The Recursive Index Calculus and Its Translation From Rω

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1 μIx, The Recursive Index Calculus

1.1 Syntax

$$\text{Term variables } x \qquad \text{Type variables } \alpha$$

Kinds	κ ::=	$\text{Nat} \mid \star \mid \kappa \to \kappa$
Types	τ, υ ::=	$\top \mid \alpha \mid (\to) \mid \forall \alpha : \kappa.\tau \mid \exists \alpha : \kappa.\tau \mid \lambda \alpha : \kappa.\tau \mid \tau\,\tau$
Terms	M, N ::=	$x \mid \lambda x : \tau.M \mid M\,N \mid \Lambda \alpha : \kappa.M \mid M\,[\tau] \mid ...$
Environments	Γ ::=	$\varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau$

Figure 1: Syntax

1.2 Kinding

Presume a set of type-level constants C containing at least the following:

$$
\begin{aligned}
(\to) \ &::= \ \star \to \star \to \star \\
\text{Zero} \ &::= \ \text{Nat} \\
\text{Suc} \ &::= \ \text{Nat} \to \text{Nat}
\end{aligned}
$$

$$\boxed{\vdash \Gamma}$$

$$(\text{C-EMP})\ \frac{}{\vdash \varepsilon} \qquad (\text{C-TVAR})\ \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \qquad (\text{C-VAR})\ \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}$$

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$(\text{K-}\top)\ \frac{\vdash \Gamma}{\Gamma \vdash \top : \star} \qquad (\text{K-VAR})\ \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}$$

$$(\text{K-}\forall)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa.\tau : \star} \qquad (\text{K-}\exists)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \exists \alpha : \kappa.\tau : \star}$$

$$(\text{K-}{\to}I)\ \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1.\tau : \kappa_1 \to \kappa_2} \qquad (\text{K-}{\to}E)\ \frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\, \tau_2 : \kappa_2}$$

$$(\text{K-}\sim)\ \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma \vdash \tau_2 : \star}{\Gamma \vdash \tau_1 \sim \tau_2 : \star} \quad (\text{K-IX})\ \frac{\Gamma \vdash \tau : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\,\tau : \star}$$

$$(\text{K-}\mu)\ \frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \mu\tau : \star} \qquad (\text{K-}\nu)\ \frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \nu\tau : \star}$$

Figure 2: Contexts and kinding.

1.3   Typing

2   Adding Recursion to Rω

The static semantics of Rω, as defined in **?**, are given in Appendix A for reference. We define only the syntax and rules necessary for least- and greatest-fixed points with general term-level recursion, which are routine.

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \mu\tau : \star} \qquad \frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \nu\tau : \star}$$

$$\boxed{\Gamma \vdash M : \tau}$$

...

3   Translating μIx from Rω

Rules for the System F_ω fragment of Rω have a trivial correspondence to the F_ω fragment of μIx and are omitted. The syntax and typing judgments on the left are that of Rω (see Appendix A); on the right are μIx.

$$\boxed{[\![\kappa]\!]}$$

$$[\![\star]\!] = \star$$
$$[\![\mathsf{L}]\!] = \star$$
$$[\![\kappa_1 \to \kappa_2]\!] = [\![\kappa_1]\!] \to [\![\kappa_2]\!]$$
$$[\![\mathsf{R}^\kappa]\!] = \mathrm{Nat} \to [\![\kappa]\!]$$

$$\boxed{[\![\Gamma \vdash \tau : \kappa]\!]}$$

$$\left[\!\!\left[\frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Pi\rho : \kappa}\right]\!\!\right] = \forall n : \mathrm{Nat}. Ix\ n \to [\![\Gamma \vdash \rho : \mathsf{R}^\kappa]\!]\ n$$

$$\left[\!\!\left[\frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Sigma\rho : \kappa}\right]\!\!\right] = \exists n : \mathrm{Nat}. [\![\Gamma \vdash \rho : \mathsf{R}^\kappa]\!]\ n$$

$$\left[\!\!\left[\frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}}\right]\!\!\right] = \top$$

$$\left[\!\!\left[\frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor\xi\rfloor : \star}\right]\!\!\right] = \top$$

$$\left[\!\!\left[\frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}\right]\!\!\right] = [\![\Gamma \vdash \tau : \kappa]\!]$$

$$\left[\!\!\left[\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\,\tau : \mathsf{R}^{\kappa_2}}\right]\!\!\right] = \lambda n : \mathrm{Nat}. [\![\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2}]\!]\ n\ [\![\Gamma \vdash \tau : \kappa_1]\!]$$

$$\left[\!\!\left[\frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\,\rho : \mathsf{R}^{\kappa_2}}\right]\!\!\right] = \lambda n : \mathrm{Nat}. [\![\Gamma \vdash \phi : \kappa_1 \to \kappa_2]\!]\ n\ [\![\Gamma \vdash \rho : \mathsf{R}^{\kappa_1}]\!]$$

$$\left[\!\!\left[\frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}\right]\!\!\right] = [\![\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa]\!]$$

$$\left[\!\!\left[\frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}\right]\!\!\right] = [\![\pi]\!] \to [\![\tau]\!]$$

$$\boxed{[\![\Gamma \vdash \pi]\!]}$$

...

Figure 3: A compositional translation of Rω to μIx

A The static semantics of $\mathrm{R}\omega$

A.1 Syntax

The syntax of $\mathrm{R}\omega(\mathcal{T})$ is given in Figure 4.

Term variables x Type variables α Labels ℓ Directions $d \in \{\mathsf{L}, \mathsf{R}\}$

$$
\begin{array}{lrl}
\text{Kinds} & \kappa &::= \star \mid \mathsf{L} \mid \mathsf{R}^\kappa \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi &::= \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi &::= \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa.\tau \mid \lambda\alpha : \kappa.\tau \mid \tau\,\tau \\
& & \mid \quad \ell \mid \lfloor \xi \rfloor \mid \xi \rhd \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi\rho \mid \Sigma\rho \\
\text{Terms} & M, N &::= x \mid \lambda x : \tau.M \mid M\,N \mid \Lambda\alpha : \kappa.M \mid M\,[\tau] \\
& & \mid \quad \ell \mid M \rhd M \mid M/M \mid \mathsf{prj}_d\,M \mid M \mathbin{+\!\!+} M \mid \mathsf{inj}_d\,M \mid M \bigtriangledown M \\
& & \mid \quad \mathsf{syn}_\phi\,M \mid \mathsf{ana}_\phi\,M \mid \mathsf{fold}\,M\,M\,M\,M \\
\text{Environments} & \Gamma &::= \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}
$$

Figure 4: Syntax

A.2 Types and Kinds

Figure 5 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

$$\boxed{\vdash \Gamma}$$

$$(\text{C-EMP}) \ \frac{}{\vdash \varepsilon} \qquad (\text{C-TVAR}) \ \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \qquad (\text{C-VAR}) \ \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau} \qquad (\text{C-PRED}) \ \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$$

$$\boxed{\Gamma \vdash \tau : \kappa} \ \boxed{\Gamma \vdash \pi}$$

$$(\text{K-VAR}) \ \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \qquad (\text{K-}(\to)) \ \frac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star} \qquad (\text{K-}\Rightarrow) \ \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$$

$$(\text{K-}\forall) \ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa.\tau : \star} \qquad (\text{K-}{\to}I) \ \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1.\tau : \kappa_1 \to \kappa_2} \qquad (\text{K-}{\to}E) \ \frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1 \, \tau_2 : \kappa_2}$$

$$(\text{K-LAB}) \ \frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}} \qquad (\text{K-SING}) \ \frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star} \qquad (\text{K-LTY}) \ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa} \qquad (\text{K-ROW}) \ \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}$$

$$(\text{K-}\Pi) \ \frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Pi \rho : \kappa} \qquad (\text{K-}\Sigma) \ \frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Sigma \rho : \kappa} \qquad (\text{K-LIFT}_1) \ \frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho \, \tau : \mathsf{R}^{\kappa_2}}$$

$$(\text{K-LIFT}_2) \ \frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi \, \rho : \mathsf{R}^{\kappa_2}} \qquad (\text{K-}\lesssim_d) \ \frac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \lesssim_d \rho_2} \qquad (\text{K-}\odot) \ \frac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$$

Figure 5: Contexts and kinding.

$$\boxed{\tau \equiv \tau}\;\;\boxed{\pi \equiv \pi}$$

$$\left(\text{E-REFL}\right)\; \frac{}{\tau \equiv \tau} \qquad \left(\text{E-SYM}\right)\; \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad \left(\text{E-TRANS}\right)\; \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad \left(\text{E-}\beta\right)\; \frac{}{(\lambda\alpha:\kappa.\tau)\,\upsilon \equiv \tau[\upsilon/\alpha]}$$

$$\left(\text{E-}\xi_\Rightarrow\right)\; \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad \left(\text{E-}\xi_\forall\right)\; \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha:\kappa.\tau \equiv \forall\beta:\kappa.\upsilon}\;(\gamma \notin fv(\tau,\upsilon)) \qquad \left(\text{E-}\xi_{\text{APP}}\right)\; \frac{\tau_i \equiv \upsilon_i}{\tau_1\,\tau_2 \equiv \upsilon_1\,\upsilon_2}$$

$$\left(\text{E-}\xi_\triangleright\right)\; \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \qquad \left(\text{E-ROW}\right)\; \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_\tau \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \qquad \left(\text{E-}\xi_{\lfloor\cdot\rfloor}\right)\; \frac{\xi_1 \equiv \xi_2}{\lfloor \xi_1 \rfloor \equiv \lfloor \xi_2 \rfloor}$$

$$\left(\text{E-LIFT}_1\right)\; \frac{}{\{\xi \triangleright \phi\}\,\tau \equiv \{\xi \triangleright \phi\,\tau\}} \qquad \left(\text{E-LIFT}_2\right)\; \frac{}{\phi\,\{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi\,\tau\}}$$

$$\left(\text{E-}\xi_{\Pi\Sigma}\right)\; \frac{\rho_1 \equiv \rho_2}{K\,\rho_1 \equiv K\,\rho_2} \qquad \left(\text{E-LIFT}_3\right)\; \frac{}{(K\rho)\,\tau \equiv K(\rho\,\tau)} \qquad \left(\text{E-SING}\right)\; \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau}\qquad (K \in \{\Pi, \Sigma\})$$

$$\left(\text{E-}\xi_{\lesssim_d}\right)\; \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad \left(\text{E-}\xi_\odot\right)\; \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Figure 6: Type and predicate equivalence

A.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$(\text{T-VAR}) \dfrac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$
$\quad (\text{T-}\!\!\rightarrow\!\! I) \dfrac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1.M : \tau_1 \rightarrow \tau_2}$
$\quad (\text{T-}\!\!\rightarrow\!\! E) \dfrac{\Gamma \vdash M_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash M_2}{\Gamma \vdash M_1\, M_2 : \tau_2}$

$(\text{T-}\!\!\equiv\!) \dfrac{\Gamma \vdash M : \tau \quad \tau \equiv \upsilon}{\Gamma \vdash M : \upsilon}$
$\quad (\text{T-}\!\!\Rightarrow\!\! I) \dfrac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau}$
$\quad (\text{T-}\!\!\Rightarrow\!\! E) \dfrac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$

$(\text{T-}\forall I) \dfrac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa.M : \forall \alpha : \kappa.\tau}$
$\quad (\text{T-}\forall E) \dfrac{\Gamma \vdash M : \forall \alpha : \kappa.\tau \quad \Gamma \vdash \upsilon : \kappa}{\Gamma \vdash M\,[\upsilon] : \tau[\upsilon/\alpha]}$

$(\text{T-SING}) \dfrac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor}$
$\quad (\text{T-}\!\triangleright\! I) \dfrac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau}$
$\quad (\text{T-}\!\triangleright\! E) \dfrac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1/M_2 : \tau}$

$(\text{T-}\Pi E) \dfrac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\, M : \Pi \rho_2}$
$\quad (\text{T-}\Pi I) \dfrac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 +\!\!\!+ M_2 : \Pi \rho_3}$

$(\text{T-}\Sigma I) \dfrac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim_d \rho_2}{\Gamma \vdash \mathsf{inj}_d\, M : \Sigma \rho_2}$
$\quad (\text{T-}\Sigma E) \dfrac{\Gamma \vdash M_1 : \Sigma \rho_1 \rightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim}{\Gamma \vdash M_1 \triangledown M_2 : \Sigma \rho_3 \rightarrow \tau}$

$(\text{T-ana}) \dfrac{\begin{array}{c} \Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \rightarrow \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa.\,(y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \rightarrow \phi\,u \rightarrow \tau \end{array}}{\Gamma \vdash \mathsf{ana}_\phi\, M : \Sigma(\phi\,\rho) \rightarrow \tau}$

$(\text{T-syn}) \dfrac{\begin{array}{c} \Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \rightarrow \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa.\,(y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \rightarrow \phi\,u \end{array}}{\Gamma \vdash \mathsf{syn}_\phi\, M : \Pi(\phi\,\rho)}$

$(\text{T-fold}) \dfrac{\begin{array}{c} M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^\kappa.\,(y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \rightarrow t \rightarrow \upsilon \\ \Gamma \vdash M_2 : \upsilon \rightarrow \upsilon \rightarrow \upsilon \quad \Gamma \vdash M_3 : \upsilon \quad \Gamma \vdash N : \Pi \rho \end{array}}{\Gamma \vdash \mathsf{fold}\, M_1\, M_2\, M_3\, N : \upsilon}$

Figure 7: Typing

Minimal Rows

Figure 8 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathsf{m}} \rho : \kappa} \quad \boxed{\rho \equiv_{\mathsf{m}} \rho}$$

$$(\text{K-MROW}) \; \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathsf{m}} \{\xi \triangleright \tau\} : \mathsf{R}^\kappa} \qquad (\text{E-MROW}) \; \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathsf{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathsf{m}} \pi}$$

$$(\text{N-AX}) \; \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathsf{m}} \pi} \qquad (\text{N-REFL}) \; \frac{}{\Gamma \Vdash_{\mathsf{m}} \rho \lesssim_d \rho} \qquad (\text{N-TRANS}) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\text{N-}\equiv) \; \frac{\Gamma \Vdash_{\mathsf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathsf{m}} \pi_2} \qquad (\text{N-}\lesssim\text{LIFT}_1) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \lesssim_d \phi\,\rho_2} \qquad (\text{N-}\lesssim\text{LIFT}_2) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \lesssim_d \rho_2\,\tau}$$

$$(\text{N-}\odot\text{LIFT}_1) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \odot \rho_2\,\tau \sim \rho_3\,\tau} \qquad (\text{N-}\odot\text{LIFT}_2) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \odot \phi\,\rho_2 \sim \phi\,\rho_3}$$

$$(\text{N-}\odot\lesssim_{\mathsf{L}}) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\text{N-}\odot\lesssim_{\mathsf{R}}) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Figure 8: Minimal row theory $\mathcal{M} = \langle \vdash_{\mathsf{m}}, \equiv_{\mathsf{m}}, \Vdash_{\mathsf{m}} \rangle$