Discrete Mathematical Structures – UCS405

MST Solutions

Q1(a): Let A, B, C, D, E, and F are sets such that:

A = {1, {4}, {2}, 3, 4, 5}, B = {{{1, 4, 5, 3, 1}}}, C = {1, {3}, 2, 1}, D = {1, 1, 3}, E = {1, 4, {5}, {3}}, F = {1, 8, {1, 2, 3, 4}} Calculate for the following sets.

- **i.** **A ∩ C**
- **ii.** **B ∩ F**
- **iii.** **D ∪ C**
- **iv.** **C ∩ E**
- **v.** **C ∪ (D ∩ F)**
- **vi.** **A ∩ E**

Ans:

- i. A ∩ C = {1}
- ii. B ∩ F = Φ
- iii. D ∪ C = {1, 2, 3, {3}}
- iv. C ∩ E = {1, {3}}
- v. C ∪ (D ∩ F) = {1, 2, {3}}
- vi. A ∩ E = {1, 4}

Q1(b): Set A comprises all three digit numbers that are multiples of 5, Set B comprises all three–digit even numbers that are multiples of 3 and Set C comprises all three–digit numbers that are multiples of 4. How many elements are present in A ∪ B ∪ C?

Ans: Set A = {100, 105, 110, ….995} ↦ {5 * 20, 5 * 21, ….. 5 * 199} ↦ 180 elements.

Set B = {102, 108, 114, ……996} ↦ {6 * 17, 6 * 18, 6 * 19, ….. 6 * 166} ↦ 150 elements.

Set C = {100, 104, 108, …..996} ↦ {4 * 25, 4 * 26, ….. 4 * 249} ↦ 225 elements.

A ∩ B = {120, 150, 180, …..990} ↦ All 3-digit multiples of 30 ↦ 30 elements.

B ∩ C = {108, 120, 132, …..996} ↦ All 3-digit multiples of 12 ↦ 75 elements.

C ∩ A = {120, 140, 160, …..980} ↦ All 3-digit multiples of 20 ↦ 45 elements.

A ∩ B ∩ C = {120, 180, …..960} ↦ All 3-digit multiples of 60 ↦ 15 elements.

A ∪ B ∪ C = A + B + C – A ∩ B – B ∩ C – C ∩ A + A ∩ B ∩ C

= 180 + 150 + 225 – 30 – 75 – 45 + 15 = 420

The question is "How many elements are present in A ∪ B ∪ C?"

Hence, the answer is "420".

Q2(a): Let A = {1, 2, 3.... 9} and R be the relation in A × A defined by (a, b) R (c, d) if a + d=b + c for (a, b), (c, d) in A x A. Prove that R is an equivalence relation and also obtain the equivalent class [(2, 5)].

Ans: A = {1, 2, 3...... 9}

R in A×A

(a, b) R (c, d) if (a, b) (c, d) ∈ A × A

a + b = b + c

Consider (a, b) R (a, b) (a, b) ∈ A × A

a + b = b + a

Hence R is reflexive

Consider (a, b) R (c, d) given by (a, b) (c, d) ∈ A × A

a + d = b + c => c + b = d + a

⇨ (c, d) R (a, b)

Hence R is symmetric

Let (a, b) R (c, d) and (c, d) R (e, f)

(a, b), (c, d), (e, f) ∈ A × A

a + b = b + c and c + f = d + e

a + b = b + c

⇨ a – c = b – d (1)
⇨ c + f = d + e (2)

adding (1) and (2)

a – c + c + f = b – d + d + e

a + f = b + e

⇨ (a, b) R (e, f)

R is transitive

R is an equivalence relation

we select from set A = {1, 2, 3,....9}

a and b such that

2 + b = 5 + a

Consider (1,4)

(2, 5) R (1, 4) => 2 + 4 = 5 + 1

[(2, 5)] = {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)} is the equivalent class under relation R.

Q2(b): Let a and b be positive integers, and suppose Q is defined recursively as follows:

$$\boldsymbol{Q(a,b) = \begin{cases} 0, & \textit{if } a < b \\ Q(a-b,b)+1, & \textit{if } b \le a \end{cases}}$$

- **i. Find: (a) Q (4, 10); (b) Q (21, 7).**
- **ii. What does this function Q do? Find Q (5861, 7).**

Ans: i. (a) Q (4, 10) = 0 since 4 < 10.

(b) Q (21, 7) = Q (14, 7) + 1

= [Q (7, 7) + 1] + 1 = Q (0, 7) + 3

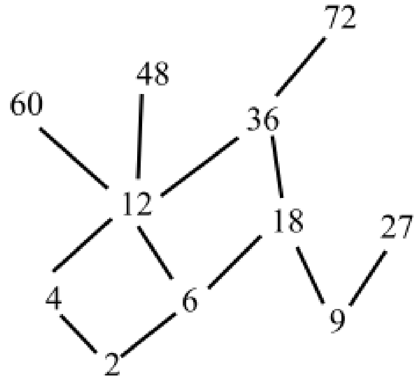
= 0 + 3 = 3

ii. Each time b is subtracted from a, the value of Q is increased by 1. Hence Q (a, b) finds the quotient when a is divided by b. Thus Q (5861, 7) = 837.

Q3: Answer these questions for the POSET ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, /):

- **i. Draw the Hasse Diagram.**
- **ii. Find all minimal and maximal elements.**
- **iii. Find the least and greatest element.**

Ans: i.



ii. Maximal Elements: 27, 48, 60, 72

Minimal Elements: 2, 9

iii. Least: No

Greatest: No

Q4(a): Using truth table find CNF and DNF for a ∧ (b ↔ c).

Ans:

a	**b**	**c**	**b ↔ c**	**a ∧ (b ↔ c)**
F	F	F	T	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	F
T	F	F	T	T
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T

DNF is:

$$(a \wedge b \wedge c) \vee \left(a \wedge \overline{b} \wedge \overline{c}\right)$$

CNF is:

$$(a \vee b \vee c) \wedge (a \vee b \vee \overline{c}) \wedge \left(a \vee \overline{b} \vee c\right) \wedge \left(a \vee \overline{b} \vee \overline{c}\right) \wedge (\overline{a} \vee b \vee \overline{c}) \wedge \left(\overline{a} \vee \overline{b} \vee c\right)$$

Q4(b): Suppose A = {a, b, c, d} and Π_1 is the following partition of A:

Π_1 = {{a, b, c}, {d}}

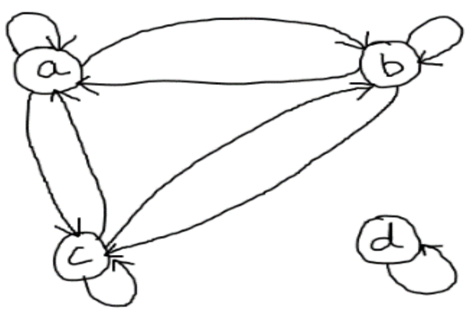
i. **List the ordered pairs of the equivalence relations induced by Π_1**
ii. **Draw the digraph of the above equivalence relation.**

Ans: i. The ordered pairs of the equivalence relations introduced by Π_1 is:

R = (a, b, c) × (a, b, c) ∪ (d) × (d)

R= {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d)}

ii. The diagraph of the above equivalence relation is as follows:



Q5(a): Give a big-O estimate for each of these functions. For the function g in your estimate f (x) is O(g(x)), use a simple function g of smallest order.

i. **$(n^3 + n^2 \log n)(\log n + 1) + (17 \log n + 19)(n^3 + 2)$**
ii. **$(2^n + n^2)(n^3 + 3^n)$**
iii. **$(n^n + n2^n + 5^n)(n! + 5^n)$**

Ans:

SOLUTION

(a) Given:

$$f(n) = (n^3 + n^2 \log n)(\log n + 1) + (17 \log n + 19)(n^3 + 2)$$

Use distributive property:

$$= n^3 \log n + n^2(\log n)^2 + n^3 + n^2 \log n + 17n^3 \log n + 19n^3 + 34 \log n + 38$$

$$= 18n^3 \log n + n^2(\log n)^2 + 20n^3 + n^2 \log n + 34 \log n + 38$$

Let us assume that $g(n) = n^3 \log n$.

For convenience sake, we will choose $k = 10^{20}$ and thus use $n > 10^{20}$. (Note: You could choose a different value of k, which will lead to a different value for C).

$$\begin{aligned}|f(n)| &= |18n^3 \log n + n^2(\log n)^2 + 20n^3 + n^2 \log n + 34 \log n + 38| \\ &= 18n^3 \log n + n^2(\log n)^2 + 20n^3 + n^2 \log n + 34 \log n + 38\end{aligned}$$

Using $\log n \leq n$ and $n > 10^{20}$ and $\log n > \log 10^{20} = 20$

$$\leq 18n^3 \log n + n^3 \log n + n^3 \log n + n^3 \log n + n \log n + n$$

$$\leq 18n^3 \log n + n^3 \log n + n^3 \log n + n^3 \log n + n^3 \log n + n^3 \log n$$

$$= 23n^3 \log n$$

$$= 23|n^3 \log n|$$

Thus we need to choose C to be at least 23. Let us then take $C = 23$.

By the definition of the Big-O notation, $f(n) = (n^3 + n^2 \log n)(\log n + 1) + (17 \log n + 19)(n^3 + 2)$ is $O(n^3 \log n)$ with $k = 10^{20}$ and $C = 23$.

(b) Given:

$$f(n) = (2^n + n^2)(n^3 + 3^n)$$

Use distributive property:

$$= n^3 2^n + n^5 + 2^n 3^n + n^2 3^n$$

$$= n^3 2^n + n^5 + 6^n + n^2 3^n$$

Let us assume that $g(n) = 6^n$.

When $n > 3$, then we have the property $n^3 < 3^n$
When $n > 2$, then we have the property $n^2 < 2^n$

For convenience sake, we will choose $k = 3$ and thus use $n > 3$. (Note: You could choose a different value of k, which will lead to a different value for C).

$$\begin{aligned} |f(n)| &= |n^3 2^n + n^5 + 6^n + n^2 3^n| \\ &= n^3 2^n + n^5 + 6^n + n^2 3^n \\ &< 3^n 2^n + 2^n 3^n + 6^n + 2^n 3^n \\ &= 6^n + 6^n + 6^n + 6^n \\ &= 4 \cdot 6^n \\ &= 4|6^n| \end{aligned}$$

Thus we need to choose C to be at least 4. Let us then take $C = 4$.

By the definition of the Big-O notation, $f(n) = (2^n + n^2)(n^3 + 3^n)$ is $O(6^n)$ with $k = 3$ and $C = 4$.

(c) Given:

$$f(n) = (n^n + n2^n + 5^n)(n! + 5^n)$$

Use distributive property:

$$= n^n n! + n2^n n! + 5^n n! + n^n 5^n + n2^n n! + 5^n 5^n$$

Let us assume that $g(n) = n^n n!$

When $n > 5$, we have the properties $5^n < n^n$ and $n2^n < n^n$

When $n \geq 12$, we also have the property $5^n < n!$

For convenience sake, we will choose $k = 12$ and thus use $x > 12$. (Note: You could choose a different value of k, which will lead to a different value for C).

$$\begin{aligned}|f(n)| &= |n^n n! + n2^n n! + 5^n n! + n^n 5^n + n2^n n! + 5^n 5^n| \\ &= n^n n! + n2^n n! + 5^n n! + n^n 5^n + n2^n n! + 5^n 5^n \\ &< n^n n! + n^n n! + n^n n! + n^n n! + n^n n! + n^n n! \\ &= 6n^n n! \\ &= 6|n^n n!|\end{aligned}$$

Thus we need to choose C to be at least 6. Let us then take $C = 6$.

By the definition of the Big-O notation, $f(n) = (n^n + n2^n + 5^n)(n! + 5^n)$ is $O(n^n n!)$ with $k = 12$ and $C = 6$.

Result:

i. **O (n^3 log n)**
ii. **O (6^n)**
iii. **O (n^n n!)**

Q5(b): Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

Ans: Let R = "It rains"

F = "It is foggy"

S = "The sailing race will be held"

D = "Life-saving demonstrations will go on"

T = "The trophy will be awarded"

We can now proceed to prove the claim:

	Step	Reason
1.	$\neg T$	Premise
2.	$S \rightarrow T$	Premise
3.	$\neg S$	Modus Tollens
4.	$\neg S \vee \neg D$	Addition
5.	$\neg (S \wedge D)$	DeMorgan's Law
6.	$(\neg R \vee \neg F) \rightarrow (S \wedge D)$	Premise
7.	$\neg(\neg R \vee \neg F)$	Modus Tollens
8.	$R \wedge F$	DeMorgan's Law
9.	R	Simplification