# 10.2 Further techniques

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| **Cornell Notes** | **Topic:**<br>10.2 Further techniques | Course: BSc Computer Science<br>Class: Discrete Mathematics-Lecture<br>Date: January 20, 2020 |

**Essential Question:**

What are the rules/strategies used when counting objects when they are sampled with or without replacement?

**Questions/Cues:**

- What is a binomial expression?
- What is the Binomial Theorem?
- What is Pascal's Identity?
- What is Pascal's Triangle?
- What are permutations with repetition?
- What are permutations without repetition?
- What are combinations with repetition?
- What are combinations without repetition?
- How do we choose which formula to use when selecting k-objects from a set with n-elements?
- How do we distribute objects into boxes?
- What is meant by distinguishable/indistinguishable?
- What is meant by with/without exclusion?

Notes

# Binomial expression

An expression consisting of two terms, connected by a + or − sign is called a binomial expression.

Examples of binomial expressions:

$x + a;\ 2x - y;\ x^2 - y^2;\ 2x - 3y, \ldots$

# Binomial theorem

$(x+y)^1 = x + y$
$(x+y)^2 = x^2 + 2xy + y^2$
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
...
$(x+y)^{30}$

Let x and y be variables, and n a non-negative integer.
The expansion of $(x+y)^n$ can be formalised as:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

# Example

What is the coefficient of $x^8y^7$ in the expansion of $(3x - y)^{15}$?

**Solution:**

- We can view the expression as $(3x + (-y))^{15}$
- By the binomial theorem:

  $(3x + (-y))^{15} = \sum_{k=0}^{15} \binom{15}{k} (3x)^k (-y)^{n-k}$

- Consequently, the coefficient of $x^8y^7$ in the expansion is obtained when k = 8:
  - $\binom{15}{8} (3)^8 (-1)^7 = -3^8 \frac{15!}{8!7!}$

# Application of the binomial theorem

Let's prove the identity $2^n = \sum_{k=0}^{n} \binom{n}{k}$

**Using binomial theorem**:

- With x = 1 and y = 1, from the binomial theorem we see that the identity is verified.

**Using Sets**:

- Consider the subsets of a set with n elements
- There are subsets with zero elements, with one element, with two elements and so on ... with n elements
- Therefore the total number of subsets is: $\sum_{k=0}^{n} \binom{n}{k}$
- Also, since we know that a set with n elements has $2^n$ subsets, we can conclude that: $2^n = \sum_{k=0}^{n} \binom{n}{k}$

# Pascal’s identity

If n and k are integers with n ≥ k ≥ 1, then:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

**Proof** :

- Let T be a set where |T| = n + 1, a ∈ T, and S = T − {a}
- There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:
  - contains a with k − 1 other elements, or
  - contains k elements of S and not a
- There are:
  - $\binom{n}{k-1}$ subsets of k elements that contain a
  - $\binom{n}{k}$ subsets of k elements of T that don’t contain a
- Hence $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

# Pascal’s triangle

Pascal's triangle is a number triangle with numbers arranged in staggered rows such that $a_{n,r}$ **is the binomial coefficient** $\binom{n}{r}$:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}
\end{array}
\qquad
\begin{array}{c}
1 \\
1\quad 1 \\
1\quad 2\quad 1 \\
1\quad 3\quad 3\quad 1 \\
1\quad 4\quad 6\quad 4\quad 1 \\
1\quad 5\quad 10\quad 10\quad 5\quad 1
\end{array}$$

$$\binom{4}{3} = \binom{3}{2} + \binom{3}{3} = 4 + 1 = 5$$

Using Pascal’s identity, we can show that the result of adding two **adjacent** coefficients in this triangle is **equal** to the binomial coefficient in the **next row between these two coefficients**.

# Permutations with repetition

The number of **r-permutations** of a set of n objects with repetition allowed is $\mathbf{n^r}$.

**Proof:**

- Since we have n choices each time, there are n possibilities for the 1st choice, n possibilities for the 2nd choice, …, and n possibilities when choosing the last number
- By the product rule, multiplying each time:

$$\underbrace{\boxed{n} \times \boxed{n} \times \boxed{n} \times \ \ldots \ \times \boxed{n}}_{\text{r times}} = \mathbf{n^r}$$

# Example

How many strings of length **r** can be formed if we are using only uppercase letters in the English alphabet?

**Solution:**

The number of such strings is $26^r$, which is the number of r-permutations with repetition of a set with 26 elements.

# Permutations without repetition

In the case of permutations without repetition, we reduce the number of available choices each time by 1. The number of **r-permutations** of a set with n objects without repetition is:

$$\underbrace{\boxed{n} \times \boxed{n-1} \times \boxed{n-2} \times \cdots \times \boxed{n-r+1}}_{\text{r times}}$$

$$P(n,r) = P_n^r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

# Example

During a running competition how many different ways can the first and the second place be awarded if 10 runners are taking part in the race?

**Solution:**

$$P(10,2) = P_{10}^2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = 90$$

# Combination with repetition

The number of ways in which k objects can be selected from n categories of objects, with repetition permitted, can be calculated as: $\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$

- It is also the total **number** of ways to put **k identical balls** into n distinct boxes.
- It is also the total **number of functions** from a set of k identical elements to a set of n distinct elements.

# Example

Let's find all multisets of size 3 from the set {1, 2, 3, 4}.

**Solution:**

- Using bars and crosses, think of the values 1, 2, 3, 4 as four categories
- We will denote each multiset of size 3 by placing three crosses in the various categories
- For instance, the multiset {1, 1, 3} is represented by × × | | × |
- This counting problem can be modelled as distributing the 3 crosses among the 3+4−1 positions, the remaining positions being occupied by bars
- Thus the number of multisets of size 3 is: $C(6,3) = \frac{6!}{3!3!} = 20$.

# Combination without repetition

- The number of ways in which r objects can be selected from n categories of objects with repetition not permitted can be calculated as: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- This counting problem is the same as the **number** of ways of **putting** k **identical balls** into n distinct boxes, where each box receives at most one ball
- It is also the **number of one-to-one functions** from a set of k identical elements into a set of n distinct elements
- It is also the number of **k-element subsets** of an n-element set.

# Choice of formulas

- We have discussed four different ways of selecting k objects from a set with n elements:
  - the order in which the choices are made may or may not matter,
  - repetition may or may not be allowed.

- The following table summarises the formula in each case:

| | **Order matters** | **Order does not matter** |
|---|---|---|
| ***Repetition is not permitted*** | $\frac{n!}{(n-k)!}$ | $\frac{n!}{k!\,(n-k)!}$ |
| ***Repetition is permitted*** | $n^k$ | $\frac{(k+n-1)!}{k!\,(n-1)!}$ |

# Example

John is the chair of a committee. In how many ways can a committee of 3 be chosen from 10 people, given that John must be one of the people selected?

**Solution:**

- Since John is already chosen, we need to choose another 2 out of 9 people.
- In choosing a committee, the order doesn't matter, so we need to apply the combination without repetition formula: $C(9,2) = \frac{9!}{2!(9-2)!} = 36$ ways.

# Distributing objects into boxes

Counting problems can be phrased in terms of distributing k **objects** into n **boxes** under various conditions:

- The **objects** can be either distinguishable or indistinguishable

- The **boxes** can be either distinguishable or indistinguishable

- The distribution can be done either with exclusion or without exclusion.

**Distinguishable =** refers to objects or boxes that are marked in some way that makes each one distinguishable from the other
**Indistinguishable =** refers to objects or boxes that are identical, so that there is no way to tell them apart

**Note** When placing indistinguishable objects into distinguishable boxes, it makes no difference which object is placed into which box
**With exclusion =** means that no box can contain more than one object
**Without exclusion =** means that a box may contain more than one object

## Distinguishable objects and distinguishable boxes with exclusion

In this case, we want to **distribute** k **balls, numbered** from 1 to k, into n **boxes**, **numbered** from 1 to n, in such a way that **no box** receives **more than one ball**.

This is equivalent to making an **ordered selection of k boxes** from **n boxes**, where the **balls** do the selecting for us:

- the ball labelled 1 chooses the first box
- the ball labelled 2 chooses the second box
- and so on ...

## Distinguishable objects and distinguishable boxes with exclusion

**Theorem:**

Distributing **k** distinguishable **balls** into n distinguishable **boxes**, with exclusion, is equivalent to forming a permutation of size $k$ from a set of size $n$.

Therefore, the number of ways of placing $k$ distinguishable balls into $n$ distinguishable boxes is as follows:

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

## Distinguishable objects and distinguishable boxes without exclusion

In this case, we want to **distribute** k **balls**, **numbered** from 1 to k, into n **boxes**, **numbered** from 1 through n, **without restrictions** on the number of **balls** in each box.

This is equivalent to making an **ordered selection of k boxes from n**, **with repetition**, where the balls do the selecting for us:

- the ball labelled 1 chooses the first box
- the ball labelled 2 chooses the second box
- and so on ...

## Distinguishable objects and distinguishable boxes without exclusion

**Theorem:**

Distributing k distinguishable balls into n distinguishable boxes, without exclusion, is equivalent to forming a permutation of size *k* from a set of size *n,* with repetition.

Therefore, there are:

$$n^k \text{ different ways.}$$

## Indistinguishable objects and distinguishable boxes with exclusion

In this case, we want to **distribute** k **balls**, into n **boxes**, **numbered** from 1 through n, in such a way that **no box** receives more **than one ball**.

**Theorem:**

Distributing k indistinguishable balls into n distinguishable boxes, with exclusion, is equivalent to forming a combination of size *k* from a set of size *n.*

Therefore, there are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ different ways}$$

## Indistinguishable objects and distinguishable boxes without exclusion

In this case, we want to **distribute** k **balls**, into n **boxes**, **numbered** from 1 through n, **without restrictions** on the number of balls in each box.

**Theorem :**

Distributing k indistinguishable balls into n distinguishable boxes, without exclusion, is equivalent to forming a combination of size *k* from a set of size *n,* with repetition.

Therefore, there are

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!} \text{ different ways}$$

# Example

How many ways are there of placing 8 indistinguishable balls into 6 distinguishable boxes?

$$\binom{8+6-1}{8} = \binom{13}{8} = \frac{13!}{8!5!} = 1{,}287$$

Summary

In this week, we learned about distinguishable & indistinguishable objects/boxes and the different permutation/combination formulas to apply in various such scenarios.