

UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

Department of Computing

B. Sc. Examination 2018

IS51002E

Mathematical Modelling for Problem Solving

Duration: 3 hours

Date and time:

This paper is in two parts: part A and part B. You should answer ALL questions from part A and THREE questions from part B. Part A carries 40 marks, and each question from part B carries 20 marks. The marks for each part of a question are indicated at the end of the part in [.] brackets.

There are 100 marks available on this paper.

Electronic calculators must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

**THIS PAPER MUST NOT BE REMOVED
FROM THE EXAMINATION ROOM**

Part A
Multiple choice

Question 1 Each question has one or more correct answers

(a) Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\}$. Which of the following sets represent A using the inclusion rules? More than one answer may apply.

- i. $\{2^{-n} : n \in \mathbb{Z} \text{ and } 0 \leq n \leq 7\}$
- ii. $\{2^{-n} : n \in \mathbb{Z} \text{ and } 0 \leq n < 8\}$
- iii. $\{\frac{1}{2^n} : n \in \mathbb{Z} \text{ and } 0 \leq n \leq 7\}$
- iv. $\{\frac{1}{2^n} : n \in \mathbb{Z} \text{ and } 0 < n < 8\}$

[2]

(b) Let $S = \{1, 2, 3\}$, which one of the following sets represents $\mathcal{P}(S)$?

- i. $\{\{1\}, \{2\}, \{3\}\}$
- ii. $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, 1, 3, \{2, 3\}\}$
- iii. $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- iv. $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

[2]

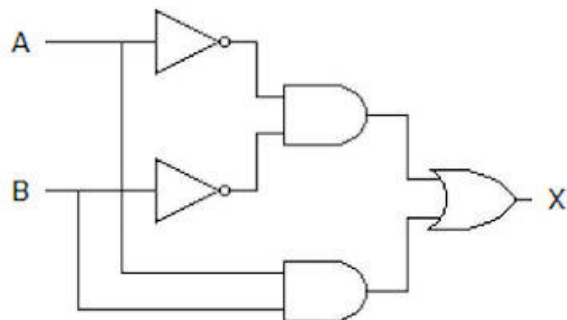
(c) Let p and q and be two propositions where p means '**Jack is happy**' and q means '**Jack paints a picture**'. Which one of the following logical expressions is a correct formalisation of the following sentence:

Jack is happy only if he paints a picture.

- i. $p \rightarrow q$
- ii. $q \rightarrow p$
- iii. $p \wedge q$
- iv. $p \rightarrow \neg q$

[2]

(d) Which one is a correct output of the following logic network:



- i. $(A \wedge B) \vee (\neg A \wedge \neg B)$
- ii. $(A \wedge B) \vee (\neg A \wedge B)$
- iii. $(A \wedge B) \vee (A \wedge \neg B)$
- iv. $(A \vee B) \wedge (\neg A \vee \neg B)$

[2]

(e) Let $f : R^+ \rightarrow R$ be a function where $f(x) = \log_2 x$. Which one of the following is the inverse function of the function f ?

- i. $f^{-1}(x) = 2^x$
- ii. $f^{-1}(x) = e^x$
- iii. $f^{-1}(x) = \sqrt{x}$
- iv. $f^{-1}(x) = \frac{x}{2}$

[2]

(f) The following sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ is

- i. arithmetic
- ii. geometric
- iii. neither geometric nor arithmetic
- iv. both arithmetic and geometric

[2]

(g) Let p and q be two propositions. Which one of the following compound statements is equivalent to $\neg(p \wedge q)$?

- i. $\neg p \wedge \neg q$
- ii. $\neg p \vee \neg q$
- iii. $p \wedge q$
- iv. $p \oplus q$

[2]

(h) Which one of the following correctly describes a complete graph G ?

- i. G is a simple graph where every two vertices has a direct link between them
- ii. G is a simple graph connected graph
- iii. G is a graph with parallel edges between every two vertices.
- iv. none of the above

[2]

(i) Which of the following statements is/are **TRUE**? More than one answer might apply.

- i. it is possible to draw a 3-regular graph with 5 vertices
- ii. it is possible to draw 3-regular graph with 6 vertices
- iii. the sum of the degree sequence of a graph is twice the number of edges in the graph
- iv. the sum of the degree sequence of a graph is twice the number of vertices in the graph.

[2]

(j) The degree of each vertex in complete graph k_n is

- i. $n-2$
- ii. $n-1$
- iii. n
- iv. $2n$

[2]

(k) What is the decimal representation of 321_8 ?

- i. 83_{10}
- ii. 418_{10}
- iii. 209_{10}
- iv. none of the above

[2]

(l) What is the multiplicative inverse of 5 in modulo 7?

- i. 1
- ii. 2
- iii. 3
- iv. 4

[2]

(m) A triangle XYZ has sides $x = 8$, $y = 7$ and angle $Y = 1.13$ radians. The size of angle X is:

- i. 0.441
- ii. 1.111
- iii. 0.913
- iv. This triangle does not exist

[2]

(n) Convert 1.7 radians to degrees

- i. 97.4°
- ii. 48.7°
- iii. 194.8°
- iv. 33.7°

[2]

(o) The frequency of $f(x) = 2 \cos(\pi + x)$ is

- i. π
- ii. 4π
- iii. 2π
- iv. $\frac{1}{2\pi}$

[2]

(p) $\log_2 6 + \log_2 \frac{1}{2}$ is equal to:

- i. 6.5
- ii. $\log_2 6.5$
- iii. $\log_2 3$
- iv. 3

[2]

(q) The graph of $\log_2 x$:

- i. has a x -intercept of 1
- ii. has a y -intercept of 0
- iii. passes through the point $(1, 2)$
- iv. passes through the point $(0, 0)$

[2]

(r) Calculate the following limit: $\lim_{x \rightarrow \infty} \frac{x^5 + x^3 - 7}{2x^5 - 3x + 1}$.

- i. -7
- ii. ∞
- iii. $\frac{1}{2}$
- iv. is not defined

[2]

(s) Given $y = x^2(x^2 + x)$

- i. $\frac{dy}{dx} = x^4 + x^3$
- ii. $\frac{dy}{dx} = 2x(2x + 1)$

- iii. $\frac{dy}{dx}=4x^3 + 3x^2$
- iv. $\frac{dy}{dx}$ is not defined

[2]

(t) Convert the vector $\vec{u}=\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in cartesian coordinates to polar coordinates

- i. (4.58, 1.19)
- ii. (5.39, 1.19)
- iii. $\sqrt{21}$
- iv. $\sqrt{29}$

[2]

Part B

Question 2 Set, Logic & Sequences

- (a) i. Describe the set A by the listing method.

$$A = \{r^3 - 1 : r \in \mathbb{Z} \text{ and } -1 < r \leq 3\}.$$

- ii. Describe the set B by the rule of inclusion method where $B = \{1, 2, 4, 8, 16, \dots, 64\}$.

[2]

- (b) Let A and B and C be subsets of a universal set \mathcal{U} .

1. Draw a labelled Venn diagram depicting A, B, C in such a way that they divide \mathcal{U} into 8 disjoint regions.
2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

[1]

A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can.

[3]

- (c) Let p and q be the following propositions concerning a positive integer n :

p : ' n has one digit'

q : ' n is less than 10'.

- i. Express each of the three following compound propositions concerning positive integers symbolically by using p , q and appropriate logical symbols.

' n has one digit if n is less than 10'

' n has one digit only if n is less than 10'

' n has one digit or greater than or equal to 10 but not both'

[3]

ii. Construct the truth table for the statement $q \rightarrow p$. [2]

iii. Write in words the contrapositive of the statement given symbolically by ' $q \rightarrow p$ '. [2]

(d) i. Express the following sum using the Σ notation [1]

$$1 + 3 + 5 + 7 + \dots + (2n - 1)$$

ii. Evaluate the following the following sum: [2]

$$\sum_{k=21}^{100} 4k$$

Hint: you might want to use the formula: $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

iii. Let $S_n = 1 + 2 + 3 + \dots + n$, for $n \geq 1$.

1. Calculate S_1, S_2 . [1]

2. Prove by induction that: $S_n = \frac{n(n+1)}{2}, \quad \forall n \geq 1$. [3]

Question 3 Graphs, Trees & Relations

- (a) i. Is it possible to construct a 3-regular graph with 7 vertices ? Explain your answer. [1]
- ii. Is it possible to construct a simple graph with the degree sequence 4,3,2,2? Explain your answer. [1]
- iii. A graph, G , with 5 vertices: a, b, c, d, e has the following adjacency list:
- $a : b, e$
 $b : a, c, d$
 $c : b, d$
 $d : b, c, e$
 $e : d, a.$
1. Draw the graph, G . [2]
2. Write down the degree sequence of G . State the relationship between the number of edges in G and its corresponding degree sequence. [1]
- Draw two non-isomorphic spanning trees of G . [2]
- (b) i. Define what a tree is. [1]
- ii. How many edges in a trees with n vertices? [1]
- iii. A binary search tree is designed to store an ordered list of 4000 records, numbered 1,2,3,...,4000 at its internal nodes. Draw levels 0, 1 and 2 of this tree, showing which number record is stored at the root and at each of the nodes at level 1 and 2, making it clear which records are at each level and find the height of this tree? [5]
- (c) Given S be the set of integers $\{1, 2, 3, 4, 5, 6\}$. Let \mathcal{R} be a relation defined on S by the following condition such that, for all $x, y \in S$, xRy if $x \bmod 3 = y \bmod 3$.
- i. Draw the digraph of \mathcal{R} . [2]
- ii. Show that \mathcal{R} is an equivalence relation. [3]
- iii. Write down the equivalence classes of \mathcal{R} . [1]

Question 4 Functions & Graph Sketching

(a) Let $f : \mathcal{R} \rightarrow \mathcal{R}$ with $f(x) = x^2 + 1$

- i. List the co-domain and the range of f .
- ii. Find the ancestors if any of 5.
- iii. Is f a one to one function? Explain your answer.
- iv. Is f an onto function? Explain your answer.

[5]

(b) Find the following limits:

- i. $\lim_{x \rightarrow 2} \frac{x^2-1}{x^3-x}$
- ii. $\lim_{x \rightarrow 0^-} \frac{x^2-1}{x^3-x}$
- iii. $\lim_{x \rightarrow 0^+} \frac{x^2-1}{x^3-x}$
- iv. $\lim_{x \rightarrow \infty} \frac{x^2-1}{x^3-x}$

[4]

(c) Given the function $f(x) = (x - 1)(x^2 + x + 1)$

- i. Find the value or values of x for which $f(x) = 0$
(note $(x^2 + x + 1) \geq 0$ for all x)
- ii. Differentiate $f(x)$.
- iii. Hence find any stationary points of $f(x)$ and determine their nature.
- iv. Sketch $f(x)$.

[6]

(d) i. Find numerical values for the following

$$\log_{10} 0.001$$

$$\log_{1000} 10$$

ii. Give the function $f(x) = 1 + \log_2 x$

Plot the graph of $f(x)$

Find the inverse function $f^{-1}(x)$

[5]

Question 5 Bases & Modular Arithmetic

- (a) i. Express the decimal number $(177)_{10}$ in base 8. [1]
ii. Express the decimal number $(11.125)_{10}$ as a binary number. [1]
iii. Express the hexadecimal number $(32.8)_{16}$ as a decimal number. [1]
iv. Express the octal number $(262.24)_8$ as
 (1) a binary number
 (2) a hexadecimal number [2]
v. Working in base 8 and showing all your working, compute the following:

$$(4763)_8 + (332)_8 - (4606)_8$$

[3]

- (b) i. Find the smallest positive integer modulo 17 that is congruent to
 (1) 271
 (2) 1277 [2]
ii. Find the remainder on division by 17 of
 (1) $271 - 1277$
 (2) 271×1277 [2]
iii. Find the following
 (1) the additive inverse of 15 modulo 17
 (2) the multiplicative inverse of 15 modulo 17 [2]
(c) i. Define what is meant by a rational number. Say whether or not the repeating decimal number $0.131313\dots$ is rational, justify your answer. [2]
ii. Give an example of an irrational number. [1]
iii. Showing all your working, express the recurring decimal $0.272727\dots$ as a fraction in its lowest form. [3]

Question 6 Probability, Vectors & Matrices

- (a) Two Friends, Jack and Charles, frequently play golf and tennis with each other. In a long run, it has been found that Jack wins 3 rounds of golf out of every 5, and 1 game of tennis out of every 4 games. If they play one round of golf and one game of tennis find the probability that Jack

- i. wins both,
- ii. loses both,
- iii. wins the round of golf only.
- iv. wins either the golf round or the tennis game but not both.

[6]

(b) Given $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

- i. Find the magnitudes of \vec{v}_1 and \vec{v}_2 .
- ii. Find the dot product of \vec{v}_1 and \vec{v}_2 .
- iii. Hence find the angle between \vec{v}_1 and \vec{v}_2 .
- iv. Find \vec{v}_3 the cross product (vector product) of \vec{v}_1 and \vec{v}_2 .

[6]

- (c) Let A be a 3x3 homogeneous transformation matrix corresponding to a scaling of the x and y-coordinates by a factor of 2 and a factor of 3 respectively, let B be a 3x3 homogeneous transformation matrix corresponding to a translation of the x and y coordinates by 1 and -1 respectively and let C be a 3x3 homogeneous transformation matrix corresponding to a clockwise rotation about the z-axis through an angle $\frac{\pi}{6}$.

- i. Find matrices A, B and C.
- ii. How would the transformation represented by the matrix B transform the following three points which represent a triangle in the Cartesian space: (1,0), (2,0) and (2,1)?
- iii. Find the inverse matrices A^{-1} and C^{-1} .

[3]

[2]

[3]