1015 Numerical Mathematics (April 2019)

Inhalt

Week 1-2: Base 2, 8 and 16

- Place value for digits:
  - $110110_2 = 1*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0 = 32+16+4+2 = 54_{10}$
  - $111111_2 = 1*2^5 + 1*2^4 + 1*2^3 + 1*2^2 + 1*2^1 + 1*2^0 = 32+16+8+4+2+1 = 63_{10}$
  - $111111_2 = 1000000_2 - 1_{10}$
- Decimals to binary: $193_{10} = 11000001_2$
  - 193/2 = 96 + 1
  - 96/2 = 48 + 0
  - 48/2 = 24 + 0
  - 24/2 = 12 + 0
  - 12/2 = 6 + 0
  - 6/2 = 3 + 0
  - 3/2 = 2 + 1
  - 1/2 = 0 + 1

	1024	512	256	128	64	32	16	8	4	2	1
199	0	0	0	1	1	0	0	0	1	1	1
313	0	0	1	0	0	1	1	1	0	0	1
488	0	0	1	1	1	1	0	1	0	0	0
1025	1	0	0	0	0	0	0	0	0	0	1

- Place value for fractional numbers
  - . = fractional point

Binary	**8**	**4**	**2**	**1**	**1/2**	**1/4**	**1/8**	**Decimal**
1101.101	1	1	0	1.	1	0	1	13 5/8

Decimal	**256**	**128**	**64**	**32**	**16**	**8**	**4**	**2**	**1**	**1/2**	**1/4**	**1/8**	**Binary**
271.25	1	0	0	0	0	1	1	1	1.	0	1		10001111.01
0.75									0.	1	1		0.11

- 0.25_{10} in binary: 0.01
  - 0.25 x 2 = 0.5
  - 0.5 x 2 = 1.0 <- stop
- 0.75_{10} in binary: 0.11
  - 0.75 x 2 = 1.5
  - 0.5 x 2 = 1.0 <- stop
- Rational and irrational numbers
  - 0.117 -> 117 / 1000
- Rational decimals to binary
  - 1/6 < 1 0
  - 1/6 x 2 = 1/3 < 1 0
  - 1/3 x 2 = 2/3 < 1 0 repeat
  - 2/3 x 2 = 4/3 > 1 1 repeat
  - 1/3 x 2 = 2/3 < 1 0
  - $1/6 = 0.0\dot{0}\dot{1}$

Total conversion:

296 (10->2)	296 (10->8)	296 (10->16)
296/2 = 148+ 0 148/2 = 74+ 0 74/2 = 37+ 0 37/2 = 18+ 1 18/2 = 9+ 0 9/2 = 4+ 1 4/2 = 2+ 0 2/2 = 1+ 0 1/2 = 0+ 1	296/8 = 37+ 0 37/8 = 4+ 5 4/8 = 0+ 4 100 101 000 100 4 101 5 000 0	296/16 = 18+ 8 18/16 = 1+ 2 1/16 = 0+ 1 1 0010 1000 1 1 0010 2 1000 8
1 0010 1000	450	128

Arithmetic in binary

- Addition

Digit 1	(27)		1	1	0	1	1
Digit 2	(11)			1	0	1	1
Carry over			1		1	1	
Result	(38)	1	0	0	1	1	0

Digit 1	(22)		1	0	1	1	0
Digit 2	(15)			1	1	1	1
Carry over			1	1	1		
Result	(37)	1	0	0	1	0	1

- Multiplication

Digit 1	(7)				1	1	1
Digit 2	(5)				1	0	1
					1	1	1
	+			0	0	0	
	+		1	1	1		
Carry over			1	1			
Result	(35)	1	0	0	0	1	1

Week 3: Modulo

- a|b
  - a divides b if there is an integer c that b=ac (a != 0)
  - then a is a divisor of b and b is a multiple of a
  - if 0 < a < b, a is a **proper divisor** of b
- **trivial divisor** of n is a divisor that equals n or 1; else nontrivial divisor
- integer values
  - if a|b and a|c, then a|(b+c);   3|6 and 3|18, then a|(18+6)
  - if a|b, then a|bc for any integer c
  - if a|b and b|c, then a|c;   3|6 and 6|12, then 3|12
- Prime
  - A positive integer n greater than 1 is called **prime**, if its only divisors are n and 1.
  - If n is an integer >= 1, then there is a prime p such that n < p <= n!+1
  - Given any real number X>=1, there exists a prime between x and 2x
  - If n is an integer >=2, then there are no primes between n!+2 and n!+n
- Composites
  - A positive integer n that is greater than 1 and is not prime is called **composite**
  - If n is a composite, then n has a prime divisor p such that p <= sqrt(n)

Theory of Congruences

- Modulo
  - Let a be an integer and n a positive integer greater than 1; r is the remainder of "a mod n";   e.g. 35 mod 12 = 11
  - "r is equal to a reduced modulo n"
  - "a is congruent to b modulo n" ($a \equiv b(\bmod n)$) if n is a divsor of a-b or if n|(a-b)

Modular addition: (A + B) mod X = A mod X + B mod X

Additive identity: A + B ≡ A mod C

- 1 + X ≡ 0 mod 5

Additive inverse: A + B ≡ 0 mod C

- -23 + B ≡ 0 mod 5; B = 2

Modular multiplication: (A * B) mod X = A mod X * B mod X

- (159 * 943) mod 5 = 159 mod 5 * 943 mod 5 = 4 mod 5 * 3 mod 5 = 12 mod 5 ≡ 2
- (-569 * -662) mod 10 = 376 678 mod 10 = 1 mod 10 * 8 mod 10 ≡ 8
- 206^9 (mod 13) ≡ ???
  206 (mod 13) ≡ 11
  206^2 (mod 13) = 11^2 (mod 13) = 121 (mod 13) ≡ 4
  206^4 (mod 13) = 4^2 (mod 13) = 16 (mod 13) = 3
  206^9 (mod 13) =206^4*206^4*206 (mod 13) = 3*3*11 (mod 13) = 99 (mod 13) ≡ 8

Multiplicative inverse: A * B ≡ 1 (mod X)

- 3 * 5 ≡ 1 mod 7
- 4 * 2 ≡ 1 mod 7
- Must not be co-prime, e.g. 2 * B ≡mod 8 (2 and 8 (2*2*2*2) are co-prime)

Week 4: Sequences

Arithmetic progressions:

-5	-3	-1	1	...	3	5	7	# 2n - 7
100	107	114	121	...	128	135	142	# 7n +93

Geometric progressions

2	4	8	16	...	32	64	128	#2^n

Week 5: Series

Arithmetic Series starting with i=0

Series with i^0

$\sum_{i=0}^{n} 3 = 3 * n$ $\sum_{i=0}^{10} 3 = 3 * 10 = 30$

Series with i^1

$\sum_{i=0}^{n} 3i = 3 * \frac{n(n+1)}{2}$ $\sum_{i=0}^{10} 3i = 3 * \frac{10(10+1)}{2} = \frac{3}{2} * 110 = 165$

Series with i^2

$\sum_{i=0}^{n} 3i^2 = 3 * \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=0}^{10} 3i^2 = 3 * \frac{10(10+1)(2*10+1)}{6} = \frac{1}{2} * 110 * 21 = 1155$

Series with compounds

$$\sum_{i=0}^{n} 3i^2 + 5i + 7 = \sum_{i=0}^{n} 3i^2 + \sum_{i=0}^{n} 5i + \sum_{i=0}^{n} 7$$

$\sum_{i=0}^{n} 3i^2 + 5i + 7 = 3 * \frac{n(n+1)(2n+1)}{6} + 5 * \frac{n(n+1)}{2} + 7 * n$

Arithmetic Series not starting with i=0

$$\sum_{i=10}^{n=20} 4i = \sum_{i=0}^{n=20} 4i - \sum_{i=0}^{n=9} 4i = 4 * \frac{20(20+1)}{2} - 4 * \frac{9(9+1)}{2} = 840 - 180 = 660$$

Geometric series

$$\sum_{i=1}^{n} a * b^{i-1} = a * \sum_{i=1}^{n} b^{i-1} = a * \frac{b^i - 1}{b - 1}$$

$$\sum_{i=1}^{5} 0.1 * -\frac{1}{2}^i = 0.1 * \sum_{i=1}^{5} -\frac{1}{2}^i = 0.1 * \frac{-\frac{1}{2}^i - 1}{-\frac{1}{2} - 1} = 0.06875$$

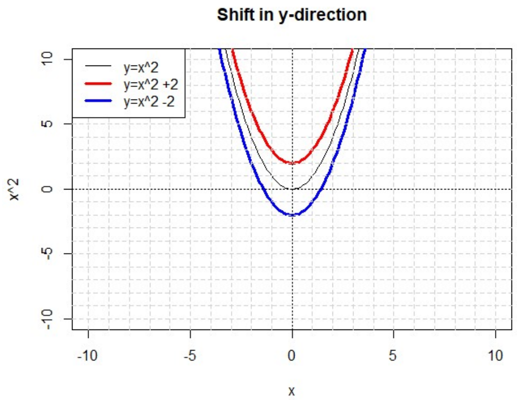
Week 6: Transformations

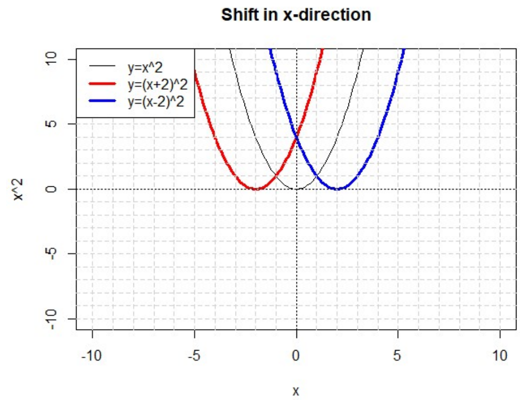
Intervals:

- Exclusive: (5, 8): {6, 7}
- Inclusive: [5, 8]: {5, 6, 7, 8}
- Mixed: [5, 8): {5, 6, 7}

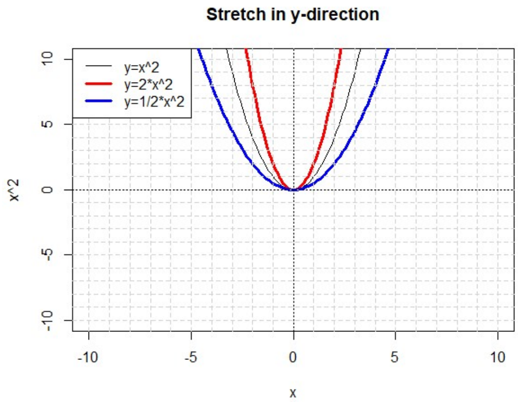
Translations:

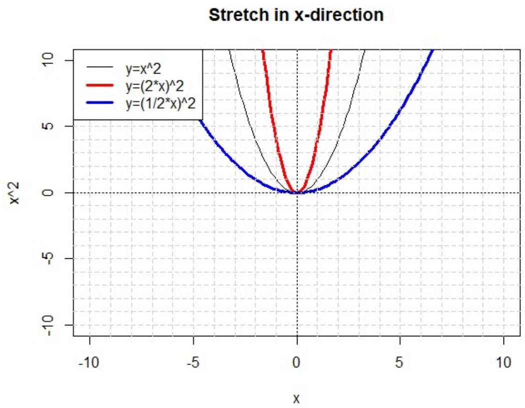
- Shift in y-direction (+2): $y=x^2$ -> $y=x^2 +2$
- Shift in y-direction (-2): $y=x^2$ -> $y=x^2 -2$
- Shift in x-direction (+2): $y=x^2$ -> $y=(x-2)^2$
- Shift in x-direction (-2): $y=x^2$ -> $y=(x+2)^2$





- Stretch in y-direction(*1/2): $y=x+2$ -> $y=(x^2)/2$
- Stretch in y-direction(*2): $y=x+2$ -> $y=2(x^2)$
- Stretch in x-direction(*1/2): $y=x^2$ -> $y=(2x)^2$
- Stretch in x-direction(*2): $y=3x+$ -> $y=(x/2)^2$





https://www.mathsisfun.com/sets/function-transformations.html

Week 7: Triangles

Angles

- Acute: biggest angle <90°
- Obtuse: biggest angle >90°
- Right: biggest angle = 90°

Degree to radians

- $1° = \frac{1°}{180°}\pi\, rad = 0.0174\, rad$
- $18° = \frac{18°}{180°}\pi\, rad = \frac{1}{10}\pi\, rad = \ 0.314\, rad$
- $180° = \frac{180°}{180°}\pi\, rad = \pi\, rad = \ 3.141\, rad$
- $360° = \frac{360°}{180°}\pi\, rad = 2\pi\, rad = \ 6.283\, rad$

Degree to radians

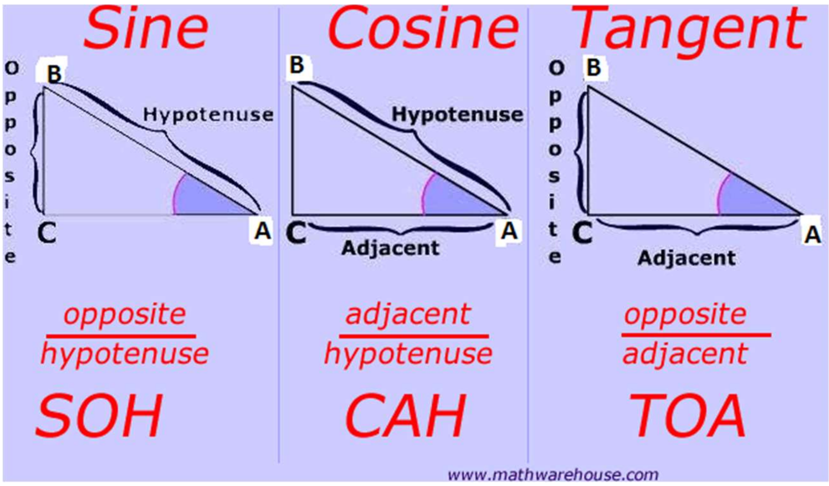
- $1\, rad = 1 * \frac{180°}{\pi} = 57.296°$
- $\pi\, rad = \pi\frac{180°}{\pi} = 180°$
- $2\pi\, rad = 2\pi\frac{180°}{\pi} = 360°$
- $\frac{5}{6}\pi\, rad = \frac{5}{6}\pi * \frac{180°}{\pi} = \frac{5}{1} * \frac{30°}{1} = 150°$
- $\frac{20}{9}\pi\, rad = \frac{20}{9}\pi * \frac{180°}{\pi} = \frac{20}{1} * \frac{20°}{1} = 400°$

Squareroots / surds

- $\sqrt{9} = 3$
- $\sqrt{n * m} = \sqrt{n} * \sqrt{m}$: $\sqrt{20} = \sqrt{2 * 2 * 5} = \sqrt{2} * \sqrt{2} * \sqrt{5} = 2\sqrt{5} =$
- $\frac{\sqrt{n}}{\sqrt{m}} = \sqrt{\frac{n}{m}}$: $\frac{\sqrt{54}}{\sqrt{6}} = \sqrt{\frac{54}{6}} = \sqrt{9} = 3$ OR $\frac{\sqrt{54}}{\sqrt{6}} = \frac{\sqrt{6*9}}{\sqrt{6}} = \frac{\sqrt{6}*\sqrt{9}}{\sqrt{6}} = \sqrt{9} = 3$
- $n\sqrt{x} + \ m\sqrt{x} = (n + m)\sqrt{x}$: $4\sqrt{5} + \ 7\sqrt{5} = 11\sqrt{5}$
- $\sqrt{18} - \sqrt{2} = \ 3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$
- $3(\sqrt{4} + \sqrt{5}) = 3\sqrt{4} + 3\sqrt{5} = \sqrt{9 * 4} + \sqrt{9 * 5} = \sqrt{36} + \sqrt{45} = 6 + 3\sqrt{5}$
- $2\sqrt{2} * 5\sqrt{10} = \sqrt{4 * 2} * \sqrt{25 * 10} = \sqrt{8} * \sqrt{250} = \sqrt{2000} = 10\sqrt{20} = 20\sqrt{5}$

Week 8: sin, cos, tan

Right triangles:



Any triangle:

- Sine rule: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
- Cosine rule: $c^2 = a^2 + b^2 - 2 * a * b * \cos(C)$

Amplitude: half distance between maximum and minimum values

- y=a*sin(bx+c); amplitude = |a|

Period: Horizontal spread of a full cycle

- y=a*sin(bx+c); period = |2pi/b|
- y=a*sin(bx+c); amplitude = |a|

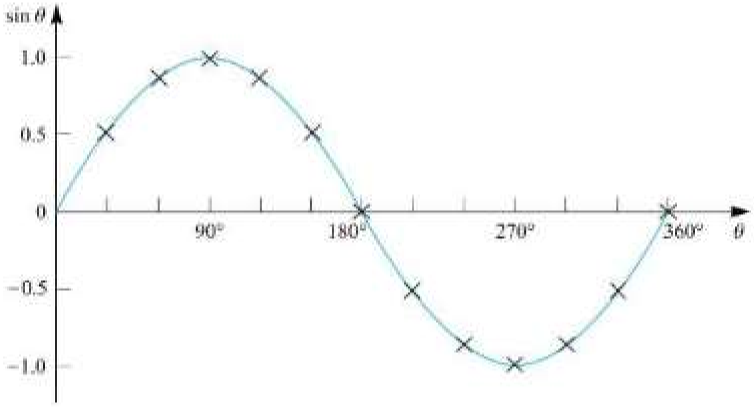
Phase: Horizontal shift

- y=a*sin(bx+c); period = c

Week 9: Trigonomic Functions

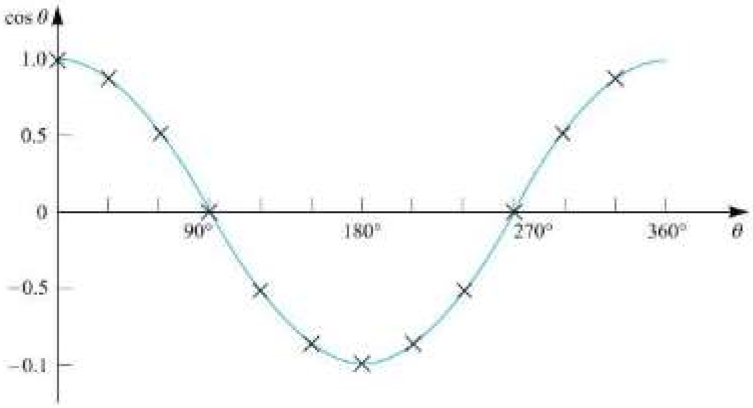
Sinus

θ	0°	30°	60°	90°	120°	150°	180°
$\sin\theta$	0	0.5	0.8660	1	0.8660	0.5	0
θ	210°	240°	270°	300°	330°	360°	
$\sin\theta$	−0.5	−0.8660	−1	−0.8660	−0.5	0	

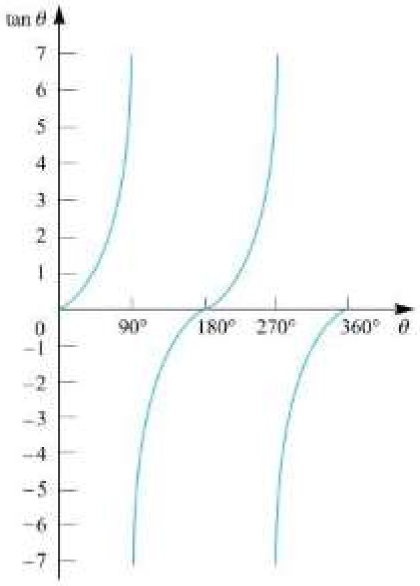


Cosinus

θ	0°	30°	60°	90°	120°	150°	180°
$\cos\theta$	1	0.8660	0.5	0	−0.5	−0.8660	−1
θ	210°	240°	270°	300°	330°	360°	
$\cos\theta$	−0.8660	−0.5	0	0.5	0.8660	1	



Tangens



Trigonometrical identities: The following equations are true for all angles:

- $\frac{sin(\theta)}{\cos(\theta)} = \tan(\theta)$
- $sin^2 A + cos^2 A = 1$
- $\theta = \frac{s}{r}$

Angle associated with coordinates

- (-3,2): Quadrant II, 180° - tan(2/3) = 146.31°
- (-3,-2): Quadrant III, 180° + tan(2/3) = 183.69°

Unit circle

- x=4/5 -> y= +/- (3/5)
- angle: -240° -> 240°-180°=60°, x=cos60°=0.5, y=sin60°=0.87

Week 10: Coordinate systems

Polar to Cartesian

- Quadrant I: P(radius *cos(angle), radius*sin(angle))
- Quadrant II: P(-radius *cos(angle), radius*sin(angle))
- Quadrant III: P(-radius *cos(angle), -radius*sin(angle))
- Quadrant IV: P(radius *cos(angle), -radius*sin(angle))
- P(11, 36.4°): x=11*cos36.4°=8.85, y=11*sin36.4°=6.53

Cartesian to Polar

- P(radius, angle)
- angle = tan(y/x)
- radius = $(x^2 + y^2)^{0.5}$
- P(3,8): theta=tan(8/3)=69.44°, r= $(3^2 + 8^2)^{0.5}$=8.54, P(8.54, 69.44°)
- P(-4,7): II, theta = 180-tan(7/4)=119.75°, r=$(4^2 + 7^2)^{0.5}$=8.06, P(8.06, 119.75°)

Week 11: Exponential functions

- a^x … a: base, x: power (or index)
- $n^2 * n^4 = n^{2+4} = n^6$
- $\frac{n^5}{n^3} = n^{5-3} \quad = n^2$
- $(n^2)^3 = n^{2*3} \quad = n^6$

Week 12: Logarithms

- y = a^x equivalent to: $\log_a y=x$
- log A + log B = log AB
- log A – log B = log A/B
- log 1 = 0
- n log A = log A^n
- $\log_a x = -\log_{ya} x$
- Calculator: $\log_n m$ = log n / log m: $\log_{16} 2$= log 16 / log 2 = 4

Week 13-14: Limits and differentiation

Limits

- $\lim_{n\to\infty} n = \infty$
- $\lim_{n\to\infty} \frac{1}{n} = 0$
- $\lim_{n\to\infty} \frac{n+1}{n} = \frac{n}{n} = 1$
- $\lim_{n\to\infty} \frac{2^n}{10^{n-1}} = 0$

Differentiation

Power rule: $f(x) = x^n$ $\quad f'(x) = \frac{1}{n}x^{n-1}$

- $f(x) = 2x^3,$ $\quad f'^{(x)} = \frac{2}{3}x^{3-1} = \frac{2}{3}x^2$
- $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $\quad f'^{(x)} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
- $f(x) = \frac{1}{x^2} = x^{-2}$ $\quad f'(x) = -\frac{1}{2}x^{-2-1} = -\frac{1}{2x^3}$

Multiplication rule: $f(x) = u * v$ $\quad f'(x) = u'v + uv'$

- $f(x) = (x^2 - 3x)(4x - 1) = uv$ $\quad f'(x) = (2x - 3)(4x - 1) + 4(x^2 - 3x)$

Division rule: $f(x) = \frac{u}{v}$ $\quad f'(x) = \frac{u'v - uv'}{v^2}$

- $f(x) = \frac{x^2 - 3x}{4x - 1} = \frac{u}{v}$ $\quad f'(x) = \frac{(2x-1)(4x-1) - 4(x^2-3x)}{(4x-1)^2}$

Chain rule: $f(x) = g(h(x))$ $\quad f'(x) = g'(h(x)) * h'(x)$

- $f(x) = g\big(h(x)\big) = (x^3 + 12x^2)^2$ $\quad g(x) = x^2, h(x) = x^3 + 12x^2$
- $f'(x) = g'(h(x)) * h'(x) = 2(x^3 + 12x^2) * (3x^2 + 24x)$

Trigonometric functions:

- $f(x) = \sin(x)$ $\quad f'(x) = \cos(x)$
- $f(x) = \cos(x)$ $\quad f'^{(x)} = -\sin(x)$
- $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ $\quad f'^{(x)} = \frac{1}{\cos^2(x)}$
- $f(x) = \sin(x^2)$ $\quad f'(x) = 2x\cos(x^2)$
- $f(x) = x^2\sin(x)$ $\quad f'^{(x)} = 2x\,sin(x) + x^2\cos(x)$
- $f(x) = \sin(x)\cos(x)$ $\quad f'^{(x)} = \cos(x)^2 - \sin(x)^2$

Exponential functions:

- $f(x) = \ln(\mathrm{x})$ $\quad f'(x) = \frac{1}{\mathrm{x}}$
- $f(x) = \ln(\mathrm{x}^2 - 2\mathrm{x}) = \ln(x) + \ln(x - 2)$ $\quad f'(x) = \frac{1}{\mathrm{x}} + \frac{1}{x-2} = \frac{2(x-1)}{x(x-2)}$
- $f(x) = \log3(\mathrm{x})$ $\quad f'(x) = \frac{1}{\mathrm{xln3}}$
- $f(x) = \mathrm{e}^{\mathrm{x}}$ $\quad f'(x) = \mathrm{e}^{\mathrm{x}}$
- $f(x) = \mathrm{e}^{\mathrm{x}^2-4\mathrm{x}+1}$ $\quad f'(x) = (2\mathrm{x} - 4)\mathrm{e}^{\mathrm{x}^2-4\mathrm{x}+1}$
- $f(x) = 0.1^{\mathrm{x}} = \left(e^{\ln(0.1)}\right)^x$ $\quad f'(x) = \ln(0.1)\,0.1^x$
- $f(x) = \frac{2^x}{3^{x-2}} = 9\left(\frac{2}{3}\right)^x$ $\quad f'(x) = 9\ln\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^x$

Extreme points

- Maxima: $f'^{(x)} = 0$ $f''^{(x)} < 0$
- Minima: $f'^{(x)} = 0$ $f''^{(x)} > 0$

Week 15: Vectors and matrices

Vectors

- $\vec{a} = \binom{3}{2}$
- $\overrightarrow{AB} = B - A = \binom{2}{2} - \binom{1}{1} = \binom{1}{1}$
- $\overrightarrow{AB} + A = B - A + A = B$
- Magnitude $|\vec{a}|$ of $\vec{a} = \binom{3}{2}$: $|\vec{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

Dot product: $u * v = \binom{\mathrm{u}_1}{\mathrm{u}_2} * \binom{\mathrm{v}_1}{\mathrm{v}_2} = \mathrm{u}_1 * \mathrm{v}_1 + \mathrm{u}_2 * \mathrm{v}_2$

- $\binom{1}{2} * \binom{3}{4} = 1 * 3 + 2 * 4 = 3 + 8 = 11$

Angle between two vectors: $\cos(\theta) = \frac{u*v}{|u|*|v|}$

- $u = \binom{6}{3}, v = \binom{5}{13}$
- $\cos(\theta) = \frac{6*5+3*13}{\sqrt{6^2+3^2}*\sqrt{5^2+13^2}} = \frac{(30+39)}{\sqrt{36+9}*\sqrt{25+1}} = \frac{69}{\sqrt{45}*\sqrt{194}}, \theta = 42°$

Matrices

- 2 by 3: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
- Transposed matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$
- Identity matrices: $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Id_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Addition: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$
- Multiplication:
  - $\mathrm{X}_1 \, by \, \mathrm{X}_2, \mathrm{Y}_1 \, by \, \mathrm{Y}_2$, multiplication possible if $\mathrm{X}_1 = \mathrm{Y}_2$
  - $\begin{pmatrix} \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 & \mathrm{u}_4 \end{pmatrix} \begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \\ \mathrm{v}_4 \end{pmatrix} = (\mathrm{u}_1 * \mathrm{v}_1 + \mathrm{u}_2 * \mathrm{v}_2 + \mathrm{u}_3 * \mathrm{v}_3 + \mathrm{u}_4 * \mathrm{v}_4)$
  - $\begin{pmatrix} 3 & -3 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 5 \end{pmatrix} = (3 * 1 + (-3) * 2 + (-1) * (-1) + 4 * 5) = (18)$
  - $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} = \begin{pmatrix} aA + bD + cG & aB + bE + cH & aC + bF + cI \\ dA + eD + fG & dB + eE + fH & dC + eF + fI \\ gA + hD + iG & gB + hE + iH & gC + hF + iI \end{pmatrix}$
  - $\begin{pmatrix} -3 & 3 & 6 \\ 2 & -1 & 5 \\ -5 & -2 & 4 \end{pmatrix} \begin{pmatrix} -1 & 3 & -2 \\ 1 & 4 & -5 \\ 6 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 42 & 21 & 27 \\ 27 & 17 & 31 \\ 27 & -11 & 44 \end{pmatrix}$

Determinants 2x2

- $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- $det\begin{pmatrix} -1 & 3 \\ 4 & 5 \end{pmatrix} = -1 * 5 - 3 * 4 = -5 - 12 = -17$
- The determinant of a 2x2 matrix is zero when one row is a multiple of the other row
- $det\begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix} = -2 * (-6) - 3 * 4 = 12 - 12 = 0$

Determinants 3x3

- $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$
- $M = \begin{pmatrix} -1 & 3 & 3 \\ 5 & 3 & -6 \\ 3 & 4 & -3 \end{pmatrix}$
- $\det(M) = (+1) * (-1) det\begin{pmatrix} 3 & -6 \\ 4 & -3 \end{pmatrix} + (-1) * 3 det\begin{pmatrix} 5 & -6 \\ 3 & -3 \end{pmatrix} + (+1) * 3 det\begin{pmatrix} 5 & 3 \\ 3 & 4 \end{pmatrix}$

Week 16: Inverting matrices, cross product of vectors

Inversion of matrices

1. Check dimensions: must be the same
2. Calculate determinant (if 0, it is not invertible)
3. Calculate determinants of smaller matrices obtained by removing one row and one column
4. Transpose co-factor matrix
5. Divide adjoint matrix by determinant
https://arndt-bruenner.de/mathe/scripts/inversematrix.htm

1. $M = \begin{pmatrix} 4 & -3 \\ 3 & 5 \end{pmatrix} : 2x2$
2. $\det(M) = 4 * 5 - (-3) + 3 = 20 + 9 = 29$
3. $\frac{1}{29}\begin{pmatrix} 5 & 3 \\ -3 & 4 \end{pmatrix}$:

Cross product of vectors

- $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, y = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$, x x y $= \begin{pmatrix} bg - cf \\ ce - ag \\ af - be \end{pmatrix}$ a b c a b c XXX e f g e f g
- $a = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, b = \begin{pmatrix} -6 \\ -6 \\ 2 \end{pmatrix}, axb = \begin{pmatrix} 3 * 2 - (-1 * (-6)) \\ -1 * (-6) - 3 * 2 \\ 3 * (-6) - 3 * (-6) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Multiplication of matrices and vectors

- $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1a + 2b + 3c \\ 1d + 2e + 3f \\ 1g + 2h + 3i \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 * 2 + 0 * 3 + 0 * 4 \\ -1 * 2 + 1 * 3 + 0 * 4 \\ 0 * 2 + 1 * 3 + 2 * 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 11 \end{pmatrix}$

Week 17: Linear transformation and matrices

Scaling

- No scaling: $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- 2x in x (dilation): $\mathrm{Id}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
- 2x in y (dilation): $\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
- ½x in x (dilation): $\mathrm{Id}_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

Reflection

- No reflection: $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection on y-axis: $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection on x-axis: $M = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection on x-axis: $M = \begin{pmatrix} cos60° & sin60° \\ sin60° & -cos60° \end{pmatrix}$

Rotation

- No rotation: $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Anticlockwise by 90°: $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cos90° & -sin90° \\ sin90° & cos90° \end{pmatrix}$
- Anticlockwise by 30°: $M = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} cos30° & -sin30° \\ sin30° & cos30° \end{pmatrix}$

Combinations

- 2x dilation in x, 3x dilation in y, rotation on y-axis: $M = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$
- 2x dilation in x, 3x dilation in y, rotation by 180°: $M = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$

Homogeneous coordinates

- $M = \begin{pmatrix} 2 \\ 3 \end{pmatrix} => M = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$
- $M = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} => M = \begin{pmatrix} \frac{2}{4} \\ \frac{3}{4} \\ \frac{4}{4} \end{pmatrix} => M = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \end{pmatrix}$
- $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} => M = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Translations

- Translation 2 in x and 3 in y direction: $M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

Week 18: Affine transformation in homogeneous coordinates

Homogeneous coordinates

- $M = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} => M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, dilation with scaling 4x x and 1/2x y direction.
- $M^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} => M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, rotation 90° anticlockwise
- $P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, rotation 90° clockwise

Combining translations and linear transformations

- $M = \begin{pmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{3} & 5 \\ 0 & 0 & 1 \end{pmatrix}$, translation by $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$, dilation x: 2, y: 1/3
- $M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 3 & -5 \\ 0 & 5 & 1 \end{pmatrix}$
- $P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, rotation 90° anticlockwise, translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solving equations using matrices

- $\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- $\begin{pmatrix} x - y = 1 \\ 2x + y = 3 \end{pmatrix}$

Week 19: Combinatorics and probability

Permutations (ordered combinations)

- Permutation of n objects is n!
  - X, Y, Z: XYZ, XZY, YXZ, YZX, ZXY, ZYX (3! = 6)
- Select m of n objects: $\frac{n!}{(n-m)!}$
  - 4 of 6: $\frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 * 5 * 4 * 3 = 360$
  - Distinct combinations of TOTTENHAM:
    - 9 letters: 3*T, 1*O, 1*E, 1*N, 1*H, 1*A, 1*M
    - $\frac{9!}{3!1!1!1!1!1!1!} = \frac{9!}{3!} = 60480$
  - Distinct combinations of LONDON:
    - 6 letters: 1*L, 2*O, 2*N, 1*D
    - $\frac{6!}{2!2!} = 180$

Combinations: Sets of objects (order does not matter)

- ${}^{n}C_{m} = \frac{n!}{m!(n-m)!}$
- 3 cards out of 52: ${}^{52}C_3 = \frac{52!}{3!(52-3)!} = \frac{52!}{3!(49)!} = \frac{52*51*50*49!}{1*2*3+49!} = 25 * 17 * 52 = 22\,100$
- 2 of 5 and 3 of 10: ${}^{15}C_2 * {}^{10}C_3 = \frac{15!}{2!13!} * \frac{10!}{3!7!} = \frac{14*15}{2!} * \frac{(8*9*10)}{3!} = 105 * 120$

Independent events:

- Having 1 Heads from coin toss: $\frac{1}{2}$
- Having 2 Heads from two coin tosses: $\frac{1}{2^2} = \frac{1}{4}$

Probability of mutually exclusive events

- Two mutually exclusive events: $P(A \cup B) = P(A) + P(B)$ …or: ∪
- Two not mutually exclusive events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ … and: ∩
  - Draw King (A) or Spades (B): $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

The complement of an event: $P(A') = 1 - P(A)$

- 15 C cakes, 10 V cakes:
  - Chance of not getting a C cake: $P(A') = 1 - P(A) = 1 - \frac{15}{25} = \frac{10}{25}$

Week 20: Conditional probability

Conditional probability

- $P(A \cap B) = P(A|B) * P(B)$
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Mean and standard deviations

- Mean:
  - sum of values divided by number of values
  - 2.4, 3.4, 5.1, 6.7, 3.0, 2.2, 3.3, 3.5 … mean = 3.7
- Median:
  - central value (or mean of two central values)
  - 2.2, 2.4, 3.0, 3.3, 3.4, 3.5, 5.1, 6.7 … median = 3.35
- Mode: most common value
- Standard deviation:
  - Mean squared distance: $s = \sqrt{(\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}}$
  - 2.2, 2.4, 3.0, 3.3, 3.4, 3.5, 5.1, 6.7 … median $s = \sqrt{(\frac{\sum_{i=1}^{N}(x_i - 3.7)^2}{N-1}} = \sqrt{(\frac{15.68}{7}} = 1.497$