UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

Department of Computing

B. Sc. Examination 2017

IS51002E / IS51002D
Mathematical Modelling for Problem Solving

Duration: 3 hours

Date and time:

This paper is in two parts: part A and part B. You should answer ALL questions from part A and THREE questions from part B. Part A carries 40 marks, and each question from part B carries 20 marks. The marks for each part of a question are indicated at the end of the part in [.] brackets.

There are 100 marks available on this paper.

Electronic calculators must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Part A

Multiple choice

Question 1 Multiple choice question

(a) Which one of the following sets is a subset of $\{2, 4, 6, 8, 10, 12\}$?

i. $\{14\}$
ii. $\{2, 3, 4\}$
iii. $\{4, 8, 12\}$
iv. $\{1, 3, 5\}$

[2]

(b) Let A, B be two subsets of a universal set U. Which of of the following describes $A - B$

i. the set of elements contained in A and in B.
ii. the set of elements contained in A or in B.
iii. the set of elements contained in A but not in B.
iv. the set of elements contained in A or in B but not in both.

[2]

(c) Let A be a set of some elements. Which of the following are correct. More than one answer may apply.

i. $\emptyset \in \mathcal{P}(A)$
ii. $A \in \mathcal{P}(A)$
iii. $A \subseteq \mathcal{P}(A)$
iv. None of the above

[2]

(d) Let p be a proposition. Which one of the following is a tautology:

i. $p \wedge F$
ii. $p \wedge T$
iii. $p \vee T$
iv. $p \vee F$

[2]

(e) The following sequence $1, 3, 5, 7, 9, \cdots$ is

i. arithmetic

ii. geometric

iii. neither geometric nor arithmetic

[2]

(f) Let p and q be two propositions. Which one of the following compound statements is equivalent to $\neg(p \vee q)$?

i. $\neg p \wedge \neg q$

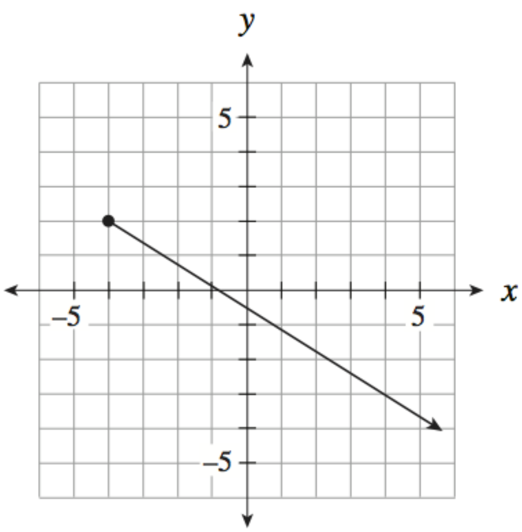
ii. $\neg p \vee \neg q$

iii. $p \wedge q$

iv. $p \oplus q$

[2]

(g) Find the range of the function graphed below:



i. $[-4, \infty[$

ii. $]-\infty, \infty[$

iii. $]-\infty, 2]$

iv. $[2, \infty[$

[2]

(h) Which one of the following correctly describes a simple graph G?

i. G has no cycles
ii. G has not parallel edges
iii. G has no loops
iv. G has neither loops nor parallel edges

[2]

(i) it is possible to draw a 3-regular graph with 5 vertices. True or False ?

i. $True$
ii. $False$

[2]

(j) A tree is a connected graph with no cycles. True or False ?

i. $True$
ii. $False$

[2]

(k) What is the decimal value of binary sequence 11111111_2?

i. 255
ii. 127
iii. 511
iv. none of the above

[2]

(l) What is the smallest positive number that is congruent to 8095×471 in modulo 256?

i. 3,812,745
ii. 14,893
iii. 137
iv. 32

[2]

(m) Convert 9^o to radians

i. $\frac{\pi}{2}$
ii. $\frac{\pi}{20}$
iii. $\frac{\pi}{4}$
iv. $\frac{\pi}{10}$

[2]

(n) Convert (5, 0) to polar coordinates

i. $(5, 0)$
ii. $(5, \pi)$
iii. $(-5, 0)$
iv. none of the above

[2]

(o) The period of $f(x) = 3\cos(x)$ is

i. 6π
ii. 3π
iii. 2π
iv. π

[2]

(p) Given $y = x^5 + 4x^3 - 2x^2$

i. $\frac{dy}{dx}$ $=5x + 12x - 4x$
ii. $\frac{dy}{dx}$ $=5x^4 + 12x^2 - 4x$
iii. $\frac{dy}{dx}$ $=13x$
iv. $\frac{dy}{dx}$ $=x^4 + 4x^2 - 2x^1$

[2]

(q) Given $y = \sin 5x$

i. $\frac{dy}{dx}$ $=5\sin 5x$
ii. $\frac{dy}{dx}$ $=5\cos 4x$
iii. $\frac{dy}{dx}$ $=\cos 5x$
iv. $\frac{dy}{dx}$ $=5\cos 5x$

[2]

(r) Rewrite the following vector in terms of standard unit vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

i. $2\vec{i}$-$\vec{j}$+$\vec{k}$

ii. $\begin{pmatrix} 2\vec{i} \\ -1\vec{j} \\ 1\vec{k} \end{pmatrix}$

iii. $2-1+1$

iv. none of the above

[2]

(s) Given W=$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Which of the following is the inverse of W

i. $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

ii. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

iii. $\begin{pmatrix} \frac{1}{2} & 0 & -1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$

iv. $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

[2]

(t) Which of the following numbers is an irrational number:

i. 2.00005

ii. π

iii. $\frac{1}{2}$

iv. $3.1212\cdots$

[2]

Part B

Question 2 Set, Logic & Sequences

(a) i. Describe the set A by the listing method.

$$A = \{r^3 - 1 : r \in Z \ \ and \ \ -1 < r \leq 3\}.$$

ii. Describe the set B by the rule of inclusion method where $B = \{1, 2, 4, 8, 16, \cdots, 128\}$

iii. Let A and B and C be subsets of a universal set $\mathcal{U}$.

1. Draw a labelled Venn diagram depicting A, B, C in such a way that they divide $\mathcal{U}$ into 8 disjoint regions.

2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can.

[6]

(b) Let p and q be the following propositions:

p : *'this animal is a cat'*

q : *'this animal is furry'*.

i. Express each of the three following compound propositions concerning positive integers symbolically by using p, q and appropriate logical symbols.

"this animal is a furry cat"

"if this animal is cat then it is furry"

"this animal is not a furry cat".

ii. Construct the truth table for the statement $q \rightarrow p$.

iii. Write in words the contrapositive of the statement given symbolically by "$q \rightarrow p$".

[7]

(c) i. Express the following sum using the $\sum$ notation

$$(2 \times 3) + (3 \times 4) + (4 \times 5) + ... + (n+1)(n+2).$$

ii. Evaluate the following the following sum:

$$\sum_{k=11}^{100} 2k$$

Hint: you might want to use the formula: $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$

iii. A sequence is determined by the recurrence relation

$$u_1 = 0 \ \ and \ \ u_{n+1} = u_n + n, \ \ for\ n \geq 1.$$

1. Calculate u_2, u_3.
2. Prove by induction that: $u_n = \frac{n(n-1)}{2}, \quad \forall\, n \geq 1.$

[7]

Question 3 Graphs, Trees & Relations

(a) i. Draw the two graphs with adjacency lists

- $a_1 : a_2, a_5$
- $a_2 : a_1, a_3, a_4, a_5$
- $a_3 : a_2, a_4, a_5$
- $a_4 : a_2, a_3, a_5$
- $a_5 : a_1, a_2, a_3, a_4$

and

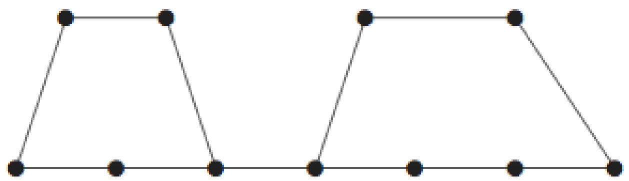
- $b_1 : b_2, b_3, b_4, b_5$
- $b_2 : b_1, b_5$
- $b_3 : b_1, b_4, b_5$
- $b_4 : b_1, b_3, b_5$
- $b_5 : b_1, b_2, b_3, b_4$

1. Write down the degree sequence for each graph above.
2. Are these graphs isomorphic? If so, show the correspondence between them.

ii. A simple connected graph has 7 vertices, all having the same degree d. Give the possible values of d and for each value of d give the number of edges of the graph.

[7]

(b) i. How many distinct spanning trees are contained in this graph?



ii. Draw two non-isomorphic spanning trees of this graph.

iii. Draw a binary search tree to hold 15 records and find it height.

[7]

(c) Given S be the set of integers $\{1, 2, 3, 4, 5, 6\}$. Let $\mathcal{R}$ be a relation defined on S by the following condition such that,
for all $x, y \in S$, xRy if $x \ mod \ 2 = y \ mod \ 2$.

i. Draw the digraph of $\mathcal{R}$.

ii. Show that $\mathcal{R}$ is an equivalence relation and find the equivalence classes.

[6]

Question 4 Functions, Probability & Trigonometry

(a) Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3, 4, 5\}$ two sets. Let f be a function defined as follows:

$f : X \to Y$

x	a	b	c	d	e
$f(x)$	1	2	3	3	5

i. Draw the arrow diagram to represent the function f .

ii. List the co-domain and the range of f.

iii. Find the ancestor (pre-image) of 3.

iv. Show that f is not a one to one function.

v. Show that f is not an onto function.

[5]

(b) i. Find numerical values for the following

(1) $\log_2 1024$

(2) $\log_{1024} 2$

(3) $\log_2(\frac{1}{2})$

ii. Sketch the graphs of

(1) $f(x) = 2^x$

(2) $g(x) = 2^{x-1}$

iii. Find the inverse functions

(1) $f^{-1}(x)$

(2) $g^{-1}(x)$

[6]

(c) Drawer A contains 7 black socks and 5 grey socks and drawer B contains 4 black socks and 8 grey socks. One sock is taken from drawer A and then one sock is taken from drawer B at random.

i. Draw a tree diagram to represent all the different outcomes of this process.

ii. What is the probability of getting 2 black socks?

iii. What is the probability of getting two socks of different colours?

[5]

(d) i. Triangle ABC is an isosceles triangle (has 2 equal sides). Side $a = 6cm$ and angle $A = 80^o$.

(1) Find all 3 possible values for angle B.

(2) Hence find all 3 possible values for the length of side b.

ii. Let $f(x) = 3\cos(x)$ and $g(x) = \sin(2x)$. By plotting the graphs of $f(x)$ and $g(x)$, or otherwise find all the values of x between $-\pi$ and π for which

$$3\cos(x) - \sin(2x) = 0$$

[4]

Question 5 Bases, Modular Arithmetic & Complex Numbers

(a) i. Express the decimal number $(347)_{10}$ in base 2.

ii. Express the binary number $(1000111.011)_2$ as a decimal number.

iii. Express the decimal number $(281.75)_{10}$ as

(1) a binary number.

(2) a hexadecimal number.

iv. Express the octal number $(574.2)_8$ as a decimal number.

v. Working in base 16 and showing all your working, compute the following:

$$(AB2)_{16} + (161)_{16} - (FF)_{16}$$

[7]

(b) i. Find the smallest positive integer modulo 13 that is congruent to

(1) 54

(2)271

ii. Find the remainder on division by 13 of

(1) $54 + 271$

(2) 54×271

(3) 271^{19}

iii. Find the following

(1) the additive inverse of 5 modulo 13

(2) the multiplicative inverse of 5 modulo 13

[6]

(c) Given complex numbers $z_1 = 3 + 2i$ and $z_2 = 5 - 2i$

i. Find

(1) $z_1 + z_2$

(2) $z_1 \times z_2$

(3) $\frac{z_1}{z_2}$

ii. Convert z_1

(1) to polar form

(2) to exponential form

iii. Hence find

(1) ${z_1}^3$

(2) All solutions to ${z_1}^{\frac{1}{3}}$

[7]

Question 6 Graph Sketching, Vectors & Matrices

(a) i. Find the following limits:

(1)$\lim_{x\to 0} \frac{x-4}{x^2-16}$
(2) $\lim_{x\to +5} \frac{x-4}{x^2-16}$
(3)$\lim_{x\to \infty} \frac{x-4}{x^2-16}$
(4)$\lim_{x\to -5} \frac{x-4}{x^2-16}$

ii. Given the following function $f(x) = x^3 - 3x^2$.

(1) Find the values of x for which $f(x) = 0$.
(2) Differentiate $f(x)$.
(3) Hence find any stationary points of $f(x)$ and determine their nature.
(4) Sketch $f(x)$.

[8]

(b) Given $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

i. Find the magnitudes of $\vec{v}_1$ and $\vec{v}_2$.
ii. Find the dot product of $\vec{v}_1$ and $\vec{v}_2$.
iii. Hence find the angle between $\vec{v}_1$ and $\vec{v}_2$.
iv. Find $\vec{v}_3$ and $\vec{v}_2$ the cross product (vector product) of $\vec{v}_1$ and $\vec{v}_2$.
v. State the angle between $\vec{v}_3$ and $\vec{v}_1$.

[5]

(c) Let A be a 3x3 matrix corresponding to a translation of 3 units in the x direction and -1 unit in the y direction. Let B be a 3x3 matrix corresponding to a scaling of factor 2 in the x direction and factor 3 in the y direction. Let C be a 3x3 homogeneous matrix transformation corresponding to an anti-clockwise rotation about the z-axis by an angle $\frac{\pi}{2}$.

i. Write down A, B and C .
ii. Find the inverse matrices A^{-1}, B^{-1} and C^{-1}.
iii. Find the single matrix T which represents the transformation represented by matrix B followed by transformation represented by matrix A .

[7]