

Recursive Rows in Rome

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1 INTRODUCTION

1.1 The expression problem, in full

1.1.1 *Seeking solutions sans encodings.*

1.2 Recursion and rows

1.2.1 *Row type systems with term- or type-level μ .* There are none.

1.2.2 *Structural typing of objects in recursive record calculi.*

1.3 Challenges to practical extensibility

1.3.1 *Polymorphic variants in OCaml.*

1.3.2 *Inheritance is not subtyping.*

2 ROME: A ROW THEORETIC FOUNDATION FOR (CO)INDUCTIVE DATA TYPES

3 $R\omega$ —HIGHER ORDERED ROWS

Recursive types have a well known semantics as the least fixed-points of type-level operators. $R\omega$ is the only row calculus (to our knowledge) to include an (explicit) type-level λ operator. Like with $F\omega$, this necessarily sacrifices desirable metatheoretic properties, such as principality of types; hence, like $F\omega$, $R\omega$ may serve as a highly expressive intermediate or target language. Correspondingly, we perceive the addition of recursive terms and types to $R\omega$ to aid the adoption of recursion in surface languages.

We review the relevant syntax and typing of $R\omega$ now.

(Todo). It will be a challenge to trim this down, as [Hubers and Morris 2023] does with [Morris and McKinna 2019].

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3.1 Syntax

The syntax of $\text{R}\omega(\mathcal{T})$ is given in Figure 6.

Term variables x	Type variables α	Labels ℓ	Directions $d \in \{\text{L}, \text{R}\}$
Kinds	$\kappa ::= \star \mid \text{L} \mid \text{R}^\kappa \mid \kappa \rightarrow \kappa$		
Predicates	$\pi, \psi ::= \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho$		
Types	$\phi, \tau, v, \rho, \xi ::= \alpha \mid (\rightarrow) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa. \tau \mid \lambda \alpha : \kappa. \tau \mid \tau \tau$ $\mid \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \dots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho$		
Terms	$H, M, N, P ::= x \mid \lambda x : \tau. M \mid MN \mid \Lambda \alpha : \kappa. M \mid M [\tau]$ $\mid \ell \mid M \triangleright M \mid M/M \mid \text{prj}_d M \mid M \# M \mid \text{inj}_d M \mid M \nabla M$ $\mid \text{syn}_\phi M \mid \text{ana}_\phi M \mid \text{fold } M M M M$		
Environments	$\Gamma ::= \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi$		

Fig. 1. Syntax

3.2 Types and Kinds

Figure 2 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory \mathcal{T} .

$\boxed{\vdash \Gamma}$			
$(\text{C-EMP}) \frac{}{\vdash \varepsilon}$	$(\text{C-TVAR}) \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa}$	$(\text{C-VAR}) \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}$	$(\text{C-PRED}) \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$
$\boxed{\Gamma \vdash \tau : \kappa} \quad \boxed{\Gamma \vdash \pi}$			
$(\text{K-VAR}) \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}$	$(\text{K-}(\rightarrow)) \frac{\vdash \Gamma}{\Gamma \vdash (\rightarrow) : \star \rightarrow \star \rightarrow \star}$	$(\text{K-}\Rightarrow) \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$	
$(\text{K-V}) \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa. \tau : \star}$	$(\text{K-}\rightarrow I) \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2}$	$(\text{K-}\rightarrow E) \frac{\Gamma \vdash \tau_1 : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1 \tau_2 : \kappa_2}$	
$(\text{K-LAB}) \frac{\vdash \Gamma}{\Gamma \vdash \ell : \text{L}}$	$(\text{K-SING}) \frac{\Gamma \vdash \xi : \text{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star}$	$(\text{K-LTY}) \frac{\Gamma \vdash \xi : \text{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}$	$(\text{K-ROW}) \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \text{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \text{R}^\kappa}$
$(\text{K-II}) \frac{\Gamma \vdash \rho : \text{R}^\kappa}{\Gamma \vdash \Pi \rho : \kappa}$	$(\text{K-}\Sigma) \frac{\Gamma \vdash \rho : \text{R}^\kappa}{\Gamma \vdash \Sigma \rho : \kappa}$	$(\text{K-LIFT}_1) \frac{\Gamma \vdash \rho : \text{R}^{\kappa_1 \rightarrow \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho \tau : \text{R}^{\kappa_2}}$	
$(\text{K-LIFT}_2) \frac{\Gamma \vdash \phi : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \rho : \text{R}^{\kappa_1}}{\Gamma \vdash \phi \rho : \text{R}^{\kappa_2}}$	$(\text{K-}\lesssim_d) \frac{\Gamma \vdash \rho_i : \text{R}^\kappa}{\Gamma \vdash \rho_1 \lesssim_d \rho_2}$	$(\text{K-}\odot) \frac{\Gamma \vdash \rho_i : \text{R}^\kappa}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$	

Fig. 2. Contexts and kinding.

$$\begin{array}{c}
\boxed{\tau \equiv \tau} \quad \boxed{\pi \equiv \pi} \\
\\
(\text{E-REFL}) \frac{}{\tau \equiv \tau} \quad (\text{E-SYM}) \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \quad (\text{E-TRANS}) \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \quad (\text{E-}\beta) \frac{}{(\lambda \alpha : \kappa. \tau) v \equiv \tau[v/\alpha]} \\
\\
(\text{E-}\xi_{\Rightarrow}) \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \quad (\text{E-}\xi_{\forall}) \frac{\tau[Y/\alpha] \equiv v[Y/\beta]}{\forall \alpha : \kappa. \tau \equiv \forall \beta : \kappa. v} (\gamma \notin \text{fv}(\tau, v)) \quad (\text{E-}\xi_{\text{APP}}) \frac{\tau_i \equiv v_i}{\tau_1 \tau_2 \equiv v_1 v_2} \\
\\
(\text{E-}\xi_{\triangleright}) \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \quad (\text{E-ROW}) \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \quad (\text{E-}\xi_{[\cdot]}) \frac{\xi_1 \equiv \xi_2}{[\xi_1] \equiv [\xi_2]} \\
\\
(\text{E-LIFT}_1) \frac{}{\{\xi \triangleright \phi\} \tau \equiv \{\xi \triangleright \phi \tau\}} \quad (\text{E-LIFT}_2) \frac{}{\phi \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi \tau\}} \\
\\
(\text{E-}\xi_{\Pi\Sigma}) \frac{\rho_1 \equiv \rho_2}{K\rho_1 \equiv K\rho_2} \quad (\text{E-LIFT}_3) \frac{}{(K\rho) \tau \equiv K(\rho \tau)} \quad (\text{E-SING}) \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \quad (K \in \{\Pi, \Sigma\}) \\
\\
(\text{E-}\xi_{\lesssim_d}) \frac{\tau_i \equiv v_i}{\tau_1 \lesssim_d \tau_2 \equiv v_1 \lesssim_d v_2} \quad (\text{E-}\xi_{\odot}) \frac{\tau_i \equiv v_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv v_1 \odot v_2 \sim v_3}
\end{array}$$

Fig. 3. Type and predicate equivalence

3.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$\begin{array}{c}
(\text{T-VAR}) \frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \quad (\text{T} \rightarrow I) \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1. M : \tau_1 \rightarrow \tau_2} \quad (\text{T} \rightarrow E) \frac{\Gamma \vdash M_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1 M_2 : \tau_2} \\
\\
(\text{T} \equiv) \frac{\Gamma \vdash M : \tau \quad \tau \equiv v}{\Gamma \vdash M : v} \quad (\text{T} \Rightarrow I) \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \quad (\text{T} \Rightarrow E) \frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau} \\
\\
(\text{T} \forall I) \frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa. M : \forall \alpha : \kappa. \tau} \quad (\text{T} \forall E) \frac{\Gamma \vdash M : \forall \alpha : \kappa. \tau \quad \Gamma \vdash v : \kappa}{\Gamma \vdash M[v] : \tau[v/\alpha]} \\
\\
(\text{T-SING}) \frac{\vdash \Gamma}{\Gamma \vdash \ell : [\ell]} \quad (\text{T} \triangleright I) \frac{\Gamma \vdash M_1 : [\ell] \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \quad (\text{T} \triangleright E) \frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : [\ell]}{\Gamma \vdash M_1 / M_2 : \tau} \\
\\
(\text{T-}\Pi E) \frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \text{prj}_d M : \Pi \rho_2} \quad (\text{T-}\Pi I) \frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \# M_2 : \Pi \rho_3} \\
\\
(\text{T-}\Sigma I) \frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \text{inj} M : \Sigma \rho_2} \quad (\text{T-}\Sigma E) \frac{\Gamma \vdash M_1 : \Sigma \rho_1 \rightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \nabla M_2 : \Sigma \rho_3 \rightarrow \tau} \\
\\
(\text{T-ana}) \frac{\Gamma \vdash \rho : R^K \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : L, u : \kappa, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u \rightarrow \tau}{\Gamma \vdash \text{ana}_{\phi} M : \Sigma(\phi \rho) \rightarrow \tau} \\
\\
(\text{T-syn}) \frac{\Gamma \vdash \rho : R^K \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : L, u : \kappa, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u}{\Gamma \vdash \text{syn}_{\phi} M : \Pi(\phi \rho)} \\
\\
(\text{T-fold}) \frac{M_1 : \forall l : L, t : \star, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow t \rightarrow v \quad \Gamma \vdash M_2 : v \rightarrow v \rightarrow v \quad \Gamma \vdash M_3 : v \quad \Gamma \vdash N : \Pi \rho}{\Gamma \vdash \text{fold } M_1 M_2 M_3 N : v}
\end{array}$$

Fig. 4. Typing

Minimal Rows

Figure 5 gives the minimal row theory \mathcal{M} .

$$\begin{array}{c}
\boxed{\Gamma \vdash_m \rho : \kappa} \quad \boxed{\rho \equiv_m \rho} \\
\\
(K\text{-MROW}) \frac{\Gamma \vdash \xi : L \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_m \{\xi \triangleright \tau\} : R^\kappa} \quad (E\text{-MROW}) \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_m \{\xi' \triangleright \tau'\}} \\
\boxed{\Gamma \Vdash_m \pi} \\
\\
(N\text{-AX}) \frac{\pi \in \Gamma}{\Gamma \Vdash_m \pi} \quad (N\text{-REFL}) \frac{}{\Gamma \Vdash_m \rho \lesssim_d \rho} \quad (N\text{-TRANS}) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_m \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_3} \\
\\
(N\text{-}\equiv) \frac{\Gamma \Vdash_m \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_m \pi_2} \quad (N\text{-}\lesssim\text{LIFT}_1) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_m \phi \rho_1 \lesssim_d \phi \rho_2} \quad (N\text{-}\lesssim\text{LIFT}_2) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_m \rho_1 \tau \lesssim_d \rho_2 \tau} \\
\\
(N\text{-}\odot\text{LIFT}_1) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_1 \tau \odot \rho_2 \tau \sim \rho_3 \tau} \quad (N\text{-}\odot\text{LIFT}_2) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \phi \rho_1 \odot \phi \rho_2 \sim \phi \rho_3} \\
\\
(N\text{-}\odot\lesssim_L) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_1 \lesssim_L \rho_3} \quad (N\text{-}\odot\lesssim_R) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_2 \lesssim_R \rho_3}
\end{array}$$

Fig. 5. Minimal row theory $\mathcal{M} = \langle \vdash_m, \equiv_m, \Vdash_m \rangle$

4 IX: THE INDEX CALCULUS

4.1 Syntax

	Term variables	$x \ \alpha$
Sorts	$\sigma ::= \star \mid \mathcal{U}$	
Terms	$M, N, T ::= x \mid \star \mid \text{Nat} \mid \text{Zero} \mid \text{Suc } M \mid$ $\text{Ix } M \mid \text{FZero} \mid \text{FSuc } M \mid$ $\top \mid \text{tt} \mid$ $\Pi \alpha : T. N \mid \lambda x : T. N \mid M N \mid$ $\Sigma \alpha : T. M \mid (\alpha : T, M) \mid M.1 \mid M.2$ $M + N \mid \text{left } M \mid \text{right } M \mid \text{case } M \ M \ M$ $M \equiv N \mid \text{refl} \mid \dots$	
Environments	$\Gamma ::= \varepsilon \mid \Gamma, \alpha : T$	

Fig. 6. Syntax

4.2 Typing

$$\begin{array}{c}
\boxed{\vdash \Gamma} \\
\\
(\text{EMP}) \frac{}{\vdash \varepsilon} \quad (\text{VAR}) \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M} \\
\\
\boxed{\Gamma \vdash M : \sigma} \\
\\
(\star) \frac{}{\Gamma \vdash \star : \mathcal{U}} \quad (\top\star) \frac{}{\Gamma \vdash \top : \star} \quad (\top\mathcal{U}) \frac{}{\Gamma \vdash \top : \mathcal{U}} \\
\\
(\text{Var}) \frac{\alpha : \sigma \in \Gamma}{\Gamma \vdash \alpha : \sigma} \quad (\text{NAT}) \frac{}{\Gamma \vdash \text{Nat} : \star} \quad (\text{Ix}) \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Ix } n : \star} \\
\\
(\text{II}) \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Pi \alpha : M. N : \star} \quad (\Sigma) \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Sigma \alpha : M. N : \star} \\
\\
(+) \frac{\Gamma \vdash M : \star \quad \Gamma \vdash N : \star}{\Gamma \vdash M + N : \star} \quad (\equiv) \frac{\Gamma \vdash M_1 : N_1 \quad \Gamma \vdash N_1 : \star \quad \Gamma \vdash M_2 : N_2 \quad \Gamma \vdash N_2 : \star}{\Gamma \vdash M_1 \equiv M_2 : \star} \\
\\
\boxed{\Gamma \vdash M : N} \\
\\
(\text{Var}) \frac{x : M \in \Gamma}{\Gamma \vdash x : M} \quad (\text{tt}) \frac{}{\Gamma \vdash \text{tt} : \top} \\
\\
(\text{Zero}) \frac{}{\Gamma \vdash \text{Zero} : \text{Nat}} \quad (\text{Suc}) \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Suc } n : \text{Nat}} \\
\\
(\text{FZero}) \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{FZero} : \text{Ix} (\text{Suc } n)} \quad (\text{FSuc}) \frac{\Gamma \vdash n : \text{Nat} \quad \Gamma \vdash i : \text{Ix } n}{\Gamma \vdash \text{FSuc } i : \text{Ix} (\text{Suc } n)} \\
\\
(\text{PII}) \frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T. M : \Pi(x : T). N} \quad (\text{PIE}) \frac{\Gamma \vdash M : \Pi(x : T_1). T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M N : T_2[N/x]} \\
\\
(\Sigma I) \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1). T_2} \quad (\Sigma E_1) \frac{\Gamma \vdash M : \Sigma(x : T_1). T_2}{\Gamma \vdash M.1 : T_1} \quad (\Sigma E_2) \frac{\Gamma \vdash M : \Sigma(x : T_1). T_2}{\Gamma \vdash M.2 : T_2[M.1/x]} \\
\\
(\equiv I) \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \text{refl} : M \equiv M} \quad (\text{CONV}) \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma}{\Gamma \vdash M : T_2} \\
\\
\begin{array}{c}
\Gamma \vdash P : T_1 \equiv T_2 \\
\Gamma \vdash M : T_1 \\
\Gamma \vdash N : T_1 \\
\Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\
\Gamma, z : T_1 \vdash H : T[z/x, z/y, \text{refl}/p] \\
(\equiv E) \frac{}{\Gamma \vdash \mathcal{J} H M N P : T[M/x, N/y, P/p]}
\end{array}
\end{array}$$

Fig. 7. Context formation and typing rules for Ix terms

Let the meta-syntax τ denote both sort and term.

$$\begin{array}{c}
\boxed{\Gamma \vdash M = N : \sigma} \\
\text{(E-REFL)} \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \quad \text{(E-SYM)} \frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \quad \text{(E-TRANS)} \frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma} \\
\boxed{\Gamma \vdash M = N : T} \\
\text{(C-REFL)} \frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \quad \text{(C-SYM)} \frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \quad \text{(C-TRANS)} \frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}
\end{array}$$

Fig. 8. Definitional equality & computational laws

5 TRANSLATION FROM $R\omega$

5.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of $R\omega$ types. Figure 9 describe the untyped translation, which is used to show translational soundness of the typed translation.

$$\begin{array}{l}
\boxed{(\kappa)^\bullet} \\
(\star)^\bullet = \star \\
(\mathsf{L})^\bullet = \top \\
(\kappa_1 \rightarrow \kappa_2)^\bullet = \Pi(\alpha : (\kappa_1)^\bullet).(\kappa_2)^\bullet \\
(\mathsf{R}^K)^\bullet = \Sigma(n : \mathsf{Nat}).\Pi(j : \mathsf{Ix } n).(\kappa)^\bullet
\end{array}$$

$$\begin{array}{l}
\boxed{(\sigma)^\bullet} \\
(\alpha)^\bullet = \alpha \\
(\tau_1 \rightarrow \tau_2)^\bullet = \Pi(\alpha : (\tau_1)^\bullet).(\tau_2)^\bullet \\
(\forall \alpha : \kappa. \tau)^\bullet = \Pi(\alpha : (\kappa)^\bullet).(\tau)^\bullet \\
(\lambda \alpha : \kappa. \tau)^\bullet = \Pi(\alpha : (\kappa)^\bullet).(\tau)^\bullet
\end{array}$$

$$\boxed{(M)^\bullet}$$

...

$$\boxed{(\pi)^\bullet}$$

...

Fig. 9. A compositional translation of $R\omega$ judgments to (untyped) Ix terms

5.2 Typed translation

$$\begin{array}{c}
 \boxed{\Gamma \vdash \tau \rightsquigarrow v : \kappa} \\
 \\
 \text{(C-FOO)} \frac{A}{B} \\
 \boxed{\Gamma \vdash M \rightsquigarrow N : \tau} \\
 \\
 \text{(C-FOO)} \frac{A}{B} \\
 \boxed{\Gamma \Vdash \pi \rightsquigarrow N} \\
 \\
 \text{(C-FOO)} \frac{A}{B} \\
 \boxed{\tau \equiv v \rightsquigarrow P} \\
 \\
 \text{(C-FOO)} \frac{A}{B}
 \end{array}$$

Fig. 10. Translation of $R\omega$ derivations to Ix derivations

5.3 Properties of Translation

THEOREM 1 (TRANSLATIONAL SOUNDNESS (TYPES)). *if $\Gamma \vdash \tau : \kappa$ such that $\Gamma \vdash \tau \rightsquigarrow v : \kappa$ then $(\Gamma)^\bullet \vdash v : (\kappa)^\bullet$.*

The following is bullshit w.r.t. definitional equality.

THEOREM 2 (TRANSLATIONAL SOUNDNESS (TYPE EQUIVALENCE)). *if $\Gamma \vdash \tau_1 \rightsquigarrow v_1 : \kappa_1$ and $\Gamma \vdash \tau_2 \rightsquigarrow v_2 : \kappa_2$ such that $\tau_1 \equiv \tau_2$ is derivable in $R\omega$, then $(\Gamma)^\bullet \vdash v_1 \equiv v_2$.*

The next theorems presume an $R\omega$ instantiation of the simple row theory.

THEOREM 3 (TRANSLATIONAL SOUNDNESS (ROW COMBINATION)). *if $\Gamma \Vdash \rho_1 \cdot \rho_2 \sim \rho_3 \rightsquigarrow N$ then $(\Gamma)^\bullet \vdash N : \text{foobar}$.*

THEOREM 4 (TRANSLATIONAL SOUNDNESS (ROW CONTAINMENT)). *if $\Gamma \Vdash \rho_1 \lesssim \rho_2 \rightsquigarrow N$ then $(\Gamma)^\bullet \vdash N : \text{foobar}$.*

Finally,

THEOREM 5 (TRANSLATIONAL SOUNDNESS). *if $\Gamma \vdash M : \tau$ such that $\Gamma \vdash M \rightsquigarrow N : \tau$ then $(\Gamma)^\bullet \vdash M : (\tau)^\bullet$.*

6 OPERATIONAL SEMANTICS OF IX

7 RECURSION

This section will later be incorporated into earlier sections.

7.1 Rome, or, $R\omega$ with μ

todo

$$(C\text{-}FOO) \frac{A}{B}$$

Fig. 11. Additional $R\omega$ judgments for recursion

7.2 Mix, the recursive index calculus

todo

$$(C\text{-}FOO) \frac{A}{B}$$

Fig. 12. Additional lx judgments for recursion

7.3 Translation and properties of translation

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