

Recursive Rows in Rome

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1 IX: THE INDEX CALCULUS

1.1 Syntax

| | |
|--------------|---|
| Sorts | $\sigma ::= \star \mid \mathcal{U}$ |
| Terms | $M, N, T ::= \star \mid x \mid$ $\mathbb{N} \mid Z \mid S M \mid$ $\text{case}_{\mathbb{N}} M N T \mid$ $\text{Ix } M \mid \text{I}_0 \mid \text{I}_S M \mid$ $\text{case}_{\text{Ix}} M N \mid \text{case}_{\text{Ix}} M N T \mid$ $\{\{M_1, \dots, M_n\}\} \mid \lambda() \mid$ $\tau \mid \text{tt} \mid$ $\forall \alpha : T. N \mid \lambda x : T. N \mid M N \mid$ $\exists \alpha : T. M \mid \langle \langle \alpha : T, M \rangle \rangle \mid \text{case}_{\exists} M N \mid$ $M + N \mid \text{left } M \mid \text{right } M \mid$ $\text{case}_+ M N T \mid$ $M \equiv N \mid \text{refl } T M N \mid$ |
| Environments | $\Gamma ::= \varepsilon \mid \Gamma, \alpha : T$ |

Fig. 1. Syntax

1.1.1 Meta-syntax & syntactic sugar. Let

- (1) $\tau \rightarrow v$ denote the non-dependent universal quantification $\forall(_ : \tau).v$;
- (2) $\tau \times v$ denote the non-dependent existential quantification $\exists(_ : \tau).v$;
- (3) $0, 1, 2, \dots$ denote object-level natural numbers in the intuitive fashion;
- (4) i_n denote the index obtained by n applications of I_S to I_0 ; and

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(5) the syntax

$$\{\!\{M_1, \dots, M_n\}\!\}$$

denote the large elimination of a known, finite quantity of indices to types M_1, \dots, M_n , elaborated by the equations:

$$\begin{aligned}\{\!\{M_1\}\!\} &:= \lambda(i : \text{Ix } 1).\text{case}_{\text{Ix}} i M_1 \\ \{\!\{M_1, \dots, M_n\}\!\} &:= \lambda(i : \text{Ix } n).\text{case}_{\text{Ix}} i M_1 \{\!\{M_2, \dots, M_n\}\!\}\end{aligned}$$

1.2 Typing

$$\begin{array}{c} \boxed{\vdash \Gamma} \\[10pt] (\text{EMP}) \frac{}{\vdash \varepsilon} \quad (\text{VAR}) \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M} \\[10pt] \boxed{\Gamma \vdash M : \sigma} \\[10pt] \begin{array}{llll} (\star) \frac{\vdash \Gamma}{\Gamma \vdash \star : \mathcal{U}} & (\top) \frac{\vdash \Gamma}{\Gamma \vdash \top : \sigma} & (\text{NAT}) \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{N} : \star} & (\text{IX}) \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \text{Ix } n : \star} \\[10pt] (\forall) \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \forall \alpha : M.N : \sigma_2} & (\exists) \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \exists \alpha : M.N : \sigma_2} & & \\[10pt] (+) \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} & (\equiv) \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \equiv N : \sigma} & & \end{array} \end{array}$$

Fig. 2. Context and type well-formedness

$$\boxed{\Gamma \vdash M : N}$$

$$\begin{array}{c}
(\text{VAR}) \frac{\vdash \Gamma \quad x : M \in \Gamma}{\Gamma \vdash x : M} \quad (\text{tt}) \frac{\vdash \Gamma}{\Gamma \vdash \text{tt} : \top} \\
(Z) \frac{\vdash \Gamma}{\Gamma \vdash Z : \mathbb{N}} \quad (S) \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash S n : \mathbb{N}} \quad (\mathbb{N}E) \frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \mathbb{N} \quad \Gamma \vdash N : A \quad \Gamma \vdash P : \mathbb{N} \rightarrow A}{\Gamma \vdash \text{case}_{\mathbb{N}} M N P : A} \\
(l_0) \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash l_0 : \text{Ix}(S n)} \quad (l_S) \frac{\Gamma \vdash n : \mathbb{N} \quad \Gamma \vdash i : \text{Ix } n}{\Gamma \vdash l_S i : \text{Ix}(S n)} \quad (\lambda()) \frac{\Gamma \vdash A : \sigma}{\Gamma \vdash \lambda() : \text{Ix } 0 \rightarrow A} \\
(l_S E) \frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \text{Ix}(S n) \quad \Gamma \vdash N : A \quad \Gamma \vdash P : \text{Ix } n \rightarrow A}{\Gamma \vdash \text{case}_{\text{Ix}} M N P : A} \\
(\forall I) \frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T. M : \forall(x : T). N} \quad (\forall E) \frac{\Gamma \vdash M : \forall(x : T_1). T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M N : T_2[N/x]} \\
(\exists I) \frac{\Gamma \vdash T_1 : \sigma \quad \Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash \langle\langle M : T_1, N \rangle\rangle : \exists(x : T_1). T_2} \\
(\exists E) \frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \exists(x : T_1). T_2 \quad \Gamma \vdash N : \forall(x : T_1). T_2 \rightarrow A}{\Gamma \vdash \text{case}_{\exists} M N : A} \\
(+_1 I) \frac{\Gamma \vdash M : A}{\Gamma \vdash \text{left } M : A + B} \quad (+_2 I) \frac{\Gamma \vdash N : B}{\Gamma \vdash \text{right } N : A + B} \\
(+E) \frac{\Gamma \vdash C : \sigma \quad \Gamma \vdash M : A + B \quad \Gamma \vdash N : A \rightarrow C \quad \Gamma \vdash P : B \rightarrow C}{\Gamma \vdash \text{case}_+ M N P : C} \\
(\equiv I) \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \text{refl} : M \equiv M} \quad (\text{CONV}) \frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma}{\Gamma \vdash M : T_2} \\
\begin{array}{c}
\Gamma \vdash P : T_1 \equiv T_2 \\
\Gamma \vdash M : T_1 \\
\Gamma \vdash N : T_1 \\
\Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\
\Gamma, z : T_1 \vdash H : T[z/x, z/y, \text{refl}/p] \\
(\equiv E) \frac{}{\Gamma \vdash \mathcal{J} H M N P : T[M/x, N/y, P/p]}
\end{array}
\end{array}$$

Fig. 3. Typing lx terms

$$\boxed{\Gamma \vdash M = N : \sigma}$$

$$\begin{array}{c}
(\text{E-REFL}) \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \quad (\text{E-SYM}) \frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \quad (\text{E-TRANS}) \frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma} \\
\boxed{\Gamma \vdash M = N : T} \\
(\text{C-REFL}) \frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \quad (\text{C-SYM}) \frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \quad (\text{C-TRANS}) \frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}
\end{array}$$

Fig. 4. Definitional equality & computational laws

1.3 Properties

THEOREM 1 (WELL-SORTEDNESS). *if $\Gamma \vdash M : N$ then $\vdash \Gamma$ and there exists σ such that $\Gamma \vdash N : \sigma$.*

1.4 Elaborating Ix to the CoC + Fin

Following the above, I believe Ix to a smaller calculus with the syntax below. This is effectively the calculus of constructions with primitive naturals and finite indices.

| | |
|--------------|--|
| Sorts | $\sigma ::= \star \mid \mathcal{U}$ |
| Terms | $M, N, T ::= \star \mid x \mid$ $\mathbb{N} \mid Z \mid S M \mid$ $\text{case}_{\mathbb{N}} M N T \mid$ $\text{Ix } M \mid l_0 \mid l_S M \mid$ $\text{case}_{\text{Ix}} M N T \mid$ $\forall \alpha : T. N \mid \lambda x : T. N \mid M N \mid$ $M \equiv N \mid \text{refl } T M N \mid$ |
| Environments | $\Gamma ::= \varepsilon \mid \Gamma, \alpha : T$ |

Fig. 5. Syntax

Elaboration is given below. One would also expect a rule for bot elimination—or you could simply encode Ix 0 as $\forall X : \star. X$.

$$\begin{aligned}
& \top \rightsquigarrow \text{Ix } 1 \\
& \text{tt} \rightsquigarrow l_0 \\
& \perp \rightsquigarrow \text{Ix } 0 \\
& A \rightarrow B \rightsquigarrow \forall (x : A). B \\
& \exists (x : A). B \rightsquigarrow \forall (C : \mathcal{U}). (\forall (x : A). B \rightarrow C) \rightarrow C \\
& \langle\langle M : T, N \rangle\rangle \rightsquigarrow \lambda (C : \mathcal{U}). \lambda (f : (\forall (x : A). B \rightarrow C)). f M N \\
& A + B \rightsquigarrow \exists (i : \text{Ix } 2). \{A, B\} i \\
& \text{left } M \rightsquigarrow \langle\langle i_0 : \text{Ix } 2, M \rangle\rangle \\
& \text{right } N \rightsquigarrow \langle\langle i_1 : \text{Ix } 2, N \rangle\rangle \\
& A \times B \rightsquigarrow \exists (x : A). B
\end{aligned}$$

One could also translate naturals away using your favorite functional encoding. (I imagine there are encodings for Fin, too.)

2 TRANSLATION FROM $R\omega$

2.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of $R\omega$ types. Figure 7 describe the untyped translation, which is used to show translational soundness of the typed translation (Figure 7).

$$\boxed{\llbracket \kappa \rrbracket}$$

$$\llbracket \star \rrbracket = \star$$

$$\llbracket \mathbf{L} \rrbracket = \top$$

$$\llbracket \kappa_1 \rightarrow \kappa_2 \rrbracket = \llbracket \kappa_1 \rrbracket \rightarrow \llbracket \kappa_2 \rrbracket$$

$$\llbracket \mathbf{R}^\kappa \rrbracket = \exists (n : \mathbb{N}). \text{Ix } n \rightarrow \llbracket \kappa \rrbracket$$

$$\boxed{\llbracket \Delta \vdash \tau : \kappa \rrbracket}$$

$$\llbracket \Delta \vdash \alpha : \kappa \rrbracket = \alpha$$

$$\llbracket \Delta \vdash \tau_1 \rightarrow \tau_2 : \star \rrbracket = \llbracket \tau_1 \rrbracket \rightarrow \llbracket \tau_2 \rrbracket$$

$$\llbracket \Delta \vdash \forall \alpha : \kappa. \tau : \star \rrbracket = \forall (\alpha : \llbracket \kappa \rrbracket). \llbracket \tau \rrbracket$$

$$\llbracket \Delta \vdash \lambda \alpha : \kappa. \tau : \kappa \rightarrow \kappa' \rrbracket = \forall (\alpha : \llbracket \kappa \rrbracket). \llbracket \tau \rrbracket$$

$$\llbracket \Delta \vdash \pi \Rightarrow \tau : \kappa \rrbracket = \llbracket \pi \rrbracket \rightarrow \llbracket \tau \rrbracket$$

$$\llbracket \Delta \vdash \tau v : \kappa \rrbracket = \llbracket \tau \rrbracket \llbracket v \rrbracket$$

$$\llbracket \Delta \vdash \ell : \mathbf{L} \rrbracket = \top$$

$$\llbracket \Delta \vdash [\xi] : \star \rrbracket = \top$$

$$\llbracket \Delta \vdash (\xi \triangleright \tau) : \kappa \rrbracket = \llbracket \tau \rrbracket$$

$$\llbracket \Delta \vdash \Pi \rho : \star \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda P : \text{Ix } n \rightarrow \star. \forall (i : \text{Ix } n). P i)$$

$$\llbracket \Delta \vdash \Sigma \rho : \star \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda P : \text{Ix } n \rightarrow \star. \exists (i : \text{Ix } n). P i)$$

$$\llbracket \Delta \vdash \epsilon : \mathbf{R}^\kappa \rrbracket = \langle \langle 0 : \mathbb{N}, \lambda () \rangle \rangle$$

$$\llbracket \Delta \vdash \rho [v] : \mathbf{R}^{\kappa_2} \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa_1 \rrbracket \rightarrow \llbracket \kappa_2 \rrbracket). \langle \langle n : \mathbb{N}, \lambda (j : \text{Ix } n). (P j) \llbracket \tau \rrbracket \rangle \rangle)$$

$$\llbracket \Delta \vdash [\tau] \rho : \mathbf{R}^{\kappa_2} \rrbracket = \text{case}_{\exists} \llbracket \rho \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa_1 \rrbracket). \langle \langle n : \mathbb{N}, \lambda (j : \text{Ix } n). \llbracket \tau \rrbracket (P j) \rangle \rangle)$$

$$\llbracket \Delta \vdash (\xi \triangleright_{\mathbf{R}} \tau) : \mathbf{R}^\kappa \rrbracket = \langle \langle 1 : \mathbb{N}, \llbracket \tau \rrbracket \rangle \rangle$$

Fig. 6. Translating kinding derivations to untyped Ix terms

$$\begin{aligned}
& \boxed{\llbracket \Delta \vdash \pi : \kappa \rrbracket} \\
& \text{case } \llbracket \rho_1 \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa \rrbracket)). \\
\llbracket \Delta \vdash \rho_1 \lesssim \rho_2 : \kappa \rrbracket &= \text{case } \llbracket \rho_2 \rrbracket (\lambda m : \mathbb{N}. \lambda (Q : \text{Ix } m \rightarrow \llbracket \kappa \rrbracket)). \\
& \forall (i : \text{Ix } n). \exists (j : \text{Ix } m). P \, i \equiv Q \, j) \\
& \text{case } \llbracket \rho_1 \rrbracket (\lambda n : \mathbb{N}. \lambda (P : \text{Ix } n \rightarrow \llbracket \kappa \rrbracket)). \\
& \text{case } \llbracket \rho_2 \rrbracket (\lambda m : \mathbb{N}. \lambda (Q : \text{Ix } m \rightarrow \llbracket \kappa \rrbracket)). \\
& \text{case } \llbracket \rho_3 \rrbracket (\lambda l : \mathbb{N}. \lambda (R : \text{Ix } l \rightarrow \llbracket \kappa \rrbracket)). \\
& (\forall (i : \text{Ix } n). \exists (k : \text{Ix } l). P \, i \equiv R \, k) \\
\llbracket \Delta \vdash \rho_1 \odot \rho_2 \sim \rho_3 : \kappa \rrbracket &= \times (\forall (j : \text{Ix } m). \exists (k : \text{Ix } l). Q \, j \equiv R \, k) \\
& \times (\forall (k : \text{Ix } l). \\
& \quad (\exists (i : \text{Ix } n). P \, i \equiv R \, k) \\
& \quad + (\exists (j : \text{Ix } m). Q \, j \equiv R \, k))))
\end{aligned}$$

Fig. 7. Translating predicate well-formedness judgments

2.2 Typed translation

There is some subtlety in mechanizing environments. Environments in $R\omega$ store kinds, *typing derivations*, and *predicate well-formedness derivations*. If we are to simply translate derivations to untyped syntax, we are losing a bit of information. I am not sure, however, it is possible to translate derivations (in $R\omega$) to derivations (in Ix) without a de facto type checker for Ix . I think we will have to perform the former: let derivations in $R\omega$ environments translate to untyped types and sorts in Ix environments. Then, argue as metatheory that $\vdash \Delta \rightsquigarrow \Gamma$ implies $\vdash \Gamma$.

$$\begin{array}{c}
\boxed{\vdash \Delta \rightsquigarrow \Gamma} \\
\\
\text{(C-}\epsilon\text{)} \frac{}{\vdash \epsilon \rightsquigarrow \epsilon} \quad \text{(C-TVAR)} \frac{\vdash \Delta \rightsquigarrow \Gamma}{\vdash \Delta, \alpha : \kappa \rightsquigarrow \Gamma, \alpha : \llbracket \kappa \rrbracket} \\
\\
\text{(C-VAR)} \frac{\vdash \Delta \rightsquigarrow \Gamma}{\vdash \Delta, x : \tau \rightsquigarrow \Gamma, x : \llbracket \tau \rrbracket} \quad \text{(C-PRED)} \frac{\vdash \Delta \rightsquigarrow \Gamma}{\vdash \Delta, \pi : \kappa \rightsquigarrow \Gamma, p : \llbracket \pi \rrbracket} (p \text{ fresh}) \\
\\
\boxed{\Delta \vdash M \rightsquigarrow N : \tau} \\
\\
\text{(T-VAR)} \frac{x : \tau \in \Delta}{\Delta \vdash x \rightsquigarrow x : \tau} \\
\\
\text{(T-}\rightarrow\text{I)} \frac{\Delta, x : \tau \vdash M \rightsquigarrow N : v}{\Delta \vdash \lambda x : \tau. M \rightsquigarrow \lambda x : \llbracket \tau \rrbracket. N : \tau \rightarrow v} \quad \text{(T-}\rightarrow\text{I)} \frac{\Delta \vdash M \rightsquigarrow F : \tau \rightarrow v \quad \Delta \vdash N \rightsquigarrow E : \tau}{\Delta \vdash M N \rightsquigarrow F E : v} \\
\\
\text{(T-}\Rightarrow\text{I)} \frac{\Delta, \pi \vdash M \rightsquigarrow N : \tau}{\Delta \vdash M \rightsquigarrow \lambda(p : \llbracket \pi \rrbracket). N : \pi \Rightarrow \tau} \quad \text{(T-}\Rightarrow\text{E)} \frac{\Delta \vdash M \rightsquigarrow F : \pi \Rightarrow \tau \quad \Delta \Vdash \pi \rightsquigarrow E}{\Delta \vdash M \rightsquigarrow F E : \tau} \\
\\
\text{(T-}\forall\text{I)} \frac{\Delta \vdash M \rightsquigarrow N : \tau}{\Delta \vdash \Lambda \alpha : \kappa. M \rightsquigarrow \lambda(\alpha : \llbracket \kappa \rrbracket). N : \forall \alpha : \kappa. \tau} \quad \text{(T-}\forall\text{E)} \frac{\Delta \vdash M \rightsquigarrow N : \forall \alpha : \kappa. \tau}{\Delta \vdash M[v] \rightsquigarrow N \llbracket v \rrbracket : \tau[v/\alpha]} \\
\\
\text{(T-SING)} \frac{}{\Delta \vdash \ell \rightsquigarrow \text{tt} : [\ell]} \quad \text{(T-FOO)} \frac{A}{B} \\
\\
\text{(C-FOO)} \frac{A}{B} \quad \text{(C-FOO)} \frac{A}{B} \quad \text{(C-FOO)} \frac{A}{B}
\end{array}$$

Fig. 8. Translation of $R\omega$ environments and typing derivations

$$\begin{array}{c}
\text{(C-FOO)} \frac{A}{B} \\
\\
\boxed{\Delta \Vdash \pi \rightsquigarrow N} \\
\\
\text{(C-FOO)} \frac{A}{B} \\
\\
\boxed{\tau \equiv v \rightsquigarrow P} \\
\\
\text{(C-FOO)} \frac{A}{B}
\end{array}$$

Fig. 9. Translation of $R\omega$ derivations to Ix derivations

2.3 Properties of Translation

THEOREM 2 (TRANSLATIONAL SOUNDNESS (ENVIRONMENTS)). *if $\vdash \Delta \rightsquigarrow \Gamma$ then $\vdash \Gamma$.*

THEOREM 3 (TRANSLATIONAL SOUNDNESS (TYPES)). *if $\Delta \vdash \tau : \kappa$ and $\vdash \Delta \rightsquigarrow \Gamma$ then $\Gamma \vdash \llbracket \tau \rrbracket : \llbracket \kappa \rrbracket$.*

THEOREM 4 (TRANSLATIONAL SOUNDNESS (TYPE EQUIVALENCE)). *if*

- (1) $\Gamma \vdash \tau_1 \rightsquigarrow v_1 : \kappa_1$;
- (2) $\Gamma \vdash \tau_2 \rightsquigarrow v_2 : \kappa_2$; *and*
- (3) $\tau_1 \equiv \tau_2 \rightsquigarrow P$,

then $\llbracket \Gamma \rrbracket \vdash P : v_1 \equiv v_2$.

THEOREM 5 (TRANSLATIONAL SOUNDNESS (PREDICATES)). *if* $\Gamma \Vdash \pi$ *such that* $\Gamma \Vdash \pi \rightsquigarrow N$ *then* $\llbracket \Gamma \rrbracket \vdash N : \llbracket \pi \rrbracket$.

Finally,

THEOREM 6 (TRANSLATIONAL SOUNDNESS). *if* $\Gamma \vdash M : \tau$ *such that* $\Gamma \vdash M \rightsquigarrow N : \tau$ *then* $\llbracket \Gamma \rrbracket \vdash N : \llbracket \tau \rrbracket$.

3 OPERATIONAL SEMANTICS

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