5.1 The Basics

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Essential Question:

What is Boolean Algebra, its postulates and/or Boolean functions?

Questions/Cues:

- What is Two-valued Boolean algebra?
- What are the Operations of Boolean Algebra?
- What are Huntington's Postulates for Boolean Algebra?
- What are some other useful theorems derived from Huntington's Postulates?
- What are De Morgan's Theorems for Boolean Algebra?
- What is principle of duality?
- What are the 4 ways of proving theorems in Boolean Algebra?
- What is a Boolean Function?
- What are algebraic forms of a Boolean function?
- What are the standardized forms of a Boolean function?
- What are the steps to building a sum of products form?
- What are some other useful Boolean functions?

Notes

Two-valued Boolean algebra

The most well-known form of Boolean algebra is a two-valued system, where:

- variables take values on the set {0, 1}
- the operators (+) and (.) correspond to (OR) and (AND) respectively

It could be used to describe and design digital circuits.

Operations of Boolean algebra

Boolean algebra is based on three fundamental operations:

AND

- logical product, intersection or conjunction
- represented as $x \,.\, y$, $x \cap y$ or $x \wedge y$

OR

- logical sum, union or disjunction
- represented as $x + y$, $x \cup y$ or $x \vee y$

NOT

- logical complement or negation
- represented as x', $\bar{x}$ or $\neg x$

When parentheses are not used, these operators have the following order of precedence:

NOT > AND > OR

The truth tables for the three operations can be represented as follows:

AND

$x \,.\, y$ is true if both x and y are true

x	y	x · y
0	0	0
0	1	0
1	0	0
1	1	1

OR

$x + y$ is true if either x or y is true

x	y	x + y
0	0	0
0	1	1
1	0	1
1	1	1

NOT

x' is true if x is not true

x	x′
0	1
1	0

Huntington's postulates

Huntington's postulates define **6 axioms** that must be satisfied by any Boolean algebra:

- **closure** with respect to the operators:
  - any result of logical operation belongs to the set {0, 1}
- **identity** elements with respect to the operators:
  - $x + 0 = x$, $x \,.\, 1 = x$
- **commutativity** with respect to the operators:
  - $x + y = y + x$, $x \,.\, y = y \,.\, x$
- **distributivity**:
  - $x(y + z) = (x \,.\, y) + (x \,.\, z)$, $x + (y \,.\, z) = (x + y) \,.\, (x + z)$
- **complements** exist for all the elements:
  - $x + x' = 1$, $x \,.\, x' = 0$
- **Distinct elements**:
  - $0 \neq 1$

Basic theorems

Using the 6 axioms of Boolean algebra, we can establish other useful theorems for analysing and designing circuits:

- theorem 1: **idempotent laws**
  $x + x = x\ ,\ x\ .\ x = x$
- theorem 2: **tautology and contradiction**
  $x + 1 = 1\ ,\ x\ .\ 0 = 0$
- theorem 3: **involution**
  $(x')' = x$
- theorem 4: **associative laws**
  $(x + y) + z = x + (y + z)\ ,\ (x\ .\ y)\ .\ z = x\ .\ (y\ .\ z)$
- theorem 5: **absorption laws**
  $x + (x\ .\ y) = x\ ,\ x\ .\ (x + y) = x$
- theorem 6: **uniqueness of complement**
  if $y + x = 1$ and $y\ .\ x = 0$, then $x = y'$
- theorem 7: **inversion law**
  $0' = 1\ ,\ 1' = 0.$

De Morgan's theorems

Theorem 1
– The complement of a product of variables is equal to the sum of the complements of the variables : $\overline{x\ .\ y} = \bar{x} + \bar{y}$.

Theorem 2
– The complement of a sum of variables is equal to the product of the complements of the variables : $\overline{x + y} = \bar{x}\ .\ \bar{y}$

x	y	$\overline{x\ .\ y}$	$\bar{x} + \bar{y}$	$\overline{x + y}$	$\bar{x}\ .\ \bar{y}$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	1	1	0	0
1	1	0	0	0	0

Proof of distributivity of + over .

Let's prove the distributivity of + over . using truth tables.

x	y	z	y . z	x+y	x+z	x + (y . z)	(x+y).(x+z)
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Principle of duality

Starting with a Boolean relation, we can build another equivalent Boolean relation by:

- changing each **OR (+)** sign to an **AND (.)** sign
- changing each **AND (.)** sign to an **OR (+)** sign
- changing each 0 to 1 and each 1 to 0.

Example

Since A + BC = (A+B)(A + C) (by distributive law), we can build another relation using the duality principle: A(B + C) = AB + AC.

Examples

Let's consider the Boolean equations:

- e1: $(a\,.\,1)\,.\,(0+\bar{a})=0$
- e2: $a+\bar{a}\,.\,b=\mathrm{a}+\mathrm{b}$

The dual equations of e1 and e2 are obtained by interchanging + and . , and interchanging 0 and 1, as follows:

- dual of e1: $(\mathrm{a}+0)+(1.\,\bar{a})=1$
- dual of e2: $\mathrm{a}.\,(\bar{a}+b)=a\,.\,b$

Ways of proving theorems

In general, there are 4 ways to prove the equivalence of Boolean relations:

- **perfect induction**: by showing the two expressions have identical truth tables. This is can be tedious if there are more than 3 or 4 variables
- **axiomatic proof**: by applying Huntington's postulates or theorems (that have already been proven) to the expressions, until identical expressions are found
- **duality principle**: every theorem in Boolean algebra remains valid if we interchange all AND's and OR's and interchange all 0's and 1's
- **contradiction**: by assuming that the hypothesis is ***false*** and then proving that the conclusion is false.

Examples

Let's consider proving the absorption theorem.

x	y	x+(x.y)
0	0	0
0	1	0
1	0	1
1	1	1

- The absorption theorem can be proved using perfect induction, by writing a truth table.

- It can also be proved directly as follows:

$$x + (x \,.\, y) = (x \,.\, 1) + (x \,.\, y) \text{ by } x \,.\, 1 = x$$
$$= x \,.\, (1 + y) \textit{ by distributivity}$$
$$= x \,.\, (y + 1) \textit{ by commutativity}$$
$$= x \,.\, 1 \text{ by } x + 1 = x$$
$$= x \text{ by } x \,.\, 1 = x$$

- From $x + (x \,.\, y) = x$, if we apply the duality principle, we can deduce: $x \,.\, (x + y) = x$

Definition

- A **function** defines a **mapping** from one or multiple Boolean input values to a Boolean output value

- For **n** Boolean input values, there are $\mathbf{2^n}$ possible combinations.

- **For example**:
  a 3-input function **f** can be completely defined with an 8-row truth table.

x	y	z	f(x,y,z)
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Algebraic forms

- There is **only one** way to represent a Boolean function in a truth table

- In algebraic form, a function can be expressed in a **variety** of ways

x	y	f(x,y)
0	0	0
0	1	1
1	0	1
1	1	1

- **For example:**
  $f(x) = x + x' \,.\, y$ and $f(x) = x + y$ are both algebraic representations of the same truth Table.

Standardised forms of a function

The two most common standardised forms are the **sum-of-products** form and the **product-of-sums** form

- The **sum-of-products form**:
  such as: $f(x, y, z) = xy + xz + yz$
- The **product-of-sums form**:
  such as: $f(x, y, z) = (x + y)(x + z)(y + z)$
- The sum-of-products form is **easier** to use and to simplify.

Building a sum of products form

1. We focus on the values of the variables that make the function equal to 1
2. Then if an input equals 1, it appears uncomplemented in the expression and stays as it is
3. If an input equals 0, it appears complemented in the expression and its corresponding complement is used
4. The function f is then expressed as the sum of products of all the terms for which f = 1

Example

- Let's consider the function f represented by following truth table.
- can be expressed as

$$f(x, y) = x'y + xy' + xy$$

x	y	f(x,y)
0	0	0
0	1	1
1	0	1
1	1	1

Useful functions

The '**exclusive-or**' function: $x \oplus y$:

- defined as "true if either x or y is true, but not both"
- represented by the following truth table
- can be expressed as:

$$x \oplus y = x'y + xy'$$

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

The '**implies**' function: $x \rightarrow y$:

- defined as "if x then y"
- represented by the following truth table
- can be expressed as:

$$x \rightarrow y = x' + y$$

x	y	$x \rightarrow y$
0	0	1
0	1	1
1	0	0
1	1	1

Summary

In this week, we learned what Boolean Algebra is, the postulates of Boolean Algebra and what is Boolean function is.