

# Recursive Rows in Rome

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## 1 INTRODUCTION

### 1.1 The expression problem, in full

1.1.1 *Seeking solutions sans encodings.*

### 1.2 Recursion and rows

1.2.1 *Row type systems with term- or type-level  $\mu$ .* There are none.

1.2.2 *Structural typing of objects in recursive record calculi.*

### 1.3 Challenges to practical extensibility

1.3.1 *Polymorphic variants in OCaml.*

1.3.2 *Inheritance is not subtyping.*

## 2 ROME: A ROW THEORETIC FOUNDATION FOR (CO)INDUCTIVE DATA TYPES

## 3 $R\omega$ —HIGHER ORDERED ROWS

We review the relevant syntax and typing of  $R\omega$  now.

(Todo). It will be a challenge to trim this down, as [Hubers and Morris 2023] does with [Morris and McKinna 2019].

### 3.1 Syntax

The syntax of  $R\omega(\mathcal{T})$  is given in Figure 6.

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Term variables	$x$	Type variables	$\alpha$	Labels	$\ell$	Directions	$d \in \{L, R\}$
Kinds	$\kappa ::= \star \mid L \mid R^K \mid \kappa \rightarrow \kappa$						
Predicates	$\pi, \psi ::= \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho$						
Types	$\phi, \tau, v, \rho, \xi ::= \alpha \mid (\rightarrow) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa. \tau \mid \lambda \alpha : \kappa. \tau \mid \tau \tau$ $\mid \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \dots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho$						
Terms	$H, M, N, P ::= x \mid \lambda x : \tau. M \mid MN \mid \Lambda \alpha : \kappa. M \mid M [\tau]$ $\mid \ell \mid M \triangleright M \mid M/M \mid \text{prj}_d M \mid M \# M \mid \text{inj}_d M \mid M \nabla M$ $\mid \text{syn}_\phi M \mid \text{ana}_\phi M \mid \text{fold } M M M M$						
Environments	$\Gamma ::= \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi$						

Fig. 1. Syntax

### 3.2 Types and Kinds

Figure 2 gives rules for context formation ( $\vdash \Gamma$ ), kinding ( $\Gamma \vdash \tau : \kappa$ ), and predicate formation ( $\Gamma \vdash \pi$ ), parameterized by row theory  $\mathcal{T}$ .

$\boxed{\vdash \Gamma}$			
$(C\text{-EMP}) \frac{}{\vdash \varepsilon}$	$(C\text{-TVAR}) \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa}$	$(C\text{-VAR}) \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}$	$(C\text{-PRED}) \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$
$\boxed{\Gamma \vdash \tau : \kappa} \quad \boxed{\Gamma \vdash \pi}$			
$(K\text{-VAR}) \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}$	$(K\text{-}(\rightarrow)) \frac{\vdash \Gamma}{\Gamma \vdash (\rightarrow) : \star \rightarrow \star \rightarrow \star}$	$(K\text{-}\Rightarrow) \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$	
$(K\text{-}\forall) \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa. \tau : \star}$	$(K\text{-}\rightarrow I) \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2}$	$(K\text{-}\rightarrow E) \frac{\Gamma \vdash \tau_1 : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1 \tau_2 : \kappa_2}$	
$(K\text{-LAB}) \frac{\vdash \Gamma}{\Gamma \vdash \ell : L}$	$(K\text{-SING}) \frac{\Gamma \vdash \xi : L}{\Gamma \vdash \lfloor \xi \rfloor : \star}$	$(K\text{-LTY}) \frac{\Gamma \vdash \xi : L \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}$	$(K\text{-ROW}) \frac{\Gamma \vdash \tau \{ \overline{\xi \triangleright \tau} \} : R^K}{\Gamma \vdash \{ \overline{\xi \triangleright \tau} \} : R^K}$
$(K\text{-}\Pi) \frac{\Gamma \vdash \rho : R^K}{\Gamma \vdash \Pi \rho : \kappa}$	$(K\text{-}\Sigma) \frac{\Gamma \vdash \rho : R^K}{\Gamma \vdash \Sigma \rho : \kappa}$	$(K\text{-LIFT}_1) \frac{\Gamma \vdash \rho : R^{\kappa_1 \rightarrow \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho \tau : R^{\kappa_2}}$	
$(K\text{-LIFT}_2) \frac{\Gamma \vdash \phi : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \rho : R^{\kappa_1}}{\Gamma \vdash \phi \rho : R^{\kappa_2}}$	$(K\text{-}\lesssim_d) \frac{\Gamma \vdash \rho_i : R^K}{\Gamma \vdash \rho_1 \lesssim_d \rho_2}$	$(K\text{-}\odot) \frac{\Gamma \vdash \rho_i : R^K}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$	

Fig. 2. Contexts and kinding.

$$\begin{array}{c}
\boxed{\tau \equiv \tau} \quad \boxed{\pi \equiv \pi} \\
\\
(\text{E-REFL}) \frac{}{\tau \equiv \tau} \quad (\text{E-SYM}) \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \quad (\text{E-TRANS}) \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \quad (\text{E-}\beta) \frac{}{(\lambda \alpha : \kappa. \tau) v \equiv \tau[v/\alpha]} \\
\\
(\text{E-}\xi_{\Rightarrow}) \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \quad (\text{E-}\xi_{\forall}) \frac{\tau[y/\alpha] \equiv v[y/\beta]}{\forall \alpha : \kappa. \tau \equiv \forall \beta : \kappa. v} (\gamma \notin \text{fv}(\tau, v)) \quad (\text{E-}\xi_{\text{APP}}) \frac{\tau_i \equiv v_i}{\tau_1 \tau_2 \equiv v_1 v_2} \\
\\
(\text{E-}\xi_{\triangleright}) \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \quad (\text{E-ROW}) \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \quad (\text{E-}\xi_{[\cdot]}) \frac{\xi_1 \equiv \xi_2}{[\xi_1] \equiv [\xi_2]} \\
\\
(\text{E-LIFT}_1) \frac{}{\{\xi \triangleright \phi\} \tau \equiv \{\xi \triangleright \phi \tau\}} \quad (\text{E-LIFT}_2) \frac{}{\phi \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi \tau\}} \\
\\
(\text{E-}\xi_{\Pi\Sigma}) \frac{\rho_1 \equiv \rho_2}{K\rho_1 \equiv K\rho_2} \quad (\text{E-LIFT}_3) \frac{}{(K\rho) \tau \equiv K(\rho \tau)} \quad (\text{E-SING}) \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \quad (K \in \{\Pi, \Sigma\}) \\
\\
(\text{E-}\xi_{\lesssim_d}) \frac{\tau_i \equiv v_i}{\tau_1 \lesssim_d \tau_2 \equiv v_1 \lesssim_d v_2} \quad (\text{E-}\xi_{\odot}) \frac{\tau_i \equiv v_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv v_1 \odot v_2 \sim v_3}
\end{array}$$

Fig. 3. Type and predicate equivalence

### 3.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$\begin{array}{c}
(\text{T-VAR}) \frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \quad (\text{T} \rightarrow I) \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1. M : \tau_1 \rightarrow \tau_2} \quad (\text{T} \rightarrow E) \frac{\Gamma \vdash M_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1 M_2 : \tau_2} \\
\\
(\text{T} \equiv) \frac{\Gamma \vdash M : \tau \quad \tau \equiv v}{\Gamma \vdash M : v} \quad (\text{T} \Rightarrow I) \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \quad (\text{T} \Rightarrow E) \frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau} \\
\\
(\text{T} \forall I) \frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa. M : \forall \alpha : \kappa. \tau} \quad (\text{T} \forall E) \frac{\Gamma \vdash M : \forall \alpha : \kappa. \tau \quad \Gamma \vdash v : \kappa}{\Gamma \vdash M[v] : \tau[v/\alpha]} \\
\\
(\text{T-SING}) \frac{\vdash \Gamma}{\Gamma \vdash \ell : [\ell]} \quad (\text{T} \triangleright I) \frac{\Gamma \vdash M_1 : [\ell] \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \quad (\text{T} \triangleright E) \frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : [\ell]}{\Gamma \vdash M_1 / M_2 : \tau} \\
\\
(\text{T-}\Pi E) \frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \text{prj}_d M : \Pi \rho_2} \quad (\text{T-}\Pi I) \frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \# M_2 : \Pi \rho_3} \\
\\
(\text{T-}\Sigma I) \frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \text{inj} M : \Sigma \rho_2} \quad (\text{T-}\Sigma E) \frac{\Gamma \vdash M_1 : \Sigma \rho_1 \rightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \nabla M_2 : \Sigma \rho_3 \rightarrow \tau} \\
\\
(\text{T-ana}) \frac{\Gamma \vdash \rho : R^K \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : L, u : \kappa, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u \rightarrow \tau}{\Gamma \vdash \text{ana}_{\phi} M : \Sigma(\phi \rho) \rightarrow \tau} \\
\\
(\text{T-syn}) \frac{\Gamma \vdash \rho : R^K \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : L, u : \kappa, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u}{\Gamma \vdash \text{syn}_{\phi} M : \Pi(\phi \rho)} \\
\\
(\text{T-fold}) \frac{M_1 : \forall l : L, t : \star, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow t \rightarrow v \quad \Gamma \vdash M_2 : v \rightarrow v \rightarrow v \quad \Gamma \vdash M_3 : v \quad \Gamma \vdash N : \Pi \rho}{\Gamma \vdash \text{fold } M_1 M_2 M_3 N : v}
\end{array}$$

Fig. 4. Typing

Minimal Rows

Figure 5 gives the minimal row theory  $\mathcal{M}$ .

$$\begin{array}{c}
\boxed{\Gamma \vdash_m \rho : \kappa} \quad \boxed{\rho \equiv_m \rho} \\
\\
(\text{K-MROW}) \frac{\Gamma \vdash \xi : L \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_m \{\xi \triangleright \tau\} : R^\kappa} \quad (\text{E-MROW}) \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_m \{\xi' \triangleright \tau'\}} \\
\\
\boxed{\Gamma \Vdash_m \pi} \\
\\
(\text{N-AX}) \frac{\pi \in \Gamma}{\Gamma \Vdash_m \pi} \quad (\text{N-REFL}) \frac{}{\Gamma \Vdash_m \rho \lesssim_d \rho} \quad (\text{N-TRANS}) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_m \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_3} \\
\\
(\text{N-}\equiv) \frac{\Gamma \Vdash_m \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_m \pi_2} \quad (\text{N-}\lesssim\text{LIFT}_1) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_m \phi \rho_1 \lesssim_d \phi \rho_2} \quad (\text{N-}\lesssim\text{LIFT}_2) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_m \rho_1 \tau \lesssim_d \rho_2 \tau} \\
\\
(\text{N-}\odot\text{LIFT}_1) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_1 \tau \odot \rho_2 \tau \sim \rho_3 \tau} \quad (\text{N-}\odot\text{LIFT}_2) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \phi \rho_1 \odot \phi \rho_2 \sim \phi \rho_3} \\
\\
(\text{N-}\odot\lesssim_L) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_1 \lesssim_L \rho_3} \quad (\text{N-}\odot\lesssim_R) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_2 \lesssim_R \rho_3}
\end{array}$$

Fig. 5. Minimal row theory  $\mathcal{M} = \langle \vdash_m, \equiv_m, \Vdash_m \rangle$ 

## 4 IX: THE INDEX CALCULUS

### 4.1 Syntax

	Term variables	$x \ \alpha$
Sorts		$\sigma ::= \star \mid \star_\perp \mid \mathcal{U} \mid \mathcal{U}_\perp$
Terms		$M, N, T ::= x \mid \star \mid \text{Nat} \mid \text{Zero} \mid \text{Suc } M \mid$ $\text{Ix } M \mid \text{FZero} \mid \text{FSuc } M \mid$ $\top \mid \text{tt} \mid \perp$ $\Pi \alpha : T.N \mid \lambda x : T.N \mid MN \mid$ $\Sigma \alpha : T.M \mid (\alpha : T, M) \mid M.1 \mid M.2 \mid$ $M + N \mid \text{left } M \mid \text{right } M \mid \text{case } M \ M \ M \mid$ $M \equiv N \mid \text{refl } T \ M \ N \mid$ $\mu \alpha : T.M$
Environments		$\Gamma ::= \varepsilon \mid \Gamma, \alpha : T$

Fig. 6. Syntax



## 4.2 Typing

$$\begin{array}{c}
\boxed{\vdash \Gamma} \\
\\
(\text{EMP}) \frac{}{\vdash \varepsilon} \quad (\text{VAR}) \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M} \\
\\
\boxed{\sigma \text{ Safe}} \quad \boxed{\Gamma \vdash M \text{ Safe}} \\
\\
\boxed{\text{(I am not sure these two rules are necessary.)}} \\
\\
\frac{}{\mathcal{U} \text{ Safe}} \quad \frac{\sigma \text{ Safe} \quad \Gamma \vdash M : \sigma}{\Gamma \vdash M \text{ Safe}} \\
\\
\boxed{\Gamma \vdash M : \sigma} \\
\\
(\star) \frac{}{\Gamma \vdash \star : \mathcal{U}} \quad (\text{CUM}) \frac{\Gamma \vdash T : \star}{\Gamma \vdash T : \mathcal{U}} \\
\\
(\star_{\perp}) \frac{}{\Gamma \vdash \star_{\perp} : \mathcal{U}_{\perp}} \quad (\text{WHY-NOT}) \frac{\Gamma \vdash T : \mathcal{U}}{\Gamma \vdash T : \mathcal{U}_{\perp}} \\
\\
(\top) \frac{}{\Gamma \vdash \top : \sigma} \quad (\perp) \frac{}{\Gamma \vdash \perp : \star_{\perp}} \\
\\
(\text{VAR}) \frac{\alpha : \sigma \in \Gamma}{\Gamma \vdash \alpha : \sigma} \quad (\text{NAT}) \frac{}{\Gamma \vdash \text{Nat} : \star} \quad (\text{IX}) \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Ix } n : \star} \\
\\
(\text{II}) \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \sigma}{\Gamma \vdash \Pi \alpha : M. N : \sigma} \quad (\Sigma) \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \sigma}{\Gamma \vdash \Sigma \alpha : M. N : \sigma} \\
\\
(+ ) \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} \quad (\equiv) \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma \quad \sigma \text{ Safe}}{\Gamma \vdash M \equiv N : \sigma} \\
\\
\boxed{\Gamma \vdash M : N} \\
\\
(\text{Var}) \frac{x : M \in \Gamma}{\Gamma \vdash x : M} \quad (\text{tt}) \frac{}{\Gamma \vdash \text{tt} : \top} \\
\\
(\text{Zero}) \frac{}{\Gamma \vdash \text{Zero} : \text{Nat}} \quad (\text{Suc}) \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Suc } n : \text{Nat}} \\
\\
(\text{FZero}) \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{FZero} : \text{Ix } (\text{Suc } n)} \quad (\text{FSuc}) \frac{\Gamma \vdash n : \text{Nat} \quad \Gamma \vdash i : \text{Ix } n}{\Gamma \vdash \text{FSuc } i : \text{Ix } (\text{Suc } n)} \\
\\
(\text{PII}) \frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T. M : \Pi(x : T). N} \quad (\text{PIE}) \frac{\Gamma \vdash M : \Pi(x : T_1). T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M N : T_2[N/x]} \\
\\
(\Sigma I) \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1). T_2} \quad (\Sigma E_1) \frac{\Gamma \vdash M : \Sigma(x : T_1). T_2}{\Gamma \vdash M.1 : T_1} \quad (\Sigma E_2) \frac{\Gamma \vdash M : \Sigma(x : T_1). T_2}{\Gamma \vdash M.2 : T_2[M.1/x]} \\
\\
(\equiv I) \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \text{refl} : M \equiv M} \quad (\text{CONV}) \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma \quad \sigma \text{ Safe}}{\Gamma \vdash M : T_2} \\
\\
\begin{array}{c}
\Gamma \vdash P : T_1 \equiv T_2 \\
\Gamma \vdash M : T_1 \\
\Gamma \vdash N : T_1 \\
\Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\
\Gamma, z : T_1 \vdash H : T[z/x, y/\text{refl}, p/\text{refl}] \\
\end{array} \\
(\equiv E) \frac{}{\Gamma \vdash \mathcal{J} H M N P : T[M/x, N/y, P/p]}
\end{array}$$

Fig. 7. Context formation and typing rules for Ix terms

Let the meta-syntax  $\tau$  denote both sort and term.

$$\begin{array}{c}
 \boxed{\Gamma \vdash M = N : \sigma} \\
 \text{(E-REFL)} \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \quad \text{(E-SYM)} \frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \quad \text{(E-TRANS)} \frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma} \\
 \boxed{\Gamma \vdash M = N : T} \\
 \text{(C-REFL)} \frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \quad \text{(C-SYM)} \frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \quad \text{(C-TRANS)} \frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}
 \end{array}$$

Fig. 8. Definitional equality & computational laws

#### 4.3 A Comparison to $\lambda^{\Pi\mathcal{U}\mathbb{N}}$ [Abel et al. 2018]

### 5 TRANSLATION FROM $R\omega$

#### 5.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of  $R\omega$  types. Figure 9 describe the untyped translation, which is used to show translational soundness of the typed translation.

$$\begin{array}{c}
 \boxed{(\kappa)^{\bullet}} \\
 (\star)^{\bullet} = \star \\
 (\mathsf{L})^{\bullet} = \top \\
 (\kappa_1 \rightarrow \kappa_2)^{\bullet} = \Pi(\alpha : (\kappa_1)^{\bullet}).(\kappa_2)^{\bullet} \\
 (\mathsf{R}^K)^{\bullet} = \Sigma(n : \mathsf{Nat}).\Pi(j : \mathsf{Ix } n).(\kappa)^{\bullet}
 \end{array}$$

$$\begin{array}{c}
 \boxed{(\sigma)^{\bullet}} \\
 (\alpha)^{\bullet} = \alpha \\
 (\tau_1 \rightarrow \tau_2)^{\bullet} = \Pi(\alpha : (\tau_1)^{\bullet}).(\tau_2)^{\bullet} \\
 (\forall \alpha : \kappa. \tau)^{\bullet} = \Pi(\alpha : (\kappa)^{\bullet}).(\tau)^{\bullet} \\
 (\lambda \alpha : \kappa. \tau)^{\bullet} = \Pi(\alpha : (\kappa)^{\bullet}).(\tau)^{\bullet}
 \end{array}$$

$$\boxed{(M)^{\bullet}}$$

...

$$\boxed{(\pi)^{\bullet}}$$

...

Fig. 9. A compositional translation of  $R\omega$  judgments to (untyped)  $\mathsf{Ix}$  terms



## 5.2 Typed translation

$$\begin{array}{c}
 \boxed{\Gamma \vdash \tau \rightsquigarrow v : \kappa} \\
 \\
 (C\text{-FOO}) \frac{A}{B} \\
 \boxed{\Gamma \vdash M \rightsquigarrow N : \tau} \\
 \\
 (C\text{-FOO}) \frac{A}{B} \\
 \boxed{\Gamma \Vdash \pi \rightsquigarrow N} \\
 \\
 (C\text{-FOO}) \frac{A}{B} \\
 \boxed{\tau \equiv v \rightsquigarrow P} \\
 \\
 (C\text{-FOO}) \frac{A}{B}
 \end{array}$$

Fig. 10. Translation of  $R\omega$  derivations to  $Ix$  derivations

## 5.3 Properties of Translation

**THEOREM 1 (TRANSLATIONAL SOUNDNESS (TYPES)).** *if  $\Gamma \vdash \tau : \kappa$  such that  $\Gamma \vdash \tau \rightsquigarrow v : \kappa$  then  $(\Gamma)^\bullet \vdash v : (\kappa)^\bullet$ .*

The following is bullshit w.r.t. definitional equality.

**THEOREM 2 (TRANSLATIONAL SOUNDNESS (TYPE EQUIVALENCE)).** *if  $\Gamma \vdash \tau_1 \rightsquigarrow v_1 : \kappa_1$  and  $\Gamma \vdash \tau_2 \rightsquigarrow v_2 : \kappa_2$  such that  $\tau_1 \equiv \tau_2$  is derivable in  $R\omega$ , then  $(\Gamma)^\bullet \vdash v_1 \equiv v_2$ .*

The next theorems presume an  $R\omega$  instantiation of the simple row theory.

**THEOREM 3 (TRANSLATIONAL SOUNDNESS (ROW COMBINATION)).** *if  $\Gamma \Vdash \rho_1 \cdot \rho_2 \sim \rho_3 \rightsquigarrow N$  then  $(\Gamma)^\bullet \vdash N : \text{foobar}$ .*

**THEOREM 4 (TRANSLATIONAL SOUNDNESS (ROW CONTAINMENT)).** *if  $\Gamma \Vdash \rho_1 \lesssim \rho_2 \rightsquigarrow N$  then  $(\Gamma)^\bullet \vdash N : \text{foobar}$ .*

Finally,

**THEOREM 5 (TRANSLATIONAL SOUNDNESS).** *if  $\Gamma \vdash M : \tau$  such that  $\Gamma \vdash M \rightsquigarrow N : \tau$  then  $(\Gamma)^\bullet \vdash M : (\tau)^\bullet$ .*

## 6 OPERATIONAL SEMANTICS OF IX

## 7 RECURSION

This section will later be incorporated into earlier sections.

### 7.1 Rome, or, $R\omega$ with $\mu$

todo

$$(C\text{-}FOO) \frac{A}{B}$$

Fig. 11. Additional  $R\omega$  judgments for recursion

### 7.2 Mix, the recursive index calculus

todo

$$(C\text{-}FOO) \frac{A}{B}$$

Fig. 12. Additional  $lx$  judgments for recursion

### 7.3 Translation and properties of translation

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