Recursive Rows in Rome

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1 IX: THE INDEX CALCULUS

1.1 Syntax

$$
\begin{array}{llcl}
\text{Sorts} & \sigma & ::= & \star \mid \mathcal{U} \\
\text{Terms} & M, N, T & ::= & \star \mid x \mid \\
 & & & \mathbb{N} \mid Z \mid \mathsf{S}\, M \mid \\
 & & & \mathrm{case}_{\mathbb{N}}\, M\, N\, T \mid \\
 & & & \mathrm{Ix}\, M \mid \mathsf{I}_0 \mid \mathsf{I}_S\, M \mid \\
 & & & \mathrm{case}_{\mathrm{Ix}}\, M\, N \mid \mathrm{case}_{\mathrm{Ix}}\, M\, N\, T \mid \\
 & & & \{\!\{M_1, \ldots, M_n\}\!\} \mid \lambda\!\!\!\lambda() \mid \\
 & & & \top \mid \mathtt{tt} \mid \\
 & & & \forall \alpha : T.N \mid \lambda x : T.N \mid M\, N \mid \\
 & & & \exists \alpha : T.M \mid \langle\!\langle \alpha : T, M \rangle\!\rangle \mid \mathrm{case}_{\exists}\, M\, N \mid \\
 & & & M + N \mid \mathrm{left}\, M \mid \mathrm{right}\, M \mid \\
 & & & \mathrm{case}_{+}\, M\, N\, T \mid \\
 & & & M \equiv N \mid \mathrm{refl}\, T\, M\, N \mid \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : T
\end{array}
$$

Fig. 1. Syntax

1.1.1 *Meta-syntax & syntactic sugar.* Let

(1) $\tau \longrightarrow \upsilon$ denote the non-dependent universal quantification $\forall(\_ : \tau).\upsilon$;
(2) $\tau \times \upsilon$ denote the non-dependent existential quantification $\exists(\_ : \tau).\upsilon$;
(3) 0, 1, 2, … denote object-level natural numbers in the intuitive fashion;
(4) i_n denote the index obtained by n applications of I_S to I_0; and

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(5) the syntax

$$\{\!\{M_1, \ldots, M_n\}\!\}$$

denote the large elimination of a known, finite quantity of indices to types $M_1, \ldots, M_n$, elaborated by the equations:

$$\begin{aligned}\{\!\{M_1\}\!\} &:= \lambda(i : \mathrm{Ix}\ 1).\mathrm{case}_{\mathrm{Ix}}\ i\ M_1 \\ \{\!\{M_1, \ldots, M_n\}\!\} &:= \lambda(i : \mathrm{Ix}\ n).\mathrm{case}_{\mathrm{Ix}}\ i\ M_1\ \{\!\{M_2, \ldots, M_n\}\!\}\end{aligned}$$

1.2 Typing

Many of these are fucked or in need of repair; refer to the translation as the SSOT.

$$\boxed{\vdash \Gamma}$$

$$(\textsc{emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{var})\ \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\star)\ \frac{\vdash \Gamma}{\Gamma \vdash \star : \mathcal{U}} \qquad (\top)\ \frac{\vdash \Gamma}{\Gamma \vdash \top : \sigma} \qquad (\textsc{Nat})\ \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{N} : \star} \qquad (\mathrm{Ix})\ \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{Ix}\ n : \star}$$

$$(\forall)\ \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \forall \alpha : M.N : \sigma_2} \qquad (\exists)\ \frac{\Gamma \vdash M : \sigma_1 \quad \Gamma, \alpha : M \vdash N : \sigma_2}{\Gamma \vdash \exists \alpha : M.N : \sigma_2}$$

$$(+)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M + N : \sigma} \qquad (\equiv)\ \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \equiv N : \sigma}$$

Fig. 2. Context and type well-formedness

$$\boxed{\Gamma \vdash M : N}$$

$$(\textsc{Var})\,\frac{\vdash \Gamma \quad x : M \in \Gamma}{\Gamma \vdash x : M} \qquad (\mathtt{tt})\,\frac{\vdash \Gamma}{\Gamma \vdash \mathtt{tt} : \top}$$

$$(Z)\,\frac{\vdash \Gamma}{\Gamma \vdash Z : \mathbb{N}} \qquad (\mathrm{S})\,\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{S}\, n : \mathbb{N}} \qquad (\mathbb{N}E)\,\frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \mathbb{N} \quad \Gamma \vdash N : A \quad \Gamma \vdash P : \mathbb{N} \to A}{\Gamma \vdash \mathrm{case}_{\mathbb{N}}\, M\, N\, P : A}$$

$$(\mathsf{I}_0)\,\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathsf{I}_0 : \mathrm{Ix}\,(\mathrm{S}\, n)} \qquad (\mathsf{I}_{\mathrm{S}})\,\frac{\Gamma \vdash n : \mathbb{N} \quad \Gamma \vdash i : \mathrm{Ix}\, n}{\Gamma \vdash \mathsf{I}_{\mathrm{S}}\, i : \mathrm{Ix}\,(\mathrm{S}\, n)} \qquad (\lambda\!\!\!^{-}())\,\frac{\Gamma \vdash A : \sigma}{\Gamma \vdash \lambda\!\!\!^{-}() : \mathrm{Ix}\, 0 \to A}$$

$$(\mathsf{I}_{\mathrm{S}} E)\,\frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \mathrm{Ix}\,(\mathrm{S}\, n) \quad \Gamma \vdash N : A \quad \Gamma \vdash P : \mathrm{Ix}\, n \to A}{\Gamma \vdash \mathrm{case}_{\mathrm{Ix}}\, M\, N\, P : A}$$

$$(\forall I)\,\frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \forall(x : T).N} \qquad (\forall E)\,\frac{\Gamma \vdash M : \forall(x : T_1).T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\, N : T_2[N/x]}$$

$$(\exists I)\,\frac{\Gamma \vdash T_1 : \sigma \quad \Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash \langle\!\langle M : T_1, N \rangle\!\rangle : \exists(x : T_1).T_2}$$

$$(\exists E)\,\frac{\Gamma \vdash A : \sigma \quad \Gamma \vdash M : \Sigma(x : T_1).T_2 \quad \Gamma \vdash N : \forall(x : T_1).T_2 \to A}{\Gamma \vdash \mathrm{case}_{\exists}\, M\, N : A}$$

$$(+_1 I)\,\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{left}\, M : A + B} \qquad (+_2 I)\,\frac{\Gamma \vdash N : B}{\Gamma \vdash \mathrm{right}\, N : A + B}$$

$$(+E)\,\frac{\Gamma \vdash C : \sigma \quad \Gamma \vdash M : A + B \quad \Gamma \vdash N : A \to C \quad \Gamma \vdash P : B \to C}{\Gamma \vdash \mathrm{case}_{+}\, M\, N\, P : C}$$

$$(\equiv I)\,\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathrm{refl} : M \equiv M} \qquad (\textsc{conv})\,\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash T_1 = T_2 : \sigma}{\Gamma \vdash M : T_2}$$

$$(\equiv E)\,\frac{\begin{array}{c}\Gamma \vdash P : T_1 \equiv T_2 \\ \Gamma \vdash M : T_1 \\ \Gamma \vdash N : T_1 \\ \Gamma, x : T_1, y : T_1, p : x \equiv y \vdash T : \star \\ \Gamma, z : T_1 \vdash H : T[z/x, z/y, \mathrm{refl}/p]\end{array}}{\Gamma \vdash \mathcal{J}\, H\, M\, N\, P : T[M/x, N/y, P/p]}$$

Fig. 3. Typing Ix terms

$$\boxed{\Gamma \vdash M = N : \sigma}$$

$$(\textsc{e-refl})\,\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M = M : \sigma} \qquad (\textsc{e-sym})\,\frac{\Gamma \vdash N = M : \sigma}{\Gamma \vdash M = N : \sigma} \qquad (\textsc{e-trans})\,\frac{\Gamma \vdash M = P : \sigma \quad \Gamma \vdash P = N : \sigma}{\Gamma \vdash M = N : \sigma}$$

$$\boxed{\Gamma \vdash M = N : T}$$

$$(\textsc{c-refl})\,\frac{\Gamma \vdash M : T}{\Gamma \vdash M = M : T} \qquad (\textsc{c-sym})\,\frac{\Gamma \vdash N = M : T}{\Gamma \vdash M = N : T} \qquad (\textsc{c-trans})\,\frac{\Gamma \vdash M = P : T \quad \Gamma \vdash P = N : T}{\Gamma \vdash M = N : T}$$

Fig. 4. Definitional equality & computational laws

1.3 Properties

Theorem 1 (Well-sortedness). *if* $\Gamma \vdash M : N$ *then* $\vdash \Gamma$ *and there exists* σ *such that* $\Gamma \vdash N : \sigma$.

2 TRANSLATION FROM Rω

2.1 Untyped Translation

We follow the approach of [Morris and McKinna 2019] and give both typed and untyped translations of Rω types. Figure 5 describe the untyped translation, which is used to show translational soundness of the typed translation (Figure 5).

$$\boxed{[\![\kappa]\!]}$$

$$\begin{aligned}
[\![\star]\!] &= \star \\
[\![\mathsf{L}]\!] &= \top \\
[\![\kappa_1 \to \kappa_2]\!] &= [\![\kappa_1]\!] \to [\![\kappa_2]\!] \\
[\![\mathsf{R}^\kappa]\!] &= \exists(n : \mathbb{N}).\mathrm{Ix}\, n \to [\![\kappa]\!]
\end{aligned}$$

$$\boxed{[\![\Gamma \vdash \tau : \kappa]\!]}$$

$$\begin{aligned}
[\![\Gamma \vdash \alpha : \kappa]\!] &= \alpha \\
[\![\Gamma \vdash \tau_1 \to \tau_2 : \star]\!] &= [\![\tau_1]\!] \to [\![\tau_2]\!] \\
[\![\Gamma \vdash \forall\alpha : \kappa.\tau : \star]\!] &= \forall(\alpha : [\![\kappa]\!]).[\![\tau]\!] \\
[\![\Gamma \vdash \lambda\alpha : \kappa.\tau : \kappa \to \kappa']\!] &= \forall(\alpha : [\![\kappa]\!]).[\![\tau]\!] \\
[\![\Gamma \vdash \pi \Rightarrow \tau : \kappa]\!] &= \forall(\alpha : [\![\pi]\!]).[\![\tau]\!] \\
[\![\Gamma \vdash \tau\, v : \kappa]\!] &= [\![\tau]\!]\ [\![v]\!] \\
[\![\Gamma \vdash \ell : \mathsf{L}]\!] &= \top \\
[\![\Gamma \vdash \lfloor\xi\rfloor : \star]\!] &= \top \\
[\![\Gamma \vdash (\xi \triangleright \tau) : \kappa]\!] &= [\![\tau]\!] \\
[\![\Gamma \vdash \Pi\rho : \star]\!] &= \mathrm{case}_\exists\ [\![\rho]\!]\ (\lambda n : \mathbb{N}.\lambda P : \mathrm{Ix}\, n \to \star.\forall(i : \mathrm{Ix}\, n).P\, i) \\
[\![\Gamma \vdash \Sigma\rho : \star]\!] &= \mathrm{case}_\exists\ [\![\rho]\!]\ (\lambda n : \mathbb{N}.\lambda P : \mathrm{Ix}\, n \to \star.\exists(i : \mathrm{Ix}\, n).P\, i) \\
[\![\Gamma \vdash \epsilon : \mathsf{R}^\kappa]\!] &= \langle\!\langle 0 : \mathbb{N}, \lambda\!\!\!^{-}() \rangle\!\rangle \\
[\![\Gamma \vdash \rho\, \lceil v \rceil : \mathsf{R}^{\kappa_2}]\!] &= \mathrm{case}_\exists\ [\![\rho]\!]\ (\lambda n : \mathbb{N}.\lambda(P : \mathrm{Ix}\, n \to [\![\kappa_1]\!] \to [\![\kappa_2]\!]).\langle\!\langle n : \mathbb{N}, \lambda(j : \mathrm{Ix}\, n).(P\, j)[\![\tau]\!] \rangle\!\rangle) \\
[\![\Gamma \vdash \lceil \tau \rceil\, \rho : \mathsf{R}^{\kappa_2}]\!] &= \mathrm{case}_\exists\ [\![\rho]\!]\ (\lambda n : \mathbb{N}.\lambda(P : \mathrm{Ix}\, n \to [\![\kappa_1]\!]).\langle\!\langle n : \mathbb{N}, \lambda(j : \mathrm{Ix}\, n).[\![\tau]\!](P\, j) \rangle\!\rangle) \\
[\![\Gamma \vdash (\xi \triangleright_{\mathsf{R}} \tau) : \mathsf{R}^\kappa]\!] &= \langle\!\langle 1 : \mathbb{N}, \{\!\{ [\![\tau]\!] \}\!\} \rangle\!\rangle
\end{aligned}$$

$$\boxed{[\![\Gamma \vdash \pi : \kappa]\!]\ \text{(Predicate well-formedness)}}$$

$$[\![\Gamma \vdash \rho_1 \lesssim \rho_2 : \kappa]\!] = \begin{array}{l}
\mathrm{case}\ [\![\rho_1]\!](\lambda n : \mathbb{N}.\lambda(P : \mathrm{Ix}\, n \to [\![\kappa]\!]). \\
\mathrm{case}\ [\![\rho_2]\!](\lambda m : \mathbb{N}.\lambda(Q : \mathrm{Ix}\, m \to [\![\kappa]\!]). \\
\forall(i : \mathrm{Ix}\, n).\exists(j : \mathrm{Ix}\, m).P\, i \equiv Q\, j))
\end{array}$$

$$[\![\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3 : \kappa]\!] = \begin{array}{l}
\quad \mathrm{case}\ [\![\rho_1]\!](\lambda n : \mathbb{N}.\lambda(P : \mathrm{Ix}\, n \to [\![\kappa]\!]). \\
\quad \mathrm{case}\ [\![\rho_2]\!](\lambda m : \mathbb{N}.\lambda(Q : \mathrm{Ix}\, m \to [\![\kappa]\!]). \\
\quad \mathrm{case}\ [\![\rho_3]\!](\lambda l : \mathbb{N}.\lambda(R : \mathrm{Ix}\, l \to [\![\kappa]\!]). \\
\quad (\forall(i : \mathrm{Ix}\, n).\exists(k : \mathrm{Ix}\, l).P\, i \equiv R\, k) \\
\times (\forall(j : \mathrm{Ix}\, m).\exists(k : \mathrm{Ix}\, l).Q\, j \equiv R\, k) \\
\times (\forall(k : \mathrm{Ix}\, l). \\
\quad (\exists(i : \mathrm{Ix}\, n).P\, i \equiv R\, k) \\
+ (\exists(j : \mathrm{Ix}\, m).Q\, j \equiv R\, k)))))
\end{array}$$

Fig. 5. A compositional translation of typed Rω kinds and predicates to untyped Ix terms

2.2 Typed translation

$$\boxed{\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \vdash M \rightsquigarrow N : \tau}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\Gamma \Vdash \pi \rightsquigarrow N}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

$$\boxed{\tau \equiv \upsilon \rightsquigarrow P}$$

$$(\textsc{c-foo})\ \frac{A}{B}$$

Fig. 6. Translation of Rω derivations to Ix derivations

2.3 Properties of Translation

Presume an Rω instantiation of the simple row theory. A lot of this is likely bullshit.

Theorem 2 (Translational Soundness (Types)). *if* $\Gamma \vdash \tau : \kappa$ *such that* $\Gamma \vdash \tau \rightsquigarrow \upsilon : \kappa$ *then* $[\![\Gamma]\!] \vdash \upsilon : [\![\kappa]\!]$.

Theorem 3 (Translational Soundness (Type Equivalence)). *if*

(1) $\Gamma \vdash \tau_1 \rightsquigarrow \upsilon_1 : \kappa_1$*;*
(2) $\Gamma \vdash \tau_2 \rightsquigarrow \upsilon_2 : \kappa_2$*; and*
(3) $\tau_1 \equiv \tau_2 \rightsquigarrow P$*,*

then $[\![\Gamma]\!] \vdash P : \upsilon_1 \equiv \upsilon_2$.

Theorem 4 (Translational Soundness (Predicates)). *if* $\Gamma \Vdash \pi$ *such that* $\Gamma \Vdash \pi \rightsquigarrow N$ *then* $[\![\Gamma]\!] \vdash N : [\![\pi]\!]$.

Finally,

Theorem 5 (Translational Soundness). *if* $\Gamma \vdash M : \tau$ *such that* $\Gamma \vdash M \rightsquigarrow N : \tau$ *then* $[\![\Gamma]\!] \vdash N : [\![\tau]\!]$.

3 OPERATIONAL SEMANTICS

REFERENCES

J. Garrett Morris and James McKinna. 2019. Abstracting extensible data types: or, rows by any other name. *Proc. ACM Program. Lang.* 3, POPL (2019), 12:1–12:28. https://doi.org/10.1145/3290325