The Index Calculus and its translation from Rω

AH & JGM

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1 Ix: The Index Calculus

1.1 Syntax

Term variables x α

Sorts	σ	$::=$	$\star \mid \mathcal{U}$
Terms	M, N, T	$::=$	$x \mid \star \mid \mathrm{Nat} \mid \mathrm{Zero} \mid \mathrm{Suc}\, M \mid$
			$\mathrm{Ix}\, M \mid \mathrm{FZero} \mid \mathrm{FSuc}\ M \mid$
			$\top \mid \mathtt{tt} \mid$
			$\Pi\alpha : T.N \mid \lambda x : T.N \mid M\ N \mid$
			$\Sigma\alpha : T.M \mid (\alpha : T, M) \mid M.1 \mid M.2$
			$M + N \mid \mathrm{left}\, M \mid \mathrm{right}\, M \mid \mathrm{case}\ M\ \mathrm{of}\ \{\mathrm{left} \mapsto M; \mathrm{right} \mapsto M\}$
			$M \equiv N \mid \mathrm{refl} \mid ...$
Environments	Γ	$::=$	$\varepsilon \mid \Gamma, \alpha : T$

Figure 1: Syntax

1.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(\textsc{c-emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{c-var})\ \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\textsc{t-}\star)\ \frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad \frac{}{\Gamma \vdash \top : \star} \qquad \frac{}{\Gamma \vdash \top : \mathcal{U}}$$

$$\frac{}{\Gamma \vdash \mathrm{Nat} : \star} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\, n : \star}$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Pi\alpha : M . N : \star} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Sigma\alpha : M . N : \star}$$

$$\frac{\Gamma \vdash M : \star \quad \Gamma \vdash N : \star}{\Gamma \vdash M + N : \star} \qquad \frac{\Gamma \vdash M_1 : N_1 \quad \Gamma \vdash N_1 : \sigma \quad \Gamma \vdash M_2 : N_2 \quad \Gamma \vdash N_2 : \sigma}{\Gamma \vdash M_1 \equiv M_2 : \star}$$

$$\boxed{\Gamma \vdash M : N}$$

$$\frac{x : M \in \Gamma}{\Gamma \vdash x : M} \qquad \frac{}{\Gamma \vdash \mathtt{tt} : \top}$$

$$\frac{}{\Gamma \vdash \mathrm{Zero} : \mathrm{Nat}} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Suc}\, n : \mathrm{Nat}} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{FZero} : \mathrm{Ix}\,(\mathrm{Suc}\, n)} \qquad \frac{\Gamma \vdash n : \mathrm{Nat} \quad \Gamma \vdash i : \mathrm{Ix}\, n}{\Gamma \vdash \mathrm{FSuc}\, i : \mathrm{Ix}\,(\mathrm{Suc}\, n)}$$

$$\frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T . M : \Pi(x : T) . N} \qquad \frac{\Gamma \vdash M : \Pi(x : T_1) . T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\ N : T_2[N/x]}$$

$$\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1) . T_2} \qquad \frac{\Gamma \vdash M : \Sigma(x : T_1) . T_2}{\Gamma \vdash M.1 : T_1} \qquad \frac{\Gamma \vdash M : \Sigma(x : T_1) . T_2}{\Gamma \vdash M.2 : T_2[M.1/x]}$$

...

2 Translation From Rω

2.1 Untyped Translation

$$\boxed{[\![\kappa]\!]}$$

$$\begin{aligned}
[\![\star]\!] &= \star \\
[\![\mathsf{L}]\!] &= \top \\
[\![\kappa_1 \to \kappa_2]\!] &= \Pi(\alpha : [\![\kappa_1]\!]).[\![\kappa_2]\!] \\
[\![\mathsf{R}^\kappa]\!] &= \Sigma(n : \mathrm{Nat}).\Pi(j : \mathrm{Ix}\, n).[\![\kappa]\!]
\end{aligned}$$

$$\boxed{[\![\tau]\!]}$$

$$\ldots$$

$$\boxed{[\![M]\!]}$$

$$\ldots$$

$$\boxed{[\![\pi]\!]}$$

$$\ldots$$

Figure 2: A compositional translation of Rω judgments to (untyped) Ix terms

2.2 Typed translation

$$\boxed{\Delta \vdash \tau \rightsquigarrow \upsilon : \kappa}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Delta \vdash M \rightsquigarrow N : \tau}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Delta \Vdash \pi \rightsquigarrow N}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\tau \equiv \upsilon \rightsquigarrow P}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Figure 3: Translation of Rω derivations to Ix derivations

- $\Delta \vdash \tau \rightsquigarrow \upsilon : \kappa$ denotes the translation of judgment $\Delta \vdash \tau : \kappa$ to term $[\![\Delta]\!] \vdash [\![\tau]\!] : [\![\kappa]\!]$.
-

Actually, I am now starting to wonder if I can give an inductive definition of the translation and then an operational semantics over translations. One could then define an erasure to both Rω and Ix terms. A preservation theorem would now be like a bisimulation argument—that well-typed pairs of Rω and Ix terms step to well-typed pairs of Rω and Ix terms. Doing this this way gets rid of the untyped translation and obligation to prove translational soundness.

To do this, you would really need *strong* bisimulation; I doubt that Rω steps and Ix steps are 1-to-1. So you may have cases where an Rω term "steps" to itself while the Ixterm steps. Maybe actually this is a bad idea.

Actually... I should be able to mimic the approach of MorrisM19, which is a mix of the two.

1. Give a translation of types, predicates, and typing environments, e.g., $[\![\tau]\!]$. (In MorrisM19, this is $(\tau)^\bullet$).
2. Give a typed translation $\Gamma \vdash M \rightsquigarrow N : \tau$, etc.

3. Prove soundness separately. That is, if $\Gamma \vdash M \rightsquigarrow N : \tau$ then $(\Gamma)^\bullet \vdash N : \tau^\bullet$.

I like this approach because I am not obligated to specify the (untyped) translation of M to N, which means mechanizing things like substitution in the untyped syntax.

Most of these notes need to be erased. These are stream of consciousness notes.

2.3 Example translations of Rω terms and types

Record selection. In Rω,

$$\forall \rho : \mathsf{R}^\star, \, \ell : \mathsf{L}, \, \tau : \star . \{\ell \triangleright \tau\} \lesssim \rho \Rightarrow \lfloor \ell \rfloor \to \Pi \rho \to \tau$$

translates to

$$\Pi(\rho : \mathrm{Row} \star).\Pi(\ell : \top).\Pi(\tau : \star).[\![\{\ell \triangleright \tau\} \lesssim \rho]\!].\Pi(\_ : \top).\Pi(i : \mathrm{Ix}\, \rho.1).\, \rho.2\; i$$

where

$$\begin{aligned}
\mathrm{Row} \kappa &: = \Sigma(n : \mathrm{Nat}).\Pi(i : \mathrm{Ix}\, n).\kappa \\
[\![\{\ell \triangleright \tau\} \lesssim \rho]\!] &= \Pi(i : \mathrm{Ix}\, [\![\{\ell \triangleright \tau\}]\!].1).\Sigma(j : \mathrm{Ix}\, \rho.1).[\![\{\ell \triangleright \tau\}]\!].1\; i \equiv \rho.2\; j \\
[\![\{\ell \triangleright \tau\}]\!] &= (\mathrm{Suc}\, \mathrm{Zero} : \mathrm{Nat}, \lambda(i : \mathrm{Ix}\, (\mathrm{Suc}\, \mathrm{Zero})).[\![\tau]\!])
\end{aligned}$$

Putting this all together:

$$
\begin{aligned}
&\Pi(\rho : (\Sigma(n : \mathrm{Nat}).\Pi(i : \mathrm{Ix}\, n).\star)).\\
&\Pi(\ell : \top).\\
&\Pi(\tau : \star).\\
&\Pi(P :\\
&\quad \Pi(i : \mathrm{Ix}\,(\mathrm{Suc}\,\mathrm{Zero} : \mathrm{Nat}, \lambda(i : \mathrm{Ix}\,(\mathrm{Suc}\,\mathrm{Zero})).[\![\tau]\!]).1).\\
&\quad \Sigma(j : \mathrm{Ix}\,\rho.1).\\
&\quad (\mathrm{Suc}\,\mathrm{Zero} : \mathrm{Nat}, \lambda(i : \mathrm{Ix}\,(\mathrm{Suc}\,\mathrm{Zero})).[\![\tau]\!]).2\ i \equiv \rho.2\ j)\\
&\Pi(\_ : \top).\\
&\Pi(i : \mathrm{Ix}\,\rho.1).\ \rho.2\, i
\end{aligned}
$$

which should normalize to

$$
\begin{aligned}
&\Pi(\rho : (\Sigma(n : \mathrm{Nat}).\Pi(i : \mathrm{Ix}\, n).\star)).\\
&\Pi(\ell : \top).\\
&\Pi(\tau : \star).\\
&\Pi(P :\\
&\quad \Pi(i : \mathrm{Ix}\,1).\\
&\quad \Sigma(j : \mathrm{Ix}\,\rho.1).\\
&\quad [\![\tau]\!] \equiv \rho.2\ j).\\
&\Pi(\_ : \top).\\
&\Pi(i : \mathrm{Ix}\,\rho.1).\ \rho.2\, i
\end{aligned}
$$

A The static semantics of Rω

A.1 Syntax

The syntax of R$\omega(\mathcal{T})$ is given in Figure 4.

Term variables x Type variables α Labels ℓ Directions $d \in \{\mathsf{L}, \mathsf{R}\}$

$$
\begin{array}{llcl}
\text{Kinds} & \kappa & ::= & \star \mid \mathsf{L} \mid \mathsf{R}^{\kappa} \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi & ::= & \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi & ::= & \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa.\tau \mid \lambda \alpha : \kappa.\tau \mid \tau\, \tau \\
 & & \mid & \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho \\
\text{Terms} & M, N & ::= & x \mid \lambda x : \tau.M \mid M\, N \mid \Lambda \alpha : \kappa.M \mid M\, [\tau] \\
 & & \mid & \ell \mid M \triangleright M \mid M / M \mid \mathsf{prj}_d\, M \mid M +\!\!+\, M \mid \mathsf{inj}_d\, M \mid M \triangledown M \\
 & & \mid & \mathsf{syn}_{\phi}\, M \mid \mathsf{ana}_{\phi}\, M \mid \mathsf{fold}\, M\, M\, M\, M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}
$$

Figure 4: Syntax

A.2 Types and Kinds

Figure 5 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

$$\boxed{\vdash \Gamma}$$

$$(\textsc{c-emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{c-tvar})\ \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \qquad (\textsc{c-var})\ \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau} \qquad (\textsc{c-pred})\ \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$$

$$\boxed{\Gamma \vdash \tau : \kappa}\ \boxed{\Gamma \vdash \pi}$$

$$(\textsc{k-var})\ \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \qquad (\textsc{k-}(\to))\ \frac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star} \qquad (\textsc{k-}\Rightarrow)\ \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$$

$$(\textsc{k-}\forall)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa . \tau : \star} \qquad (\textsc{k-}\to I)\ \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1 . \tau : \kappa_1 \to \kappa_2} \qquad (\textsc{k-}\to E)\ \frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\, \tau_2 : \kappa_2}$$

$$(\textsc{k-lab})\ \frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}} \qquad (\textsc{k-sing})\ \frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star} \qquad (\textsc{k-lty})\ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa} \qquad (\textsc{k-row})\ \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}$$

$$(\textsc{k-}\Pi)\ \frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Pi \rho : \kappa} \qquad (\textsc{k-}\Sigma)\ \frac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Sigma \rho : \kappa} \qquad (\textsc{k-lift}_1)\ \frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\, \tau : \mathsf{R}^{\kappa_2}}$$

$$(\textsc{k-lift}_2)\ \frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\, \rho : \mathsf{R}^{\kappa_2}} \qquad (\textsc{k-}\lesssim_d)\ \frac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \lesssim_d \rho_2} \qquad (\textsc{k-}\odot)\ \frac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$$

Figure 5: Contexts and kinding.

$$\boxed{\tau \equiv \tau} \ \boxed{\pi \equiv \pi}$$

$$(\text{E-REFL})\ \frac{}{\tau \equiv \tau} \qquad (\text{E-SYM})\ \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad (\text{E-TRANS})\ \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad (\text{E-}\beta)\ \frac{}{(\lambda\alpha : \kappa.\tau)\, \upsilon \equiv \tau[\upsilon/\alpha}$$

$$(\text{E-}\xi_{\Rightarrow})\ \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad (\text{E-}\xi_{\forall})\ \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha : \kappa.\tau \equiv \forall\beta : \kappa.\upsilon}\ (\gamma \notin fv(\tau, \upsilon)) \qquad (\text{E-}\xi_{\text{APP}})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1\, \tau_2 \equiv \upsilon_1\, \upsilon_2}$$

$$(\text{E-}\xi_{\triangleright})\ \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \qquad (\text{E-ROW})\ \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_T \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \qquad (\text{E-}\xi_{\lfloor\cdot\rfloor})\ \frac{\xi_1 \equiv \xi_2}{\lfloor\xi_1\rfloor \equiv \lfloor\xi_2\rfloor}$$

$$(\text{E-LIFT}_1)\ \frac{}{\{\xi \triangleright \phi\}\, \tau \equiv \{\xi \triangleright \phi\, \tau\}} \qquad (\text{E-LIFT}_2)\ \frac{}{\phi\, \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi\, \tau\}}$$

$$(\text{E-}\xi_{\Pi\Sigma})\ \frac{\rho_1 \equiv \rho_2}{K\rho_1 \equiv K\rho_2} \qquad (\text{E-LIFT}_3)\ \frac{}{(K\rho)\, \tau \equiv K(\rho\, \tau)} \qquad (\text{E-SING})\ \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \qquad (K \in \{\Pi, \Sigma\})$$

$$(\text{E-}\xi_{\lesssim_d})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad (\text{E-}\xi_{\odot})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Figure 6: Type and predicate equivalence

A.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$(\textsc{t-var})\ \frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad (\textsc{t-}{\to}I)\ \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . M : \tau_1 \to \tau_2} \qquad (\textsc{t-}{\to}E)\ \frac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2}{\Gamma \vdash M_1\, M_2 : \tau_2}$$

$$(\textsc{t-}{\equiv})\ \frac{\Gamma \vdash M : \tau \quad \tau \equiv \upsilon}{\Gamma \vdash M : \upsilon} \qquad (\textsc{t-}{\Rightarrow}I)\ \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \qquad (\textsc{t-}{\Rightarrow}E)\ \frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$$

$$(\textsc{t-}\forall I)\ \frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa . M : \forall \alpha : \kappa . \tau} \qquad (\textsc{t-}\forall E)\ \frac{\Gamma \vdash M : \forall \alpha : \kappa . \tau \quad \Gamma \vdash \upsilon : \kappa}{\Gamma \vdash M\, [\upsilon] : \tau[\upsilon / \alpha]}$$

$$(\textsc{t-sing})\ \frac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor} \qquad (\textsc{t-}{\triangleright}I)\ \frac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \qquad (\textsc{t-}{\triangleright}E)\ \frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1 / M_2 : \tau}$$

$$(\textsc{t-}\Pi E)\ \frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\, M : \Pi \rho_2} \qquad (\textsc{t-}\Pi I)\ \frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 +\!\!+ M_2 : \Pi \rho_3}$$

$$(\textsc{t-}\Sigma I)\ \frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \mathsf{inj}\, M : \Sigma \rho_2} \qquad (\textsc{t-}\Sigma E)\ \frac{\Gamma \vdash M_1 : \Sigma \rho_1 \to \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \to \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim}{\Gamma \vdash M_1 \triangledown M_2 : \Sigma \rho_3 \to \tau}$$

$$(\textsc{t-}\mathsf{ana})\ \frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa . \, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u \to \tau\end{array}}{\Gamma \vdash \mathsf{ana}_\phi\, M : \Sigma(\phi\, \rho) \to \tau}$$

$$(\textsc{t-}\mathsf{syn})\ \frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa . \, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u\end{array}}{\Gamma \vdash \mathsf{syn}_\phi\, M : \Pi(\phi\, \rho)}$$

$$(\textsc{t-}\mathsf{fold})\ \frac{\begin{array}{c}M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^\kappa . \, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to t \to \upsilon \\ \Gamma \vdash M_2 : \upsilon \to \upsilon \to \upsilon \quad \Gamma \vdash M_3 : \upsilon \quad \Gamma \vdash N : \Pi \rho\end{array}}{\Gamma \vdash \mathsf{fold}\, M_1\, M_2\, M_3\, N : \upsilon}$$

Figure 7: Typing

Minimal Rows

Figure 8 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathsf{m}} \rho : \kappa} \; \boxed{\rho \equiv_{\mathsf{m}} \rho}$$

$$(\textsc{k-mrow})\ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathsf{m}} \{\xi \triangleright \tau\} : \mathsf{R}^{\kappa}} \qquad (\textsc{e-mrow})\ \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathsf{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathsf{m}} \pi}$$

$$(\textsc{n-ax})\ \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathsf{m}} \pi} \qquad (\textsc{n-refl})\ \frac{}{\Gamma \Vdash_{\mathsf{m}} \rho \lesssim_d \rho} \qquad (\textsc{n-trans})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\textsc{n-}\equiv)\ \frac{\Gamma \Vdash_{\mathsf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathsf{m}} \pi_2} \qquad (\textsc{n-}\lesssim\textsc{lift}_1)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \phi\, \rho_1 \lesssim_d \phi\, \rho_2} \qquad (\textsc{n-}\lesssim\textsc{lift}_2)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \rho_1\, \tau \lesssim_d \rho_2\, \tau}$$

$$(\textsc{n-}\odot\textsc{lift}_1)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1\, \tau \odot \rho_2\, \tau \sim \rho_3\, \tau} \qquad (\textsc{n-}\odot\textsc{lift}_2)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \phi\, \rho_1 \odot \phi\, \rho_2 \sim \phi\, \rho_3}$$

$$(\textsc{n-}\odot\lesssim_{\mathsf{L}})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\textsc{n-}\odot\lesssim_{\mathsf{R}})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Figure 8: Minimal row theory $\mathcal{M} = \langle \vdash_{\mathsf{m}}, \equiv_{\mathsf{m}}, \Vdash_{\mathsf{m}} \rangle$