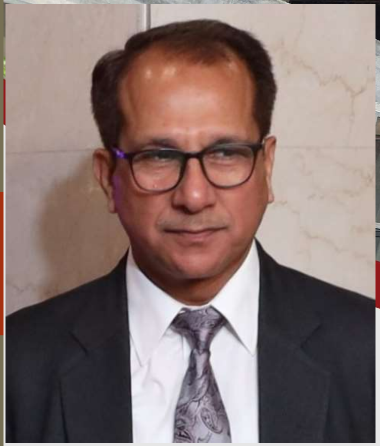
Representation of Forces in Cartesian Coordinates



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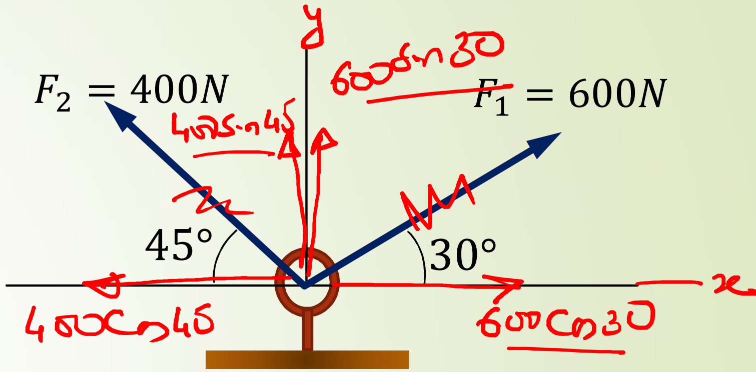
Problem: The link shown in the figure is subjected to two forces ***F1*** and ***F2***. Determine the magnitude and direction of the resultant force.



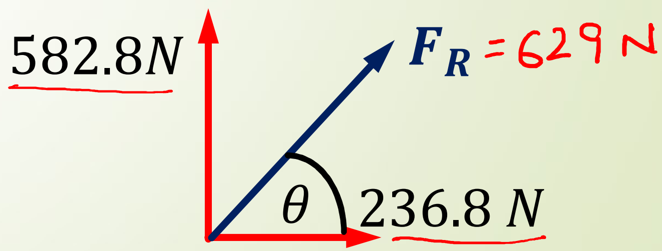
Solution: Resolve the forces ***F1*** and ***F2*** into their horizontal and vertical components and sum these algebraically.

$$F_{Rx} = \Sigma F_x = 600\cos 30° - 400\cos 45° \quad = 236.8\ N$$

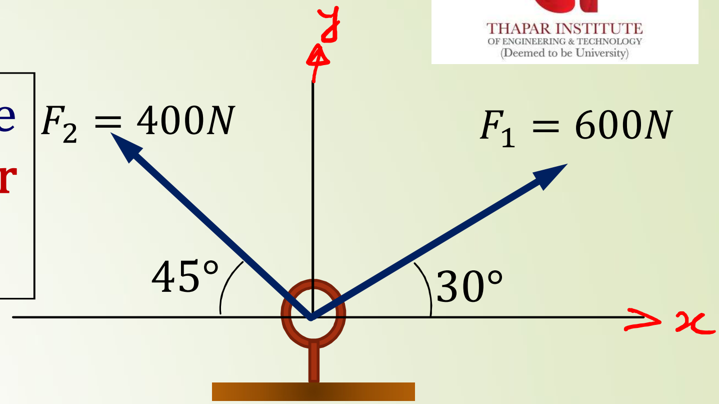
$$F_{Ry} = \Sigma F_y = 600\sin 30° + 400\sin 45° \quad = 582.8\ N$$

The resultant force ***FR*** will have a magnitude of $\boldsymbol{F_R} = \sqrt{(236.8)^2 + (582.8)^2} \quad = \quad \mathbf{629\ N}$

$$\tan\theta = \frac{F_{Ry}}{F_{Rx}} \rightarrow \boldsymbol{\theta} = \tan^{-1}\left(\frac{582.8}{236.8}\right) = \mathbf{67.9°}$$



Problem: Also determine the magnitude and direction of the resultant of the two forces ***F1*** and ***F2***, using the **Cartesian vector notations.**



Solution: Each force is first expressed as a Cartesian vector.

$$\overrightarrow{\boldsymbol{F}_1} = \{600\cos 30^0\boldsymbol{i} + 600\sin 30^0\boldsymbol{j}\}\ N$$

$$\overrightarrow{\boldsymbol{F}_2} = \{-400\cos 45^0\boldsymbol{i} + 400\sin 45^0\boldsymbol{j}\}\ N$$

$$\overrightarrow{\boldsymbol{F}_R} = \Sigma\overrightarrow{\boldsymbol{F}} = \overrightarrow{\boldsymbol{F}_1} + \overrightarrow{\boldsymbol{F}_2}$$

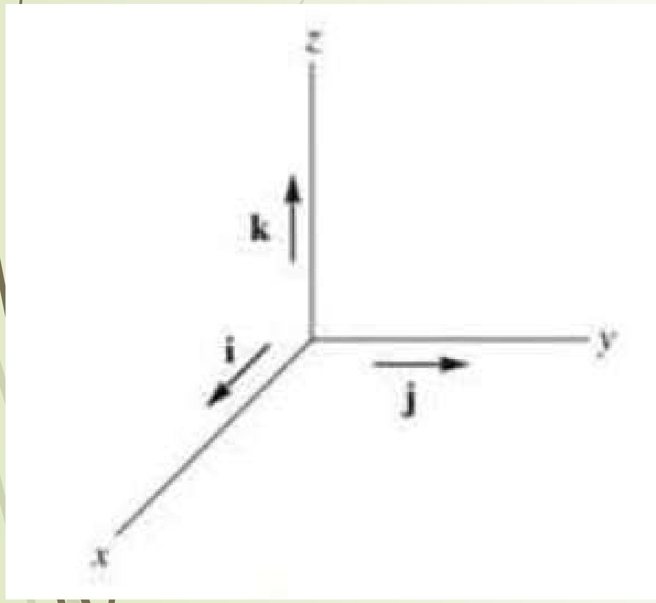
$$\overrightarrow{\boldsymbol{F}_R} = (600\cos 30^0 - 400\cos 45^0)\boldsymbol{i} + (600\sin 30^0 + 400\sin 45^0)\boldsymbol{j}$$

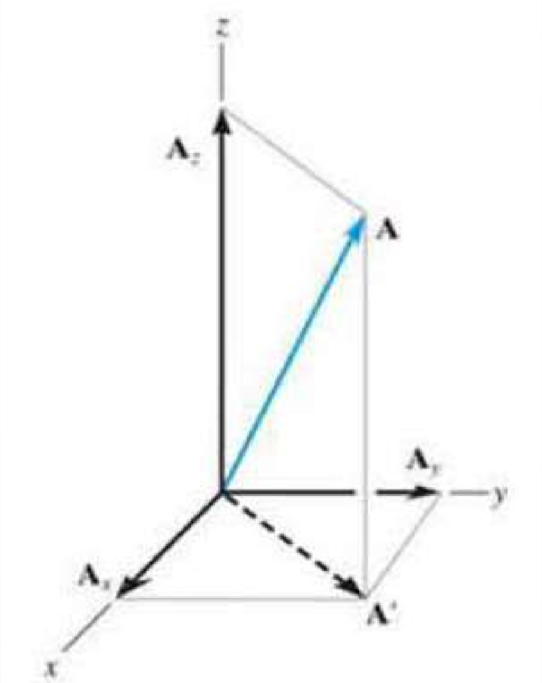
$$\overrightarrow{\boldsymbol{F}_R} = \{\overset{F_x}{236.8}\boldsymbol{i} + \overset{F_y}{582.8}\boldsymbol{j}\}\ N$$

$$F_R = \sqrt{(236.8)^2 + (582.8)^2} = \mathbf{629\ N}$$

Cartesian Vectors

Cartesian Unit Vectors: In three dimensions the set of Cartesian unit vectors, ***i***, ***j***, ***k***, is used to designate the directions of the ***x***, ***y***, ***z*** axes.

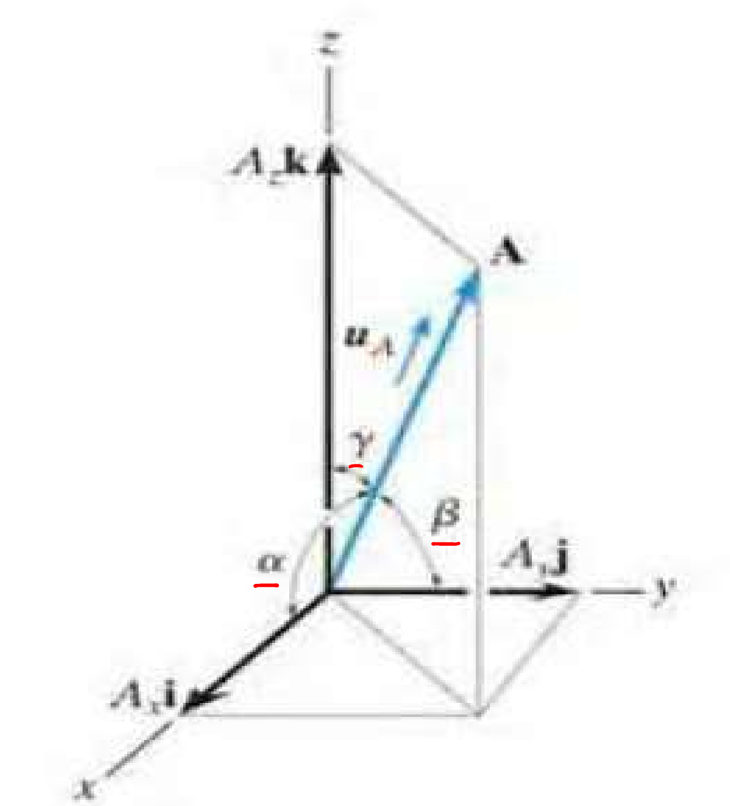




$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

$$A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$$

Cartesian Vectors



$$\mathbf{u}_A = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$$

$$\boldsymbol{u_A} = \frac{\boldsymbol{A}}{A} = \frac{A_x}{A}\boldsymbol{i} + \frac{A_y}{A}\boldsymbol{j} + \frac{A_z}{A}\boldsymbol{k}$$

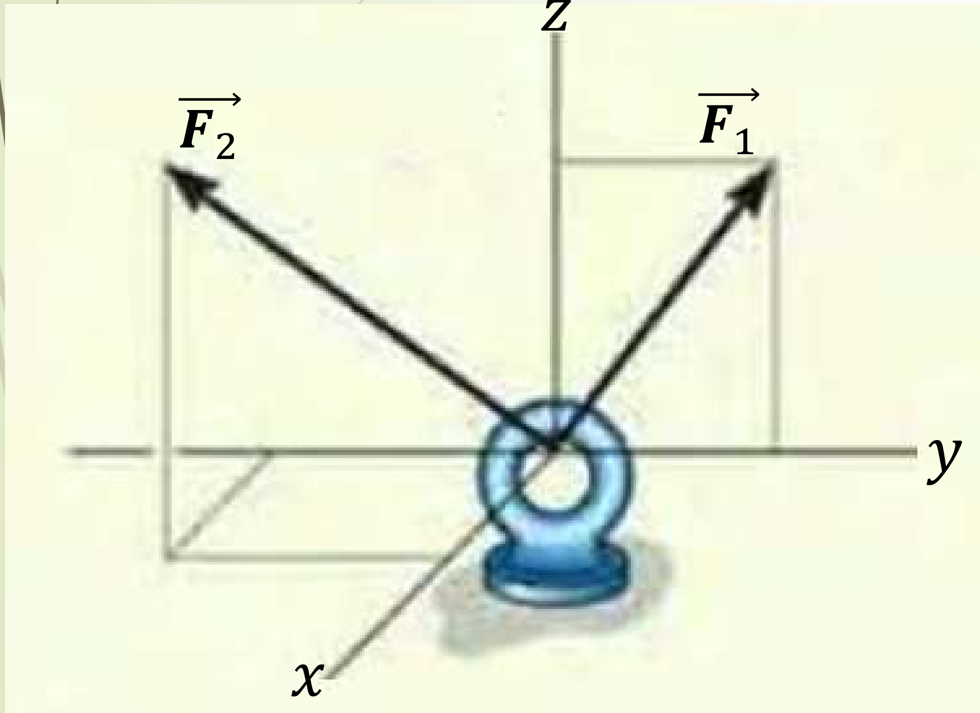
$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

The **Force Vector** is obtained by multiplying magnitude of the force and unit vector.

$$\boldsymbol{A} = A \times \mathbf{u}_A$$

Example: Determine the magnitude and coordinate direction angles of the resultant force acting on the ring, when $\overrightarrow{\boldsymbol{F}_1} = \{60\boldsymbol{j} + 80\boldsymbol{k}\}\ lb$ and $\overrightarrow{\boldsymbol{F}_2} = \{50\boldsymbol{i} - 100\boldsymbol{j} + 100\boldsymbol{k}\}\ lb$.



$$\overrightarrow{\boldsymbol{F}_R} = \Sigma\overrightarrow{\boldsymbol{F}} = \overrightarrow{\boldsymbol{F}_1} + \overrightarrow{\boldsymbol{F}_2}$$

$$\overrightarrow{\boldsymbol{F}_R} = \{60\boldsymbol{j} + 80\boldsymbol{k}\} + \{50\boldsymbol{i} - 100\boldsymbol{j} + 100\boldsymbol{k}\}$$

$$= \{50\boldsymbol{i} - 40\boldsymbol{j} + 180\boldsymbol{k}\}\ lb$$

Magnitude of the resultant force

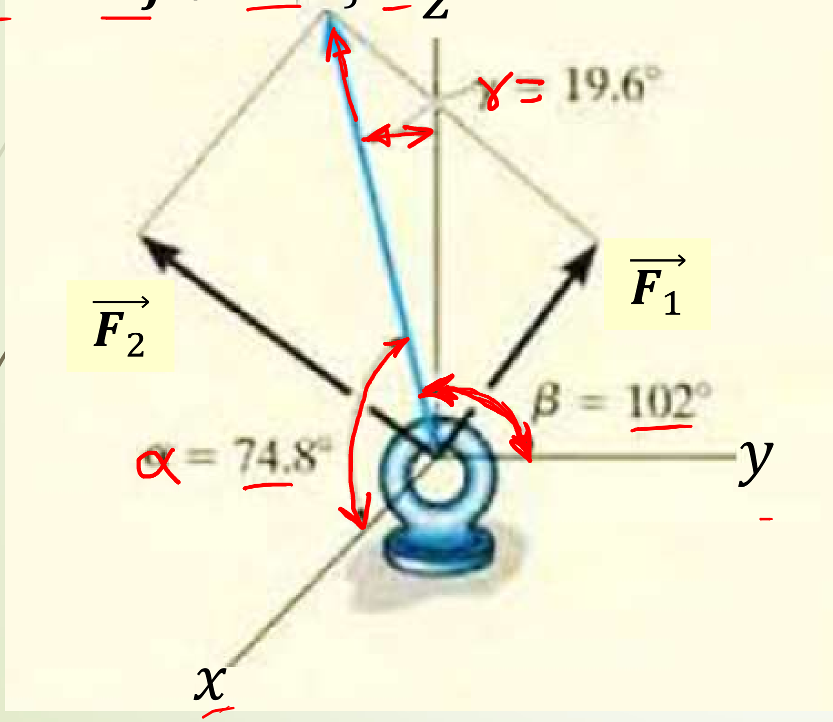
$$F_R = \sqrt{(50)^2 + (-40)^2 + (180)^2} = \mathbf{191}\ \boldsymbol{lb}$$

$$\boldsymbol{u}_{\boldsymbol{F}_R} = \frac{\overrightarrow{\boldsymbol{F}_R}}{F_R} = \frac{50}{191} - \frac{40}{191} + \frac{180}{191}$$

$$= 0.2617\boldsymbol{i} - 0.2094\boldsymbol{j} + 0.9422\boldsymbol{k}$$

To find coordinate direction angles or direction cosines

$\overrightarrow{F_R} = \{50\boldsymbol{i} - 40\boldsymbol{j} + 180\boldsymbol{k}\}\ lb$



$$= 0.2617\boldsymbol{i} - 0.2094\boldsymbol{j} + 0.9422\boldsymbol{k}$$

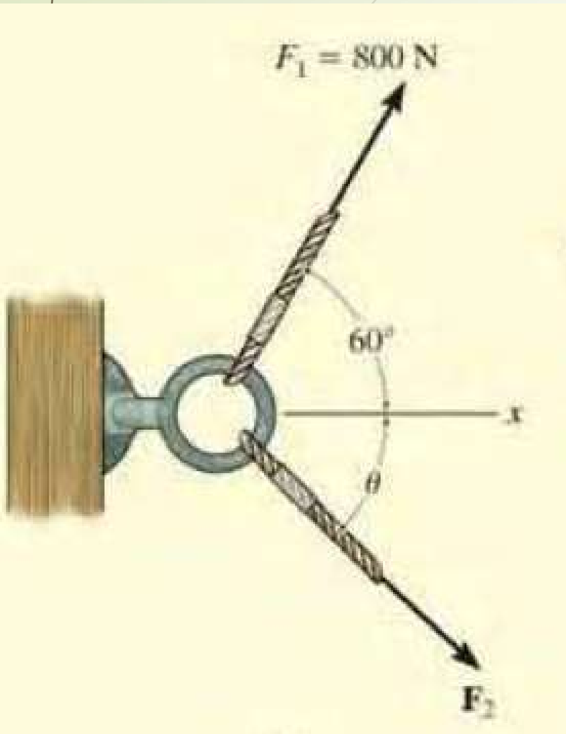
$$\boldsymbol{u}_{\boldsymbol{F}_R} = \ cos \propto i + cos\, \beta\, j + \ cos\, \gamma\, k$$

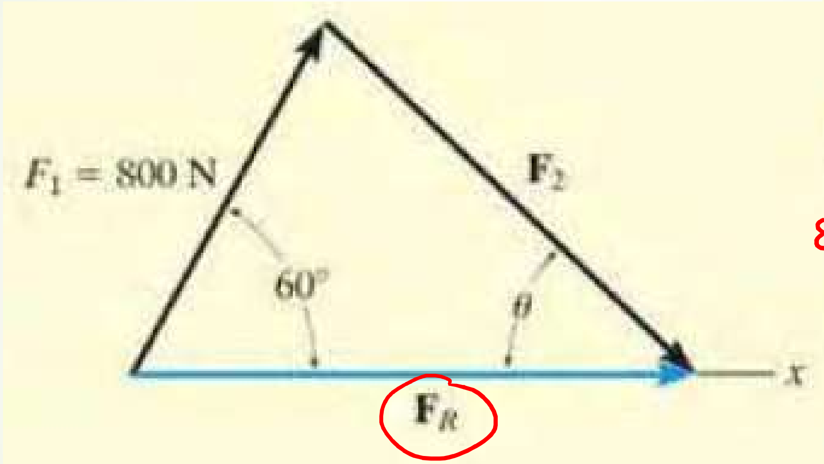
$\cos\alpha = 0.2617 \qquad \alpha = 74.8^o$

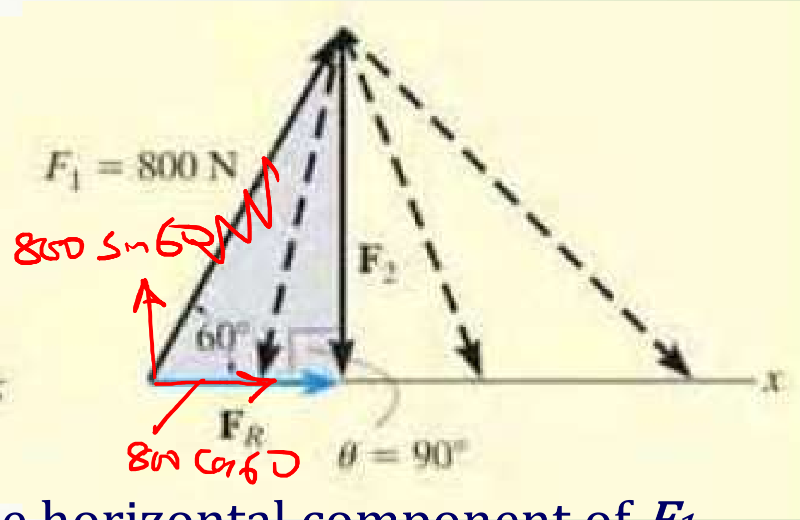
$\cos\beta = -0.2094 \qquad \beta = 102^o$

$\cos\gamma = 0.9422 \qquad \gamma = 19.6^o$

Problem: It is required that the resultant force acting on the eyebolt in the figure be directed along the positive $\boldsymbol{x}$ axes and that $\boldsymbol{F_2}$ has a minimum magnitude. Determine this magnitude, angle $\boldsymbol{\theta}$, and the corresponding resultant force.







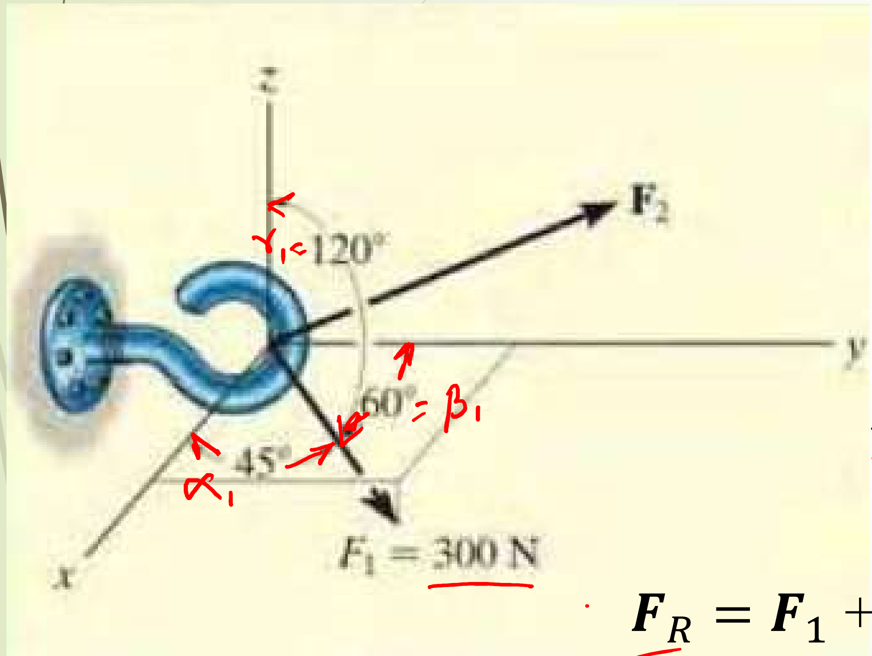
Under this condition, $\boldsymbol{F_R}$ will be the horizontal component of $\boldsymbol{F_1}$.

$$F_R = 800\cos 60 = 400N$$

$\boldsymbol{F_2}$ will be equal to the vertical component of $\boldsymbol{F_1}$.

$$F_2 = 800\sin 60 = 693N$$

Example: Two forces act on the hook. Specify the magnitude of force $\boldsymbol{F_2}$ and its coordinate direction angles, such that the resultant force $\boldsymbol{F_R}$ acts along the positive $\boldsymbol{y}$ axis and has a magnitude of **800N**.



$$\boldsymbol{F}_1 = F_1 \cos\alpha_1 \boldsymbol{i} + F_1 \cos\beta_1 \boldsymbol{j} + F_1 \cos\gamma_1 \boldsymbol{k}$$

$$\boldsymbol{F}_1 = 300 \cos 45° \, \boldsymbol{i} + 300 \cos 60° \, \boldsymbol{j} + 300 \cos 120° \, \boldsymbol{k}$$

$$= \{212.1\boldsymbol{i} + 150\boldsymbol{j} - 150\boldsymbol{k}\} \, N$$

$$\boldsymbol{F}_2 = F_{2x} \, \boldsymbol{i} + F_{2y} \, \boldsymbol{j} + F_{2z} \boldsymbol{k}$$

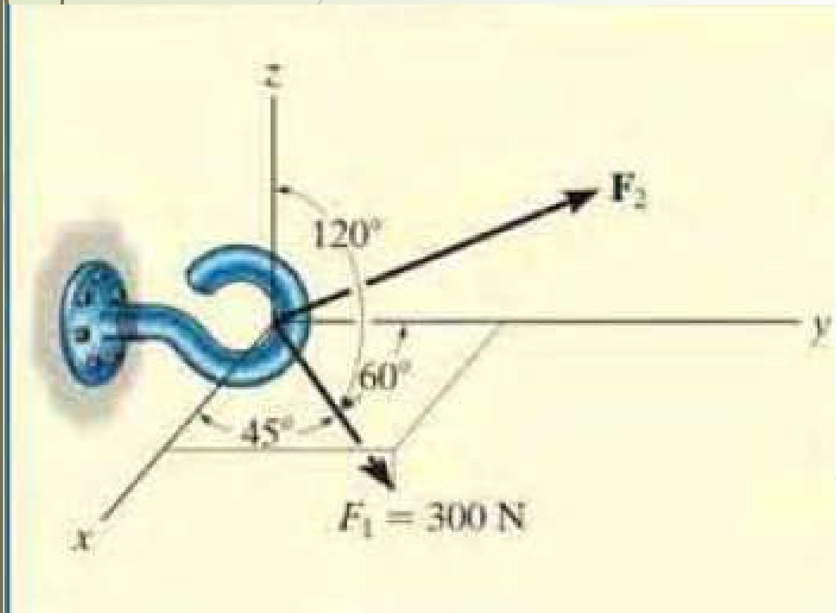
$$\boldsymbol{F}_R = (800 \, N)(+\boldsymbol{j}) = \{800\boldsymbol{j}\} \, N$$

$$\boldsymbol{F}_R = \boldsymbol{F}_1 + \boldsymbol{F}_2$$

$$800\boldsymbol{j} = 212.1\boldsymbol{i} + 150\boldsymbol{j} - 150\boldsymbol{k} + F_{2x} \, \boldsymbol{i} + F_{2y} \, \boldsymbol{j} + F_{2z} \boldsymbol{k}$$

$$800\boldsymbol{j} = (212.1 + F_{2x})\boldsymbol{i} + (150 + F_{2y})\boldsymbol{j} - (150 + F_{2z})\boldsymbol{k}$$

$$800\boldsymbol{j} = (212.1 + F_{2x})\boldsymbol{i} + (150 + F_{2y})\boldsymbol{j} - (150 + F_{2z})\boldsymbol{k}$$

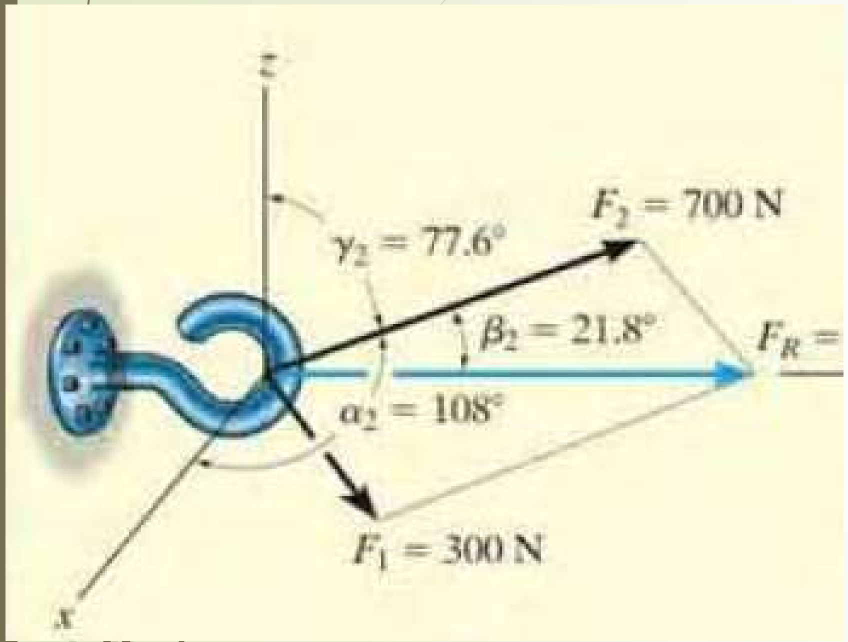


$$212.1 + F_{2x} = 0 \quad \rightarrow \quad F_{2x} = -212.1\ N$$

$$150 + F_{2y} = 800 \quad \rightarrow \quad F_{2y} = 650\ N$$

$$-150 + F_{2z} = 0 \quad \rightarrow \quad F_{2z} = 150\ N$$

$$F_2 = \sqrt{(-212.1)^2 + (650)^2 + (150)^2} = \mathbf{700\ N}$$

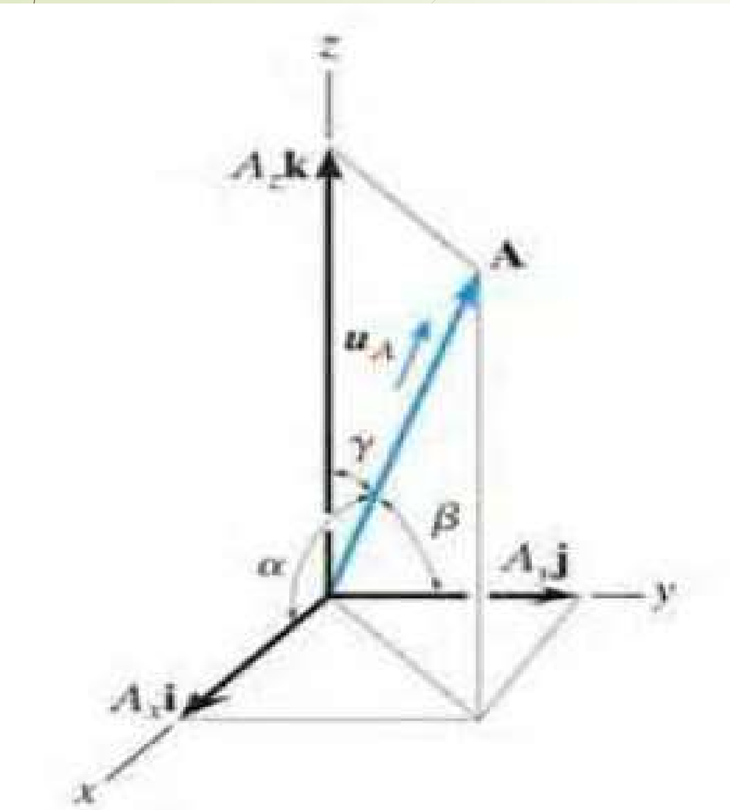


$$\cos\alpha_2 = \frac{-212.1}{700}; \qquad \alpha_2 = 108^o$$

$$\cos\beta_2 = \frac{650}{700}; \qquad \beta_2 = 21.8^o$$

$$\cos\gamma_2 = \frac{150}{700}; \qquad \gamma_2 = 77.6^o$$

Cartesian Vectors



$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

$$A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$$

$$\boldsymbol{u_A} = \frac{\boldsymbol{A}}{A} = \frac{A_x}{A}\boldsymbol{i} + \frac{A_y}{A}\boldsymbol{j} + \frac{A_z}{A}\boldsymbol{k}$$

$$\mathbf{u}_A = \cos\alpha\ \boldsymbol{i} + \cos\beta\ \boldsymbol{j} + \cos\gamma\ \boldsymbol{k}$$

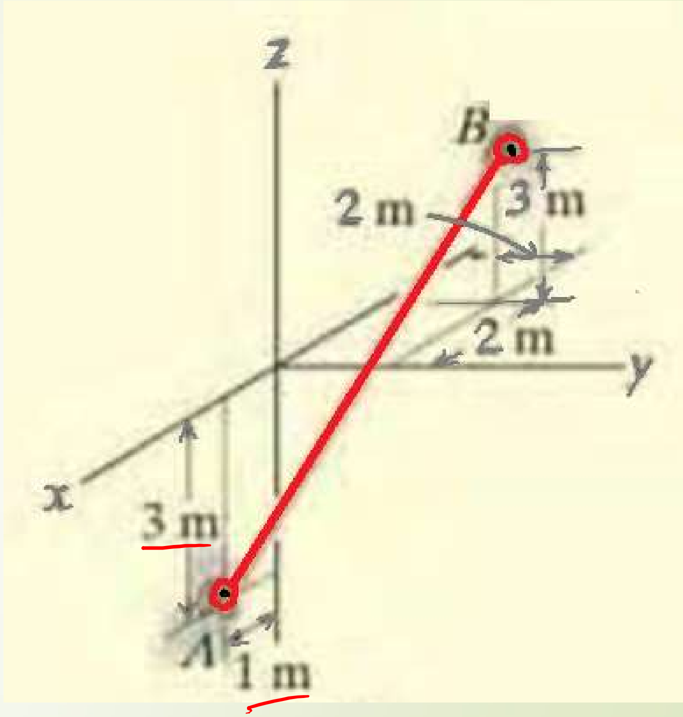
$$\cos\alpha = \frac{A_x}{A}; \qquad \cos\beta = \frac{A_y}{A}; \qquad \cos\gamma = \frac{A_z}{A}$$

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

The **Force Vector** is obtained by multiplying magnitude of the force and unit vector.

$$\boldsymbol{A} = A \times \mathbf{u}_A$$

Example: An elastic rubber band is attached to point ***A*** and ***B*** as shown. Determine its length and its direction measured from ***A*** towards ***B***.



Solution: The coordinates of points A(1, 0, -3) and B(-2, 2, 3),

Position vector from ***A*** to ***B***,

$$\bar{\boldsymbol{r}} = \{(-2-1)\boldsymbol{i} + (2-0)\boldsymbol{j} + (3-(-3))\boldsymbol{k}\};$$

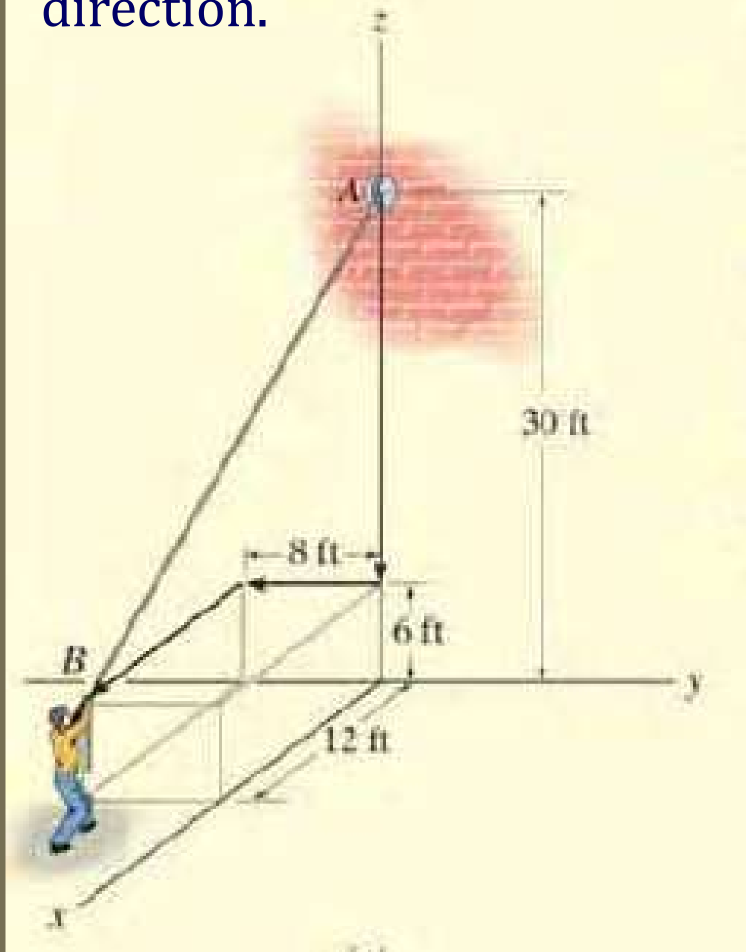
$$= \{-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}\}m$$

$$r = \sqrt{(-3)^2 + (2)^2 + (6)^2} = \boldsymbol{7m}$$

$$\overline{\boldsymbol{u}_{AB}} = \frac{\bar{\boldsymbol{r}}}{r} = -\frac{3}{7}\boldsymbol{i} + \frac{2}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k}$$

$\cos\alpha = \frac{-3}{7};\quad \alpha = 115^o \qquad \cos\beta = \frac{2}{7};\quad \beta = 73.4^o \qquad \cos\gamma = \frac{6}{7};\quad \gamma = 31^o$

Example: The man shown in the figure pulls on the cord ***AB*** with a force of 70 *lb*. Represent this force acting on the support ***A*** as a Cartesian vector and determine its direction.



Solution: The coordinates of points A(0, 0, 30) and B(12, -8, 6);

Position vector from ***B*** to ***A,***

$$\bar{\boldsymbol{r}} = \{(12-0)\boldsymbol{i} + (-8-0)\boldsymbol{j} + \big(6-(30)\big)\boldsymbol{k}\};$$

$$= \{12\boldsymbol{i} - 8\boldsymbol{j} - 24\boldsymbol{k}\}\ ft$$

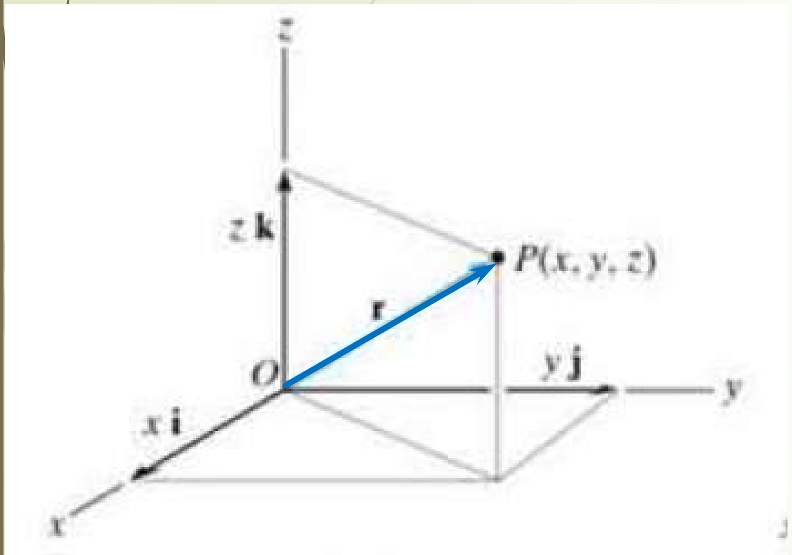
$$r = \sqrt{(12)^2 + (-8)^2 + (-24)^2} = \boldsymbol{28\ ft}$$

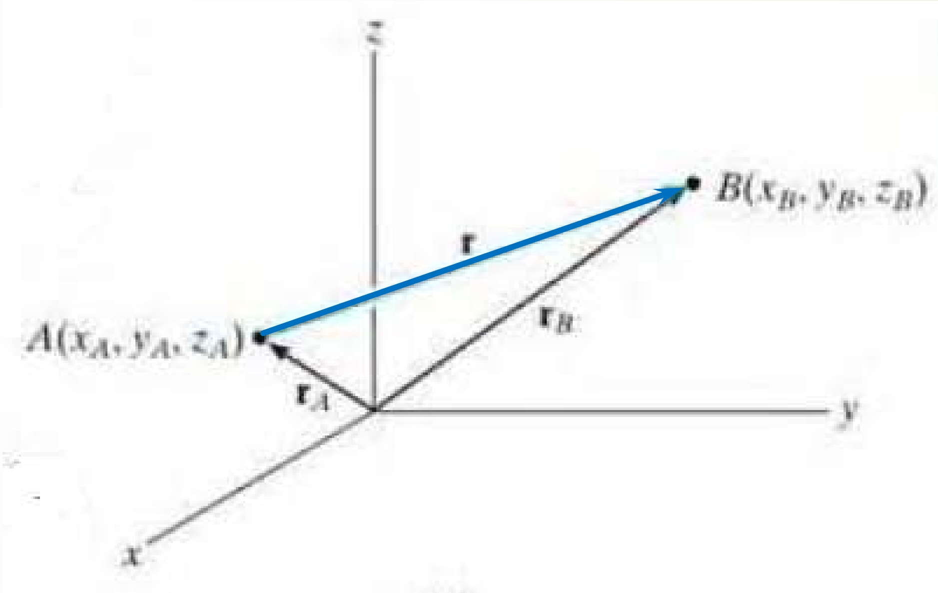
$$\bar{\boldsymbol{u}} = \frac{\bar{\boldsymbol{r}}}{r} = \frac{12}{28}\boldsymbol{i} - \frac{8}{28}\boldsymbol{j} - \frac{24}{28}\boldsymbol{k}$$

$$\bar{\boldsymbol{F}} = F\bar{\boldsymbol{u}} = 70\left(\frac{12}{28}\boldsymbol{i} - \frac{8}{28}\boldsymbol{j} - \frac{24}{28}\boldsymbol{k}\right) = \{30\boldsymbol{i} - 20\boldsymbol{j} - 60\boldsymbol{k}\}\ lbs$$

Position vector

A position vector '$\boldsymbol{r}$' is defined as a fixed vector which locates a point in space relative to another point

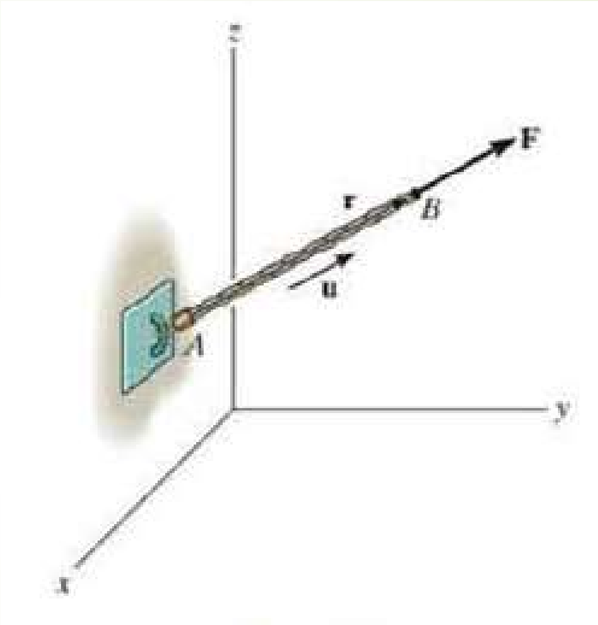




$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k};$$

$$\boldsymbol{r} = \boldsymbol{r}_B - \boldsymbol{r}_A = (x_B\boldsymbol{i} + y_B\boldsymbol{j} + z_B\boldsymbol{k}) - (x_A\boldsymbol{i} + y_A\boldsymbol{j} + z_A\boldsymbol{k});$$

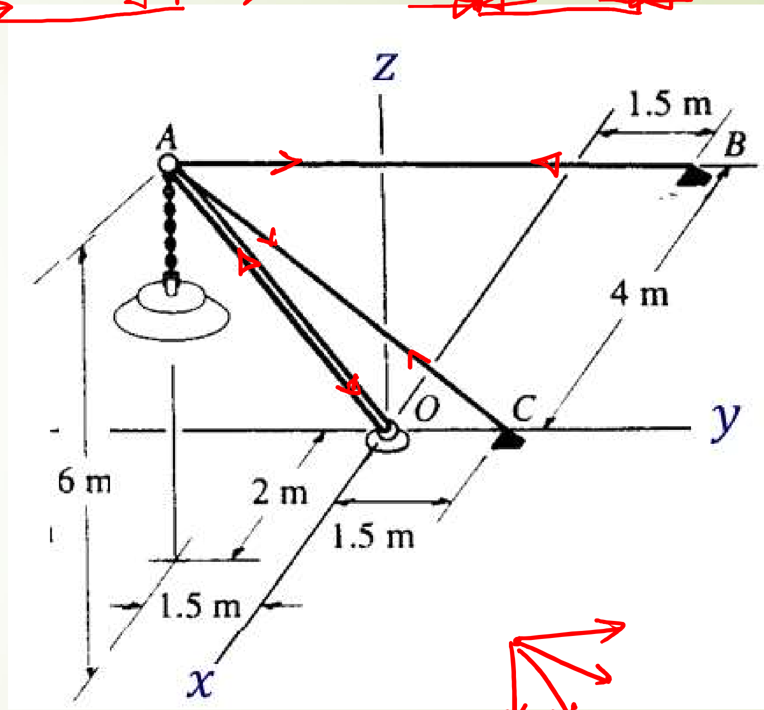
Force Vector Directed Along a Line



$$\boldsymbol{F} = F\boldsymbol{u} = F\left(\frac{\boldsymbol{r}}{r}\right) = F\left(\frac{(x_B - x_A)\boldsymbol{i} + (y_B - y_A)\boldsymbol{j} + (z_B - z_A)\boldsymbol{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Example: The lamp has a mass of 15 kg and is supported by a pole *AO* and cables *AB* and AC. If the force in the pole acts along its axis, determine the forces in *AO*, *AC* and *AB* for equilibrium.

3d $O(0,0,0)$, $A(2,-1.5,6)$, $B(-4,1.5,0)$, $C(0,1.5,0)$

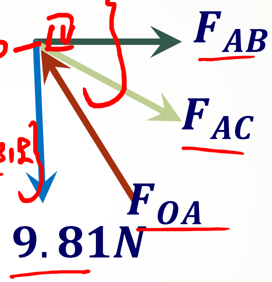


$\Sigma F_x = 0.3077 F_{OA} - 0.6667 F_{AB} - 0.2857 F_{AC} = 0$ — I

$\Sigma F_y = -0.2308 F_{OA} + 0.3333 F_{AB} + 0.4286 F_{AC} = 0$ — II

$\Sigma F_z = 0.9231 F_{OA} - 0.6667 F_{AB} - 0.8517 F_{AC} - 147.15 = 0$ — III

FBD at A



Force vector F_{OA},

$$= F_{OA}\frac{(2i - 1.5j + 6k)}{\sqrt{2^2 + 1.5^2 + 6^2}} = F_{AO}\{0.3077i - 0.2308j + 0.9231k\}$$

$F_{OA} = 319$ N
$F_{AB} = 110$ N
$F_{AC} = 85.8$ N

Force vector F_{AB},

$$= F_{AB}\frac{(-6i + 3j - 6k)}{\sqrt{6^2 + 3^2 + 6^2}} = F_{AB}\{-0.6667i + 0.3333j - 0.6667k$$

Force vector F_{AC},

$$= F_{AC}\frac{(-2i + 3j - 6k)}{\sqrt{2^2 + 3^2 - 6^2}} = F_{AC}\{-0.2857i + 0.4286j - 0.8517k)$$

$\bar{W} = \{-15 \times 9.81k\}$ N

$\Sigma F = \bar{F}_{OA} + \bar{F}_{AB} + \bar{F}_{AC} + \bar{W} = 0$

$\overline{F_{AO}} = F_{AO}\{0.3077i - 2308j + 0.9231k\}$, $\overline{F_{AB}} = F_{AB}\{-0.6667i + 0.3333j - 0.6667k\}$

$\overline{F_{AC}} = F_{AC}\{-0.2857i + 0.4286j - 0.8571k\}$ and $\overline{W} = \{-147.15k\}N$

$$\Sigma F = \overline{F_{AO}} + \overline{F_{AB}} + \overline{F_{AC}} + \overline{W} = 0$$

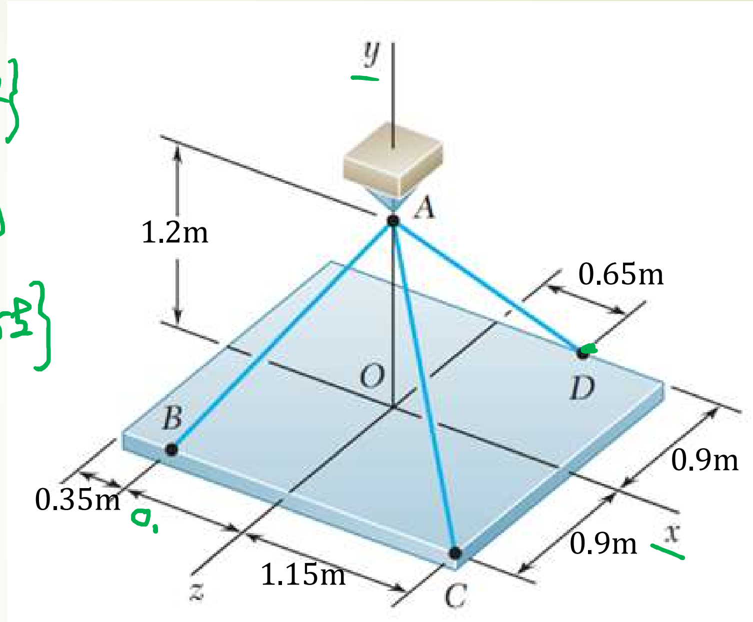
$$\Sigma F_x = 0 \quad 0.3077 F_{AO} - 0.6667 F_{AB} - 0.2857 F_{AC} = 0 \quad - \mathrm{I}$$

$$\Sigma F_y = 0 \quad -0.2308 F_{AO} + 0.3333 F_{AB} + 0.4286 F_{AC} = 0 \quad - \mathrm{II}$$

$$\Sigma F_z = 0 \quad 0.9231 F_{AO} - 0.6667 F_{AB} - 0.8571 F_{AC} - 147.15 = 0 \quad - \mathrm{III}$$

$$\boxed{\begin{aligned} F_{AO} &= 319\ N \\ F_{AB} &= 110\ N \\ F_{AC} &= 85.8\ N \end{aligned}}$$

Example: A rectangular plate is supported by three cables. Knowing that the tension in cable AB is 540 N, determine the weight of the plate.



$1.15 - 0.35$

Sol : $A(0, 1.2, 0)$, $B(-0.8, 0, 0.9)$ $C(1.15, 0, 0.9)$ $D(0.65, 0, -0.9)$

$$\bar{F}_{AB} = \underset{540}{F_{AB}} \frac{(-0.8i - 1.2j + 0.9k)}{\sqrt{0.8^2 + 1.2^2 + 0.9^2}} = \{-254.12i - 381.18j + 281.88k\}$$

$$\bar{F}_{AC} = F_{AC} \frac{(1.15i - 1.2j + 0.9k)}{\sqrt{1.15^2 + 1.2^2 + 0.9^2}} = F_{AC}\{0.6084i - 0.635j + 0.4762k\}$$

$$\bar{F}_{AD} = F_{AD} \frac{(0.65i - 1.2j - 0.9k)}{\sqrt{0.65^2 + 1.2^2 + 0.9^2}} = F_{AD}\{0.3976i - 0.7340j - 0.5505k\}$$

Weight of the Plate $= -Wj$

$$\Sigma F = \bar{F}_{AB} + \bar{F}_{AC} + \bar{F}_{AD} + \bar{W}$$

$$-254.12 + 0.6084 F_{AC} + 0.3976 F_{AD} = 0 \quad \text{— I}$$

$$-381.18 - 0.635 F_{AC} - 0.7340 F_{AD} - W = 0 \quad \text{— II}$$

$$-281.88 + 0.4762 F_{AC} - 0.5505 F_{AD} = 0 \quad \text{— III}$$

$F_{AC} = 50N$

$F_{AD} = 562.60N$

$$\boxed{W = 826N}$$

THANK YOU