Augmented Lagrangian Tutorial

Canonical probem

$$\min_x \quad f(x)$$

$$\text{s.t.} \quad c(x) = 0$$

$$h(x) \le 0$$

our QP

$$\min_x \quad \frac{1}{2} x^T Q x + g^T x$$

$$\text{s.t.} \quad Ax - b = 0$$

$$Cx - d \le 0$$

Lagrangian

$$L(x, \lambda, \mu) = f(x) + \lambda^T c(x) + \mu^T h(x)$$

KKT conditions

$$\nabla_x L = \nabla_x f + \left(\frac{\partial c}{\partial x}\right)^T \lambda + \left(\frac{\partial h}{\partial x}\right)^T \mu = 0 \qquad \text{Stationarity}$$

$$Ax - b = 0$$
$$Cx - d \le 0 \qquad \text{primal feasibility}$$

any x, λ, μ that satisfies these is a globally optimal (this is only true of convex problems)

$$\mu \ge 0 \qquad \text{dual feasability}$$

$$\mu_i h_i(x) = 0 \qquad \text{Complementarity}$$

Augmented Lagrangian

$$L_\rho(x, \lambda, \mu, \rho) = L(x, \lambda, \mu) + \frac{\rho}{2} c(x)^T c(x) + \frac{1}{2} h(x)^T I_\rho h(x)$$

I_ρ is a diagonal matrix

if $\quad h_i(x) < 0 \;\&\&\; \mu_i = 0$

$$I_\rho[i,i] = 0$$

else

$$I_\rho[i,i] = \rho$$

this is how we make sure to only penalize active inequality constraints

AL alg

init $x = 0$, $\lambda = 0$, $\mu = 0$, $\rho = 1$

hold these constant

Loop

1. Solve $\min_x L_\rho(x, \lambda, \mu, \rho)$

   w/ Newton's method, update x w/ the solution

2. update dual variables

   $$\lambda = \lambda + \rho c(x)$$

   $$\mu = \max(0, \mu + \rho \cdot h(x))$$

   element-wise max (max.() in Julia)

3. Update penalty

   $$\rho = \rho \cdot \phi$$

4. Check convergence (KKT conditions)