

Syntax & Static Semantics of $R\omega$

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1 $R\omega$ —HIGHER ORDERED ROWS

1.1 Syntax

The syntax of $R\omega(\mathcal{T})$ is given in Figure 1.

Term variables	x	Type variables	α	Labels	ℓ	Directions	$d \in \{L, R\}$
Kinds		$\kappa ::= \star \mid L \mid R^\kappa \mid \kappa \rightarrow \kappa$					
Predicates		$\pi, \psi ::= \rho \lesssim_d \rho \mid \rho \odot \rho \sim \rho$					
Types		$\phi, \tau, v, \rho, \xi ::= \alpha \mid (\rightarrow) \mid \pi \Rightarrow \tau \mid \forall \alpha : \kappa. \tau \mid \lambda \alpha : \kappa. \tau \mid \tau \tau$ $\mid \ell \mid \lfloor \xi \rfloor \mid \xi \triangleright \tau \mid \{\tau_1, \dots, \tau_n\} \mid \Pi \rho \mid \Sigma \rho$					
Terms		$H, M, N, P ::= x \mid \lambda x : \tau. M \mid MN \mid \Lambda \alpha : \kappa. M \mid M [\tau]$ $\mid \ell \mid M \triangleright M \mid M/M \mid \text{prj}_d M \mid M \# M \mid \text{inj}_d M \mid M \nabla M$ $\mid \text{syn}_\phi M \mid \text{ana}_\phi M \mid \text{fold } M M M M$					
Environments		$\Gamma ::= \varepsilon \mid \Gamma, \alpha : \kappa \mid \Gamma, x : \tau \mid \Gamma, \pi$					

Fig. 1. Syntax

1.2 Types and Kinds

Figure 2 gives rules for context formation ($\vdash \Gamma$), kinding ($\Gamma \vdash \tau : \kappa$), and predicate formation ($\Gamma \vdash \pi$), parameterized by row theory \mathcal{T} .

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$$\begin{array}{c}
\boxed{\vdash \Gamma} \\
\\
\text{(C-EMP)} \frac{}{\vdash \varepsilon} \quad \text{(C-TVAR)} \frac{\vdash \Gamma}{\vdash \Gamma, \alpha : \kappa} \quad \text{(C-VAR)} \frac{\vdash \Gamma \quad \Gamma \vdash \tau : \star}{\vdash \Gamma, x : \tau} \quad \text{(C-PRED)} \frac{\vdash \Gamma \quad \Gamma \vdash \pi}{\vdash \Gamma, \pi} \\
\\
\boxed{\Gamma \vdash \tau : \kappa} \quad \boxed{\Gamma \vdash \pi} \\
\\
\text{(K-VAR)} \frac{\vdash \Gamma \quad \alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \quad \text{(K-}\rightarrow\text{))} \frac{\vdash \Gamma}{\Gamma \vdash (\rightarrow) : \star \rightarrow \star \rightarrow \star} \quad \text{(K-}\Rightarrow\text{))} \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash \tau : \star}{\Gamma \vdash \pi \Rightarrow \tau : \star} \\
\\
\text{(K-}\forall\text{))} \frac{\Gamma, \alpha : \kappa \vdash \tau : \star}{\Gamma \vdash \forall \alpha : \kappa. \tau : \star} \quad \text{(K-}\rightarrow I\text{))} \frac{\Gamma, \alpha : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda \alpha : \kappa_1. \tau : \kappa_2} \quad \text{(K-}\rightarrow E\text{))} \frac{\Gamma \vdash \tau_1 : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \tau_2 : \kappa_1}{\Gamma \vdash \tau_1 \tau_2 : \kappa_2} \\
\\
\text{(K-LAB)} \frac{\vdash \Gamma}{\Gamma \vdash \ell : L} \quad \text{(K-SING)} \frac{\Gamma \vdash \xi : L}{\Gamma \vdash \lfloor \xi \rfloor : \star} \quad \text{(K-LTY)} \frac{\Gamma \vdash \xi : L \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash \xi \triangleright \tau : \kappa} \quad \text{(K-ROW)} \frac{\Gamma \vdash_{\mathcal{T}} \{\overline{\xi \triangleright \tau}\} : R^K}{\Gamma \vdash \{\overline{\xi \triangleright \tau}\} : R^K} \\
\\
\text{(K-}\Pi\text{))} \frac{\Gamma \vdash \rho : R^K}{\Gamma \vdash \Pi \rho : \kappa} \quad \text{(K-}\Sigma\text{))} \frac{\Gamma \vdash \rho : R^K}{\Gamma \vdash \Sigma \rho : \kappa} \quad \text{(K-LIFT}_1\text{))} \frac{\Gamma \vdash \rho : R^{\kappa_1 \rightarrow \kappa_2} \quad \Gamma \vdash \tau : \kappa_1}{\Gamma \vdash \rho \tau : R^{\kappa_2}} \\
\\
\text{(K-LIFT}_2\text{))} \frac{\Gamma \vdash \phi : \kappa_1 \rightarrow \kappa_2 \quad \Gamma \vdash \rho : R^{\kappa_1}}{\Gamma \vdash \phi \rho : R^{\kappa_2}} \quad \text{(K-}\lesssim_d\text{))} \frac{\Gamma \vdash \rho_i : R^K}{\Gamma \vdash \rho_1 \lesssim_d \rho_2} \quad \text{(K-}\odot\text{))} \frac{\Gamma \vdash \rho_i : R^K}{\Gamma \vdash \rho_1 \odot \rho_2 \sim \rho_3}
\end{array}$$

Fig. 2. Contexts and kinding.

$$\begin{array}{c}
\boxed{\tau \equiv \tau} \quad \boxed{\pi \equiv \pi} \\
\\
\text{(E-REFL)} \frac{}{\tau \equiv \tau} \quad \text{(E-SYM)} \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1} \quad \text{(E-TRANS)} \frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3} \quad \text{(E-}\beta\text{))} \frac{}{(\lambda \alpha : \kappa. \tau) v \equiv \tau[v/\alpha]} \\
\\
\text{(E-}\xi \Rightarrow\text{))} \frac{\pi_1 \equiv \pi_2 \quad \tau_1 \equiv \tau_2}{\pi_1 \Rightarrow \tau_1 \equiv \pi_2 \Rightarrow \tau_2} \quad \text{(E-}\xi_{\forall}\text{))} \frac{\tau[\gamma/\alpha] \equiv v[\gamma/\beta]}{\forall \alpha : \kappa. \tau \equiv \forall \beta : \kappa. v} \quad (\gamma \notin \text{fv}(\tau, v)) \quad \text{(E-}\xi_{\text{APP}}\text{))} \frac{\tau_i \equiv v_i}{\tau_1 \tau_2 \equiv v_1 v_2} \\
\\
\text{(E-}\xi_{\triangleright}\text{))} \frac{\xi_1 \equiv \xi_2 \quad \tau_1 \equiv \tau_2}{\xi_1 \triangleright \tau_1 \equiv \xi_2 \triangleright \tau_2} \quad \text{(E-ROW)} \frac{\{\overline{\xi_i \triangleright \tau_i}\} \equiv_{\mathcal{T}} \{\overline{\xi'_j \triangleright \tau'_j}\}}{\{\overline{\xi_i \triangleright \tau_i}\} \equiv \{\overline{\xi'_j \triangleright \tau'_j}\}} \quad \text{(E-}\xi_{\lfloor \cdot \rfloor}\text{))} \frac{\xi_1 \equiv \xi_2}{\lfloor \xi_1 \rfloor \equiv \lfloor \xi_2 \rfloor} \\
\\
\text{(E-LIFT}_1\text{))} \frac{}{\{\xi \triangleright \phi\} \tau \equiv \{\xi \triangleright \phi \tau\}} \quad \text{(E-LIFT}_2\text{))} \frac{}{\phi \{\xi \triangleright \tau\} \equiv \{\xi \triangleright \phi \tau\}} \\
\\
\text{(E-}\xi_{\Pi\Sigma}\text{))} \frac{\rho_1 \equiv \rho_2}{K\rho_1 \equiv K\rho_2} \quad \text{(E-LIFT}_3\text{))} \frac{}{(K\rho) \tau \equiv K(\rho \tau)} \quad \text{(E-SING)} \frac{}{K\{\xi \triangleright \tau\} \equiv \xi \triangleright \tau} \quad (K \in \{\Pi, \Sigma\}) \\
\\
\text{(E-}\xi_{\lesssim_d}\text{))} \frac{\tau_i \equiv v_i}{\tau_1 \lesssim_d \tau_2 \equiv v_1 \lesssim_d v_2} \quad \text{(E-}\xi_{\odot}\text{))} \frac{\tau_i \equiv v_i}{\tau_1 \odot \tau_2 \sim \tau_3 \equiv v_1 \odot v_2 \sim v_3}
\end{array}$$

Fig. 3. Type and predicate equivalence

1.3 Terms

$$\boxed{\Gamma \vdash M : \tau}$$

$$\begin{array}{c}
(\text{T-VAR}) \frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \quad (\text{T} \rightarrow I) \frac{\Gamma \vdash \tau_1 : \star \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1. M : \tau_1 \rightarrow \tau_2} \quad (\text{T} \rightarrow E) \frac{\Gamma \vdash M_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1 M_2 : \tau_2} \\
\\
(\text{T} \equiv) \frac{\Gamma \vdash M : \tau \quad \tau \equiv v}{\Gamma \vdash M : v} \quad (\text{T} \Rightarrow I) \frac{\Gamma \vdash \pi \quad \Gamma, \pi \vdash M : \tau}{\Gamma \vdash M : \pi \Rightarrow \tau} \quad (\text{T} \Rightarrow E) \frac{\Gamma \vdash M : \pi \Rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \pi}{\Gamma \vdash M : \tau} \\
\\
(\text{T} \forall I) \frac{\Gamma, \alpha : \kappa \vdash M : \tau}{\Gamma \vdash \Lambda \alpha : \kappa. M : \forall \alpha : \kappa. \tau} \quad (\text{T} \forall E) \frac{\Gamma \vdash M : \forall \alpha : \kappa. \tau \quad \Gamma \vdash v : \kappa}{\Gamma \vdash M[v] : \tau[v/\alpha]} \\
\\
(\text{T-SING}) \frac{\vdash \Gamma}{\Gamma \vdash \ell : [\ell]} \quad (\text{T} \triangleright I) \frac{\Gamma \vdash M_1 : [\ell] \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \triangleright M_2 : \ell \triangleright \tau} \quad (\text{T} \triangleright E) \frac{\Gamma \vdash M_1 : \ell \triangleright \tau \quad \Gamma \vdash M_2 : [\ell]}{\Gamma \vdash M_1 / M_2 : \tau} \\
\\
(\text{T-}\Pi E) \frac{\Gamma \vdash M : \Pi \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_2 \lesssim_d \rho_1}{\Gamma \vdash \text{prj}_d M : \Pi \rho_2} \quad (\text{T-}\Pi I) \frac{\Gamma \vdash M_1 : \Pi \rho_1 \quad \Gamma \vdash M_2 : \Pi \rho_2 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \# M_2 : \Pi \rho_3} \\
\\
(\text{T-}\Sigma I) \frac{\Gamma \vdash M : \Sigma \rho_1 \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \lesssim \rho_2}{\Gamma \vdash \text{inj} M : \Sigma \rho_2} \quad (\text{T-}\Sigma E) \frac{\Gamma \vdash M_1 : \Sigma \rho_1 \rightarrow \tau \quad \Gamma \vdash M_2 : \Sigma \rho_2 \rightarrow \tau \quad \Gamma \Vdash_{\mathcal{T}} \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \vdash M_1 \nabla M_2 : \Sigma \rho_3 \rightarrow \tau} \\
\\
(\text{T-ana}) \frac{\Gamma \vdash \rho : R^K \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : L, u : \kappa, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u \rightarrow \tau}{\Gamma \vdash \text{ana}_{\phi} M : \Sigma(\phi \rho) \rightarrow \tau} \\
\\
(\text{T-syn}) \frac{\Gamma \vdash \rho : R^K \quad \Gamma \vdash \phi : \kappa \rightarrow \star \quad \Gamma \vdash M : \forall l : L, u : \kappa, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow \phi u}{\Gamma \vdash \text{syn}_{\phi} M : \Pi(\phi \rho)} \\
\\
(\text{T-fold}) \frac{M_1 : \forall l : L, t : \star, y_1, z, y_2 : R^K. (y_1 \odot \{l \triangleright u\} \sim z, z \odot y_2 \sim \rho) \Rightarrow [l] \rightarrow t \rightarrow v \quad \Gamma \vdash M_2 : v \rightarrow v \rightarrow v \quad \Gamma \vdash M_3 : v \quad \Gamma \vdash N : \Pi \rho}{\Gamma \vdash \text{fold } M_1 M_2 M_3 N : v}
\end{array}$$

Fig. 4. Typing

Minimal Rows

Figure 5 gives the minimal row theory \mathcal{M} .

$$\begin{array}{c}
\boxed{\Gamma \vdash_m \rho : \kappa} \quad \boxed{\rho \equiv_m \rho} \\
\\
(\text{K-MROW}) \frac{\Gamma \vdash \xi : \mathbb{L} \quad \Gamma \vdash \tau : \kappa}{\Gamma \vdash_m \{\xi \triangleright \tau\} : \mathbb{R}^\kappa} \quad (\text{E-MROW}) \frac{\xi \equiv \xi' \quad \tau \equiv \tau'}{\{\xi \triangleright \tau\} \equiv_m \{\xi' \triangleright \tau'\}} \\
\\
\boxed{\Gamma \Vdash_m \pi} \\
\\
(\text{N-AX}) \frac{\pi \in \Gamma}{\Gamma \Vdash_m \pi} \quad (\text{N-REFL}) \frac{}{\Gamma \Vdash_m \rho \lesssim_d \rho} \quad (\text{N-TRANS}) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2 \quad \Gamma \Vdash_m \rho_2 \lesssim_d \rho_3}{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_3} \\
\\
(\text{N-}\equiv) \frac{\Gamma \Vdash_m \pi_1 \quad \pi_1 \equiv \pi_2}{\Gamma \Vdash_m \pi_2} \quad (\text{N-}\lesssim_{\text{LIFT}_1}) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_m \phi \rho_1 \lesssim_d \phi \rho_2} \quad (\text{N-}\lesssim_{\text{LIFT}_2}) \frac{\Gamma \Vdash_m \rho_1 \lesssim_d \rho_2}{\Gamma \Vdash_m \rho_1 \tau \lesssim_d \rho_2 \tau} \\
\\
(\text{N-}\odot_{\text{LIFT}_1}) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_1 \tau \odot \rho_2 \tau \sim \rho_3 \tau} \quad (\text{N-}\odot_{\text{LIFT}_2}) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \phi \rho_1 \odot \phi \rho_2 \sim \phi \rho_3} \\
\\
(\text{N-}\odot_{\lesssim_{\text{L}}}) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_1 \lesssim_{\text{L}} \rho_3} \quad (\text{N-}\odot_{\lesssim_{\text{R}}}) \frac{\Gamma \Vdash_m \rho_1 \odot \rho_2 \sim \rho_3}{\Gamma \Vdash_m \rho_2 \lesssim_{\text{R}} \rho_3}
\end{array}$$

Fig. 5. Minimal row theory $\mathcal{M} = \langle \vdash_m, \equiv_m, \Vdash_m \rangle$