Solutions

Tutorial Sheet-13

1. Here the $n = 12$ months are the pigeonholes, and $k + 1 = 3$ so $k = 2$. Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.
(**Generalized Pigeonhole Principle:** If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.)

2. Draw two lines between the opposite sides of S which partitions S into four sub squares each whose sides have length one inch. By the Pigeonhole Principle, two of the points lie in one of the sub squares. The diagonal of each sub square is $\sqrt{2}$ inches, so the distance between the two points is less than $\sqrt{2}$ inches.

3. Here the pigeonholes are the five sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5\}$. Thus any choice of six elements (pigeons) of S will guarantee that two of the numbers add up to ten.

4. If department contains 13 professors, then two of the professors (pigeons) were born in the same month (pigeonholes).

5. $P(n) : 1 + 3 + 5 + \cdot \quad \cdot \quad \cdot + (2n - 1) = n^2$

Observe that $P(n)$ is true for $n = 1$; namely,
$P(1) = 1^2$
Assuming $P(k)$ is true, we add $2k + 1$ to both sides of $P(k)$, obtaining
$1 + 3 + 5 + \cdot \quad \cdot \quad \cdot + (2k - 1) + (2k + 1) - k^2 + (2k + 1) = (k + 1)^2$
which is $P(k + 1)$. In other words, $P(k + 1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.

6. Let a_n denote the number of bit strings of length n that do not have two consecutive 0s.

To obtain a recurrence relation for $\{a_n\}$, note that by the sum rule, the number of bit strings of length n that do not have two consecutive 0s equals the number of such bit strings ending with a 0 plus the number of such bit strings ending with a 1.We will assume that $n \geq 3$, so that the bit string has at least three bits.

The bit strings of length n ending with 1 that do not have two consecutive 0s are precisely the bit strings of length $n - 1$ with no two consecutive 0s with a 1 added at the end. Consequently, there are a_n-1 such bit strings.

Bit strings of length n ending with a 0 that do not have two consecutive 0s must have 1 as their $(n - 1)$st bit; otherwise they would end with a pair of 0s. It follows that the bit strings of length n ending with a 0 that have no two consecutive 0s are precisely the bit strings of length $n - 2$ with no two consecutive 0s with 10 added at the end. Consequently, there are a_n-2 such bit strings.

We conclude, that
$a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.
The initial conditions are $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 11.

7. Basis Step: $P(0)$ is true since $1 = 2^1 - 1$.

Inductive Step: Assuming $P(k)$ is true, we add $2k+1$ to both sides of $P(k)$, obtaining
$1 + 2 + 2^2 + 2^3 + \cdot \quad \cdot \quad \cdot +2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1$
which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the principle of induction, $P(n)$ is true for all n.

8. Z is the set of all positive or negative integers. If a and b are such that $a \neq b$ and $a^2 = b^2$, then the statement is disproved.
Choosing any integer for a and then choosing $b=-a$ will accomplish this.
Let $a=4$ and $b=-4$.
In this case, $a^2 = 16$ and $b^2 = 16$ and so we have found an example where $a^2 = b^2$ but $a \neq b$ and thus disproving the statement.

9. There are three cases:
Case 1: if $n=3p$, then $n^3 = 27p^3$, which is a multiple of 3.

Case 2: if $n=3p+1$,
Then $n^3=27p^3 + 27p^2 + 9p+1$ which is 1 more than a multiple of 9.
(for instance: if $n=4$, $n^3 = 64 = 9*7+1$)

Case 3: if $n=3p-1$
Then $n^3=27p^3 - 27p^2 + 9p-1$ (for instance: if $n=5$, $n^3 = 125 = 9*14-1$)