The Index Calculus and its translation from Rω

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1 Ix: The Index Calculus

1.1 Syntax

<div align="center">

Term variables x α

</div>

Sorts	σ ::=	$\star \mid \mathcal{U}$
Terms	M, N ::=	$x \mid \star \mid \mathrm{Nat} \mid \mathrm{Zero} \mid \mathrm{Suc}\ M \mid$
		$\mathrm{Ix}\ M \mid \mathrm{FZero} \mid \mathrm{FSuc}\ M \mid$
		$\top \mid \mathtt{tt} \mid$
		$\Pi\alpha : M.N \mid \lambda x : M.N \mid M\ N \mid$
		$\Sigma\alpha : M.M \mid (\alpha : M, M) \mid M.1 \mid M.2$
		$M + N \mid \mathrm{left}\ M \mid \mathrm{right}\ M \mid \mathrm{case}\ M\ \mathrm{of}\ \{\mathrm{left} \mapsto M; \mathrm{right} \mapsto M\}$
		$M \equiv N \mid \mathrm{refl} \mid \ldots$
Environments	Γ ::=	$\varepsilon \mid \Gamma, \alpha : M$

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Figure 1: Syntax

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1.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(\text{C-EMP}) \; \frac{}{\vdash \varepsilon} \qquad (\text{C-VAR}) \; \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\text{T-}\star) \; \frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad \frac{}{\Gamma \vdash \top : \star} \qquad \frac{}{\Gamma \vdash \text{Nat} : \star} \qquad \frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Ix}\, n : \star}$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Pi\alpha : M.N : \star} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Sigma\alpha : M.N : \star}$$

$$\frac{\Gamma \vdash M : \star \quad \Gamma \vdash N : \star}{\Gamma \vdash M + N : \star} \qquad \frac{\Gamma \vdash M_1 : N_1 \quad \Gamma \vdash N_1 : \sigma_1 \quad \Gamma \vdash M_2 : N_2 \quad \Gamma \vdash N_2 : \sigma_2}{\Gamma \vdash M_1 \equiv M_2 : \star}$$

$$\boxed{\Gamma \vdash M : N}$$

$$\frac{x : M \in \Gamma}{\Gamma \vdash x : M} \qquad \frac{}{\Gamma \vdash \mathtt{tt} : \top}$$

2 Translation From Rω

2.1 Example translations of Rω terms and types

2.2 Formal specification

3 Recursion

A The static semantics of Rω

A.1 Syntax

The syntax of R$\omega(\mathcal{T})$ is given in Figure 2.

Term variables x Type variables α Labels ℓ Directions $d \in \{\mathsf{L}, \mathsf{R}\}$

$$
\begin{array}{llll}
\text{Kinds} & \kappa & ::= & \star \mid \mathsf{L} \mid \mathsf{R}^\kappa \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi & ::= & \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi & ::= & \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa.\tau \mid \lambda \alpha : \kappa.\tau \mid \tau\,\tau \\
& & \mid & \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi\rho \mid \Sigma\rho \\
\text{Terms} & M, N & ::= & x \mid \lambda x : \tau.M \mid M\,N \mid \Lambda \alpha : \kappa.M \mid M\,[\tau] \\
& & \mid & \ell \mid M \triangleright M \mid M/M \mid \mathsf{prj}_d\,M \mid M \mathbin{+\!\!+} M \mid \mathsf{inj}_d\,M \mid M \triangledown M \\
& & \mid & \mathsf{syn}_\phi\,M \mid \mathsf{ana}_\phi\,M \mid \mathsf{fold}\,M\,M\,M\,M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}
$$

Figure 2: Syntax

A.2   Types and Kinds

Figure 3 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

3

$$\boxed{\vdash \Gamma}$$

$\text{(C-EMP)} \ \dfrac{}{\vdash \varepsilon}$
$\qquad$
$\text{(C-TVAR)} \ \dfrac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa}$
$\qquad$
$\text{(C-VAR)} \ \dfrac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau}$
$\qquad$
$\text{(C-PRED)} \ \dfrac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$

$$\boxed{\Gamma \vdash \tau : \kappa} \ \boxed{\Gamma \vdash \pi}$$

$\text{(K-VAR)} \ \dfrac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa}$
$\qquad$
$\text{(K-}(\to)) \ \dfrac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star}$
$\qquad$
$\text{(K-}\Rightarrow) \ \dfrac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$

$\text{(K-}\forall) \ \dfrac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa.\tau : \star}$
$\qquad$
$\text{(K-}\to I) \ \dfrac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1.\tau : \kappa_1 \to \kappa_2}$
$\qquad$
$\text{(K-}\to E) \ \dfrac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1 \, \tau_2 : \kappa_2}$

$\text{(K-LAB)} \ \dfrac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}}$
$\qquad$
$\text{(K-SING)} \ \dfrac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star}$
$\qquad$
$\text{(K-LTY)} \ \dfrac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa}$
$\qquad$
$\text{(K-ROW)} \ \dfrac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^\kappa}$

$\text{(K-}\Pi) \ \dfrac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Pi\rho : \kappa}$
$\qquad$
$\text{(K-}\Sigma) \ \dfrac{\Gamma \vdash \rho : \mathsf{R}^\kappa}{\Gamma \vdash \Sigma\rho : \kappa}$
$\qquad$
$\text{(K-LIFT}_1) \ \dfrac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho \, \tau : \mathsf{R}^{\kappa_2}}$

$\text{(K-LIFT}_2) \ \dfrac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi \, \rho : \mathsf{R}^{\kappa_2}}$
$\qquad$
$\text{(K-}\lesssim_d) \ \dfrac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \lesssim_d \rho_2}$
$\qquad$
$\text{(K-}\odot) \ \dfrac{\Gamma \vdash \rho_i : \mathsf{R}^\kappa}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$

Figure 3: Contexts and kinding.

$$\boxed{\tau \equiv \tau} \quad \boxed{\pi \equiv \pi}$$

$$\left(\text{E-REFL}\right) \frac{}{\tau \equiv \tau} \qquad \left(\text{E-SYM}\right) \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad \left(\text{E-TRANS}\right) \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad \left(\text{E-}\beta\right) \frac{}{(\lambda\alpha : \kappa.\tau)\, \upsilon \equiv \tau[\upsilon/\alpha]}$$

$$\left(\text{E-}\xi_\Rightarrow\right) \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad \left(\text{E-}\xi_\forall\right) \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha : \kappa.\tau \equiv \forall\beta : \kappa.\upsilon}\, (\gamma \notin fv(\tau, \upsilon)) \qquad \left(\text{E-}\xi_{\text{APP}}\right) \frac{\tau_i \equiv \upsilon_i}{\tau_1\, \tau_2 \equiv \upsilon_1\, \upsilon_2}$$

$$\left(\text{E-}\xi_\rhd\right) \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \rhd \tau_1 \equiv \xi_2 \rhd \tau_2} \qquad \left(\text{E-ROW}\right) \frac{\{\overline{\xi_i \rhd \tau_i}\} \equiv_\mathcal{T} \{\overline{\xi'_j \rhd \tau'_j}\}}{\{\overline{\xi_i \rhd \tau_i}\} \equiv \{\overline{\xi'_j \rhd \tau'_j}\}} \qquad \left(\text{E-}\xi_{\lfloor\cdot\rfloor}\right) \frac{\xi_1 \equiv \xi_2}{\lfloor\xi_1\rfloor \equiv \lfloor\xi_2\rfloor}$$

$$\left(\text{E-LIFT}_1\right) \frac{}{\{\xi \rhd \phi\}\, \tau \equiv \{\xi \rhd \phi\, \tau\}} \qquad \left(\text{E-LIFT}_2\right) \frac{}{\phi\, \{\xi \rhd \tau\} \equiv \{\xi \rhd \phi\, \tau\}}$$

$$\left(\text{E-}\xi_{\Pi\Sigma}\right) \frac{\rho_1 \equiv \rho_2}{K\, \rho_1 \equiv K\, \rho_2} \qquad \left(\text{E-LIFT}_3\right) \frac{}{(K\rho)\, \tau \equiv K(\rho\, \tau)} \qquad \left(\text{E-SING}\right) \frac{}{K\{\xi \rhd \tau\} \equiv \xi \rhd \tau}\, (K \in \{\Pi, \Sigma\})$$

$$\left(\text{E-}\xi_{\lesssim_d}\right) \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad \left(\text{E-}\xi_\odot\right) \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Figure 4: Type and predicate equivalence

A.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$(\text{T-VAR}) \dfrac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$
$\qquad (\text{T-}{\to}I) \dfrac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1. M : \tau_1 \to \tau_2}$
$\qquad (\text{T-}{\to}E) \dfrac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2}{\Gamma \vdash M_1\, M_2 : \tau_2}$

$(\text{T-}{\equiv}) \dfrac{\Gamma \vdash M : \tau \quad \tau \equiv \upsilon}{\Gamma \vdash M : \upsilon}$
$\qquad (\text{T-}{\Rightarrow}I) \dfrac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau}$
$\qquad (\text{T-}{\Rightarrow}E) \dfrac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$

$(\text{T-}\forall I) \dfrac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa. M : \forall \alpha : \kappa. \tau}$
$\qquad (\text{T-}\forall E) \dfrac{\Gamma \vdash M : \forall \alpha : \kappa. \tau \quad \Gamma \vdash \upsilon : \kappa}{\Gamma \vdash M\,[\upsilon] : \tau[\upsilon/\alpha]}$

$(\text{T-SING}) \dfrac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor}$
$\qquad (\text{T-}{\triangleright}I) \dfrac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau}$
$\qquad (\text{T-}{\triangleright}E) \dfrac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1 / M_2 : \tau}$

$(\text{T-}\Pi E) \dfrac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\, M : \Pi \rho_2}$
$\qquad (\text{T-}\Pi I) \dfrac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 +\!\!\!+\, M_2 : \Pi \rho_3}$

$(\text{T-}\Sigma I) \dfrac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \mathsf{inj}\, M : \Sigma \rho_2}$
$\qquad (\text{T-}\Sigma E) \dfrac{\Gamma \vdash M_1 : \Sigma \rho_1 \to \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \to \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim}{\Gamma \vdash M_1 \bigtriangledown M_2 : \Sigma \rho_3 \to \tau}$

$(\text{T-ana}) \dfrac{\Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \to \star \quad\quad\quad}{\dfrac{\Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa.\, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u \to \tau}{\Gamma \vdash \mathsf{ana}_\phi\, M : \Sigma(\phi\, \rho) \to \tau}}$

$(\text{T-syn}) \dfrac{\Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \to \star \quad\quad\quad}{\dfrac{\Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa.\, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u}{\Gamma \vdash \mathsf{syn}_\phi\, M : \Pi(\phi\, \rho)}}$

$(\text{T-fold}) \dfrac{\begin{array}{c} M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^\kappa.\, (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to t \to \upsilon \\ \Gamma \vdash M_2 : \upsilon \to \upsilon \to \upsilon \quad \Gamma \vdash M_3 : \upsilon \quad \Gamma \vdash N : \Pi \rho \end{array}}{\Gamma \vdash \mathsf{fold}\, M_1\, M_2\, M_3\, N : \upsilon}$

Figure 5: Typing

Minimal Rows

Figure 6 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathsf{m}} \rho : \kappa} \quad \boxed{\rho \equiv_{\mathsf{m}} \rho}$$

$$(\text{K-MROW}) \; \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathsf{m}} \{\xi \triangleright \tau\} : \mathsf{R}^{\kappa}} \qquad (\text{E-MROW}) \; \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathsf{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathsf{m}} \pi}$$

$$(\text{N-AX}) \; \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathsf{m}} \pi} \qquad (\text{N-REFL}) \; \frac{}{\Gamma \Vdash_{\mathsf{m}} \rho \lesssim_d \rho} \qquad (\text{N-TRANS}) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\text{N-}\equiv) \; \frac{\Gamma \Vdash_{\mathsf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathsf{m}} \pi_2} \qquad (\text{N-}\lesssim\text{LIFT}_1) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \lesssim_d \phi\,\rho_2} \qquad (\text{N-}\lesssim\text{LIFT}_2) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \lesssim_d \rho_2\,\tau}$$

$$(\text{N-}\odot\text{LIFT}_1) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \odot \rho_2\,\tau \sim \rho_3\,\tau} \qquad (\text{N-}\odot\text{LIFT}_2) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \odot \phi\,\rho_2 \sim \phi\,\rho_3}$$

$$(\text{N-}\odot\lesssim_{\mathsf{L}}) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\text{N-}\odot\lesssim_{\mathsf{R}}) \; \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Figure 6: Minimal row theory $\mathcal{M} = \langle \vdash_{\mathsf{m}}, \equiv_{\mathsf{m}}, \Vdash_{\mathsf{m}} \rangle$