Past paper 2018:

Question 1 Each question has one or more correct answers

(a) Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\}$. Which of the following sets represent A using the inclusion rules? More than one answer may apply.

i. $\{2^{-n} : n \in \mathcal{Z}$ **and** $0 \leq n \leq 7\}$
ii. $\{2^{-n} : n \in \mathcal{Z}$ **and** $0 \leq n < 8\}$
iii. $\{\frac{1}{2n} : n \in \mathcal{Z}$ **and** $0 \leq n \leq 7\}$
iv. $\{\frac{1}{2n} : n \in \mathcal{Z}$ **and** $0 < n < 8\}$

[2]

i, ii

(b) Let $S = \{1, 2, 3\}$, which one of the following sets represents $\mathcal{P}(S)$?

i. $\{\{1\}, \{2\}, \{3\}\}$
ii. $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, 1, 3, \{2, 3\}\}$
iii. $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
iv. $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

[2]

iv

(c) Let p and q and be two propositions where p means '**Jack is happy**' and q means '**Jack paints a picture**'. Which one of the following logical expressions is a correct formalisation of the following sentence:

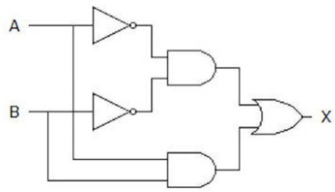
Jack is happy only if he paints a picture.

i. $p \rightarrow q$
ii. $q \rightarrow p$
iii. $p \wedge q$
iv. $p \rightarrow \neg q$

[2]

i

(d) Which one is a correct output of the following logic network:



i. $(A \wedge B) \vee (\neg A \wedge \neg B)$
ii. $(A \wedge B) \vee (\neg A \wedge B)$
iii. $(A \wedge B) \vee (A \wedge \neg B)$
iv. $(A \vee B) \wedge (\neg A \vee \neg B)$

[2]

i

(e) Let $f : R^+ \to R$ be a function where $f(x) = \log_2 x$. Which one of the following is the inverse function of the function f?

i. $f^{-1}(x) = 2^x$
ii. $f^{-1}(x) = e^x$
iii. $f^{-1}(x) = \sqrt{x}$
iv. $f^{-1}(x) = \frac{x}{2}$

[2]

i

(f) The following sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ is

i. arithmetic
ii. geometric
iii. neither geometric nor arithmetic
iv. both arithmetic and geometric

[2]

ii (probably not relevant to us)

(g) Let p and q be two propositions. Which one of the following compound statements is equivalent to $\neg(p \wedge q)$?

i. $\neg p \wedge \neg q$
ii. $\neg p \vee \neg q$
iii. $p \wedge q$
iv. $p \oplus q$

[2]

ii

(h) Which one of the following correctly describes a complete graph G?

i. G is a simple graph where every two vertices has a direct link between them
ii. G is a simple graph connected graph
iii. G is a graph with parallel edges between every two vertices.
iv. none of the above

[2]

i

(i) Which of the following statements is/are **TRUE**? More than one answer might apply.

i. it is possible to draw a 3-regular graph with 5 vertices
ii. it is possible to draw 3-regular graph with 6 vertices
iii. the sum of the degree sequence of a graph is twice the number of edges in the graph
iv. the sum of the degree sequence of a graph is twice the number of vertices in the graph.

[2]

ii, iii

(j) The degree of each vertex in complete graph k_n is

i. n-2
ii. n-1
iii. n
iv. 2n

ii

Question 2 Set, Logic & Sequences

(a) i. Describe the set A by the listing method.

$$A = \{r^3 - 1 : r \in \mathcal{Z} \textbf{ and } -1 < r \leq 3\}.$$

ii. Describe the set B by the rule of inclusion method where $B = \{1, 2, 4, 8, 16, \cdots, 64\}$.

[2]

A = {-1, 0, 7, 26}

$B = \{2^r : r \in Z,\ 0 \leq r \leq 6\}$

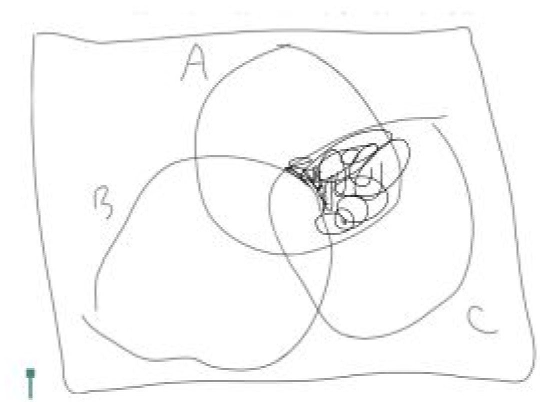
(b) Let A and B and C be subsets of a universal set $\mathcal{U}$.

1. Draw a labelled Venn diagram depicting A, B, C in such a way that they divide $\mathcal{U}$ into 8 disjoint regions.

2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can.



(c) Let p and q be the following propositions concerning a positive integer n:

$$p : \text{`}n \text{ has one digit'}$$
$$q : \text{`}n \text{ is less than } 10\text{'}.$$

i. Express each of the three following compound propositions concerning positive integers symbolically by using p, q and appropriate logical symbols.

'n has one digit if n is less than 10'
'n has one digit only if n is less than 10'
'n has one digit or greater than or equal to 10 but not both'

$q \rightarrow p$
$p \rightarrow q$
$p \oplus \neg q$

ii. Construct the truth table for the statement $q \rightarrow p$. [2]

q	p	$q \rightarrow p$
T	T	T
F	T	T

F	F	T
T	F	F

iii. Write in words the contrapositive of the statement given symbolically by
‘$q \to p$’.

If n has more than one digit, then n is equal or greater to 10

Question 3 Graphs, Trees & Relations

(a) i. Is it possible to construct a 3-regular graph with 7 vertices ? Explain your answer.

No, $\textit{number of edges} = \frac{rn}{2}$, where r=3 and n=7, $\frac{21}{2}$ is not an integer
(see https://www.coursera.org/learn/uol-discrete-mathematics/lecture/vfFhv/7-109-special-graphs-simple-r-regular-and-complete-graphs)

ii. Is it possible to construct a simple graph with the degree sequence 4,3,2,2? Explain your answer.

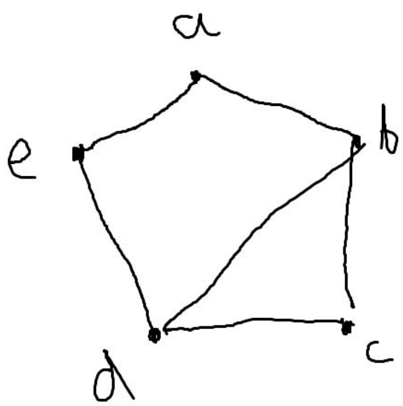
It has a 4 degree sequence, which means it has 4 vertices. Because for the first vertex, there is a degree of 4, meaning it would need to connect to four other vertices. As there are only three other vertices available, it would have to loop on itself or create a parallel edge with another vertex. Therefore it is not a simple graph. Also, the sum of the degree sequence is odd, while for a simple graph it must be even.

iii. A graph, G, with 5 vertices: a, b, c, d, e has the following adjacency list:

$a : b,\ e$
$b : a,\ c,\ d$
$c : b,\ d$
$d : b,\ c,\ e$
$e : d,\ a.$

1. Draw the graph, G.
2. Write down the degree sequence of G. State the relationship between the number of edges in G and its corresponding degree sequence.

Draw two non-isomorphic spanning trees of G.



1.
2. 3,3,2,2,2 The number of edges in G is equal to the sum of its degree sequence, divided by 2

TABLE 1 **Graph Terminology.**

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

(b) i. Define what a tree is.

A tree is a simple acyclic graph with no circuits.
NOTE: A **circuit** is path that begins and ends at the same vertex. A **circuit** that doesn't repeat vertices is called a **cycle**.

13.3.5 Definition of a cycle

A cycle is a closed path in which a vertex is reachable from itself.

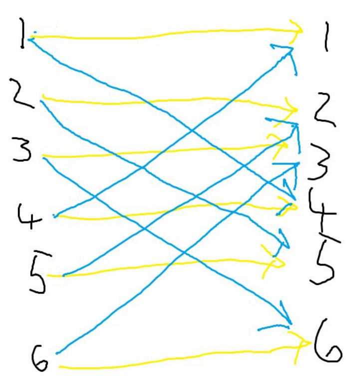
ii. How many edges in a trees with n vertices?

$n - 1$

iii. A binary search tree is designed to store an ordered list of 4000 records, numbered 1,2,3,...,4000 at its internal nodes. Draw levels 0, 1 and 2 of this tree, showing which number record is stored at the root and at each of the nodes at level 1 and 2, making it clear which records are at each level and find the height of this tree?

$ceil(log_2 4001) = 12$

(c) Given S be the set of integers $\{1, 2, 3, 4, 5, 6\}$. Let $\mathcal{R}$ be a relation defined on S by the following condition such that, for all $x, y \in S$, xRy if $x \ mod \ 3 = y \ mod \ 3$.



i. Draw the digraph of $\mathcal{R}$.

ii. Show that $\mathcal{R}$ is an equivalence relation.

(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,5),(3,6),(4,1),(5,2),(6,3),(1,4),

R is reflexive as we have (1,1) (2,2) etc, xRx where x is an element of S

R is symmetric as $\forall (x,y) \in R \ xRy \ and \ yRx$

R is transitive as $\forall x,y,z \in S \ xRy \land yRz \rightarrow xRz$

R is reflective as x mod 3 is equivalent to x mod 3 for any x in Z

R is symmetric as if x mod 3 = y mod 3, then y mod 3 = x mod 3 for any x, y in Z

R is transitive as if x = y and y = z then x = z for any x,y,z in Z

Symmetric: A relation R on a set S is said to be symmetric if and only if

- $\forall a, b \in S$, if $a\ R\ b$ then $b\ R\ a$

Example:

- Let R be a relation of elements in $\mathbb{Z}$:
  - ⋆ $R = \{(a, b) \in \mathbb{Z}^2 | a \bmod 2 = b \bmod 2$

Proof:

- Let $a, b \in \mathbb{Z}$ with $a\ R\ b$:
  - ⋆ $a \bmod 2 = b \bmod 2$
  - ⋆ $b \bmod 2 = a \bmod 2$
  - ⋆ $b\ R\ a$
- Therefore R is symmetric $\square$

iii. Write down the equivalence classes of $\mathcal{R}$.

[1]: {1, 4}
[2]: {2, 5}
[3]: {3, 6}

Question 4 Functions & Graph Sketching

(a) Let $f : \mathcal{R} \to \mathcal{R}$ with $f(x) = x^2 + 1$

i. List the co-domain and the range of f.

Co-domain: R, range: R+ \ {0}

ii. Find the ancestors if any of 5. (ancestor = pre-image)

+/- 2

iii. Is f a one to one function? Explain your answer.

No, because every image does not have one unique pre-image, as every square range value represents the squaring of two numbers: the positive and negative of the same absolute value. Ex: 4 = sqrt2 and sqrt-2

iv. Is f an onto function? Explain your answer.

f: R(domain)-R(co-domain)

A function is onto if for every element y of the co-domain (R), there exists an element in the domain, x in R(domain) such that f(x) = y, (this would make the range equal to the co-domain). There is not a pre-image for every element in the co-domain as all of the negative integers are not mapped to. The range is not equal to the co-domain.

Past Paper 2017:

Part a:

Question 1 Multiple choice question

(a) Which one of the following sets is a subset of $\{2, 4, 6, 8, 10, 12\}$?

i. $\{14\}$

ii. $\{2, 3, 4\}$

iii. $\{4, 8, 12\}$

iv. $\{1, 3, 5\}$

[2]

iii

(b) Let A, B be two subsets of a universal set U. Which of of the following describes $A - B$

i. the set of elements contained in A and in B.

ii. the set of elements contained in A or in B.

iii. the set of elements contained in A but not in B.

iv. the set of elements contained in A or in B but not in both.

[2]

iii

(c) Let A be a set of some elements. Which of the following are correct. More than one answer may apply.

i. $\emptyset \in \mathcal{P}(A)$

ii. $A \in \mathcal{P}(A)$

iii. $A \subseteq \mathcal{P}(A)$

iv. None of the above

[2]

i, ii

iii is not true as all of the elements of P(A) are sets. Ex:

Given a set S={1, 2, 3}
The subsets of S are:
∅,{1},{2},{3},
{1,2},{1,3},{2,3},
{1,2,3}
P(S) = {∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}

Let's create sets out of P(A) (like P(S) above):
Subset of P(A): {{1},{2,3}}
Can A {1,2,3} be a subset of P(S)? No. A subset of P(A) B={{1,2,3}}, so then {1,2,3} is an element of B

(d) Let p be a proposition. Which one of the following is a tautology:

i. $p \wedge F$

ii. $p \wedge T$

iii. $p \vee T$

iv. $p \vee F$

[2]

p or T because if we have the option to go with true, then it's always true! (Domination rule)

(f) Let p and q be two propositions. Which one of the following compound statements is equivalent to $\neg(p \vee q)$?

i. $\neg p \wedge \neg q$

ii. $\neg p \vee \neg q$

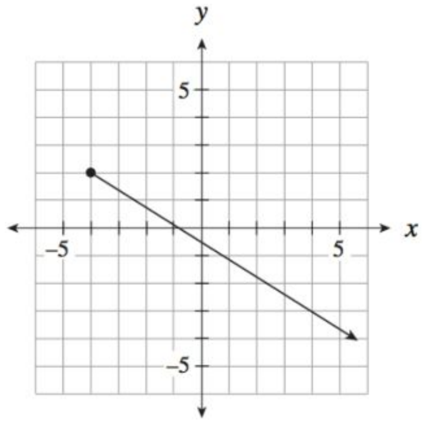
iii. $p \wedge q$

iv. $p \oplus q$

[2]

i: Using DeMorgan's law: not p and not q

(g) Find the range of the function graphed below:



i. $[-4, \infty[$
ii. $]-\infty, \infty[$
iii. $]-\infty, 2]$
iv. $[2, \infty[$

[2]

iii (list the negative infinity term first as it is less than 2)

(h) Which one of the following correctly describes a simple graph G?

i. G has no cycles
ii. G has not parallel edges
iii. G has no loops
iv. G has neither loops nor parallel edges

[2]

iv

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

(i) it is possible to draw a 3-regular graph with 5 vertices. True or False ?

i. *True*

ii. *False*

[2]

False, because (3 x 5)/2 it is not an integer (the number of vertices in a graph is equal to the sum of the degrees divided by 2)

13.5.3 Regular graphs

A graph is said to be regular if all local degrees are the same number.

A graph G where all vertices have the same degree r is called an r-regular graph.

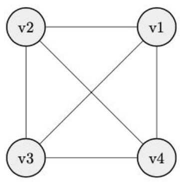


Figure 32: 3-regular graph

13.5.4 Properties of regular graphs

Given an r-regular graph G with n vertices, the following is true:

- Degree sequence of G is $r, r, r, \ldots$ (repeated n times)
- Sum of degree sequence is $r \cdot n$
- Number of edges in G is $\frac{r \cdot n}{2}$

(j) A tree is a connected graph with no cycles. True or False ?

i. *True*

ii. *False*

[2]

True

Question 2 Set, Logic & Sequences

(a) i. Describe the set A by the listing method.

$$A = \{r^3 - 1 : r \in Z \ \ and \ \ -1 < r \leq 3\}.$$

ii. Describe the set B by the rule of inclusion method where $B = \{1, 2, 4, 8, 16, \cdots, 128\}$

iii. Let A and B and C be subsets of a universal set $\mathcal{U}$.

1. Draw a labelled Venn diagram depicting A, B, C in such a way that they divide $\mathcal{U}$ into 8 disjoint regions.

2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can.

[6]

i. A= {-1,0,7,26}

ii. $B = \{2^n \mid n \in Z, 0 \leq n \leq 7\}$

iii and iv done in the above exam

(b) Let p and q be the following propositions:

$$p \;:\; \text{'this animal is a cat'}$$
$$q \;:\; \text{'this animal is furry'.}$$

i. Express each of the three following compound propositions concerning positive integers symbolically by using p, q and appropriate logical symbols.

"*this animal is a furry cat*"
"*if this animal is cat then it is furry*"
"*this animal is not a furry cat*".

ii. Construct the truth table for the statement $q \to p$.

iii. Write in words the contrapositive of the statement given symbolically by "$q \to p$".

[7]

i. 1. $p \wedge q$ 2. $p \to q$ 3. $\neg(p \wedge q) \equiv \neg p \vee \neg q$

ii.

p	q	p->q
T	T	T
T	F	F
F	T	T
F	F	T

iii. $\neg p \to \neg q$ **which in words is: If the animal is not a cat, then it is not furry**

iii. A sequence is determined by the recurrence relation

$$u_1 = 0 \;\; and \;\; u_{n+1} = u_n + n, \;\; for\ n \geq 1.$$

1. Calculate u_2, u_3.
2. Prove by induction that: $u_n = \frac{n(n-1)}{2}, \quad \forall\, n \geq 1.$

1. u_2 = 1 (we have been given u_1 as 0. If we make our n = 1, then our n+1 is u_2. So u_2 (u_n+1) = u_1 (u_n = 0) + 1 (the n index of u_n which is u_1). So u_2 = 1.
   So for u_3 we use u_2 as the n. u_3 (u_n+1) = u_2 (u_n) + 2 (the n index of u_n which is u_2) . u_2's value is 1, so u_3 = 1 + 2 = 3.
2. Let P = the above
   a. Base case P(n) for n=1: 0=1(1-1)/2 = 0

b. Inductive hypothesis: u_k = k(k-1)/2
c. Show that P(k) -> P(k+1)
    i. P(k) = k(k-1)/2
    ii. P(k+1) = (k+1(k+1-1))/2 =(k+1(k))/2 = (k+1(k))/2 <- this is what we want to show through algebra in the next steps
    iii. k(k-1)/2 = k_n + 1 + n
    iv.

$$
\begin{aligned}
u_{k+1} = u_k + k &= \frac{k(k+1)}{2} \\
\frac{k(k-1)}{2} + k &= \frac{k(k+1)}{2} \\
\frac{k(k-1)+2k}{2} &= \frac{k(k+1)}{2} \\
\frac{k^2-k+2k}{2} &= \frac{k(k+1)}{2} \\
\frac{k^2+k}{2} &= \frac{k(k+1)}{2} \\
\frac{k(k+1)}{2} &= \frac{k(k+1)}{2}
\end{aligned}
$$

    v.

Part 2 is relevant to the course.

P(k), P(k) + (k+1) = P(k+1)

Question 3 Graphs, Trees & Relations

(a) i. Draw the two graphs with adjacency lists

- $a_1 : a_2, a_5$
- $a_2 : a_1, a_3, a_4, a_5$
- $a_3 : a_2, a_4, a_5$
- $a_4 : a_2, a_3, a_5$
- $a_5 : a_1, a_2, a_3, a_4$

and

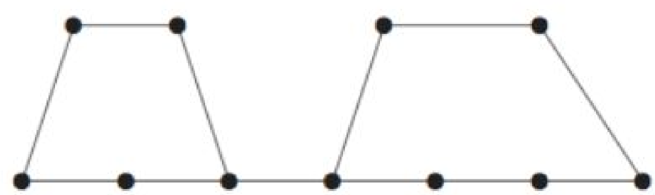
- $b_1 : b_2, b_3, b_4, b_5$
- $b_2 : b_1, b_5$
- $b_3 : b_1, b_4, b_5$
- $b_4 : b_1, b_3, b_5$
- $b_5 : b_1, b_2, b_3, b_4$

1. Write down the degree sequence for each graph above.
2. Are these graphs isomorphic? If so, show the correspondence between them.

ii. A simple connected graph has 7 vertices, all having the same degree d. Give the possible values of d and for each value of d give the number of edges of the graph.

[7]

(b) i. How many distinct spanning trees are contained in this graph?



ii. Draw two non-isomorphic spanning trees of this graph.

iii. Draw a binary search tree to hold 15 records and find it height.

[7]

(c) Given S be the set of integers $\{1, 2, 3, 4, 5, 6\}$. Let $\mathcal{R}$ be a relation defined on S by the following condition such that,
for all $x, y \in S$, xRy if $x \text{ mod } 2 = y \text{ mod } 2$.

i. Draw the digraph of $\mathcal{R}$.

ii. Show that $\mathcal{R}$ is an equivalence relation and find the equivalence classes.

[6]

Question 4 Functions, Probability & Trigonometry

(a) Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3, 4, 5\}$ two sets. Let f be a function defined as follows:

$f : X \to Y$

x	a	b	c	d	e
$f(x)$	1	2	3	3	5

i. Draw the arrow diagram to represent the function f .

ii. List the co-domain and the range of f.

iii. Find the ancestor (pre-image) of 3.

iv. Show that f is not a one to one function.

v. Show that f is not an onto function.

[5]