

## 10.1 The Basics

**Notebook:** Discrete Mathematics [CM1020]

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Cornell Notes	Topic: 10.1 The Basics	Course: BSc Computer Science
		Class: Discrete Mathematics-Lecture
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Essential Question:		
What are the rules/strategies used when counting objects when they are sampled with or without replacement?		
Questions/Cues:		
<ul style="list-style-type: none"><li>• What is the Product Rule?</li><li>• What is the Product Rule in terms of Sets?</li><li>• What is the Addition Rule?</li><li>• What is the Addition/Sum Rule in terms of Sets?</li><li>• What is the Subtraction Rule?</li><li>• What is the Division Rule?</li><li>• What is the Pigeonhole principle?</li><li>• What is the generalized pigeonhole principle?</li><li>• What is a permutation on a set?</li><li>• What is the number of permutations possible on a set?</li><li>• What is a combination on a set?</li><li>• What is the number of combinations possible on a set?</li></ul>		
Notes		

# Product rule

To determine the **number** of different **possible outcomes** in a complex process, we can break the problem into a sequence of two independent tasks:

- if there are **n** ways of doing the first task
- for each of these ways of doing the first task, there are **m** ways of doing the second task
- then there are **n·m** different ways of doing the whole process.

## Example

Let's consider a restaurant offering a **combination meal** where a person can order one from each of the following categories: 2 salads, 3 main dishes, 4 side dishes and 3 desserts.

How many different combination meals are possible?

## Solution

### Solution

The problem can be **broken** down into **4 independent events**:

- selecting a salad, selecting a main dish, selecting a side dish and selecting a dessert.

**For each event**, the **number** of available options is:

- 2 for the first event
- 3 for the second event
- 4 for the third event
- 3 for the fourth event

Thus, there are  $2 \cdot 3 \cdot 4 \cdot 3 = 72$  possible combination meals.

## Product rule in terms of sets

Let **A** be the set of ways to do the first task and **B** the set of ways to do second task. If A and B are disjoint, then:

The number ways to do both task 1 and 2 can be represented as  $|A \times B| = |A| \cdot |B|$

In other words, the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.

# Addition rule

- Suppose a **task 1** can be done **n** ways and a **task 2** can be done in **m** ways
- Assume that both tasks are independent, that is, performing task 1 doesn't mean performing task 2 and vice versa
- In this case, the number of ways of executing task 1 or task 2 is equal to  $n + m$ .

## Example

- The computing department must choose either a student or a member of academic staff as a representative for a university committee
- How many ways of choosing this representative are there if there are 10 academic staff and 77 mathematics students, and no one is both a member of academic staff and a student?

### Solution:

- By the addition rule, there are  $10 + 77$  ways of choosing this representative.

## The sum rule in terms of sets

Let **A** be the set of ways to do **task 1** and **B** the set of ways to do **task 2**, where **A** and **B** are disjoint sets

- The sum rule can be phrased in terms of sets
- $|A \cup B| = |A| + |B|$  as long as **A** and **B** are disjoint sets.

## Combining the sum and product rules

Combining the sum and product rules allows us to solve more complex problems.

### Example:

- Suppose a label in a programming language can be either a single letter or a letter followed by two digits. What is the number of possible labels?

### Solution:

- The number of labels with one letter only is 26
- Using the product rule the number of labels with a letter followed by 2 digits is  $26 \times 10 \times 10$
- Using the sum rule the total number of labels is  $26 + 26 \cdot 10 \cdot 10 = 2,626$ .

# Subtraction rule

- Suppose a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways.
- Then the total number of ways to do the task is  $n_1 + n_2$  minus the number of ways common to the two different ways.
- This is also known as the **principle of inclusion-exclusion**

$$|A \cup B| = |A| + |B| - |A \cap B|$$

## Example

How many binary bit strings of length eight either start with a 1 bit or end with the two bits 00?

**Solution:**

- Number of bit strings of length eight that start with a 1 bit:  $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$
- Number of bit strings of length eight that end with the two bits 00:  $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 is  $2^5 = 32$
- Using the subtraction rule:
  - the number of bit strings either starting with a 1 or ending with 00 is  $128 + 64 - 32 = 160$ .

## Division rule

- Suppose a task can be done using a procedure that can be carried out in  $n$  ways, and for every way  $w$ , exactly  $d$  of the  $n$  ways correspond to  $w$ . Then this task can be done in  $n/d$  ways
- **In terms of sets:** if the finite set  $A$  is the union of  $n$  pairwise disjoint subsets each with  $d$  elements, then  $n = |A|/d$
- **In terms of functions:** if  $f$  is a function from  $A$  to  $B$ , where both are finite sets, and for every value  $y \in B$  there are exactly  $d$  values  $x \in A$  such that  $f(x) = y$ , then  $|B| = |A|/d$



# Example

In how many ways can we **seat 4 people** around a table, where two seating arrangements are considered the same when each person has the same left and right neighbour?

**Solution:**

Let's first number the seats around the table from 1 to 4 proceeding clockwise:

- There are **four ways** to select the person for seat 1, three for seat 2, two for seat 3, and one for seat 4
- Thus there are  $4 \cdot 3 \cdot 2 \cdot 1 = 24$  ways to order the four people
- Since two seating arrangements are the same when each person has the same left and right neighbour, for every choice for seat 1, we get the same seating
- Therefore, **by the division rule**, there are  $24/4 = 6$  different seating arrangements.

# Pigeonhole principle

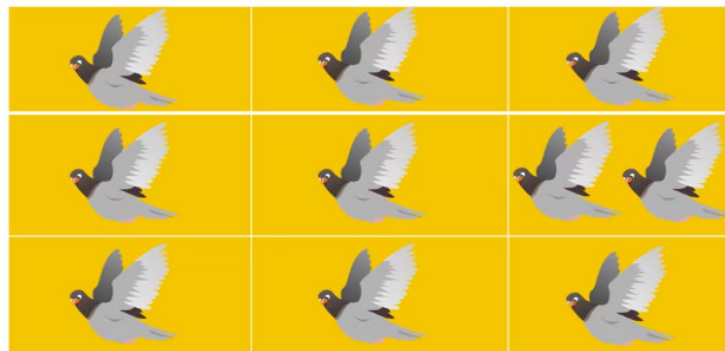
If  $k$  is a positive integer and  $k + 1$  objects are placed into  $k$  boxes, then at least one box contains two or more objects.

**Proof by contrapositive:**

- Let's suppose none of the  $k$  boxes has more than one object
- Then the total number of objects would be at most  $k$
- Which contradicts the statement that we have  $k + 1$  objects.

# Example

If a flock of 10 pigeons roosts in a set of 9 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



# Exercise

Prove that a function  $f$  from a set with  $k + 1$  elements to a set with  $k$  elements is not one-to-one.

**Solution:** We can prove this using the pigeonhole principle as follows:

- Create a box <sub>$y$</sub>  for each element  $y$  in the co-domain of  $f$
- Put all of the elements  $x$  from the domain in the box for  $y$  such that  $f(x) = y$
- Because there are  $k + 1$  elements and only  $k$  boxes, at least one box has two or more elements
- Hence,  $f$  can't be one-to-one.

## The generalised pigeonhole principle

If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects, where  $\lceil x \rceil$  is called the ceiling function, which represents the round-up value of  $x$ .

**Let's prove it by contrapositive:**

- Suppose that none of the boxes contains more than  $\lceil N/k \rceil - 1$  objects
- Then the total number of objects is at most
$$k(\lceil \frac{N}{k} \rceil - 1) < k((\frac{N}{k} + 1) - 1) = N$$
- This is a contradiction because there is a total of  $N$  objects.

## Example

How many cards must be selected from a standard deck of **52 cards** to guarantee that **at least four cards of the same suit** are chosen?

**Solution:**

- We assume four boxes, one for each suit
- Using the generalised pigeonhole principle, at least one box contains at least  $\lceil \frac{N}{4} \rceil$  cards, where  $N$  is the number of cards selected
- At least four cards of one suit are selected if  $\lceil \frac{N}{4} \rceil \geq 4$
- The smallest integer  $N$  such that  $\lceil \frac{N}{4} \rceil \geq 4$  is equal to 13.

# Definition of a permutation

- A permutation of a set of distinct objects is an **ordered arrangement** of these objects
- An ordered arrangement of  $r$  elements of a set is called an  **$r$ -permutation**
- The number of  $r$ -permutations of a set with  $n$  elements is denoted by  **$P(n,r)$** .

## Example

Let  $S = \{1,2,3\}$

- The ordered arrangement **3,1,2 is a 3-permutation** of  $S$
- The ordered arrangement **3,2 is a 2-permutation** of  $S$
- The 2-permutations of  $S = \{1,2,3\}$  are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2
- Hence,  $P(3,2) = 6$ .

## Number of permutations

If  $n$  is a positive integer and  $r$  is an integer with  $r \leq n$ , then there are  $P(n,r) = n(n-1)(n-2) \cdots (n-(r-1))$   $r$ -permutations of a set with  $n$  distinct elements.

$$P(n,r) = \frac{n!}{(n-r)!}$$

**Proof:**

- By the product rule:
  - there are  $n$  different ways for choosing the 1<sup>st</sup> element
  - $n-1$  ways for choosing the 2<sup>nd</sup> element
  - $n-2$  ways for choosing the 3<sup>rd</sup> element, and so on
  - there are  $(n-(r-1))$  ways to choose the last element
  - hence,  $P(n,r) = n(n-1)(n-2) \cdots (n-(r-1))$
  - $P(n,0) = 1$ , since there is only one way to order zero.

## Example

How many possible ways are there of selecting a **first prize** winner, a **second prize** winner and a **third-prize** winner from 50 different people?

**Solution:**

$$P(50,3) = 50 \cdot 49 \cdot 48 = 117,600$$

# Definition of combinations

- An **r-combination** of elements of a set is an **unordered** selection of  $r$  elements from the set
- An  $r$ -combination is a **subset** of the set with  $r$  elements
- The number of  $r$ -combinations of a set with  $n$  distinct elements is denoted by  $C(n,r) = \binom{n}{r}$
- The notation used is also called a **binomial coefficient**.

## Number of combinations

- The number of  $r$ -combinations of a set with  $n$  distinct elements can be formulated as:

$$C(n, r) = \frac{n!}{(n-r)!r!} = \frac{P(n, r)}{r!}$$

- $C(n, r)$  can be referred to as  $n$  choose  $r$
- It follows that  $C(n, r) = C(n, n - r)$ .

## Example

How many ways are there of selecting **six players** from a **20-member** tennis team to make a trip to an international competition?

**Solution:**

$$C(20,6) = \frac{20!}{6!14!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 38,760$$

### Summary

In this week, we learned about the Product, Addition, Subtraction & Division rules pertaining to counting. Alongside this, we explored the pigeonhole principle, combinations & permutations.



