Divide-and-Conquer algorithsm for matrix multiplication

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \quad B = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix} \quad C = A \times B = \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix}$$

Formulas for $C^{11}, C^{12}, C^{21}, C^{22}$:

$$\begin{aligned} C^{11} &= A^{11}B^{11} + A^{12}B^{21} \qquad & C^{12} &= A^{11}B^{12} + A^{12}B^{22} \\ C^{21} &= A^{21}B^{11} + A^{22}B^{21} \qquad & C^{22} &= A^{21}B^{12} + A^{22}B^{22} \end{aligned}$$

The First Attempt Straightforward from the formulas above (assuming that n is a power of 2):

MMult(A, B, n)

If $n = 1$ Output $A \times B$
Else
    Compute $A^{11}, B^{11}, \ldots, A^{22}, B^{22}$ % by computing $m = n/2$
    $X_1 \leftarrow MMult(A^{11}, B^{11}, n/2)$
    $X_2 \leftarrow MMult(A^{12}, B^{21}, n/2)$
    $X_3 \leftarrow MMult(A^{11}, B^{12}, n/2)$
    $X_4 \leftarrow MMult(A^{12}, B^{22}, n/2)$
    $X_5 \leftarrow MMult(A^{21}, B^{11}, n/2)$
    $X_6 \leftarrow MMult(A^{22}, B^{21}, n/2)$
    $X_7 \leftarrow MMult(A^{21}, B^{12}, n/2)$
    $X_8 \leftarrow MMult(A^{22}, B^{22}, n/2)$
    $C^{11} \leftarrow X_1 + X_2$
    $C^{12} \leftarrow X_3 + X_4$
    $C^{21} \leftarrow X_5 + X_6$
    $C^{22} \leftarrow X_7 + X_8$
    Output C
End If

Strassen's algorithm is based on the following observation:

$$C^{11} = P_5 + P_4 - P_2 + P_6 \qquad C^{12} = P_1 + P_2$$
$$C^{21} = P_3 + P_4 \qquad C^{22} = P_1 + P_5 - P_3 - P_7$$

where

$$\begin{aligned}
P_1 &= A^{11}(B^{12} - B^{22}) \\
P_2 &= (A^{11} + A^{12})B^{22} \\
P_3 &= (A^{21} + A^{22})B^{11} \\
P_4 &= A^{22}(B^{21} - B^{11}) \\
P_5 &= (A^{11} + A^{22})(B^{11} + B^{22}) \\
P_6 &= (A^{12} - A^{22})(B^{21} + B^{22}) \\
P_7 &= (A^{11} - A^{21})(B^{11} + B^{12})
\end{aligned}$$

Exercise Verify that $C^{11}, \ldots, C^{22}$ can be computed as above.

The above formulas can be used to compute $A \times B$ recursively as follows:

Strassen(A, B)

If $n = 1$ Output $A \times B$
Else
    Compute $A^{11}, B^{11}, \ldots, A^{22}, B^{22}$ % by computing $m = n/2$
    $P_1 \leftarrow Strassen(A^{11}, B^{12} - B^{22})$
    $P_2 \leftarrow Strassen(A^{11} + A^{12}, B^{22})$
    $P_3 \leftarrow Strassen(A^{21} + A^{22}, B^{11})$
    $P_4 \leftarrow Strassen(A^{22}, B^{21} - B^{11})$
    $P_5 \leftarrow Strassen(A^{11} + A^{22}, B^{11} + B^{22})$
    $P_6 \leftarrow Strassen(A^{12} - A^{22}, B^{21} + B^{22})$
    $P_7 \leftarrow Strassen(A^{11} - A^{21}, B^{11} + B^{12})$
    $C^{11} \leftarrow P_5 + P_4 - P_2 + P_6$
    $C^{12} \leftarrow P_1 + P_2$
    $C^{21} \leftarrow P_3 + P_4$
    $C^{22} \leftarrow P_1 + P_5 - P_3 - P_7$
    Output C
End If