Recursive Rows in Rome

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1 INTRODUCTION

1.1 The expression problem, in full

1.1.1 Seeking solutions sans encodings.

1.2 Recursion and rows

1.2.1 Row type systems with term- or type-level μ. There are none.

1.2.2 Structural typing of objects in recursive record calculi.

1.3 Challenges to practical extensibility

1.3.1 Polymorphic variants in OCaml.

1.3.2 Inheritance is not subtyping.

2 ROME: A ROW THEORETIC FOUNDATION FOR (CO)INDUCTIVE DATA TYPES

3 Rω—HIGHER ORDERED ROWS

We review the relevant syntax and typing of Rω now.

(Todo). It will be a challenge to trim this down, as [Hubers and Morris 2023] does with [Morris and McKinna 2019].

3.1 Syntax

The syntax of R$\omega(\mathcal{T})$ is given in Figure 6.

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Term variables x Type variables α Labels ℓ Directions $d \in \{\mathsf{L}, \mathsf{R}\}$

$$
\begin{array}{llcl}
\text{Kinds} & \kappa & ::= & \star \mid \mathsf{L} \mid \mathsf{R}^{\kappa} \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi & ::= & \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi & ::= & \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa.\tau \mid \lambda \alpha : \kappa.\tau \mid \tau\,\tau \\
& & \mid & \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho \\
\text{Terms} & H, M, N, P & ::= & x \mid \lambda x : \tau.M \mid M\,N \mid \Lambda \alpha : \kappa.M \mid M\,[\tau] \\
& & \mid & \ell \mid M \triangleright M \mid M/M \mid \mathsf{prj}_d\,M \mid M \mathbin{+\!\!\!+} M \mid \mathsf{inj}_d\,M \mid M \mathbin{\triangledown} M \\
& & \mid & \mathsf{syn}_\phi\,M \mid \mathsf{ana}_\phi\,M \mid \mathsf{fold}\,M\,M\,M\,M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}
$$

Fig. 1. Syntax

3.2 Types and Kinds

Figure 2 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

$$\boxed{\vdash \Gamma}$$

$$
(\textsc{c-emp})\,\frac{}{\vdash \varepsilon}
\qquad
(\textsc{c-tvar})\,\frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa}
\qquad
(\textsc{c-var})\,\frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}
\qquad
(\textsc{c-pred})\,\frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}
$$

$$\boxed{\Gamma \vdash \tau : \kappa}\ \boxed{\Gamma \vdash \pi}$$

$$
(\textsc{k-var})\,\frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}
\qquad
(\textsc{k-}(\to))\,\frac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star}
\qquad
(\textsc{k-}\Rightarrow)\,\frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}
$$

$$
(\textsc{k-}\forall)\,\frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa.\tau : \star}
\qquad
(\textsc{k-}{\to}I)\,\frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1.\tau : \kappa_1 \to \kappa_2}
\qquad
(\textsc{k-}{\to}E)\,\frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\,\tau_2 : \kappa_2}
$$

$$
(\textsc{k-lab})\,\frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}}
\qquad
(\textsc{k-sing})\,\frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star}
\qquad
(\textsc{k-lty})\,\frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}
\qquad
(\textsc{k-row})\,\frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^{\kappa}}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^{\kappa}}
$$

$$
(\textsc{k-}\Pi)\,\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa}}{\Gamma \vdash \Pi \rho : \kappa}
\qquad
(\textsc{k-}\Sigma)\,\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa}}{\Gamma \vdash \Sigma \rho : \kappa}
\qquad
(\textsc{k-lift}_1)\,\frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\,\tau : \mathsf{R}^{\kappa_2}}
$$

$$
(\textsc{k-lift}_2)\,\frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\,\rho : \mathsf{R}^{\kappa_2}}
\qquad
(\textsc{k-}\lesssim_d)\,\frac{\Gamma \vdash \rho_i : \mathsf{R}^{\kappa}}{\Gamma \vdash \rho_1 \lesssim_d \rho_2}
\qquad
(\textsc{k-}\odot)\,\frac{\Gamma \vdash \rho_i : \mathsf{R}^{\kappa}}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}
$$

Fig. 2. Contexts and kinding.

$$\boxed{\tau \equiv \tau} \quad \boxed{\pi \equiv \pi}$$

$$(\textsc{e-refl})\ \frac{}{\tau \equiv \tau} \qquad (\textsc{e-sym})\ \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad (\textsc{e-trans})\ \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad (\textsc{e-}\beta)\ \frac{}{(\lambda\alpha : \kappa.\tau)\, \upsilon \equiv \tau[\upsilon/\alpha]}$$

$$(\textsc{e-}\xi_{\Rightarrow})\ \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad (\textsc{e-}\xi_{\forall})\ \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha : \kappa.\tau \equiv \forall\beta : \kappa.\upsilon}\ (\gamma \notin fv(\tau, \upsilon)) \qquad (\textsc{e-}\xi_{\textsc{app}})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1\ \tau_2 \equiv \upsilon_1\ \upsilon_2}$$

$$(\textsc{e-}\xi_{\triangleright})\ \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \qquad (\textsc{e-row})\ \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \qquad (\textsc{e-}\xi_{\lfloor\cdot\rfloor})\ \frac{\xi_1 \equiv \xi_2}{\lfloor\xi_1\rfloor \equiv \lfloor\xi_2\rfloor}$$

$$(\textsc{e-lift}_1)\ \frac{}{\{\xi \triangleright \phi\}\ \tau \equiv \{\xi \triangleright \phi\ \tau\}} \qquad (\textsc{e-lift}_2)\ \frac{}{\phi\ \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi\ \tau\}}$$

$$(\textsc{e-}\xi_{\Pi\Sigma})\ \frac{\rho_1 \equiv \rho_2}{K\rho_1 \equiv K\rho_2} \qquad (\textsc{e-lift}_3)\ \frac{}{(K\rho)\ \tau \equiv K(\rho\ \tau)} \qquad (\textsc{e-sing})\ \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \qquad (K \in \{\Pi, \Sigma\})$$

$$(\textsc{e-}\xi_{\lesssim_d})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad (\textsc{e-}\xi_{\odot})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Fig. 3. Type and predicate equivalence

3.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$(\text{T-VAR})\,\frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad (\text{T-}{\longrightarrow}I)\,\frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . M : \tau_1 \longrightarrow \tau_2} \qquad (\text{T-}{\longrightarrow}E)\,\frac{\Gamma \vdash M_1 : \tau_1 \longrightarrow \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1\, M_2 : \tau_2}$$

$$(\text{T-}{\equiv})\,\frac{\Gamma \vdash M : \tau \quad \tau \equiv v}{\Gamma \vdash M : v} \qquad (\text{T-}{\Longrightarrow}I)\,\frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \qquad (\text{T-}{\Longrightarrow}E)\,\frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$$

$$(\text{T-}\forall I)\,\frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa . M : \forall \alpha : \kappa . \tau} \qquad (\text{T-}\forall E)\,\frac{\Gamma \vdash M : \forall \alpha : \kappa . \tau \quad \Gamma \vdash v : \kappa}{\Gamma \vdash M\,[v] : \tau[v/\alpha]}$$

$$(\text{T-SING})\,\frac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor} \qquad (\text{T-}{\triangleright}I)\,\frac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \qquad (\text{T-}{\triangleright}E)\,\frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1 / M_2 : \tau}$$

$$(\text{T-}\Pi E)\,\frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\, M : \Pi \rho_2} \qquad (\text{T-}\Pi I)\,\frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \mathbin{+\!\!+} M_2 : \Pi \rho_3}$$

$$(\text{T-}\Sigma I)\,\frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \mathsf{inj}\, M : \Sigma \rho_2} \qquad (\text{T-}\Sigma E)\,\frac{\Gamma \vdash M_1 : \Sigma \rho_1 \longrightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \longrightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \triangledown M_2 : \Sigma \rho_3 \longrightarrow \tau}$$

$$(\text{T-ana})\,\frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^{\kappa} \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^{\kappa}.\, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u \to \tau\end{array}}{\Gamma \vdash \mathsf{ana}_{\phi}\, M : \Sigma(\phi\, \rho) \to \tau}$$

$$(\text{T-syn})\,\frac{\begin{array}{c}\Gamma \vdash \rho : \mathsf{R}^{\kappa} \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^{\kappa}.\, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u\end{array}}{\Gamma \vdash \mathsf{syn}_{\phi}\, M : \Pi(\phi\, \rho)}$$

$$(\text{T-fold})\,\frac{\begin{array}{c}M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^{\kappa}.\, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to t \to v \\ \Gamma \vdash M_2 : v \to v \to v \quad \Gamma \vdash M_3 : v \quad \Gamma \vdash N : \Pi \rho\end{array}}{\Gamma \vdash \mathsf{fold}\, M_1\, M_2\, M_3\, N : v}$$

Fig. 4. Typing

Minimal Rows

Figure 5 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathsf{m}} \rho : \kappa} \; \boxed{\rho \equiv_{\mathsf{m}} \rho}$$

$$(\text{K-MROW})\ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathsf{m}} \{\xi \triangleright \tau\} : \mathsf{R}^{\kappa}} \qquad (\text{E-MROW})\ \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathsf{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathsf{m}} \pi}$$

$$(\text{N-AX})\ \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathsf{m}} \pi} \qquad (\text{N-REFL})\ \frac{}{\Gamma \Vdash_{\mathsf{m}} \rho \lesssim_d \rho} \qquad (\text{N-TRANS})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\text{N-}\equiv)\ \frac{\Gamma \Vdash_{\mathsf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathsf{m}} \pi_2} \qquad (\text{N-}\lesssim\text{LIFT}_1)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \phi\, \rho_1 \lesssim_d \phi\, \rho_2} \qquad (\text{N-}\lesssim\text{LIFT}_2)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \rho_1\, \tau \lesssim_d \rho_2\, \tau}$$

$$(\text{N-}\odot\text{LIFT}_1)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1\, \tau \odot \rho_2\, \tau \sim \rho_3\, \tau} \qquad (\text{N-}\odot\text{LIFT}_2)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \phi\, \rho_1 \odot \phi\, \rho_2 \sim \phi\, \rho_3}$$

$$(\text{N-}\odot\lesssim_{\mathsf{L}})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\text{N-}\odot\lesssim_{\mathsf{R}})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Fig. 5. Minimal row theory $\mathcal{M} = \langle \vdash_{\mathsf{m}}, \equiv_{\mathsf{m}}, \Vdash_{\mathsf{m}} \rangle$

4 IX: THE INDEX CALCULUS

4.1 Syntax

Term variables $x\ \alpha$

$$
\begin{array}{llcl}
\text{Sorts} & \sigma & ::= & \star \mid \star_{\perp} \mid \mathcal{U} \mid \mathcal{U}_{\perp} \\
\text{Terms} & M, N, T & ::= & x \mid \star \mid \text{Nat} \mid \text{Zero} \mid \text{Suc}\, M \mid \\
 & & & \text{Ix}\, M \mid \text{FZero} \mid \text{FSuc}\, M \mid \\
 & & & \top \mid \mathtt{tt} \mid \perp \\
 & & & \Pi\alpha : T.N \mid \lambda x : T.N \mid M\, N \mid \\
 & & & \Sigma\alpha : T.M \mid (\alpha : T, M) \mid \text{fst}\, M \mid \text{snd}\, M \mid \\
 & & & M + N \mid \text{left}\, M \mid \text{right}\, M \mid \text{case}\ M\ M\ M \mid \\
 & & & M \equiv N \mid \text{refl}\ T\, M\, N \mid \\
 & & & \mu\alpha : T.M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : T
\end{array}
$$

Fig. 6. Syntax

Let 0, 1, 2, ... denote (as meta-syntax) object-level natural numbers in the intuitive fashion and let i_n denote finite natural obtained by n applications of FSuc to FZero.

4.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(\textsc{emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{var})\ \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\sigma \text{ Safe}}\ \boxed{\Gamma \vdash M \text{ Safe}}$$

(I am not sure these two rules are necessary.)

$$\frac{}{\mathcal{U} \text{ Safe}} \qquad \frac{\sigma \text{ Safe} \quad \Gamma \vdash M : \sigma}{\Gamma \vdash M \text{ Safe}}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\star)\ \frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad (\textsc{cum})\ \frac{\Gamma \vdash T : \star}{\Gamma \vdash T : \mathcal{U}}$$

$$(\star_\perp)\ \frac{}{\Gamma \vdash \star_\perp : \mathcal{U}_\perp} \qquad (\textsc{why-not})\ \frac{\Gamma \vdash T : \mathcal{U}}{\Gamma \vdash T : \mathcal{U}_\perp}$$

$$(\top)\ \frac{}{\Gamma \vdash \top : \sigma} \qquad (\perp)\ \frac{}{\Gamma \vdash \perp : \star_\perp}$$

$$(\textsc{Var})\ \frac{\alpha : \sigma \in \Gamma}{\Gamma \vdash \alpha : \sigma} \qquad (\textsc{Nat})\ \frac{}{\Gamma \vdash \mathrm{Nat} : \star} \qquad (\mathrm{Ix})\ \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\ n : \star}$$

$$(\Pi)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \sigma}{\Gamma \vdash \Pi\alpha : M.N : \sigma} \qquad (\Sigma)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \sigma}{\Gamma \vdash \Sigma\alpha : M.N : \sigma}$$

$$(+)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} \qquad (\equiv)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma \quad \sigma \text{ Safe}}{\Gamma \vdash M \equiv N : \sigma}$$

$$\boxed{\Gamma \vdash M : N}$$

$$(\mathit{Var})\ \frac{x : M \in \Gamma}{\Gamma \vdash x : M} \qquad (\mathtt{tt})\ \frac{}{\Gamma \vdash \mathtt{tt} : \top}$$

$$(\mathrm{Zero})\ \frac{}{\Gamma \vdash \mathrm{Zero} : \mathrm{Nat}} \qquad (\mathrm{Suc})\ \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Suc}\ n : \mathrm{Nat}}$$

$$(\mathrm{FZero})\ \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{FZero} : \mathrm{Ix}\ (\mathrm{Suc}\ n)} \qquad (\mathrm{FSuc})\ \frac{\Gamma \vdash n : \mathrm{Nat} \quad \Gamma \vdash i : \mathrm{Ix}\ n}{\Gamma \vdash \mathrm{FSuc}\ i : \mathrm{Ix}\ (\mathrm{Suc}\ n)}$$

$$(\Pi I)\ \frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \Pi(x : T).N} \qquad (\Pi E)\ \frac{\Gamma \vdash M : \Pi(x : T_1).T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\,N : T_2[N/x]}$$

$$(\Sigma I)\ \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1).T_2} \qquad (\Sigma E_1)\ \frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash \mathrm{fst}\ M : T_1} \qquad (\Sigma E_2)\ \frac{\Gamma \vdash M : \Sigma(x : T_1).T_2}{\Gamma \vdash \mathrm{snd}\ M : T_2[(\mathrm{fst}\ M)/x]}$$

$$(\equiv I)\ \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathrm{refl} : M \equiv M} \qquad (\textsc{conv})\ \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma \quad \sigma \text{ Safe}}{\Gamma \vdash M : T_2}$$

$$(\equiv E)\ \frac{\begin{array}{c}\Gamma \vdash P : T_1 \equiv T_2 \\ \Gamma \vdash M : T_1 \\ \Gamma \vdash N : T_1 \\ \Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\ \Gamma, z : T_1 \vdash H : T[z/x, z/y, \mathrm{refl}/p]\end{array}}{\Gamma \vdash \mathcal{J}\ H\ M\ N\ P : T[M/x, N/y, P/p]}$$

Fig. 7. Context formation and typing rules for Ix terms

$$\boxed{\Gamma \vdash M = N : \sigma}$$

$$(\text{E-REFL})\,\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \qquad (\text{E-SYM})\,\frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \qquad (\text{E-TRANS})\,\frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma}$$

$$\boxed{\Gamma \vdash M = N : T}$$

$$(\text{C-REFL})\,\frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \qquad (\text{C-SYM})\,\frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \qquad (\text{C-TRANS})\,\frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}$$

Fig. 8. Definitional equality & computational laws

4.3 A Comparison to $\lambda^{\Pi\mathcal{U}\mathbb{N}}$ [Abel et al. 2018]

5 TRANSLATION FROM Rω

5.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of Rω types. Figure 9 describe the untyped translation, which is used to show translational soundness of the typed translation.

$$\boxed{(\kappa)^\bullet}$$

$$\begin{aligned}
(\star)^\bullet &= \star \\
(\mathsf{L})^\bullet &= \top \\
(\kappa_1 \longrightarrow \kappa_2)^\bullet &= \Pi(\alpha : (\kappa_1)^\bullet).(\kappa_2)^\bullet \\
(\mathsf{R}^\kappa)^\bullet &= \Sigma(n : \mathrm{Nat}).\Pi(j : \mathrm{Ix}\, n).(\kappa)^\bullet
\end{aligned}$$

$$\boxed{(\Gamma \vdash \tau : \kappa)^\bullet}$$

$$\begin{aligned}
(\alpha)^\bullet &= \alpha \\
(\tau_1 \longrightarrow \tau_2)^\bullet &= \Pi(\alpha : (\tau_1)^\bullet).(\tau_2)^\bullet \\
(\forall \alpha : \kappa.\tau)^\bullet &= \Pi(\alpha : (\kappa)^\bullet).(\tau)^\bullet \\
(\lambda \alpha : \kappa.\tau)^\bullet &= \Pi(\alpha : (\kappa)^\bullet).(\tau)^\bullet \\
(\pi \Rightarrow \tau)^\bullet &= \Pi(\alpha : (\pi)^\bullet).(\tau)^\bullet \\
(\tau\, \upsilon)^\bullet &= (\tau)^\bullet\, (\upsilon)^\bullet \\
(\ell)^\bullet &= \top \\
(\lfloor \xi \rfloor)^\bullet &= \top \\
(\xi \triangleright \tau)^\bullet &= (1,\ \lambda(i : \mathrm{Ix}\, 1).(\tau)^\bullet) \\
(\Pi \rho)^\bullet &= \Pi(i : \mathrm{Ix}\, (\mathrm{fst}\, (\rho)^\bullet)).(\mathrm{snd}\, (\rho)^\bullet)\; i \\
(\Sigma \rho)^\bullet &= \Sigma(i : \mathrm{Ix}\, (\mathrm{fst}\, (\rho)^\bullet)).(\mathrm{snd}\, (\rho)^\bullet)\; i
\end{aligned}$$

$$\boxed{(\Gamma \vdash \pi : \kappa)^\bullet}$$

$$\ldots$$

$$\boxed{(\Gamma \Vdash \pi)^\bullet}$$

$$\ldots$$

$$\boxed{(\Gamma \vdash M : \tau)^\bullet}$$

$$\ldots$$

Fig. 9. A compositional translation of Rω judgments to (untyped) Ix terms

5.2 Typed translation

$$\boxed{\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \vdash M \rightsquigarrow N : \tau}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \Vdash \pi \rightsquigarrow N}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\tau \equiv \upsilon \rightsquigarrow P}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Fig. 10. Translation of Rω derivations to Ix derivations

5.3 Properties of Translation

Presume an Rω instantiation of the simple row theory. A lot of this is likely bullshit.

Theorem 1 (Translational Soundness (Types)). *if* $\Gamma \vdash \tau : \kappa$ *such that* $\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa$ *then* $(\Gamma)^\bullet \vdash \upsilon : (\kappa)^\bullet$.

Theorem 2 (Translational Soundness (Type Equivalence)). *if*

(1) $\Gamma \vdash \tau_1 \rightsquigarrow \upsilon_1 : \kappa_1$;
(2) $\Gamma \vdash \tau_2 \rightsquigarrow \upsilon_2 : \kappa_2$; *and*
(3) $\tau_1 \equiv \tau_2 \rightsquigarrow P$,

then $(\Gamma)^\bullet \vdash P : \upsilon_1 \equiv \upsilon_2$.

Theorem 3 (Translational Soundness (Of Predicates)). *if* $\Gamma \Vdash \pi$ *such that* $\Gamma \Vdash \pi \rightsquigarrow N$ *then* $(\Gamma)^\bullet \vdash N : (\pi)^\bullet$.

Finally,

Theorem 4 (Translational Soundness). *if* $\Gamma \vdash M : \tau$ *such that* $\Gamma \vdash M \rightsquigarrow N{:}\tau$ *then* $(\Gamma)^\bullet \vdash N : (\tau)^\bullet$.

6 OPERATIONAL SEMANTICS OF IX

7 RECURSION

This section will later be incorporated into earlier sections.

7.1 Rome, or, Rω with μ

$$\boxed{\text{todo}}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Fig. 11. Additional Rω judgments for recursion

7.2 Mix, the recursive index calculus

$$\boxed{\text{todo}}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Fig. 12. Additional Ix judgments for recursion

7.3 Translation and properties of translation

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