CS 754 - Advanced Image Processing
Assignment 3 - Report

Shaan ul Haque - 180070053

Mantri Krishna Sri Ipsit - 180070032

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Question 2

The original images of the slices used are as follows:

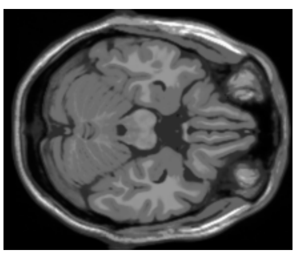

Figure 1: Slice 50

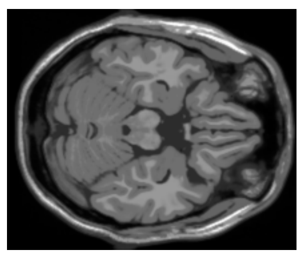

Figure 2: Slice 51

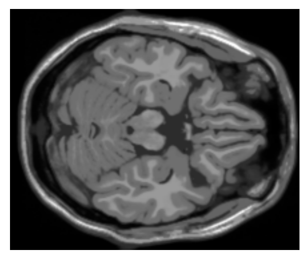

Figure 3: Slice 52

Part a

Here the tomographic reconstruction is performed using the **Filtered Backprojection** using the **Ram-Lak** filter. The reconstruction results for slices 50 and 51 are as follows:

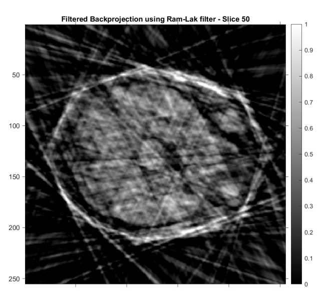

Figure 4: Ram-Lak FBP - Slice 50

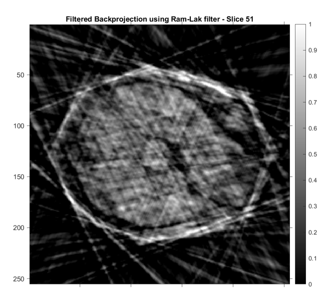

Figure 5: Ram-Lak FBP - Slice 51

Part b

Here the tomographic reconstruction is performed by using the **Independent Compressive Sensing** based reconstruction for each slice by solving an optimization problem of the form

$$J(x) = \|y - Ax\|_2^2 + \lambda \|x\|_1$$

The reconstruction results for slices 50 and 51 are as follows:

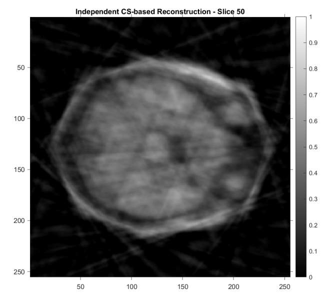

Figure 6: Independent CS - Slice 50

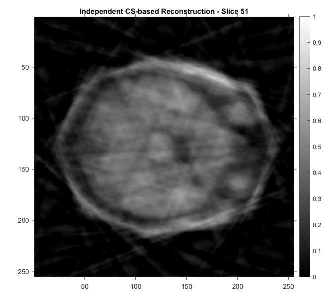

Figure 7: Independent CS - Slice 51

Part c

Here the tomographic reconstruction is performed by using the **Coupled Compressive Sensing** based reconstruction by solving an optimization problem of the form:

$$J\left(\begin{bmatrix} \beta_1 \\ \Delta\beta_1 \end{bmatrix} \right) = \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} R_1 U & 0 \\ R_2 U & R_2 U \end{bmatrix} \begin{bmatrix} \beta_1 \\ \Delta\beta_1 \end{bmatrix} \right\|_2^2 + \lambda \left\| \begin{bmatrix} \beta_1 \\ \Delta\beta_1 \end{bmatrix} \right\|_1$$

The reconstruction results for slices 50 and 51 are as follows:

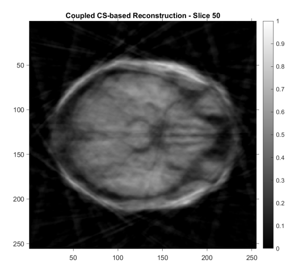

Figure 8: Coupled CS - Slice 50

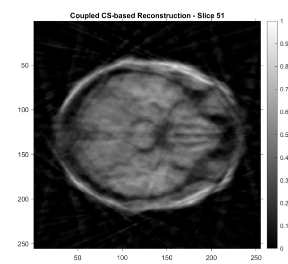

Figure 9: Coupled CS - Slice 51

Part d

Here the tomographic reconstruction is performed by using the **Coupled Compressive Sensing** based reconstruction **using 3 slices** by solving an optimization problem of the form:

$$J\left(\begin{bmatrix} \beta_1 \\ \Delta\beta_1 \\ \Delta\beta_2 \end{bmatrix} \right) = \left\| \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} R_1 U & 0 & 0 \\ R_2 U & R_2 U & 0 \\ R_3 U & 0 & R_3 U \end{bmatrix} \begin{bmatrix} \beta_1 \\ \Delta\beta_1 \\ \Delta\beta_2 \end{bmatrix} \right\|_2^2 + \lambda \left\| \begin{bmatrix} \beta_1 \\ \Delta\beta_1 \\ \Delta\beta_2 \end{bmatrix} \right\|_1$$

Here

- y_1, y_2, y_3 are the radon transforms of 3 slices x_1, x_2, x_3 under consideration.
- β_1 is the 2D-DCT of first slice x_1, vectorized.
- $\beta_1 + \Delta\beta_1$ is the 2D-DCT of second slice x_2, vectorized.
- $\beta_1 + \Delta\beta_2$ is the 2D-DCT of second slice x_3, vectorized.
- R_1 is the radon transform matrix for the random angles used to compute tomographic projections of x_1.
- R_2 is the radon transform matrix for the random angles used to compute tomographic projections of x_2.
- R_3 is the radon transform matrix for the random angles used to compute tomographic projections of x_3.

The reconstruction results for slices 50, 51 and 52 are as follows:

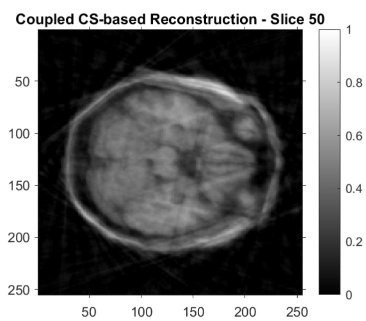

Figure 10: 3 CS - Slice 50

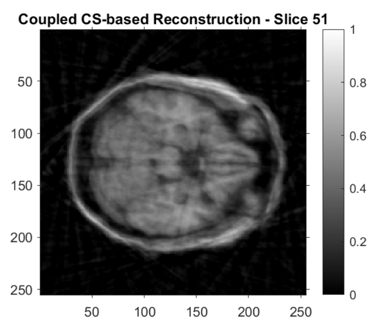

Figure 11: 3 CS - Slice 51

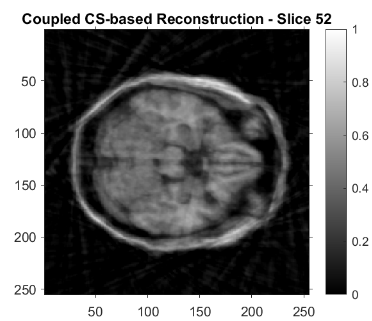

Figure 12: 3 CS - Slice 52

Question 3

For the sake of notational simplicity, denote $R(\rho, \theta)$ as the radon transform of $g(x, y)$ at a translation ρ and at an angle θ.

$$R(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, \delta(x \cos\theta + y \sin\theta - \rho)\, dx\, dy$$

Part a

$$R\big(g(x-x_0,y-y_0)\big)(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-x_0,y-y_0)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

Replace $x \to u + x_0, y \to v + y_0$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\,\delta\Big(u\cos\theta + v\sin\theta - \big(\rho - x_0\cos\theta - y_0\sin\theta\big)\Big)\,dx\,dy$$
$$= R\big(g(u,v)\big)(\rho - x_0\cos\theta - y_0\sin\theta, \theta)$$
$$= R\big(g(x,y)\big)(\rho - x_0\cos\theta - y_0\sin\theta, \theta)$$

Part b

$$g'(r,\psi) = g(r,\psi-\psi_0)$$

Consider the radon transform in polar coordinates:

$$R(g)(\rho,\theta) = \int_0^{2\pi}\int_0^{\infty} g(r,\psi)\,\delta\Big(r\cos(\psi-\theta)-\rho\Big)\,r\,dr\,d\psi$$

Now,

$$R(g')(\rho,\theta) = \int_0^{\infty}\int_0^{2\pi} g'(r,\psi)\,\delta\Big(r\cos(\psi-\theta)-\rho\Big)\,r\,dr\,d\psi$$
$$= \int_0^{\infty}\int_0^{2\pi} g(r,\psi-\psi_0)\,\delta\Big(r\cos(\psi-\theta)-\rho\Big)\,r\,dr\,d\psi$$

Replace $\psi \to \psi_0 + \alpha$

$$= \int_0^{\infty}\int_{-\psi_0}^{2\pi-\psi_0} g(r,\alpha)\,\delta\Big(r\cos(\alpha+\psi_0-\theta)-\rho\Big)\,r\,dr\,d\alpha$$
$$= \int_0^{\infty}\int_0^{2\pi} g(r,\alpha)\,\delta\Big(r\cos\big(\alpha-(\theta-\psi_0)\big)-\rho\Big)\,r\,dr\,d\alpha$$

Replace $\alpha \to \psi$

$$= \int_0^{\infty}\int_0^{2\pi} g(r,\psi)\,\delta\Big(r\cos\big(\psi-(\theta-\psi_0)\big)-\rho\Big)\,r\,dr\,d\psi$$
$$= R(g)(\rho,\theta-\psi_0)$$

Part c

The convolution of two signals $f(x,y)$ and $k(x,y)$ is given by:

$$(f*k)(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)k(x-u,y-v)\,dudv$$

The radon transform of the convolution will be

$$\begin{aligned}
R_\theta(f*k) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f*k)(x,y)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)k(x-u,y-v)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dudv\,dxdy
\end{aligned}$$

Replace $x-u \to a, y-v \to b$

$$\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)k(a,b)\,\delta((u+a)\cos\theta + (v+b)\sin\theta - \rho)\,dudv\,dadb \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)k(a,b)\,dudvdadb \int_{-\infty}^{\infty} \delta(u\cos\theta + v\sin\theta - \sigma) \\
&\qquad \delta(a\cos\theta + b\sin\theta - (\rho-\sigma))\,d\sigma \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\delta(u\cos\theta + v\sin\theta - \sigma)\,dudv\right) \\
&\qquad \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(a,b)\,\delta(a\cos\theta + b\sin\theta - (\rho-\sigma))\,dadb\right) d\sigma \\
&= \int_{-\infty}^{\infty} R(f)(\sigma,\theta)\,R(k)(\rho-\sigma,\theta)\,d\sigma \\
&= R_\theta(f) * R_\theta(k)
\end{aligned}$$