N \mid M\, N \mid \\
& & & \Sigma\alpha : T.M \mid (\alpha : T, M) \mid M.1 \mid M.2 \\
& & & M + N \mid \mathrm{left}\, M \mid \mathrm{right}\, M \mid \mathrm{case}\ M\ \mathrm{of}\ \{\mathrm{left} \mapsto M; \mathrm{right} \mapsto M\} \\
& & & M \equiv N \mid \mathrm{refl} \mid \ldots \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : T
\end{array}$$

Fig. 6. Syntax

4.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(\text{C-EMP})\ \frac{}{\vdash \varepsilon} \qquad (\text{C-VAR})\ \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\text{T-}\star)\ \frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad \frac{}{\Gamma \vdash \top : \star} \qquad \frac{}{\Gamma \vdash \top : \mathcal{U}}$$

$$\frac{}{\Gamma \vdash \mathrm{Nat} : \star} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\, n : \star}$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Pi\alpha : M.N : \star} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Sigma\alpha : M.N : \star}$$

$$\frac{\Gamma \vdash M : \star \quad \Gamma \vdash N : \star}{\Gamma \vdash M + N : \star} \qquad \frac{\Gamma \vdash M_1 : N_1 \quad \Gamma \vdash N_1 : \sigma \quad \Gamma \vdash M_2 : N_2 \quad \Gamma \vdash N_2 : \sigma}{\Gamma \vdash M_1 \equiv M_2 : \star}$$

$$\boxed{\Gamma \vdash M : N}$$

$$\frac{x : M \in \Gamma}{\Gamma \vdash x : M} \qquad \frac{}{\Gamma \vdash \mathtt{tt} : \top}$$

$$\frac{}{\Gamma \vdash \mathrm{Zero} : \mathrm{Nat}} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Suc}\, n : \mathrm{Nat}} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{FZero} : \mathrm{Ix}\,(\mathrm{Suc}\, n)} \qquad \frac{\Gamma \vdash n : \mathrm{Nat} \quad \Gamma \vdash i : \mathrm{Ix}\, n}{\Gamma \vdash \mathrm{FSuc}\, i : \mathrm{Ix}\,(\mathrm{Suc}\, n)}$$

$$\frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \Pi(x : T).N} \qquad \frac{\Gamma \vdash M : \Pi(x : T_1).T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\, N : T_2[N/x]}$$

$$\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1).T_2} \qquad \frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash M.1 : T_1} \qquad \frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash M.2 : T_2[M.1/x]}$$

...

5 TRANSLATION FROM Rω

5.1 Untyped Translation

$$\boxed{(\kappa)^{\bullet}}$$

$$
\begin{aligned}
(\star)^{\bullet} &= \star \\
(\mathsf{L})^{\bullet} &= \top \\
(\kappa_1 \longrightarrow \kappa_2)^{\bullet} &= \Pi(\alpha : (\kappa_1)^{\bullet}).(\kappa_2)^{\bullet} \\
(\mathsf{R}^{\kappa})^{\bullet} &= \Sigma(n : \mathrm{Nat}).\Pi(j : \mathrm{Ix}\ n).(\kappa)^{\bullet}
\end{aligned}
$$

$$\boxed{(\tau)^{\bullet}}$$

$$\ldots$$

$$\boxed{(M)^{\bullet}}$$

$$\ldots$$

$$\boxed{(\pi)^{\bullet}}$$

$$\ldots$$

Fig. 7. A compositional translation of Rω judgments to (untyped) Ix terms

5.2 Typed translation

$$\boxed{\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa}$$

$$(\textsc{c-foo})\,\frac{A}{B}$$

$$\boxed{\Gamma \vdash M \rightsquigarrow N : \tau}$$

$$(\textsc{c-foo})\,\frac{A}{B}$$

$$\boxed{\Gamma \Vdash \pi \rightsquigarrow N}$$

$$(\textsc{c-foo})\,\frac{A}{B}$$

$$\boxed{\tau \equiv \upsilon \rightsquigarrow P}$$

$$(\textsc{c-foo})\,\frac{A}{B}$$

Fig. 8. Translation of Rω derivations to Ix derivations

5.3 Properties of Translation

THEOREM 1 (TRANSLATIONAL SOUNDNESS (TYPES)). *if* $\Gamma \vdash \tau : \kappa$ *such that* $\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa$ *then* $(\Gamma)^\bullet \vdash \upsilon : (\kappa)^\bullet$.

THEOREM 2 (TRANSLATIONAL SOUNDNESS (TYPE EQUIVALENCE)). *if* $\Gamma \vdash \tau_1 \rightsquigarrow \upsilon_1 : \kappa_1$ *and* $\Gamma \vdash \tau_2 \rightsquigarrow \upsilon_2 : \kappa_2$ *such that* $\tau_1 \equiv \tau_2$ *is derivable in Rω, then* $(\Gamma)^\bullet \vdash \upsilon_1 \equiv \upsilon_2$.

The next theorems presume an Rω instantiation of the simple row theory.

THEOREM 3 (TRANSLATIONAL SOUNDNESS (ROW COMBINATION)). *if* $\Gamma \Vdash \rho_1 \cdot \rho_2 \sim \rho_3 \rightsquigarrow N$ *then* $(\Gamma)^\bullet \vdash N : \mathit{foobar}$.

THEOREM 4 (TRANSLATIONAL SOUNDNESS (ROW CONTAINMENT)). *if* $\Gamma \Vdash \rho_1 \lesssim \rho_2 \rightsquigarrow N$ *then* $(\Gamma)^\bullet \vdash N : \mathit{foobar}$.

Finally,

THEOREM 5 (TRANSLATIONAL SOUNDNESS). *if* $\Gamma \vdash M : \tau$ *such that* $\Gamma \vdash M \rightsquigarrow N{:}\tau$ *then* $(\Gamma)^\bullet \vdash M : (\tau)^\bullet$.

6 OPERATIONAL SEMANTICS OF IX

7 RECURSION

This section will later be incorporated into earlier sections.

7.1 Rome, or, Rω with μ

$\boxed{\text{todo}}$

$$(\text{C-FOO})\ \frac{A}{B}$$

Fig. 9. Additional Rω judgments for recursion

7.2 Mix, the recursive index calculus

$\boxed{\text{todo}}$

$$(\text{C-FOO})\ \frac{A}{B}$$

Fig. 10. Additional Ix judgments for recursion

7.3 Translation and properties of translation