The Recursive Index Calculus and Its Translation From Rω

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1 μIx, The Recursive Index Calculus

1.1 Syntax

$$\text{Term variables } x \qquad \text{Type variables } \alpha$$

$$\begin{array}{lrcl}
\text{Kinds} & \kappa & ::= & \text{Nat} \mid \star \mid \kappa \to \kappa \\
\text{Types} & \tau, \upsilon & ::= & \top \mid \alpha \mid (\to) \mid \forall \alpha : \kappa.\tau \mid \exists \alpha : \kappa.\tau \mid \lambda \alpha : \kappa.\tau \mid \tau\, \tau \\
\text{Terms} & M, N & ::= & x \mid \lambda x : \tau.M \mid M\; N \mid \Lambda \alpha : \kappa.M \mid M\; [\tau] \mid \ldots \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau
\end{array}$$

Figure 1: Syntax

1.2 Kinding

Presume a set of type-level constants C containing at least the following:

$$\begin{array}{rcl}
(\to) & ::= & \star \to \star \to \star \\
\text{Zero} & ::= & \text{Nat} \\
\text{Suc} & ::= & \text{Nat} \to \text{Nat}
\end{array}$$

$$\boxed{\vdash \Gamma}$$

$$(\textsc{c-emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{c-tvar})\ \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \qquad (\textsc{c-var})\ \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}$$

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$(\textsc{k-}\top)\ \frac{\vdash \Gamma}{\Gamma \vdash \top : \star} \qquad (\textsc{k-var})\ \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}$$

$$(\textsc{k-}\forall)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa . \tau : \star} \qquad (\textsc{k-}\exists)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \exists \alpha : \kappa . \tau : \star}$$

$$(\textsc{k-}{\to}I)\ \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1 . \tau : \kappa_1 \to \kappa_2} \qquad (\textsc{k-}{\to}E)\ \frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\, \tau_2 : \kappa_2}$$

$$(\textsc{k-}\sim)\ \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma \vdash \tau_2 : \star}{\Gamma \vdash \tau_1 \sim \tau_2 : \star} \ (\textsc{k-Ix})\ \frac{\Gamma \vdash \tau : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\, \tau : \star}$$

$$(\textsc{k-}\mu)\ \frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \mu \tau : \star} \qquad (\textsc{k-}\nu)\ \frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \nu \tau : \star}$$

Figure 2: Contexts and kinding.

1.3 Typing

2 Adding Recursion to Rω

The static semantics of Rω, as defined in **?**, are given in Appendix A for reference. We define only the syntax and rules necessary for least- and greatest-fixed points with general term-level recursion, which are routine.

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \mu \tau : \star} \qquad \frac{\Gamma \vdash \tau : \star \to \star}{\Gamma \vdash \nu \tau : \star}$$

$$\boxed{\Gamma \vdash M : \tau}$$

...

3 Translating μIx from Rω

Rules for the System F_ω fragment of Rω have a trivial correspondence to the F_ω fragment of μIx and are omitted. The syntax and typing judgments on the left are that of Rω (see Appendix A); on the right are μIx.

$$\boxed{[\![\kappa]\!]}$$

$$\begin{aligned}
[\![\star]\!] &= \star \\
[\![\mathsf{L}]\!] &= \star \\
[\![\kappa_1 \to \kappa_2]\!] &= [\![\kappa_1]\!] \to [\![\kappa_2]\!] \\
[\![\mathsf{R}^\kappa]\!] &= \mathrm{Nat} \to [\![\kappa]\!]
\end{aligned}$$

$$\boxed{[\![\Gamma \vdash \tau : \kappa]\!]}$$

$$\begin{aligned}
[\![\frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Pi\rho : \kappa}]\!] &= \forall n : \mathrm{Nat}. \mathit{Ix}\ n \to [\![\rho]\!]\ n \\
[\![\frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Sigma\rho : \kappa}]\!] &= \exists n : \mathrm{Nat}. [\![\rho]\!]\ n \\
[\![\frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}}]\!] &= \top \\
[\![\frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star}]\!] &= \top \\
[\![\frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}]\!] &= [\![\Gamma \vdash \tau : \kappa]\!] \\
[\![\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\, \tau : \mathsf{R}^{\kappa_2}}]\!] &= \lambda n : \mathrm{Nat}. [\![\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2}]\!]\ n\ [\![\Gamma \vdash \tau : \kappa_1]\!] \\
[\![\frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\, \rho : \mathsf{R}^{\kappa_2}}]\!] &= \lambda n : \mathrm{Nat}. [\![\Gamma \vdash \phi : \kappa_1 \to \kappa_2]\!]\ n\ [\![\Gamma \vdash \rho : \mathsf{R}^{\kappa_1}]\!] \\
[\![\frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}]\!] &= [\![\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa]\!] \\
[\![\frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}]\!] &= [\![\pi]\!] \to [\![\tau]\!]
\end{aligned}$$

$$\boxed{[\![\Gamma \vdash \pi]\!]}$$

...

Figure 3: A compositional translation of Rω to μIx

A The static semantics of Rω

A.1 Syntax

The syntax of R$\omega(\mathcal{T})$ is given in Figure 4.

$$\text{Term variables } x \qquad \text{Type variables } \alpha \qquad \text{Labels } \ell \qquad \text{Directions } d \in \{\mathsf{L}, \mathsf{R}\}$$

$$\begin{array}{llcl}
\text{Kinds} & \kappa & ::= & \star \mid \mathsf{L} \mid \mathsf{R}^{\kappa} \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi & ::= & \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi & ::= & \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa . \tau \mid \lambda \alpha : \kappa . \tau \mid \tau\, \tau \\
 & & \mid & \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho \\
\text{Terms} & M, N & ::= & x \mid \lambda x : \tau . M \mid M\ N \mid \Lambda \alpha : \kappa . M \mid M\ [\tau] \\
 & & \mid & \ell \mid M \triangleright M \mid M / M \mid \mathsf{prj}_d\ M \mid M +\!\!+\ M \mid \mathsf{inj}_d\ M \mid M \bigtriangledown M \\
 & & \mid & \mathsf{syn}_\phi\ M \mid \mathsf{ana}_\phi\ M \mid \mathsf{fold}\ M\ M\ M\ M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}$$

Figure 4: Syntax

A.2 Types and Kinds

Figure 5 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

$$\boxed{\vdash \Gamma}$$

$$(\textsc{c-emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{c-tvar})\ \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \qquad (\textsc{c-var})\ \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau} \qquad (\textsc{c-pred})\ \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$$

$$\boxed{\Gamma \vdash \tau : \kappa}\ \boxed{\Gamma \vdash \pi}$$

$$(\textsc{k-var})\ \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \qquad (\textsc{k-}(\to))\ \frac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star} \qquad (\textsc{k-}\Rightarrow)\ \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$$

$$(\textsc{k-}\forall)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa . \tau : \star} \qquad (\textsc{k-}\to I)\ \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1 . \tau : \kappa_1 \to \kappa_2} \qquad (\textsc{k-}\to E)\ \frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\, \tau_2 : \kappa_2}$$

$$(\textsc{k-lab})\ \frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}} \qquad (\textsc{k-sing})\ \frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star} \qquad (\textsc{k-lty})\ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa} \qquad (\textsc{k-row})\ \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}$$

$$(\textsc{k-}\Pi)\ \frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Pi \rho : \kappa} \qquad (\textsc{k-}\Sigma)\ \frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Sigma \rho : \kappa} \qquad (\textsc{k-lift}_1)\ \frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\, \tau : \mathsf{R}^{\kappa_2}}$$

$$(\textsc{k-lift}_2)\ \frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\, \rho : \mathsf{R}^{\kappa_2}} \qquad (\textsc{k-}\lesssim_d)\ \frac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \lesssim_d \rho_2} \qquad (\textsc{k-}\odot)\ \frac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$$

Figure 5: Contexts and kinding.

$$\boxed{\tau \equiv \tau} \; \boxed{\pi \equiv \pi}$$

$$(\textsc{E-Refl}) \; \frac{}{\tau \equiv \tau} \qquad (\textsc{E-Sym}) \; \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad (\textsc{E-Trans}) \; \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad (\textsc{E-}\beta) \; \frac{}{(\lambda\alpha : \kappa.\tau)\, \upsilon \equiv \tau[\upsilon/\alpha}$$

$$(\textsc{E-}\xi_{\Rightarrow}) \; \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad (\textsc{E-}\xi_{\forall}) \; \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha : \kappa.\tau \equiv \forall\beta : \kappa.\upsilon} \; (\gamma \notin fv(\tau, \upsilon)) \qquad (\textsc{E-}\xi_{\textsc{app}}) \; \frac{\tau_i \equiv \upsilon_i}{\tau_1\, \tau_2 \equiv \upsilon_1\, \upsilon_2}$$

$$(\textsc{E-}\xi_{\triangleright}) \; \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \qquad (\textsc{E-Row}) \; \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \qquad (\textsc{E-}\xi_{\lfloor \cdot \rfloor}) \; \frac{\xi_1 \equiv \xi_2}{\lfloor \xi_1 \rfloor \equiv \lfloor \xi_2 \rfloor}$$

$$(\textsc{E-Lift}_1) \; \frac{}{\{\xi \triangleright \phi\}\, \tau \equiv \{\xi \triangleright \phi\, \tau\}} \qquad (\textsc{E-Lift}_2) \; \frac{}{\phi\, \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi\, \tau\}}$$

$$(\textsc{E-}\xi_{\Pi\Sigma}) \; \frac{\rho_1 \equiv \rho_2}{K \rho_1 \equiv K \rho_2} \qquad (\textsc{E-Lift}_3) \; \frac{}{(K \rho)\, \tau \equiv K (\rho\, \tau)} \qquad (\textsc{E-Sing}) \; \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \qquad (K \in \{\Pi, \Sigma\})$$

$$(\textsc{E-}\xi_{\lesssim_d}) \; \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad (\textsc{E-}\xi_{\odot}) \; \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Figure 6: Type and predicate equivalence

A.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$(\textsc{t-var})\ \frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad (\textsc{t-}{\to}I)\ \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . M : \tau_1 \to \tau_2} \qquad (\textsc{t-}{\to}E)\ \frac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2}{\Gamma \vdash M_1\ M_2 : \tau_2}$$

$$(\textsc{t-}{\equiv})\ \frac{\Gamma \vdash M : \tau \quad \tau \equiv \upsilon}{\Gamma \vdash M : \upsilon} \qquad (\textsc{t-}{\Rightarrow}I)\ \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \qquad (\textsc{t-}{\Rightarrow}E)\ \frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$$

$$(\textsc{t-}\forall I)\ \frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa . M : \forall \alpha : \kappa . \tau} \qquad (\textsc{t-}\forall E)\ \frac{\Gamma \vdash M : \forall \alpha : \kappa . \tau \quad \Gamma \vdash \upsilon : \kappa}{\Gamma \vdash M\ [\upsilon] : \tau[\upsilon/\alpha]}$$

$$(\textsc{t-sing})\ \frac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor} \qquad (\textsc{t-}{\triangleright}I)\ \frac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \qquad (\textsc{t-}{\triangleright}E)\ \frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1 / M_2 : \tau}$$

$$(\textsc{t-}\Pi E)\ \frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\ M : \Pi \rho_2} \qquad (\textsc{t-}\Pi I)\ \frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 +\!\!+\ M_2 : \Pi \rho_3}$$

$$(\textsc{t-}\Sigma I)\ \frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim_d \rho_2}{\Gamma \vdash \mathsf{inj}_d\ M : \Sigma \rho_2} \qquad (\textsc{t-}\Sigma E)\ \frac{\Gamma \vdash M_1 : \Sigma \rho_1 \to \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \to \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim}{\Gamma \vdash M_1 \triangledown M_2 : \Sigma \rho_3 \to \tau}$$

$$(\textsc{t-}\mathsf{ana})\ \frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^{\kappa} \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^{\kappa} . (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u \to \tau\end{array}}{\Gamma \vdash \mathsf{ana}_{\phi}\ M : \Sigma(\phi\, \rho) \to \tau}$$

$$(\textsc{t-}\mathsf{syn})\ \frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^{\kappa} \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^{\kappa} . (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u\end{array}}{\Gamma \vdash \mathsf{syn}_{\phi}\ M : \Pi(\phi\, \rho)}$$

$$(\textsc{t-}\mathsf{fold})\ \frac{\begin{array}{c}M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^{\kappa} . (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to t \to \upsilon \\ \Gamma \vdash M_2 : \upsilon \to \upsilon \to \upsilon \quad \Gamma \vdash M_3 : \upsilon \quad \Gamma \vdash N : \Pi \rho\end{array}}{\Gamma \vdash \mathsf{fold}\ M_1\ M_2\ M_3\ N : \upsilon}$$

Figure 7: Typing

Minimal Rows

Figure 8 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathsf{m}} \rho : \kappa} \; \boxed{\rho \equiv_{\mathsf{m}} \rho}$$

$$(\text{K-MROW})\,\frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathsf{m}} \{\xi \triangleright \tau\} : \mathsf{R}^{\kappa}} \qquad (\text{E-MROW})\,\frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathsf{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathsf{m}} \pi}$$

$$(\text{N-AX})\,\frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathsf{m}} \pi} \qquad (\text{N-REFL})\,\frac{}{\Gamma \Vdash_{\mathsf{m}} \rho \lesssim_d \rho} \qquad (\text{N-TRANS})\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\text{N-}\equiv)\,\frac{\Gamma \Vdash_{\mathsf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathsf{m}} \pi_2} \qquad (\text{N-}\lesssim\text{LIFT}_1)\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \lesssim_d \phi\,\rho_2} \qquad (\text{N-}\lesssim\text{LIFT}_2)\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \lesssim_d \rho_2\,\tau}$$

$$(\text{N-}\odot\text{LIFT}_1)\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \odot \rho_2\,\tau \sim \rho_3\,\tau} \qquad (\text{N-}\odot\text{LIFT}_2)\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \odot \phi\,\rho_2 \sim \phi\,\rho_3}$$

$$(\text{N-}\odot\lesssim_{\mathsf{L}})\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\text{N-}\odot\lesssim_{\mathsf{R}})\,\frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Figure 8: Minimal row theory $\mathcal{M} = \langle \vdash_{\mathsf{m}}, \equiv_{\mathsf{m}}, \Vdash_{\mathsf{m}} \rangle$