The Index Calculus and its translation from Rω

AH & JGM

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1 Ix: The Index Calculus

1.1 Syntax

Term variables x α

$$
\begin{array}{lrcl}
\text{Sorts} & \sigma & ::= & \star \mid \mathcal{U} \\
\text{Terms} & M, N, T & ::= & x \mid \star \mid \text{Nat} \mid \text{Zero} \mid \text{Suc}\, M \mid \\
& & & \text{Ix}\, M \mid \text{FZero} \mid \text{FSuc}\ M \mid \\
& & & \top \mid \texttt{tt} \mid \\
& & & \Pi\alpha : T.N \mid \lambda x : T.N \mid M\ N \mid \\
& & & \Sigma\alpha : T.M \mid (\alpha : T, M) \mid M.1 \mid M.2 \\
& & & M + N \mid \text{left}\ M \mid \text{right}\ M \mid \text{case}\ M \text{ of } \{\text{left} \mapsto M; \text{right} \mapsto M\} \\
& & & M \equiv N \mid \text{refl} \mid ... \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : T
\end{array}
$$

Figure 1: Syntax

1.2 Typing

$$\boxed{\vdash \Gamma}$$

$$(\text{C-EMP})\ \frac{}{\vdash \varepsilon} \qquad (\text{C-VAR})\ \frac{\vdash \Gamma \quad \Gamma \vdash M : \sigma}{\vdash \Gamma, x : M}$$

$$\boxed{\Gamma \vdash M : \sigma}$$

$$(\text{T-}\star)\ \frac{}{\Gamma \vdash \star : \mathcal{U}} \qquad \frac{}{\Gamma \vdash \top : \star} \qquad \frac{}{\Gamma \vdash \mathrm{Nat} : \star} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Ix}\, n : \star}$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Pi\alpha : M.N : \star} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \alpha : M \vdash N : \star}{\Gamma \vdash \Sigma\alpha : M.N : \star}$$

$$\frac{\Gamma \vdash M : \star \quad \Gamma \vdash N : \star}{\Gamma \vdash M + N : \star} \qquad \frac{\Gamma \vdash M_1 : N_1 \quad \Gamma \vdash N_1 : \sigma_1 \quad \Gamma \vdash M_2 : N_2 \quad \Gamma \vdash N_2 : \sigma_2}{\Gamma \vdash M_1 \equiv M_2 : \star}$$

$$\boxed{\Gamma \vdash M : N}$$

$$\frac{x : M \in \Gamma}{\Gamma \vdash x : M} \qquad \frac{}{\Gamma \vdash \mathtt{tt} : \top}$$

$$\frac{}{\Gamma \vdash \mathrm{Zero} : \mathrm{Nat}} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{Suc}\, n : \mathrm{Nat}} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash \mathrm{FZero} : \mathrm{Ix}\,(\mathrm{Suc}\, n)} \qquad \frac{\Gamma \vdash n : \mathrm{Nat} \quad \Gamma \vdash i : \mathrm{Ix}\, n}{\Gamma \vdash \mathrm{FSuc}\, i : \mathrm{Ix}\,(\mathrm{Suc}\, n)}$$

$$\frac{\Gamma \vdash T : \sigma \quad \Gamma, x : T \vdash M : N}{\Gamma \vdash \lambda x : T.M : \Pi(x : T).N} \qquad \frac{\Gamma \vdash M : \Pi(x : T_1).\, T_2 \quad \Gamma \vdash N : T_1}{\Gamma \vdash M\ N : T_2[N/x]}$$

$$\frac{\Gamma \vdash M : T_1 \quad \Gamma \vdash N : T_2[M/x]}{\Gamma \vdash (M : T_1, N) : \Sigma(x : T_1).\, T_2} \qquad \frac{\Gamma \vdash M : \Sigma(x : T_1).\, T_2}{\Gamma \vdash M.1 : T_1} \qquad \frac{\Gamma \vdash M : \Sigma(x : T_1).\, T_2}{\Gamma \vdash M.2 : T_2[M.1/x]}$$

2 Translation From Rω

2.1 Example translations of Rω terms and types

Record selection. In Rω,

$$\forall \rho : \mathsf{R}^{\star},\ \ell : \mathsf{L},\ \tau : \star.\{\ell \triangleright \tau\} \lesssim \rho \Rightarrow \lfloor \ell \rfloor \to \Pi\rho \to \tau$$

translates to

$$\Pi(\rho : \mathrm{Row}\,\star).\Pi(\ell : \top).\Pi(\tau : \star).[\![\{\ell \triangleright \tau\} \lesssim \rho]\!].\Pi(\_ : \top).\Pi(i : \mathrm{Ix}\,\rho.1).\ \rho.2\ i$$

where

$$\begin{aligned}
\mathrm{Row}\kappa :&= \Sigma(n : \mathrm{Nat}).\Pi(i : \mathrm{Ix}\,n).\kappa \\
[\![\{\ell \triangleright \tau\} \lesssim \rho]\!] &= \Pi(i : \mathrm{Ix}\,[\![\{\ell \triangleright \tau\}]\!].1).\Sigma(j : \mathrm{Ix}\,\rho.1).[\![\{\ell \triangleright \tau\}]\!].1\ i \equiv \rho.2\ j \\
[\![\{\ell \triangleright \tau\}]\!] &= (\mathrm{Suc\,Zero} : \mathrm{Nat}, \lambda(i : \mathrm{Ix}\,(\mathrm{Suc\,Zero})).[\![\tau]\!])
\end{aligned}$$

Putting this all together:

$$\begin{aligned}
&\Pi(\rho : (\Sigma(n : \mathrm{Nat}).\Pi(i : \mathrm{Ix}\,n).\star)). \\
&\Pi(\ell : \top). \\
&\Pi(\tau : \star). \\
&\Pi(P : \\
&\quad \Pi(i : \mathrm{Ix}\,(\mathrm{Suc\,Zero} : \mathrm{Nat}, \lambda(i : \mathrm{Ix}\,(\mathrm{Suc\,Zero})).[\![\tau]\!]).1). \\
&\quad \Sigma(j : \mathrm{Ix}\,\rho.1). \\
&\quad (\mathrm{Suc\,Zero} : \mathrm{Nat}, \lambda(i : \mathrm{Ix}\,(\mathrm{Suc\,Zero})).[\![\tau]\!]).2\ i \equiv \rho.2\ j) \\
&\Pi(_ : \top). \\
&\Pi(i : \mathrm{Ix}\,\rho.1).\ \rho.2\ i
\end{aligned}$$

which should normalize to

$$\begin{aligned}
&\Pi(\rho : (\Sigma(n : \mathrm{Nat}).\Pi(i : \mathrm{Ix}\,n).\star)). \\
&\Pi(\ell : \top). \\
&\Pi(\tau : \star). \\
&\Pi(P : \\
&\quad \Pi(i : \mathrm{Ix}\,1). \\
&\quad \Sigma(j : \mathrm{Ix}\,\rho.1). \\
&\quad [\![\tau]\!] \equiv \rho.2\ j). \\
&\Pi(_ : \top). \\
&\Pi(i : \mathrm{Ix}\,\rho.1).\ \rho.2\ i
\end{aligned}$$

A The static semantics of Rω

A.1 Syntax

The syntax of R$\omega(\mathcal{T})$ is given in Figure 2.

Term variables x Type variables α Labels ℓ Directions $d \in \{\mathsf{L}, \mathsf{R}\}$

$$
\begin{array}{llcl}
\text{Kinds} & \kappa & ::= & \star \mid \mathsf{L} \mid \mathsf{R}^{\kappa} \mid \kappa \to \kappa \\
\text{Predicates} & \pi, \psi & ::= & \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho \\
\text{Types} & \phi, \tau, \upsilon, \rho, \xi & ::= & \alpha \mid (\to) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa . \tau \mid \lambda \alpha : \kappa . \tau \mid \tau \, \tau \\
 & & \mid & \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \ldots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho \\
\text{Terms} & M, N & ::= & x \mid \lambda x : \tau . M \mid M \; N \mid \Lambda \alpha : \kappa . M \mid M \, [\tau] \\
 & & \mid & \ell \mid M \triangleright M \mid M / M \mid \mathsf{prj}_d \, M \mid M +\!\!+ M \mid \mathsf{inj}_d \, M \mid M \bigtriangledown M \\
 & & \mid & \mathsf{syn}_{\phi} \, M \mid \mathsf{ana}_{\phi} \, M \mid \mathsf{fold} \, M \; M \; M \; M \\
\text{Environments} & \Gamma & ::= & \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi
\end{array}
$$

Figure 2: Syntax

A.2 Types and Kinds

Figure 3 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory $\mathcal{T}$.

$$\boxed{\vdash \Gamma}$$

$$(\textsc{c-emp})\ \frac{}{\vdash \varepsilon} \qquad (\textsc{c-tvar})\ \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \qquad (\textsc{c-var})\ \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau} \qquad (\textsc{c-pred})\ \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi}$$

$$\boxed{\Gamma \vdash \tau : \kappa}\ \boxed{\Gamma \vdash \pi}$$

$$(\textsc{k-var})\ \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \qquad (\textsc{k-}(\to))\ \frac{\vdash \Gamma}{\Gamma \vdash (\to) : \star \to \star \to \star} \qquad (\textsc{k-}\Rightarrow)\ \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star}$$

$$(\textsc{k-}\forall)\ \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa . \tau : \star} \qquad (\textsc{k-}{\to}I)\ \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1 . \tau : \kappa_1 \to \kappa_2} \qquad (\textsc{k-}{\to}E)\ \frac{\Gamma \vdash \tau_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1\, \tau_2 : \kappa_2}$$

$$(\textsc{k-lab})\ \frac{\vdash \Gamma}{\Gamma \vdash \ell : \mathsf{L}} \qquad (\textsc{k-sing})\ \frac{\Gamma \vdash \xi : \mathsf{L}}{\Gamma \vdash \lfloor \xi \rfloor : \star} \qquad (\textsc{k-lty})\ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa} \qquad (\textsc{k-row})\ \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^{\kappa}}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : \mathsf{R}^{\kappa}}$$

$$(\textsc{k-}\Pi)\ \frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa}}{\Gamma \vdash \Pi \rho : \kappa} \qquad (\textsc{k-}\Sigma)\ \frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa}}{\Gamma \vdash \Sigma \rho : \kappa} \qquad (\textsc{k-lift}_1)\ \frac{\Gamma \vdash \rho : \mathsf{R}^{\kappa_1 \to \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho\, \tau : \mathsf{R}^{\kappa_2}}$$

$$(\textsc{k-lift}_2)\ \frac{\Gamma \vdash \phi : \kappa_1 \to \kappa_2 \quad \Gamma \vdash \rho : \mathsf{R}^{\kappa_1}}{\Gamma \vdash \phi\, \rho : \mathsf{R}^{\kappa_2}} \qquad (\textsc{k-}\lesssim_d)\ \frac{\Gamma \vdash \rho_i : \mathsf{R}^{\kappa}}{\Gamma \vdash \rho_1 \lesssim_d \rho_2} \qquad (\textsc{k-}\odot)\ \frac{\Gamma \vdash \rho_i : \mathsf{R}^{\kappa}}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}$$

Figure 3: Contexts and kinding.

$$\boxed{\tau \equiv \tau}\ \boxed{\pi \equiv \pi}$$

$$(\textsc{E-REFL})\ \frac{}{\tau \equiv \tau} \qquad (\textsc{E-SYM})\ \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \qquad (\textsc{E-TRANS})\ \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \qquad (\textsc{E-}\beta)\ \frac{}{(\lambda\alpha : \kappa.\tau)\,\upsilon \equiv \tau[\upsilon/\alpha}$$

$$(\textsc{E-}\xi_{\Rightarrow})\ \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \qquad (\textsc{E-}\xi_{\forall})\ \frac{\tau[\gamma/\alpha] \equiv \upsilon[\gamma/\beta]}{\forall\alpha : \kappa.\tau \equiv \forall\beta : \kappa.\upsilon}\ (\gamma \notin fv(\tau, \upsilon)) \qquad (\textsc{E-}\xi_{\textsc{app}})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1\,\tau_2 \equiv \upsilon_1\,\upsilon_2}$$

$$(\textsc{E-}\xi_{\triangleright})\ \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \qquad (\textsc{E-ROW})\ \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_T \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \qquad (\textsc{E-}\xi_{\lfloor\cdot\rfloor})\ \frac{\xi_1 \equiv \xi_2}{\lfloor \xi_1 \rfloor \equiv \lfloor \xi_2 \rfloor}$$

$$(\textsc{E-LIFT}_1)\ \frac{}{\{\xi \triangleright \phi\}\,\tau \equiv \{\xi \triangleright \phi\,\tau\}} \qquad (\textsc{E-LIFT}_2)\ \frac{}{\phi\,\{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi\,\tau\}}$$

$$(\textsc{E-}\xi_{\Pi\Sigma})\ \frac{\rho_1 \equiv \rho_2}{K\,\rho_1 \equiv K\,\rho_2} \qquad (\textsc{E-LIFT}_3)\ \frac{}{(K\,\rho)\,\tau \equiv K\,(\rho\,\tau)} \qquad (\textsc{E-SING})\ \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \qquad (K \in \{\Pi, \Sigma\})$$

$$(\textsc{E-}\xi_{\lesssim_d})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \lesssim_d \tau_2 \equiv \upsilon_1 \lesssim_d \upsilon_2} \qquad (\textsc{E-}\xi_{\odot})\ \frac{\tau_i \equiv \upsilon_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv \upsilon_1 \odot \upsilon_2 \sim \upsilon_3}$$

Figure 4: Type and predicate equivalence

A.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$(\textsc{t-var})\ \dfrac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad (\textsc{t-}{\to}I)\ \dfrac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . M : \tau_1 \to \tau_2} \qquad (\textsc{t-}{\to}E)\ \dfrac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2}{\Gamma \vdash M_1\, M_2 : \tau_2}$$

$$(\textsc{t-}{\equiv})\ \dfrac{\Gamma \vdash M : \tau \quad \tau \equiv \upsilon}{\Gamma \vdash M : \upsilon} \qquad (\textsc{t-}{\Rightarrow}I)\ \dfrac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \qquad (\textsc{t-}{\Rightarrow}E)\ \dfrac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau}$$

$$(\textsc{t-}{\forall}I)\ \dfrac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa . M : \forall \alpha : \kappa . \tau} \qquad (\textsc{t-}{\forall}E)\ \dfrac{\Gamma \vdash M : \forall \alpha : \kappa . \tau \quad \Gamma \vdash \upsilon : \kappa}{\Gamma \vdash M\,[\upsilon] : \tau[\upsilon / \alpha]}$$

$$(\textsc{t-sing})\ \dfrac{\vdash \Gamma}{\Gamma \vdash \ell : \lfloor \ell \rfloor} \qquad (\textsc{t-}{\triangleright}I)\ \dfrac{\Gamma \vdash M_1 : \lfloor \ell \rfloor \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \qquad (\textsc{t-}{\triangleright}E)\ \dfrac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : \lfloor \ell \rfloor}{\Gamma \vdash M_1 / M_2 : \tau}$$

$$(\textsc{t-}\Pi E)\ \dfrac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \mathsf{prj}_d\, M : \Pi \rho_2} \qquad (\textsc{t-}\Pi I)\ \dfrac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \mathbin{+\!\!+} M_2 : \Pi \rho_3}$$

$$(\textsc{t-}\Sigma I)\ \dfrac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \mathsf{inj}\, M : \Sigma \rho_2} \qquad (\textsc{t-}\Sigma E)\ \dfrac{\Gamma \vdash M_1 : \Sigma \rho_1 \to \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \to \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim}{\Gamma \vdash M_1 \triangledown M_2 : \Sigma \rho_3 \to \tau}$$

$$(\textsc{t-}\mathsf{ana})\ \dfrac{\begin{gathered}\Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa . \, (y_1 \odot \{ l \triangleright u \} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u \to \tau\end{gathered}}{\Gamma \vdash \mathsf{ana}_\phi\, M : \Sigma(\phi\, \rho) \to \tau}$$

$$(\textsc{t-}\mathsf{syn})\ \dfrac{\begin{gathered}\Gamma \vdash \rho : \mathsf{R}^\kappa \quad \Gamma \vdash \phi : \kappa \to \star \\ \Gamma \vdash M : \forall l : \mathsf{L}, u : \kappa, y_1, z, y_2 : \mathsf{R}^\kappa . \, (y_1 \odot \{ l \triangleright u \} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to \phi\, u\end{gathered}}{\Gamma \vdash \mathsf{syn}_\phi\, M : \Pi(\phi\, \rho)}$$

$$(\textsc{t-}\mathsf{fold})\ \dfrac{\begin{gathered} M_1 : \forall l : \mathsf{L}, t : \star, y_1, z, y_2 : \mathsf{R}^\kappa . \, (y_1 \odot \{ l \triangleright u \} \sim z, z \odot y_2 \sim \rho) \Rightarrow \lfloor l \rfloor \to t \to \upsilon \\ \Gamma \vdash M_2 : \upsilon \to \upsilon \to \upsilon \quad \Gamma \vdash M_3 : \upsilon \quad \Gamma \vdash N : \Pi \rho\end{gathered}}{\Gamma \vdash \mathsf{fold}\, M_1\, M_2\, M_3\, N : \upsilon}$$

Figure 5: Typing

Minimal Rows

Figure 6 gives the minimal row theory $\mathcal{M}$.

$$\boxed{\Gamma \vdash_{\mathsf{m}} \rho : \kappa} \; \boxed{\rho \equiv_{\mathsf{m}} \rho}$$

$$(\textsc{k-mrow})\ \frac{\Gamma \vdash \xi : \mathsf{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_{\mathsf{m}} \{\xi \triangleright \tau\} : \mathsf{R}^{\kappa}} \qquad (\textsc{e-mrow})\ \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_{\mathsf{m}} \{\xi' \triangleright \tau'\}}$$

$$\boxed{\Gamma \Vdash_{\mathsf{m}} \pi}$$

$$(\textsc{n-ax})\ \frac{\pi \in \Gamma}{\Gamma \Vdash_{\mathsf{m}} \pi} \qquad (\textsc{n-refl})\ \frac{}{\Gamma \Vdash_{\mathsf{m}} \rho \lesssim_d \rho} \qquad (\textsc{n-trans})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_3}$$

$$(\textsc{n-}\equiv)\ \frac{\Gamma \Vdash_{\mathsf{m}} \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_{\mathsf{m}} \pi_2} \qquad (\textsc{n-}\lesssim\textsc{lift}_1)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \lesssim_d \phi\,\rho_2} \qquad (\textsc{n-}\lesssim\textsc{lift}_2)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \lesssim_d \rho_2\,\tau}$$

$$(\textsc{n-}\odot\textsc{lift}_1)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1\,\tau \odot \rho_2\,\tau \sim \rho_3\,\tau} \qquad (\textsc{n-}\odot\textsc{lift}_2)\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \phi\,\rho_1 \odot \phi\,\rho_2 \sim \phi\,\rho_3}$$

$$(\textsc{n-}\odot\lesssim_{\mathsf{L}})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_1 \lesssim_{\mathsf{L}} \rho_3} \qquad (\textsc{n-}\odot\lesssim_{\mathsf{R}})\ \frac{\Gamma \Vdash_{\mathsf{m}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_{\mathsf{m}} \rho_2 \lesssim_{\mathsf{R}} \rho_3}$$

Figure 6: Minimal row theory $\mathcal{M} = \langle \vdash_{\mathsf{m}}, \equiv_{\mathsf{m}}, \Vdash_{\mathsf{m}} \rangle$