CS 754 - Advanced Image Processing
Assignment 2 - Report

Shaan ul Haque - 180070053
Mantri Krishna Sri Ipsit - 180070032

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Question 1

1.1

If $\delta_{2s} = 1$, then for any $2s$ - sparse vector $h \in \mathbb{R}^n$, we have

$$0 \leq \|\Phi h\|_2^2 \leq 2 \ \|h\|_2^2 \tag{1}$$

If $2s$ columns of Φ are linearly dependent, then we have

$$\sum_i \alpha_i \phi_i = 0 \quad , \alpha_i \neq 0$$

Where $\{\phi_i\}$ are the $2s$ linearly dependent columns of Φ and $\{\alpha_i\}$ are some non-zero constants. This summation can be re-written as

$$\Phi v = 0, \quad \Phi = [\phi_1 | \ldots | \phi_{2s}], v^{\mathrm{T}} = [\alpha_1, \ldots, \alpha_{2s}]$$

Now, we can extend v by adding rows of zeros so that the extended vector $v_1 \in \mathbb{R}^n$. Similarly, we can extend the columns of Φ by adding random columns (this works because we want the measurement matrix to be random in Compressed Sensing) so that $\Phi \in \mathbb{R}^{m \times n}$. Even after this extension, we still have

$$\Phi v_1 = 0$$

Clearly, we can see that v_1 is a $2s$ - sparse vector and also satisfies (1) trivially because $\|\Phi v_1\|_2 = 0$. Hence, the above justifies that when $\delta_{2s} = 1$, there exists a $2s$ - sparse vector h such that $\Phi h = 0$ and $2s$ columns of Φ are linearly dependent.

1.2

We have x^* and x which satisfy the convex optimization problem

$$\min_{\tilde{x} \in \mathbb{R}^n} \|\tilde{x}\|_1 \quad \text{subject to} \ \ \|y - \Phi \tilde{x}\|_2 \leq \epsilon \tag{2}$$

Consider

$$\|\Phi(x^* - x)\|_2 = \|(\Phi x^* - y) + (y - \Phi x)\|_2$$

Then, using the triangle inequality, we have

$$\|\Phi(x^* - x)\|_2 = \|(\Phi x^* - y) + (y - \Phi x)\|_2 \leq \|\Phi x^* - y\|_2 + \|y - \Phi x\|_2$$

But using (2) we can say that

$$\|\Phi x^* - y\|_2 + \|y - \Phi x\|_2 \leq \epsilon + \epsilon = 2\epsilon$$

Hence, we have

$$\|\Phi(x^* - x)\|_2 \leq \|\Phi x^* - y\|_2 + \|y - \Phi x\|_2 \leq 2\epsilon$$

1.3

We know that each h_{T_j} is s sparse. We also know that the ℓ_∞ norm of a vector denotes the maximum element of that vector. As h_{T_j} is s sparse, we have, for each non-zero entry α_i of h_{T_j}

$$\alpha_i^2 \leq \left\|h_{T_j}\right\|_\infty^2 \quad i = 1, 2, 3, \ldots, s$$

Adding all the above s inequalities, we have

$$\left\|h_{T_j}\right\|_2^2 \leq s \ \left\|h_{T_j}\right\|_\infty^2$$

$$\implies \left\|h_{T_j}\right\|_2 \leq s^{1/2} \ \left\|h_{T_j}\right\|_\infty \tag{3}$$

We know that, by the construction of the vectors $\{h_{T_j}\}$, the largest element of h_{T_j} is smaller than the smallest element of $h_{T_{j-1}}$. In fact, the $\left\|h_{T_j}\right\|_\infty$ is smaller than every (non-zero) element of $h_{T_{j-1}}$. Re-writing this mathematically, we have

$$\left\|h_{T_j}\right\|_\infty \leq |\beta_i| \quad i = 1, 2, 3, \ldots, s$$

Adding all the above s inequalities, we get

$$s \ \left\|h_{T_j}\right\|_\infty \leq \left\|h_{T_{j-1}}\right\|_1$$

$$\implies s^{1/2} \ \left\|h_{T_j}\right\|_\infty \leq s^{-1/2} \ \left\|h_{T_{j-1}}\right\|_1 \tag{4}$$

Combining the inequalities (3) and (4), we have

$$\left\|h_{T_j}\right\|_2 \leq s^{1/2} \ \left\|h_{T_j}\right\|_\infty \leq s^{-1/2} \ \left\|h_{T_{j-1}}\right\|_1$$

1.4

Using the relation between the 1st and the 3rd expression in the above inequality, we have

$$\left\|h_{T_j}\right\|_2 \leq s^{-1/2} \ \left\|h_{T_{j-1}}\right\|_1$$

Summing up all such above inequalities for $j \geq 2$, we have

$$\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2} \left(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots \right) \tag{5}$$

By construction of $\{h_{T_j}\}$, we can say that

$$\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots = \left\|h_{T_0^c}\right\|_1$$

Hence, (5) becomes

$$\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2} \left(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots \right) \leq s^{-1/2} \ \left\|h_{T_0^c}\right\|_1 \tag{6}$$

1.5

We know that

$$h_{(T_0 \cup T_1)^c} = \sum_{j \geq 2} h_{T_j}$$

Hence, we have

$$\left\| h_{(T_0 \cup T_1)^c} \right\|_2 = \left\| \sum_{j \geq 2} h_{T_j} \right\|_2$$

Using the extended version of triangle inequality for the above equation and using the inequality (6) above, we get

$$\left\| h_{(T_0 \cup T_1)^c} \right\|_2 = \left\| \sum_{j \geq 2} h_{T_j} \right\|_2 \leq \sum_{j \geq 2} \left\| h_{T_j} \right\|_2 \leq s^{-1/2} \ \left\| h_{T_0^c} \right\|_1 \tag{7}$$

1.6

We have the *reverse triangle inequality* for any two real numbers as follows:

$$|x - y| \geq \Big| |x| - |y| \Big|$$

Using this inequality for $|x_i + h_i|, i \in T_0$, we have

$$|x_i + h_i| \geq \Big| |x_i| - |h_i| \Big| \geq |x_i| - |h_i|$$

Summing over all $i \in T_0$, we have

$$\sum_{i \in T_0} |x_i + h_i| \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 \tag{8}$$

Again using the reverse triangle inequality for $|h_i + x_i|, i \in T_0^c$, we have

$$|h_i + x_i| \geq \Big| |h_i| - |x_i| \Big| \geq |h_i| - |x_i|$$

Summing over all $i \in T_1$, we have

$$\sum_{i \in T_0^c} |x_i + h_i| \geq \left\| h_{T_0^c} \right\|_1 - \left\| x_{T_0^c} \right\|_1 \tag{9}$$

By adding the inequalities (8) and (9) we get

$$\sum_{i \in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i| \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \left\| h_{T_0^c} \right\|_1 - \left\| x_{T_0^c} \right\|_1 \tag{10}$$

1.7

As per the convex optimization problem that we are solving, we have

$$\|x\|_1 \geq \|x+h\|_1$$

Using this in the inequality (10) we have

$$\|x\|_1 \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \left\|h_{T_0^c}\right\|_1 - \left\|x_{T_0^c}\right\|_1 \tag{11}$$

By definition we have

$$\left\|x_{T_0^c}\right\|_1 = \|x - x_s\|_1$$

By using the reverse triangle inequality, we have

$$\left\|x_{T_0^c}\right\|_1 = \|x - x_s\|_1 \geq \|x\|_1 - \|x_s\|_1$$

Noting the fact that $\|x_s\| = \|x_{T_0}\|$ and using inequality (11) we have

$$\left\|x_{T_0^c}\right\|_1 \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \left\|h_{T_0^c}\right\|_1 - \left\|x_{T_0^c}\right\|_1 - \|x_{T_0}\|_1$$

which is same as

$$\left\|h_{T_0^c}\right\|_1 \leq \|h_{T_0}\|_1 + 2\, \left\|x_{T_0^c}\right\|_1 \tag{12}$$

1.8

Using inequalities (7) and (12) we have

$$\left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq s^{-1/2}\left(\, \|h_{T_0}\|_1 + 2\, \left\|x_{T_0^c}\right\|_1 \right)$$

Using the known inequality $\|v\|_1 \leq \sqrt{k}\, \|v\|_2$ for a k sparse vector v, we have

$$\left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq s^{-1/2}\left(\, \|h_{T_0}\|_1 + 2\, \left\|x_{T_0^c}\right\|_1 \right) \leq s^{-1/2}\left(\sqrt{s}\, \|h_{T_0}\|_2 + 2\, \left\|x_{T_0^c}\right\|_1 \right)$$

Hence we have

$$\left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq \|h_{T_0}\|_2 + 2e_0, \quad e_0 \equiv s^{-1/2}\, \|x - x_s\|_1 \tag{13}$$

1.9

We know that by Cauchy-Schwartz inequality, we have

$$|\langle \Phi h_{T_0 \cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0 \cup T_1}\|_2\, \|\Phi h\|_2 \tag{14}$$

As $h = x^* - x$, we have by inequality justified in section 1.3,

$$\|\Phi h\|_2 = \|\Phi(x^* - x)\|_2 \leq 2\epsilon$$

also, as $h_{T_0 \cup T_1}$ is a $2s$ sparse vector and Φ follows RIP of order $2s$, we have

$$\|\Phi h_{T_0 \cup T_1}\|_2 \leq \sqrt{1 + \delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2$$

Using the above two inequalities in (14), we have

$$|\langle \Phi h_{T_0 \cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0 \cup T_1}\|_2\, \|\Phi h\|_2 \leq 2\epsilon\sqrt{1 + \delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2 \tag{15}$$

1.10

Lemma 2.1 from the paper states

$$|\langle \Phi x, \Phi x' \rangle| \leq \delta_{s+s'} \, \|x\|_2 \, \|x'\|_2$$

for all x, x' supported on disjoint subsets $T, T' \subset \{1, 2, \ldots, n\}$ with $|T| \leq s, |T'| \leq s'$ Using this lemma for $x = h_{T_0}$ and $x' = h_{T_j}$, we have

$$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \, \|h_{T_0}\|_2 \, \left\|h_{T_j}\right\|_2 \tag{16}$$

1.11

We know that for two positive real numbers a and b, the QM-AM inequality is as follows:

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

Taking $a = \|h_{T_0}\|_2$ and $b = \|h_{T_1}\|_2$ and substituting in the above inequality, we get

$$\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2} \sqrt{\|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2} = \sqrt{2} \, \|h_{T_0 \cup T_1}\|_2 \tag{17}$$

as T_0 and T_1 are disjoint.

1.12

By replacing T_0 with T_1 in inequality (16) and adding both, we get

$$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s}(\|h_{T_0}\|_2 + \|h_{T_1}\|_2) \left\|h_{T_j}\right\|$$

Making use of triangle inequality for the LHS of above inequality, we get

$$|\langle \Phi h_{T_0} + \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s}(\|h_{T_0}\|_2 + \|h_{T_1}\|_2) \left\|h_{T_j}\right\|$$

Replacing $\Phi h_{T_0} + \Phi h_{T_1} = \Phi h_{T_0 \cup T_1}$ and above inequalities for $j \geq 2$, we get

$$\sum_{j \geq 2} |\langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s}(\|h_{T_0}\|_2 + \|h_{T_1}\|_2) \left(\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \right)$$

Again making use of triangle inequality for the LHS of the above inequality, we get

$$\left| \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| \leq \delta_{2s}(\|h_{T_0}\|_2 + \|h_{T_1}\|_2) \left(\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \right)$$

Finally, using (17), we finally have

$$\left| \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| \leq \delta_{2s} \sqrt{2} \, \|h_{T_0 \cup T_1}\| \left(\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \right) \tag{18}$$

Now, using the fact that $\Phi h = \Phi h_{T_0 \cup T_1} + \sum_{j \geq 2} \Phi h_{T_j}$ in the inequality (15) above, we get

$$\left| \left\langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_0 \cup T_1} + \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| \leq 2\epsilon \sqrt{1 + \delta_{2s}} \; \|h_{T_0 \cup T_1}\|_2$$

Using the linearity property of inner product, we have

$$\left| \|\Phi h_{T_0 \cup T_1}\|_2^2 + \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| \leq 2\epsilon \sqrt{1 + \delta_{2s}} \; \|h_{T_0 \cup T_1}\|_2 \tag{19}$$

Now, adding the inequalities (18) and (19), we get

$$\begin{aligned} \left| \|\Phi h_{T_0 \cup T_1}\|_2^2 + \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| + \left| \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| \quad &\leq \quad 2\epsilon \sqrt{1 + \delta_{2s}} \; \|h_{T_0 \cup T_1}\|_2 + \\ &\delta_{2s} \sqrt{2} \; \|h_{T_0 \cup T_1}\| \; \left(\sum_{j \geq 2} \left\| h_{T_j} \right\|_2 \right) \end{aligned}$$

We know from the triangle inequality that

$$\|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \left| \|\Phi h_{T_0 \cup T_1}\|_2^2 + \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right| + \left| - \left\langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \right\rangle \right|$$

Using this, we finally have,

$$\|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \; \left(2\epsilon \sqrt{1 + \delta_{2s}} + \sqrt{2}\, \delta_{2s} \sum_{j \geq 2} \left\| h_{T_j} \right\|_2 \right) \tag{20}$$

From the restricted isometry property, we have

$$(1 - \delta_{2s}) \; \|h_{T_0 \cup T_1}\|_2^2 \leq \|\Phi h_{T_0 \cup T_1}\|_2^2$$

Hence, we have

$$(1 - \delta_{2s}) \; \|h_{T_0 \cup T_1}\|_2^2 \leq \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \; \left(2\epsilon \sqrt{1 + \delta_{2s}} + \sqrt{2}\, \delta_{2s} \sum_{j \geq 2} \left\| h_{T_j} \right\|_2 \right) \tag{21}$$

1.13

By combining the inequality (6) with RHs of (21), we have

$$\|h_{T_0 \cup T_1}\|_2 \; \left(2\epsilon \sqrt{1 + \delta_{2s}} + \sqrt{2}\, \delta_{2s} \sum_{j \geq 2} \left\| h_{T_j} \right\|_2 \right) \leq \|h_{T_0 \cup T_1}\|_2 \left(2\epsilon \sqrt{1 + \delta_{2s}} + \sqrt{2} \delta_{2s} \, s^{-1/2} \left\| h_{T_0^c} \right\|_1 \right)$$

Using the above inequality with (21), we have

$$(1 - \delta_{2s}) \, \|h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \left(2\epsilon \sqrt{1 + \delta_{2s}} + \sqrt{2} \delta_{2s} \, s^{-1/2} \left\| h_{T_0^c} \right\|_1 \right)$$

On simplifying the above inequality, we get

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \left\| h_{T_0^c} \right\|_1, \quad \alpha \equiv \frac{2\sqrt{1 + \delta_{2s}}}{1 - \delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1 - \delta_{2s}} \tag{22}$$

1.14

We now conclude from (22) and (12) that

$$\left\|h_{T_0\cup T_1}\right\|_2 \le \alpha\epsilon + \rho s^{-1/2}\left\|h_{T_0}\right\|_1 + 2\rho e_0 \le \alpha\epsilon + \rho\left\|h_{T_0}\right\|_2 + 2\rho e_0$$

As T_0 and T_1 are disjoint and h_{T_0} and h_{T_1} are both s sparse, we can say that

$$\left\|h_{T_0}\right\|_2 \le \left\|h_{T_0} + h_{T_1}\right\|_2 = \left\|h_{T_0\cup T_1}\right\|_2$$

Using this in the above inequality, we get

$$\left\|h_{T_0\cup T_1}\right\|_2 \le \alpha\epsilon + \rho\left\|h_{T_0\cup T_1}\right\|_2 + 2\rho e_0$$

On simplifying, we get

$$\left\|h_{T_0\cup T_1}\right\|_2 \le (1-\rho)^{-1}\left(\alpha\epsilon + 2\rho e_0\right) \tag{23}$$

1.15

We can write h as

$$h = h_{T_0\cup T_1} + h_{(T_0\cup T_1)^c}$$

Hence, by triangle inequality, we have

$$\left\|h\right\|_2 \le \left\|h_{T_0\cup T_1}\right\|_2 + \left\|h_{(T_0\cup T_1)^c}\right\|_2$$

But from (13), we know that

$$\left\|h_{(T_0\cup T_1)^c}\right\|_2 \le \left\|h_{T_0}\right\|_2 + 2e_0 \le \left\|h_{T_0\cup T_1}\right\| + 2e_0$$

Combining the above two inequalities, we get

$$\left\|h\right\|_2 \le 2\left\|h_{T_0}\right\|_2 + 2e_0 \tag{24}$$

Now combining (24) with (23) we get

$$\left\|h\right\|_2 \le 2\left\|h_{T_0}\right\|_2 + 2e_0 \le 2(1-\rho)^{-1}\big(\alpha\epsilon + (1+\rho)e_0\big) \tag{25}$$

1.16

We know that

$$\left\|h_{T_0}\right\|_1 \le \rho\left\|h_{T_0^c}\right\|_1$$

Adding $\left\|h_{T_0^c}\right\|_1$ on both sides, we get

$$\left\|h\right\|_1 \le (1+\rho)\left\|h_{T_0^c}\right\|_1 \tag{26}$$

Using the result

$$\left\|h_{T_0^c}\right\|_1 \le 2(1-\rho)^{-1}\left\|x_{T_0^c}\right\|_1$$

in (26), we get

$$\left\|h\right\|_1 \le 2(1+\rho)(1-\rho)^{-1}\left\|x_{T_0^c}\right\|_1 \tag{27}$$

Question 3

3.a

The oracular solution will be nothing but the solution of the linear equation

$$\boldsymbol{y} = \boldsymbol{\Phi_S x_S}$$

which is nothing but

$$\tilde{\boldsymbol{x}} = \boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\boldsymbol{y}$$

3.b

$$\begin{aligned}
\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2 &= \left\|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\boldsymbol{y} - \boldsymbol{x}\right\|_2 \\
&= \left\|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}(\boldsymbol{\Phi x} - \boldsymbol{\eta}) - \boldsymbol{x}\right\|_2 \\
&= \left\|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\boldsymbol{\eta}\right\|_2 \leq \left\|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\right\|_2 \|\boldsymbol{\eta}\|_2
\end{aligned}$$

3.c

We know that if λ_i are the non-zero singular values of $\boldsymbol{\Phi}$, then the singular values of $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}$ will be λ_i^{-1}. Also, if the eigen values of $\boldsymbol{\Phi}$ are denoted as e_i, then we know $\lambda_i = \sqrt{e_i}$.
As $\boldsymbol{\Phi}$ satisfies restricted isometry property of order $2k$, we have

$$(1 - \delta_{2k}) \|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{\Phi x}\|_2^2 \leq (1 + \delta_{2k}) \|\boldsymbol{x}\|_2^2$$

Re-writing $\|A\|_2^2 = A^T A$, we can say that

$$1 - \delta_{2k} \leq e_i \leq 1 + \delta_{2k}$$

which is equivalent to

$$\frac{1}{\sqrt{1 + \delta_{2k}}} \leq \frac{1}{\lambda_i} \leq \frac{1}{\sqrt{1 - \delta_{2k}}}$$

Hence we can say that the largest singular value of $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}$ lies between $\frac{1}{\sqrt{1+\delta_{2k}}}$ and $\frac{1}{\sqrt{1-\delta_{2k}}}$

3.d

From Noisy recovery theorem, we have

$$\|x^* - x\|_2 \leq C_0 k^{-1/2} \|x - \tilde{x}\|_1 + C_1 \epsilon$$

We know that for a $2k$ sparse vector $x - \tilde{x}$, we have

$$\|x - \tilde{x}\|_1 \leq \sqrt{2k} \|x - \tilde{x}\|_2$$

Hence, we have

$$\|x^* - x\|_2 \leq \sqrt{2} C_0 \|x - \tilde{x}\|_2 + C_1 \epsilon$$

But we have

$$\|x - \tilde{x}\|_2 \leq \frac{\epsilon}{\sqrt{1-\delta_{2k}}}$$

Hence, we will have

$$\|x^* - x\|_2 \leq C_2 \frac{\epsilon}{\sqrt{1-\delta_{2k}}}$$

where

$$C_2 = C_0\sqrt{2} + C_1\sqrt{1-\delta_{2k}}$$

which is a constant factor.

Question 5

- Title: Boolean Compressed Sensing and Noisy Group Testing
- Link: https://arxiv.org/pdf/0907.1061.pdf
- Objective Function:
  Here, we try to maximize the likelihood of decoder which in essense minimizes the error probability. For N items and a defective set of size K, the decoder goes through all $\binom{N}{K}$ possible sets of size K and chooses the set that is most likely. If we denote the tests' outcomes as Y^T, we choose ω^* for which

  $$p(Y^T|\boldsymbol{X}_{S_{\omega^*}}) > p(Y^T|\boldsymbol{X}_{S_\omega}); \quad \forall \omega \neq \omega^*$$

  Here

  - N is the total number of items, k is the known number of defectives (or positive items), p denotes the probability that an item is part of a given test, and T is the total number of tests
  - Codewords: For the j-th term, X_j^T is a binary vector $\in \{0,1\}^T$, with the t-th entry $X_j(t) = 1$ if the j-th item is pooled in test t, and 0 otherwise. Following an information theoretic convention, we call it the j-th codeword. The observation vector Y^T is a binary vector of length T, with entries equal to 1 for the tests with positive outcome. Similarly $Y(t)$ denotes the t-th component of the vector Y^T.
  - $\boldsymbol{X} \in \{0,1\}^{N \times T}$ is the measurement matrix, or the codebook, which is a collection of N codewords defining the pool design, i.e., the assignment of items to tests.

    $$\boldsymbol{X} = [X_1^T; X_2^T; \ldots; X_N^T]$$

  - Given a subset $S \subset \{1, 2, \ldots N\}$ with cardinality $|S|$, the matrix $\boldsymbol{X}_S$ is an $|S| \times T$ matrix formed from the rows indexed by S. In other words, $\boldsymbol{X}_S$ denotes the codewords (each of length T) corresponding to the items in S. Similarly, X_S denotes a vector, whose components are restricted to the set of components indexed by S. Thus, X_S is a column of the matrix $\boldsymbol{X}_S$. When indexing by test is needed, $X_S(t)$ is used to specifically denote the t-th column of the matrix $\boldsymbol{X}_S$, and $X_j(t)$ is the t-th component of the vector X_j^T.

- Index the different sets of items of size K as S_ω with index ω. Since there are N items in total, there are $\binom{N}{K}$ such sets, hence

$$\omega \in \mathcal{I} = \left\{1, 2, \ldots, \binom{N}{K}\right\}$$

- Differences between Tapestry and the proposed method:

  1. Tapestry models the task from a linear algebra perspective whereas the porposed method models the task from an Information Theory perspective making use of channel coding concepts and probability and random processes.
  2. Tapestry solves a constrained convex optimization problem whereas the proposed method solves an un-constrained optimization problem.
  3. In Tapestry pooling, we model noise as a Poisson random variable whereas in the proposed method they model it as a Bernoulli random variable.