Electromagnetic Waves

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- Stokes' and Green's (Gauss') theorems
- Concept of Displacement current
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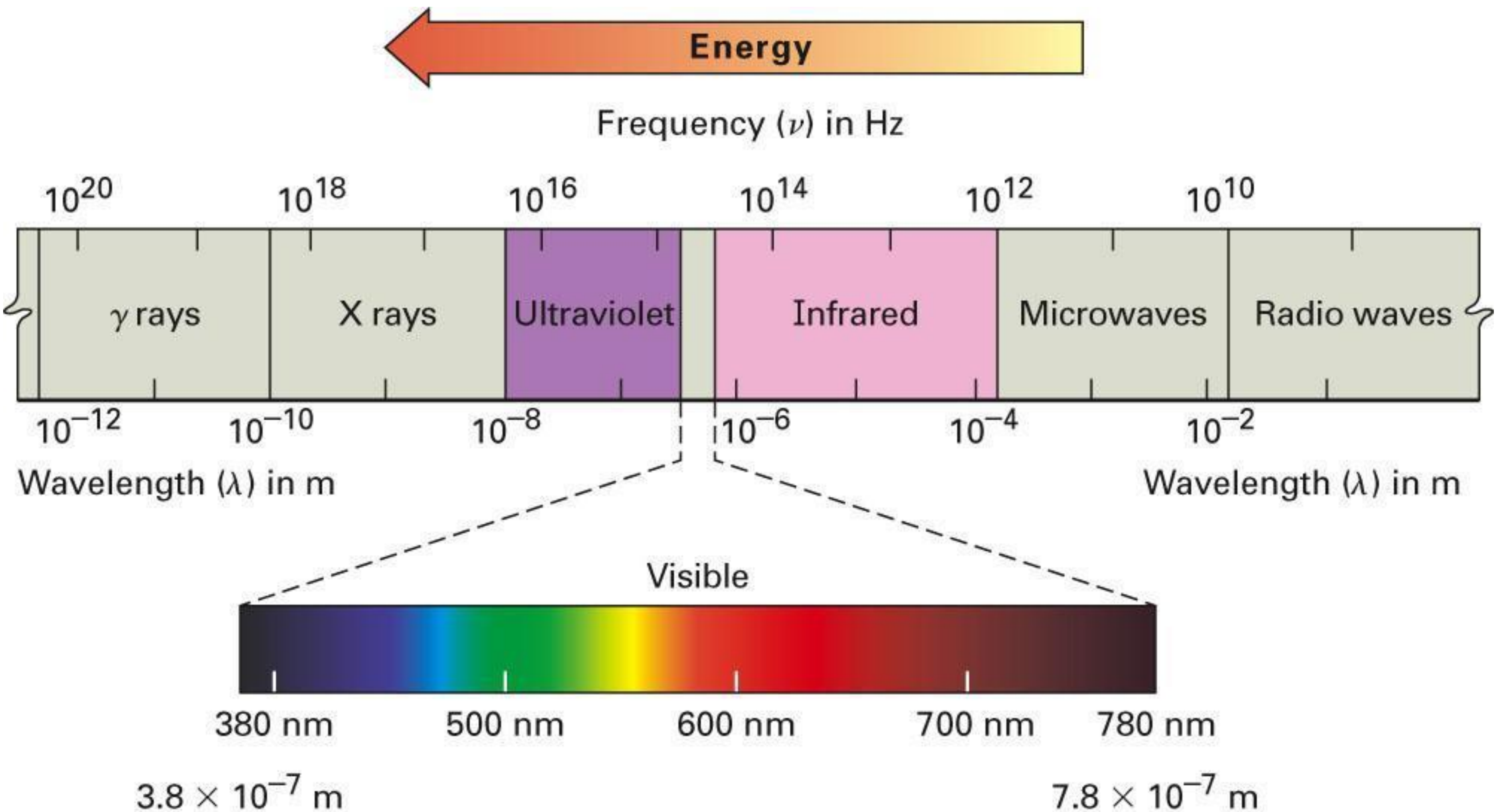
- Skin depth and its applications

Electromagnetic Waves :



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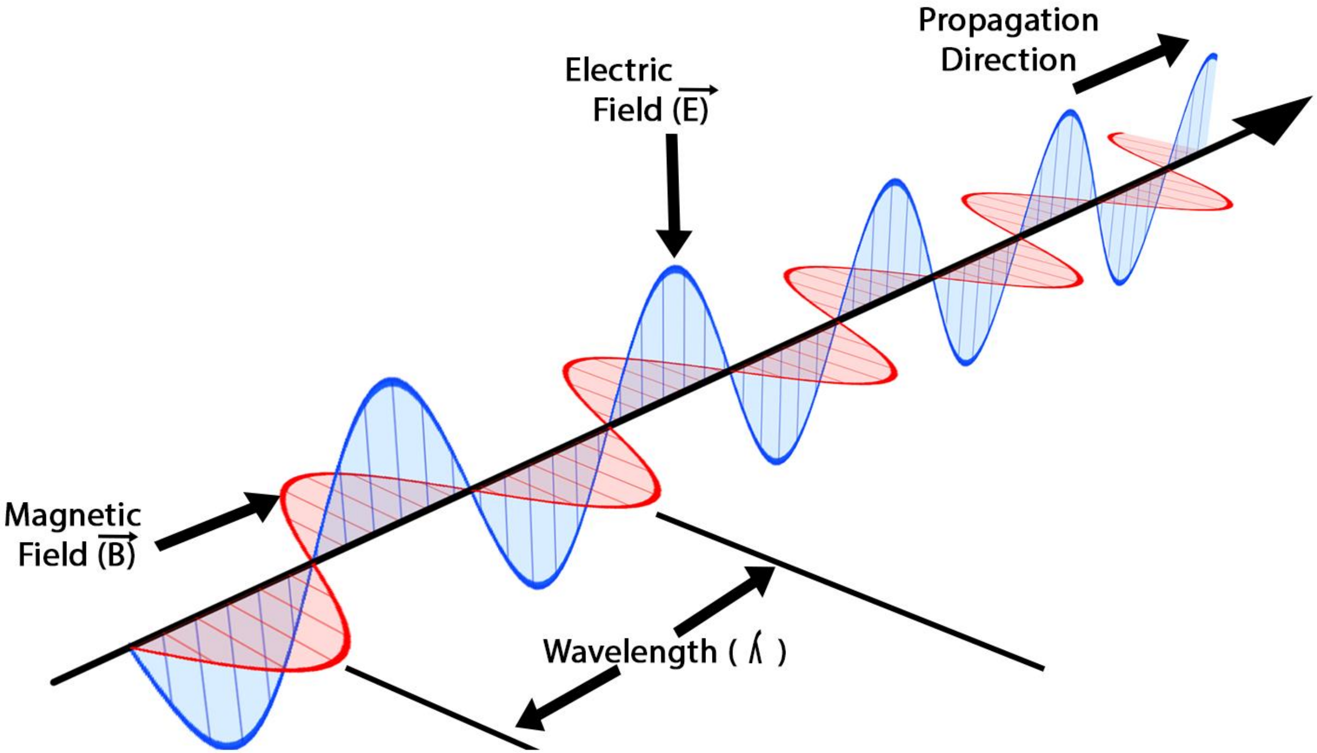
Electromagnetic Waves



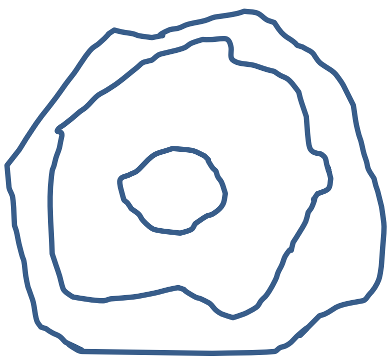
Properties of Electromagnetic Waves:

- They are transverse in nature i.e. direction of propagation ($\boldsymbol{k}$) is perpendicular to electric field (E) and magnetic field (B). $(\boldsymbol{E} \perp \boldsymbol{B} \perp \boldsymbol{k})$
- EM waves have oscillating electric and magnetic field.
- They travel with fixed speed ($3 \times 10^8\,\mathrm{m/s}$) in vacuum.
- They don't need medium to propagate.
- While travelling through medium, their speed is less than c.

Scalar Fields : like temperature, electric potential etc.

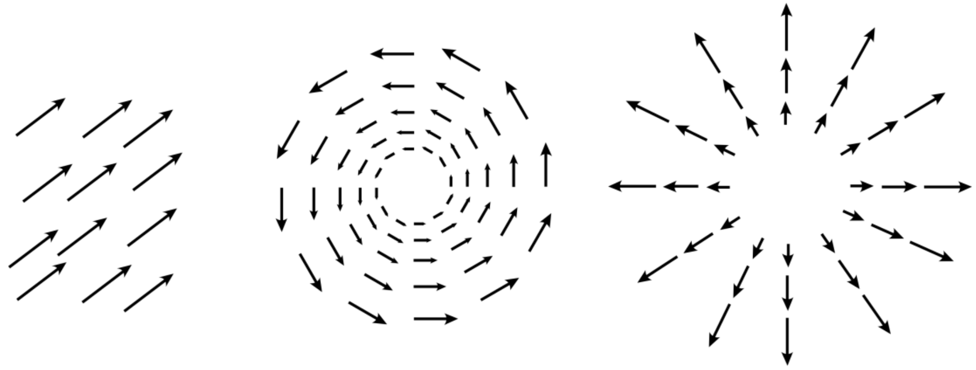
- Scalar fields are used to represent scalar quantities/functions in a region.
- These fields can be represented by contours which are imaginary surfaces drawn through all points are which field has same value (called as equipotential surfaces).
- No two equipotential surfaces cut each other.

Ex : equipotential surface :



Vector Fields : quantities with magnitude and direction like electric field, force, velocity etc.

- Vector fields are used to represent vector quantities/functions in a region.
- These fields are represented by flux or field lines drawn in such a way that tangent at any point of the line gives direction of vector field at that point.
- Lines representing vector fields can not cross each other because that would give non-unique value at that point.



(a) (b) (c)

Del operator ($\vec{\nabla}$) : Not a scalar or vector but operator.

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

It doesn't have meaning until it acts (not multiply) up on a function.

It is an instruction to differentiate what follows.

There are 3 ways $\vec{\nabla}$ can act :

1. On a scalar function T : $\vec{\nabla}$T (**gradient**)
2. On a vector function ($\vec{v}$) by dot product : $\vec{\nabla} \cdot \vec{v}$ (**divergence**)
3. On a vector function ($\vec{v}$) by cross product : $\vec{\nabla} \times \vec{v}$ (**curl**)

Gradient $\vec{\nabla}$T :

Suppose scalar quantity (Let us say temperature) T is function of (x,y,z). Theorem on partial derivatives states

$$dT = \left(\frac{\partial T}{\partial x}\right) \mathrm{dx} + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \qquad \textbf{(1)}$$

Physically this tell us how T changes when three variables are changed by infinitesimal amounts dx, dy, dz.

Equation (1) can be written as dot product

$$dT = \left(\left(\frac{\partial T}{\partial x}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial T}{\partial y}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial T}{\partial z}\right)\hat{\boldsymbol{z}}\right) \cdot (dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}})$$

$$= \left(\vec{\nabla}\mathrm{T}\right).\left(\overrightarrow{dl}\right) \qquad \textbf{(2)}$$

Where $\vec{\nabla}\mathrm{T} = \left(\frac{\partial T}{\partial x}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial T}{\partial y}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial T}{\partial z}\right)\hat{\boldsymbol{z}}$ is gradient of T. It is vector quantity with three components.

Gradient $\vec{\nabla}$T interpretation : Gradient has magnitude as well as direction.

$$dT = (\vec{\nabla}\mathrm{T}).(\overrightarrow{dl}) = |\vec{\nabla}T|\,|\overrightarrow{dl}|\cos\theta$$

Where θ is angle between $\vec{\nabla}$T and $\overrightarrow{dl}$.

If magnitude of $|\overrightarrow{dl}|$ is fixed and θ is varied, dT is maximum when θ = 0 ($\cos\theta = 1$). It means for fixed distance $|\overrightarrow{dl}|$, dT is maximum when you move in direction of $\vec{\nabla}$T.

or

Gradient $\vec{\nabla}$T points in direction of maximum increase of function T.

and

Magnitude $|\vec{\nabla}T|$ gives the slope along this maximal direction.

Gradient $\vec{\nabla}T = 0$ meaning :

If $\vec{\nabla}T = 0$ at (x, y, z) then dT = 0 for small displacements about the point (x, y, z). This is then **stationary point** of the function T(x, y, z). It could be a maximum, a minimum or a shoulder.

If you want to locate extrema of a function of three variables, set its gradient equal to 0.

Numerical on gradient :

The height of certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is distance (in miles) in north and x the distance in east.

a) Where is the top of hill located?
b) How high is the hill?
c) How steep is slope (in feet per mile) at point 1 mile north and 1 mile east?

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

$$\vec{\nabla} h(x, y, z) = \left(\frac{\partial h}{\partial x}\right)\hat{x} + \left(\frac{\partial h}{\partial y}\right)\hat{y}$$

$$\vec{\nabla} h(x, y, z) = 10(2y\ - 6x\ - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

(a) Where is the top of hill located ?

Remember to find maxima, minima, you put gradient = 0

$$2y\ - 6x\ - 18 = 0$$

$$2x - 8y + 28 = 0$$

Solving these x = -2, y = 3 (location of top of hill)

(b) How high is the hill?

Putting x = -2 and y =3,

h = 10(-12 - 12 - 36 + 36 + 84 + 12) = 720 ft

(c) How steep is slope (in feet per mile) at point 1 mile north and 1 mile east?

Remember Magnitude $\left|\vec{\nabla}T\right|$ gives the slope.

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

$$\vec{\nabla}h(x,y,z) = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

Putting x = 1 and y = 1

$$\vec{\nabla}h(x,y,z) = 10(2 - 6 - 18)\hat{x} + 10(2 - 8 + 28)\hat{y}$$

$$\vec{\nabla}h(x,y,z) = -220\,\hat{x} + 220\,\hat{y}$$

$$\left|\vec{\nabla}h\right| = 220\sqrt{2}$$

Divergence $\vec{\nabla}.\vec{v}$:

From definition of $\vec{\nabla}$, divergence will be

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Divergence of vector function is a scalar quantity.

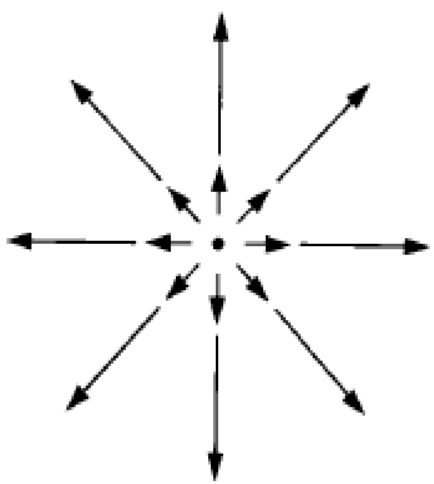
If $\vec{\nabla}\cdot\vec{v} = 0$, then it is called solenoidal field.

Ex: Calculate divergence of function $\vec{v} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

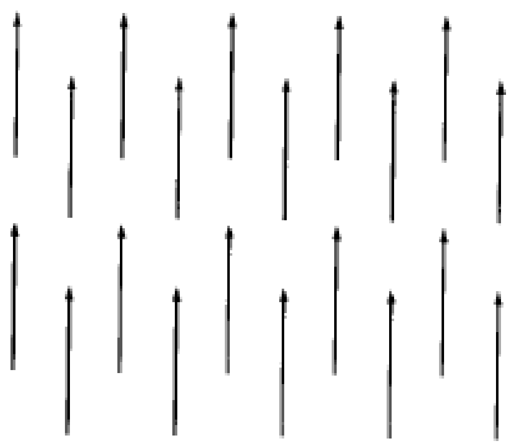
Geometrical Interpretation of divergence ($\vec{\nabla}.\vec{v}$):

$\vec{\nabla}.\vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question.

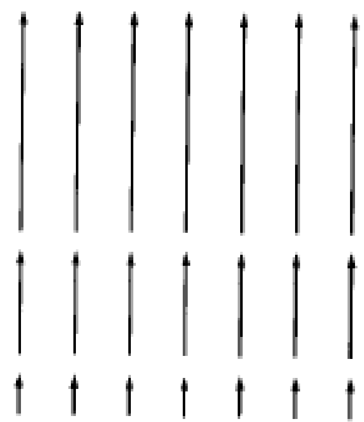
A point of positive divergence is a "source" and a point of negative divergence is a "sink" or "drain".



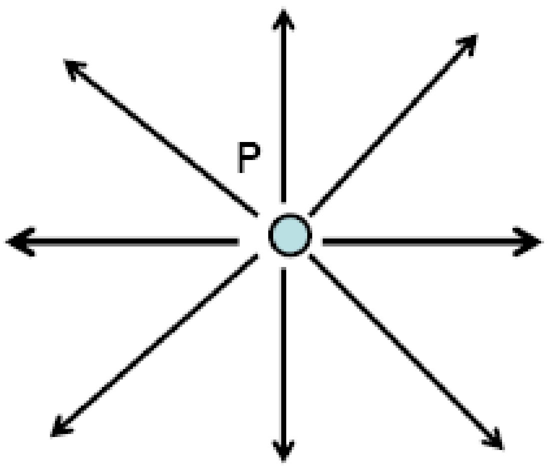
Positive divergence



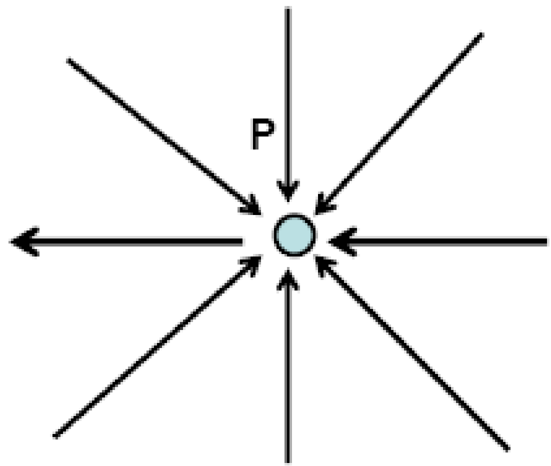
Zero divergence



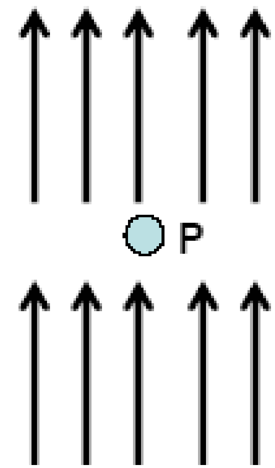
Positive divergence



Positive
Divergence



Negative
Divergence



Zero
Divergence

The curl ($\vec{\nabla} \times \vec{v}$):

From definition of $\vec{\nabla}$, curl will be

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

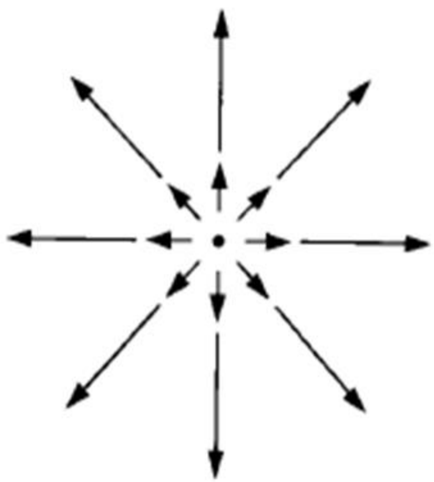
Curl of vector function $\vec{v}$ is, like any cross product, a vector.

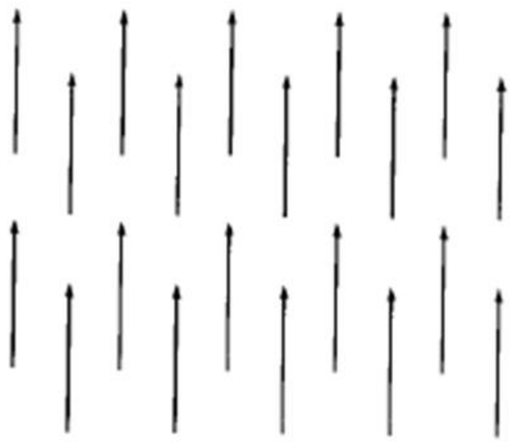
If $\vec{\nabla} \times \vec{v} = 0$, then it is called irrotational field.

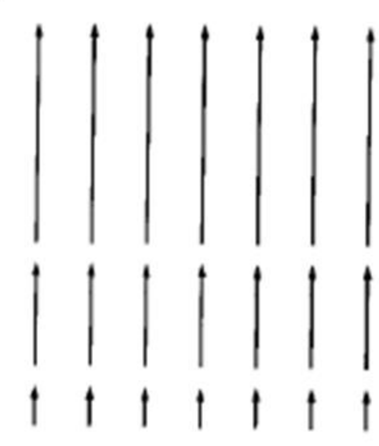
Geometrical interpretation of the curl ($\vec{\nabla} \times \vec{v}$):

$\vec{\nabla} \times \vec{v}$ is a measure of how much the vector curls/swirls around the point in question.

What is curl of these functions?



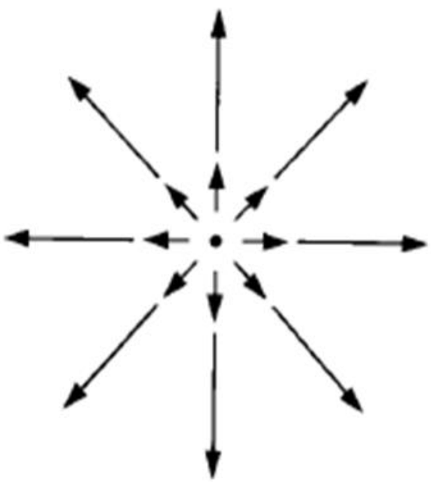


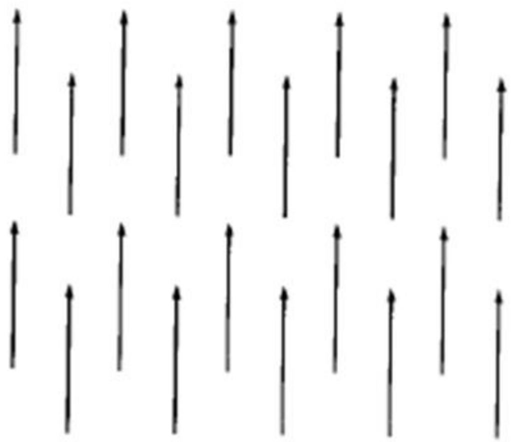


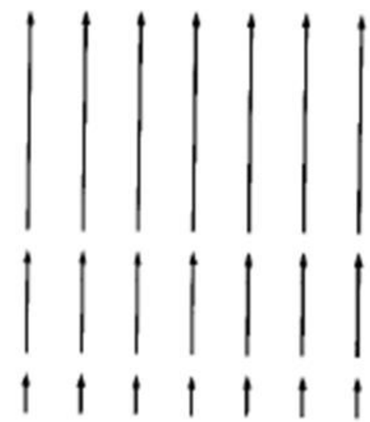
Geometrical interpretation of the curl ($\vec{\nabla} \times \vec{v}$):

$\vec{\nabla} \times \vec{v}$ is a measure of how much the vector curls/swirls around the point in question.

What is curl of these functions?



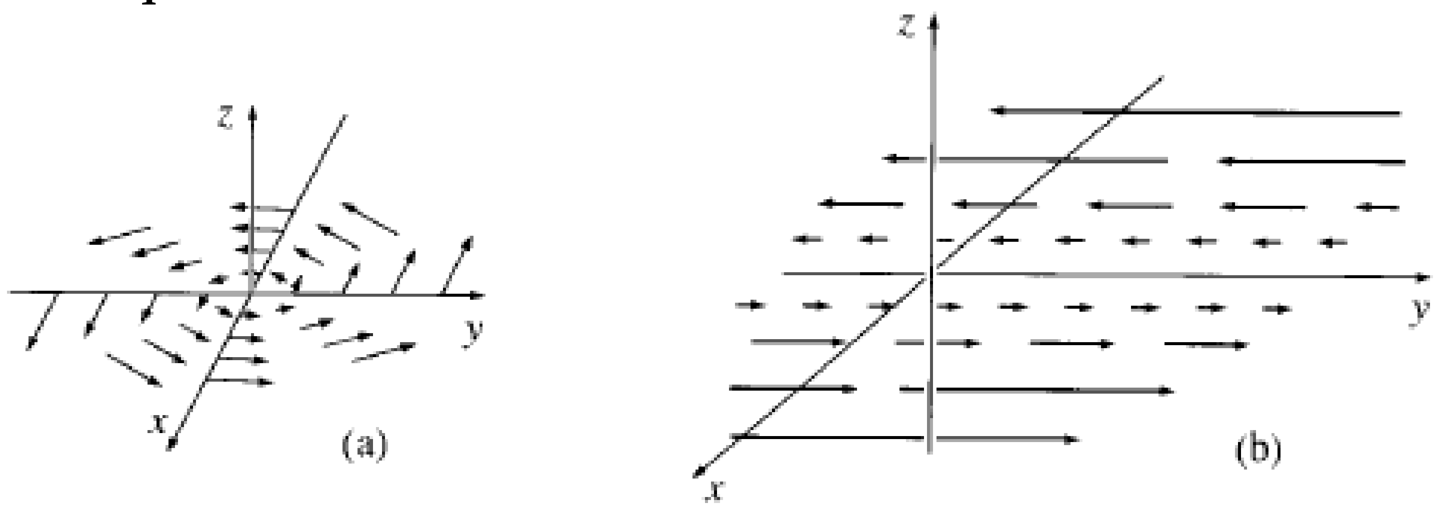




Zero

Geometrical interpretation of the curl ($\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$):

$\vec{\nabla} \times \vec{v}$ is a measure of how much the vector curls/swirls around the point in question.



These functions have curls pointing in z-direction (given by right hand rule : Curl your fingers in direction of swirl, then thumb gives direction of curl.)

Calculate curl of the function $\vec{\boldsymbol{v}} = -\boldsymbol{y}\,\hat{\boldsymbol{x}} + \boldsymbol{x}\,\hat{\boldsymbol{y}}$.

Fundamental Theorem for divergences (Green's theorem or Gauss's theorem):

This theorem states that :

$$\int_{V} (\vec{\nabla} . \vec{v})\, d\tau = \oint_{S} \vec{v} \cdot d\vec{a}$$

(V represents volume and $\vec{v}$ represents a vector)

$d\tau$ is integration over volume V ($dxdydz$).

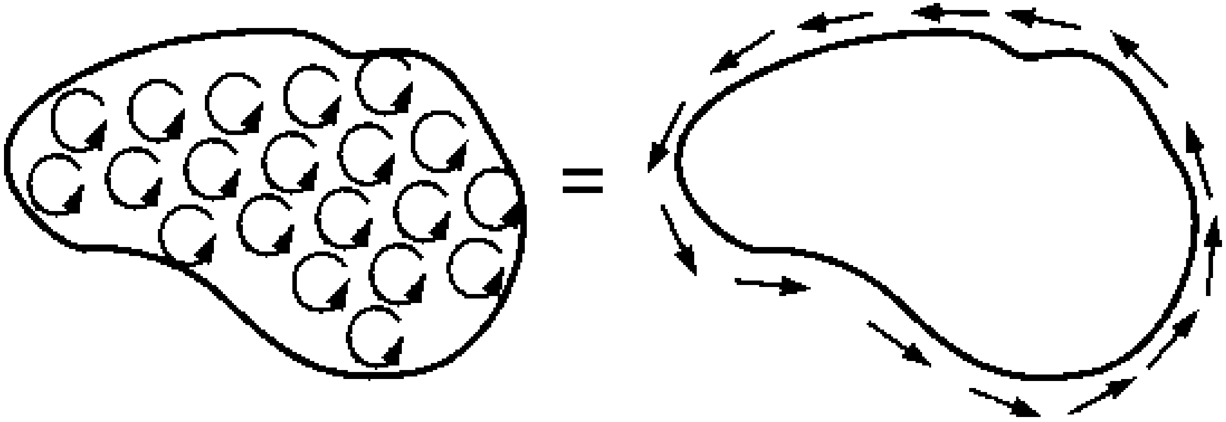
In words, it says that the integral of a divergence over a volume V is equal to the value of the function at the boundary (here, surface S that bounds volume.

The boundary of a volume is a closed surface, that of a surface is a closed line. But the boundary of a line is just two points.

Fundamental Theorem for curls (Stoke's theorem):

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

It says that integral of curl over a region of surface S is equal to the value of function at the boundary (here perimeter of surface, P).



Maxwell's first equation (Gauss's Law) :

Flux of $\vec{E}$ through a surface S is measure of "number of field lines" passing through S

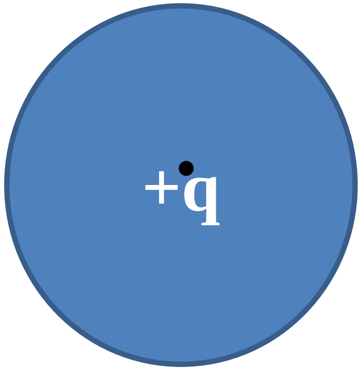
$$\varphi_E \equiv \int_S \vec{E}.d\vec{a}$$

(The dot product picks out area $d\vec{a}$ in direction of $\vec{E}$.)

This means total flux through a **closed surface** is measure of total charge inside. This is essence of **Gauss's Law**.

Maxwell's first equation (Gauss's Law) :

If there is point charge at origin, flux E through a spherical radius of r is



$$\oint \vec{E}\cdot d\vec{a} = \int \left(\frac{q}{4\pi\epsilon_0 r^2}\hat{r}\right)\cdot(r^2 sin\theta\ d\theta\ d\phi\ \hat{r})$$

$$= \frac{q}{4\pi\epsilon_0}\int sin\,\theta\ d\theta\ d\phi = \frac{q}{\epsilon_0}$$

For any closed surface, whatever its shape, would be pierced by same number of field lines. Hence, flux through any surface enclosing charge q is $q/_{\epsilon_0}$.

Maxwell's first equation (Gauss's Law) :

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \qquad \textbf{(1)}$$

Applying divergence theorem $\int_V (\vec{\nabla}.\vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$ on L.H.S., eq. (1) can be written

$$\int_V (\vec{\nabla}.\vec{E})\, d\tau = \frac{q}{\epsilon_0} \Rightarrow \int_V (\vec{\nabla}.\vec{E})\, d\tau = \int_V \frac{\rho}{\epsilon_0}\, d\tau \quad \left(\because q = \int_V \rho\, d\tau \right)$$

$$\boldsymbol{Hence, \vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}}$$ **(Maxwell's First Equation in differential form)**

Maxwell's Second equation:

Flux of $\vec{\boldsymbol{B}}$ through a surface S is measure of "number of field lines" passing through S

$$\varphi_B \equiv \int \vec{B}.d\vec{a}$$

(The dot product picks out area $d\vec{a}$ in direction of $\vec{B}$.) For a closed surface S,

$$\oint_S \vec{B} \cdot d\vec{a} = 0$$

(Because number of lines entering and leaving closed surface has to be same)

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \qquad \textbf{(2)}$$

Applying divergence theorem $\int_V (\vec{\nabla}.\vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$

$$\int_V (\vec{\nabla}.\vec{B})\, d\tau = 0$$

$$\Longrightarrow \vec{\nabla}.\vec{B} = 0$$

(Maxwell's Second Equation in differential form)

Maxwell's third equation (Faraday's Law):

Faraday's law of electromagnetic induction says that the induced emf (ε) is rate of change of magnetic flux (φ)

$$\varepsilon = -\frac{d\varphi}{dt}$$

$$\varepsilon = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a}$$

$$\varepsilon = -\int_{S} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \qquad \textbf{(2)}$$

Maxwell's third equation (Faraday's Law):

ε is actually potential or energy per unit charge. So, ε is work done in carrying a unit positive charge around a closed loop. So,

$$\varepsilon = \oint_{P} \vec{E}.d\vec{l} = \int_{S} (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} \qquad \textbf{(3)}$$

Using Stoke's theorem $\int_{S} (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint_{P} \vec{E} \cdot d\vec{l}$

From eq. (2) and (3) $\int_{S} (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = -\int_{S} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(Maxwell's Third Equation in differential form)

Maxwell's third equation (Faraday's Law):

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

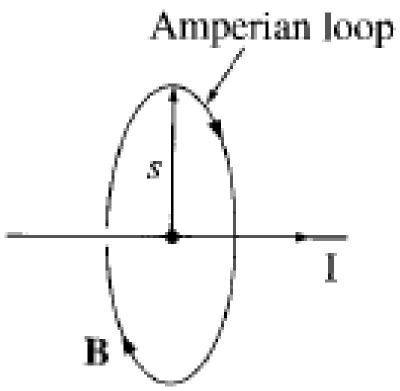
Taking divergence on both sides

$$\vec{\nabla} \cdot (\vec{\nabla} \times \overrightarrow{E}) = -\frac{\partial (\vec{\nabla} \cdot \vec{B})}{\partial t}$$

L. H. S. of above eq. is zero because divergence of curl of any vector is zero and R. H. S. is zero from Maxwell's second equation ($\vec{\nabla} \cdot \vec{B} = 0$).

Hence, everything is okay so far.

Ampere's Circuital Law:



According to Ampere's circuital law, if $d\vec{l}$ is perimeter of Amperian loop and I_{enc} is current enclosed by that loop

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \, I_{enc} \qquad \textbf{(4)}$$

Using Stoke's theorem: $\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$

And $$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

Hence, equation (4) becomes $$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

Therefore, $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$ **(Ampere's Circuital Law in differential form)**

Electrodynamics before Maxwell:

1. $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ (Gauss's Law)
2. $\vec{\nabla}.\vec{B} = 0$ (No Name)
3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)
4. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's Law)

Equation of continuity:

Current density ($\vec{J}$) : Defined as current per unit area (area being parallel to direction of flow)

$$I = \int_{S} \vec{J} \cdot d\vec{a}$$

The charge per unit time leaving a volume is

$$\oint_{S} \vec{J} \cdot d\vec{a} = \int_{V} (\vec{\nabla} \cdot \vec{J}) d\tau$$ (Using Gauss' divergence theorem)

Because charge is conserved, so whatever is flowing through surface must come at expense of what remains inside

$$\int_{V} (\vec{\nabla} \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_{V} \rho \, d\tau = -\int_{V} \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Equation of continuity:

$$\int_V (\vec{\nabla} \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_V \rho \, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

ρ is the charge density (charge per unit volume).
(-ve sign because the outward flow decreases the charge left in volume V.)

$$\Longrightarrow \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$ **(Equation of continuity)**

When a steady current (*I*) is flowing through a wire then its magnitude *I* must be same all along line; otherwise charge would be piling up somewhere. Because $\frac{\partial \rho}{\partial t} = 0$, hence $\vec{\nabla} \cdot \vec{J} = 0$.

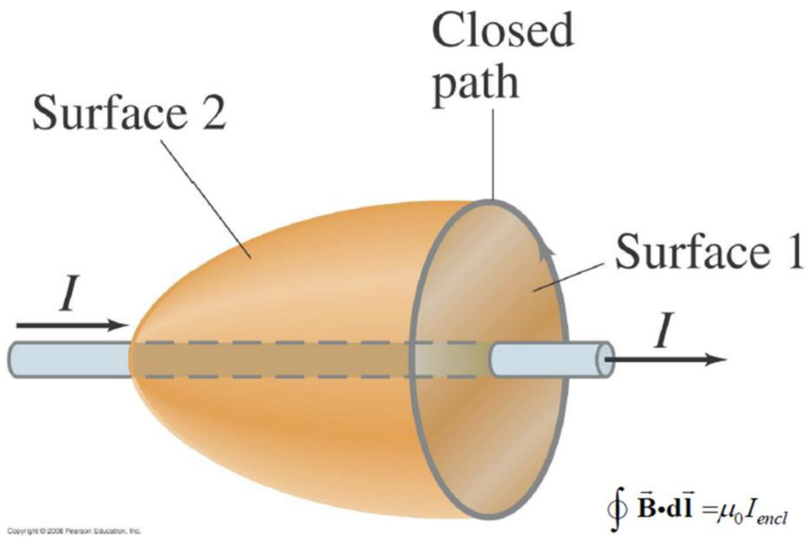
Problem with Ampere's Circuital Law:

Ampere's circuital law states that $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Taking divergence on both sides $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$

As discussed before divergence of curl for any vector is zero , so, L.H.S. is zero but R. H.S. might not be zero necessarily. Actually. R.H.S. is zero only when a steady current is flowing (As discussed in continuity equation).

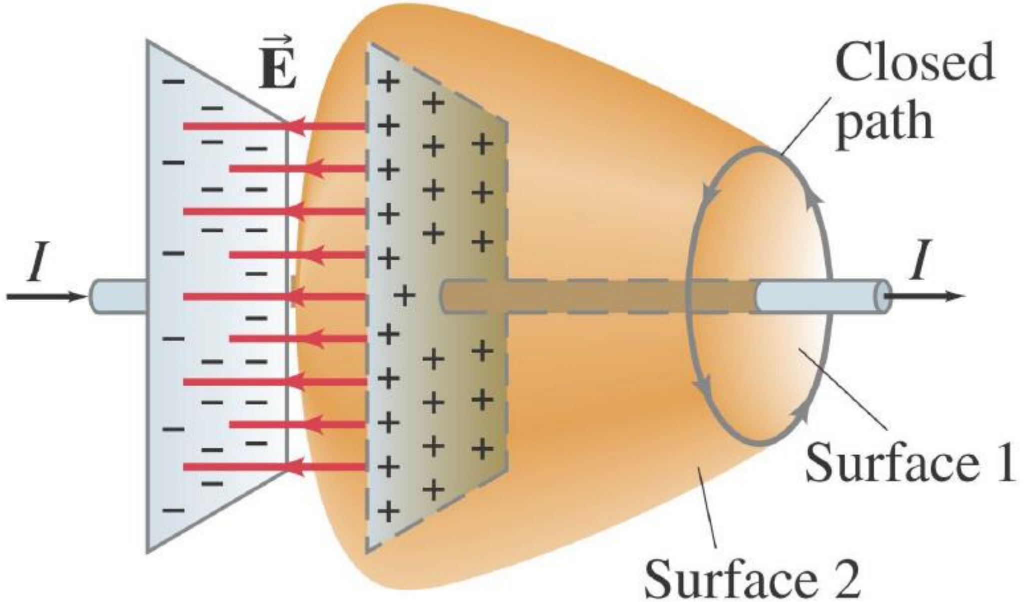
Problem with Ampere's Circuital Law:



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

There could be two surfaces for which closed path is boundary and for both surfaces same amount of current pierces the surface. So, Ampere's law works fine!!

Problem with Ampere's Circuital Law:



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

But when charge is piling up somewhere like in case of capacitor, Ampere's circuital law fails. Again for closed path as boundary, there are 2 surfaces shown in Fig. and for surface 1, current I pierces it but no current pierces surface 2 which is contradictory.

How Maxwell fixed Ampere's Circuital Law:

Maxwell fixed it by purely theoretical arguments.

Ampere's circuital law states that $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Taking divergence on both sides $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$

L.H.S. is zero but R. H.S. might not be zero which is issue. R. H. S. can be rewritten using continuity equation :

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Using Gauss Law : $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$\Longrightarrow \vec{\nabla} \cdot \vec{J} = -\frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla} \cdot \vec{E}\right) = -\vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

How Maxwell fixed Ampere's Circuital Law:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\left(\epsilon_0\,\vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\cdot\left(\epsilon_0\,\frac{\partial\vec{E}}{\partial t}\right) \qquad \textbf{(5)}$$

Ampere's circuital law states that $\vec{\nabla}\times\vec{B} = \mu_0\,\vec{J}$

Taking divergence on both sides $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\,(\vec{\nabla}\cdot\vec{J})$ **(6)**

Problem was R. H. S. of equation (6) not being zero, but if we add negative of (5) in equation (6) R. H.S., then this too will be zero i.e.

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\,\vec{\nabla}\cdot\left(\vec{J}+\epsilon_0\,\frac{\partial\vec{E}}{\partial t}\right) = 0$$

$$\Longrightarrow (\vec{\nabla}\times\vec{B}) = \mu_0\left(\vec{J}+\epsilon_0\,\frac{\partial\vec{E}}{\partial t}\right)$$

How Maxwell fixed Ampere's Circuital Law:

$$(\vec{\nabla} \times \vec{B}) = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$ **(Maxwell's fourth equation)**

Changing electric field produces magnetic field just as changing Magnetic field induces an electric field (Faraday's law)!!

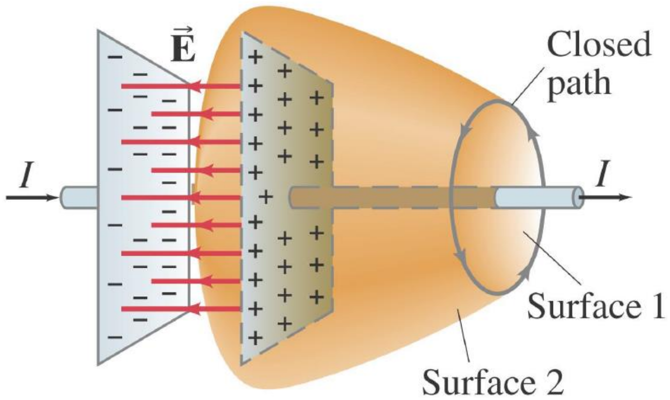
Maxwell called his extra term as displacement current

$$\vec{J}_D \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Now, Ampere's circuital law becomes

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

How Maxwell fixed Ampere's Circuital Law:



Electric field between capacitor plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \Longrightarrow \frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

Ampere's circuital law after Maxwell's change :

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{a}$$

For surface 1, E = 0 and I_{enc} = I. For surface 2, I_{enc} = 0, Hence

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{a} = \mu_0 \epsilon_0 \frac{I}{\epsilon_0} = \mu_0 I$$

Hence, we get same answer for either surface!!

Maxwell's Equations :

1. $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ (Gauss's Law)

2. $\vec{\nabla}.\vec{B} = 0$ (No Name)

3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

4. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ (Ampere's Law with Maxwell's correction)

EM wave equation in free space/vacuum :

In free space, charge density ρ = 0, current density $\vec{J} = 0$. Hence, Maxwell's equations are

1. $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ **becomes** $\vec{\nabla}.\vec{E} = 0$

2. $\vec{\nabla}.\vec{B} = 0$ **becomes** $\vec{\nabla}.\vec{B} = 0$

3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ **becomes** $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

4. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ **becomes** $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Here μ_0 is permeability and ϵ_0 is permittivity of free space.

EM wave equation in free space/vacuum :

Taking curl of Maxwell's third equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Longrightarrow \quad \vec{\nabla}\big(\vec{\nabla} \cdot \vec{E}\big) - \big(\vec{\nabla} \cdot \vec{\nabla}\big)\vec{E} = -\frac{\partial}{\partial t}\big(\vec{\nabla} \times \vec{B}\big)$$

(because $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$)

$$\Longrightarrow \quad 0 \; - \; \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Longrightarrow \quad \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Longrightarrow \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \qquad \textbf{(7)}$$

EM wave equation in free space/vacuum :

Taking curl of Maxwell's fourth equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{B}}) = \vec{\nabla} \times (\mu_0 \epsilon_0 \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}})$$

$$\Longrightarrow \ \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

(because $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$)

$$\Longrightarrow \ 0 \ - \ \nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\Longrightarrow \ \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B} \qquad \textbf{(8)}$$

EM wave equation in free space/vacuum :

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \qquad \textbf{(7)}$$

and

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B} \qquad \textbf{(8)}$$

Equation of plane wave travelling in *x* direction is given by

$$y(x,t) = A\sin(kx - \omega t)$$

And

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad \left(v = \frac{\omega}{k}\right) \qquad \textbf{(9)}$$

Comparing equation (9) with (7) and (8), speed of $\vec{E}$ and $\vec{B}$ will be given by $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \ m/sec$.

EM wave equation in free space/vacuum :

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \qquad \textbf{(7)}$$

Solution of equation (7) will be

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{E}(\vec{r},t) = \left(E_{0x}\,\hat{x} + E_{0y}\,\hat{y} + E_{0z}\,\hat{z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{k}\cdot\vec{r} = \left(k_x\,\hat{x} + k_y\,\hat{y} + k_z\hat{z}\right)\cdot(x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$$
$$= \left(k_x\,x + k_y\,y + k_z\,z\right)$$

or $\quad \vec{E}(\vec{r},t) = \left(E_x\,\hat{x} + E_y\,\hat{y} + E_z\,\hat{z}\right) \quad$ where

$$E_x = E_{0x}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\ E_y = E_{0y}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\ E_z = E_{0z}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

EM wave equation in free space/vacuum :

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B} \qquad \textbf{(8)}$$

Solution of equation (8) will be

$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B}(\vec{r}, t) = \left(B_{0x}\,\hat{x} + B_{0y}\,\hat{y} + B_{0z}\,\hat{z}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{k}\cdot\vec{r} = \left(k_x\,\hat{x} + k_y\,\hat{y} + k_z\hat{z}\right)\cdot\left(x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\right)$$
$$= \left(k_x\,x + k_y\,y + k_z\,z\right)$$

or $\vec{B}(\vec{r}, t) = \left(B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}\right)$ where

$$B_x = B_{0x}\,e^{i(\vec{k}\cdot\vec{r} - \omega t)},\ B_y = B_{0y}\,e^{i(\vec{k}\cdot\vec{r} - \omega t)},\ B_z = B_{0z}\,e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

EM wave equation in free space/vacuum :

$$\vec{E}(\vec{r},t) = \left(E_x\,\hat{x} + E_y\,\hat{y} + E_z\,\hat{z}\right)$$

$$E_x = E_{0x}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\ E_y = E_{0y}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\ E_z = E_{0z}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Putting these values in Maxwell's first equation in free space

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \Longrightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \textbf{(9)}$$

$$\frac{\partial E_x}{\partial x} = ik_x E_{0x}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\,k_x\,E_x$$

Similarly $\dfrac{\partial E_y}{\partial y} = ik_y E_y$ and $\dfrac{\partial E_z}{\partial z} = ik_z E_z$

Hence equation (9) becomes $ik_x E_x + ik_y E_y + ik_z E_z = 0$

$$\Longrightarrow i\left(\vec{k}\cdot\vec{E}\right) = 0$$

Dot product of two vectors is zero when they are perpendicular to each other. It means $\vec{k} \perp \vec{E}$.

EM wave equation in free space/vacuum :

$$\vec{B}(\vec{r},t) = \left(B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}\right)$$

$$B_x = B_{0x}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\ B_y = B_{0y}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\ B_z = B_{0z}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Putting these values in Maxwell's first equation in free space

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \Longrightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \textbf{(10)}$$

$$\frac{\partial B_x}{\partial x} = ik_x B_{0x}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\,k_x\,B_x$$

Similarly $\dfrac{\partial B_y}{\partial y} = ik_y B_y$ and $\dfrac{\partial B_z}{\partial z} = ik_z B_z$

Hence equation (10) becomes $\quad ik_x B_x + ik_y B_y + ik_z B_z = 0$

$$\Longrightarrow i\left(\vec{k}\cdot\vec{B}\right) = 0$$

It means $\vec{k} \perp \vec{B}$.

EM wave equation in free space/vacuum :

Maxwell's third equation : $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \textbf{(11)}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$= \hat{x}\left(ik_yE_z - ik_zE_y\right) - \hat{y}(ik_xE_z - ik_zE_x) + \hat{z}\left(ik_xE_y - ik_yE_x\right)$$

$$= i\,(\vec{k} \times \vec{E}) \qquad \textbf{(12)}$$

$(E_x = E_{0x}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}, E_y = E_{0y}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}, E_z = E_{0z}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\frac{\partial E_z}{\partial y} = ik_yE_{0z}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\, k_y\, E_z$ and so on……)

EM wave equation in free space/vacuum :

Maxwell's third equation : $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \textbf{(11)}$$

$$\vec{\nabla} \times \vec{E} = i\,(\vec{k} \times \vec{E})$$

And $$-\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) = i\omega\vec{B}$$

Putting value of L.H.S. and R. H.S. in equation (11)

$$i\,(\vec{k} \times \vec{E}) = i\omega\vec{B} \qquad \text{or} \qquad (\vec{k} \times \vec{E}) = \omega\vec{B} \qquad \textbf{(12)}$$

From equation (12), $\vec{B} \perp \vec{E}$ and $\vec{B} \perp \vec{k}$. We already proved $\vec{E} \perp \vec{k}$ which means $\vec{B} \perp \vec{E} \perp \vec{k}$.

Hence, EM waves are transverse in nature.

EM wave equation in free space/vacuum :

We just derived $(\vec{k} \times \vec{E}) = \omega \vec{B}$

$$|\vec{k}|\,|\vec{E}| \sin 90° = \omega\,|\vec{B}| \qquad \textbf{(13)} \qquad (\,because\ \vec{E} \perp \vec{k}\,)$$

$$(\vec{B}(\vec{r},t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \text{ and } \vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)})$$

$$|\vec{B}| = \sqrt{\vec{B}\,\vec{B}^{\,*}} = B_0 \quad \text{and similarly} \quad |\vec{E}| = \sqrt{\vec{E}\,\vec{E}^{\,*}} = E_0$$

So, equation (13) becomes

$$E_0 = \frac{\omega}{k} B_0 = c\,B_0 \qquad \textbf{(14)} \qquad \because velocity\,(c) = \frac{\omega}{k}$$

Hence, magnitude of electric field is *c* times magnitude of magnetic field. That is why direction of polarisation is denoted by electric field conventionally.

EM wave equation in free space/vacuum :

Equation (14) states that $E_0 = c\,B_0$

Space impedance (Z_0) is defined as

$$Z_0 = \left|\frac{E}{H}\right| = \mu_0 \frac{E_0}{B_0} \qquad (B = \mu\, H)$$

$$= \mu_0 c\, \frac{B_0}{B_0} = \mu_0 c$$

$$= \mu_0 \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= 376.7\ \Omega$$

EM wave equation in conducting medium :

From equation of continuity $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

$$\Longrightarrow \frac{\partial \rho}{\partial t} = -\left(\vec{\nabla} \cdot \sigma \vec{E}\right)$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} = -\sigma\left(\vec{\nabla} \cdot \vec{E}\right)$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} = -\sigma \; \frac{\rho}{\epsilon}$$

$$\Longrightarrow \frac{\partial \rho}{\rho} = -\frac{\sigma}{\epsilon} \; \partial t$$

Integrating on both sides $\ln(\rho)\,{}^{\rho}_{\rho_0} = -\frac{\sigma}{\epsilon}\;(\mathrm{t})^{\mathrm{t}}_{0}$

$$\rho = \; \rho_0 \, e^{-\left(\frac{\sigma}{\epsilon}\right)t}$$

EM wave equation in conducting medium :

$$\rho = \rho_0 \, e^{-\left(\frac{\sigma}{\epsilon}\right)t}$$

Characteristic time or charge relaxation time ($\tau = \epsilon/\sigma$) is defined as time in which charge reduces to 1/e of its initial value.

Characteristic time is a measure of how good conductor is. Smaller it is, better conductor it is.

For ex. Characteristic time for Cu is $4.5 \times 10^{-19}\, sec$.

It means you can assume there is no charge inside the conductor since it immediately flows to the surface.

EM wave equation in conducting medium :

We just said that we can assume there is no charge inside the conductor so, one can say then $\rho = 0$ inside a conductor.
Let us say that μ is permeability and ϵ is permittivity of this medium.

Maxwell's equations for conducting medium are :

1. $\vec{\nabla}.\vec{E} = 0$ (as $\rho = 0$ inside a conductor)

2. $\vec{\nabla}.\vec{B} = 0$

3. $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

4. $\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \dfrac{\partial \vec{E}}{\partial t}$

Here μ is permeability and ϵ is permittivity of conducting medium.

EM wave equation in conducting medium :

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Taking curl on both sides $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = -\frac{\partial}{\partial t}\left(\mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow 0 - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}\right) \quad (\because \vec{\nabla} \cdot \vec{E} = 0 \text{ and } \vec{J} = \sigma\vec{E})$$

$$\Rightarrow \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \textbf{(13)}$$

EM wave equation in conducting medium :

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

Taking curl on both sides

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \mu \left(\vec{\nabla} \times \vec{J}\right) + \mu\epsilon \frac{\partial}{\partial t}\left(\vec{\nabla} \times E\right)$$

$$\Longrightarrow \vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{B} = \mu \left(\vec{\nabla} \times \sigma\vec{E}\right) + \mu\epsilon \frac{\partial}{\partial t}\left(\vec{\nabla} \times E\right) \quad \left(\because \vec{J} = \sigma\vec{E}\right)$$

$$\Longrightarrow 0 - \nabla^2\vec{B} = \left(-\mu\sigma\frac{\partial \vec{B}}{\partial t} - \mu\epsilon\frac{\partial^2 \vec{B}}{\partial t^2}\right) \qquad \left(\because \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Longrightarrow \nabla^2\vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \qquad \textbf{(14)}$$

EM wave equation in conducting medium :

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Since we have established EM waves are transverse in nature. So, if EM wave is travelling along z-axis, Electric field along y-axis then magnetic field will be along x-axis i.e.

$$\vec{E} = E_{0y} e^{i(kz-\omega t)} \hat{y} \qquad \vec{B} = B_{0x} e^{i(kz-\omega t)} \hat{x}$$

Putting value of $\vec{E}$ in above equation (13) :

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(E_{0y} e^{i(kz-\omega t)}\right) =$$

$$\mu\sigma \frac{\partial}{\partial t}\left(E_{0y} e^{i(kz-\omega t)}\right) + \mu\epsilon \frac{\partial^2}{\partial t^2}\left(E_{0y} e^{i(kz-\omega t)}\right)$$

$$\Longrightarrow (ik)^2 E_{0y} e^{i(kz-\omega t)} = \mu\sigma(-i\omega) E_{0y} e^{i(kz-\omega t)} + \mu\epsilon (-i\omega)^2 E_{0y} e^{i(kz-\omega t)}$$

EM wave equation in conducting medium :

$$\Longrightarrow\ (ik)^2 E_{0y} e^{i(kz-\omega t)} = \mu\sigma(-i\omega) E_{0y} e^{i(kz-\omega t)} + \mu\epsilon\,(-i\omega)^2 E_{0y} e^{i(kz-\omega t)}$$

$$\Longrightarrow\ (ik)^2 = \mu\sigma(-i\omega) + \mu\epsilon\,(-i\omega)^2$$

$$\Longrightarrow\ -k^2 = -i\mu\sigma\omega - \mu\epsilon\,\omega^2$$

$$\Longrightarrow\ k^2 = i\mu\sigma\omega + \mu\epsilon\omega^2 \qquad \textbf{(15)}$$

You can get equation (15) by putting value of $\vec{B}$ in equation (14). Equation (15) also implies that k can be written as

$$k = k_+ + ik_- \qquad \Longrightarrow\ k^2 = {k_+}^2 - {k_-}^2 + 2ik_+k_- \qquad \textbf{(16)}$$

Comparing eq. (15) and (16)

$$\mu\epsilon\omega^2 = {k_+}^2 - {k_-}^2$$

$$\mu\sigma\omega = 2k_+k_- \qquad \Longrightarrow\ k_- = \frac{\mu\sigma\omega}{2k_+}$$

EM wave equation in conducting medium :

$$\mu\epsilon\omega^2 = {k_+}^2 - {k_-}^2 \qquad \textbf{(17)}$$

$$\mu\sigma\omega = 2k_+k_- \quad \Longrightarrow \; k_- = \frac{\mu\sigma\omega}{2k_+} \qquad \textbf{(18)}$$

Putting value of k_- from eq. (18) to (17)

$$\mu\epsilon\omega^2 = {k_+}^2 - \left(\frac{\mu\sigma\omega}{2k_+}\right)^2$$

$$\Longrightarrow {k_+}^4 - \mu\epsilon\omega^2\, {k_+}^2 - \left(\frac{\mu\sigma\omega}{2}\right)^2 = 0$$

This is quadratic equation, solving it we will get

$${k_+}^2 = \frac{\mu\epsilon\omega^2 \pm \sqrt{(\mu\epsilon\omega^2)^2 + (\mu\sigma\omega)^2}}{2}$$

Or $\quad {k_+}^2 = \frac{\mu\epsilon\omega^2}{2}\left[1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)}\right]$

EM wave equation in conducting medium :

$$\text{Or} \quad {k_+}^2 = \frac{\mu\epsilon\omega^2}{2}\left[1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\,\right]$$

On physical grounds, -ve sign is not acceptable, because ${k_+}^2$ will be negative. From previous equations :

$$\mu\epsilon\omega^2 = {k_+}^2 - {k_-}^2$$

$$\mu\sigma\omega = 2k_+k_- \qquad \Longrightarrow k_+ = \frac{\mu\sigma\omega}{2k_-}$$

Solving this time for k_- as before done for k_+, one will get

$$\text{Or} \quad {k_-}^2 = \frac{\mu\epsilon\omega^2}{2}\left[-1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\,\right]$$

Again, on physical grounds, -ve sign (± between brackets) is not acceptable, because ${k_-}^2$ will be negative

EM wave equation in conducting medium :

$$k_{\pm}{}^2 = \frac{\mu\epsilon\omega^2}{2}\left[\pm 1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\,\right]$$

$$\Longrightarrow\ k_+ = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\,\right]^{1/2} \quad \text{and} \quad k_- = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\,\right]^{1/2}$$

Both k_+ as well as k_- are frequency (ω) dependent. We had defined wave vector (k), electric field ($\vec{E}$) and magnetic field ($\vec{B}$) as

$$k = k_+ + ik_- \quad \text{and} \quad \vec{E} = E_{0y}e^{i(kz-\omega t)}\hat{y} \quad \text{and} \quad \vec{B} = B_{0x}e^{i(kz-\omega t)}\hat{x}$$

Electric field ($\vec{E}$) and magnetic field ($\vec{B}$) can be now written as :

$$\vec{E} = E_{0y}e^{i((k_+ + ik_-)z-\omega t)}\hat{y} \qquad \vec{B} = B_{0x}e^{i(((k_+ + ik_-)z-\omega t)}\hat{x}$$

or $$\vec{E} = E_{0y}\ \underbrace{e^{-k_- z}}\ e^{i(k_+ z-\omega t)}\hat{y} \qquad \vec{B} = B_{0x}\,\underbrace{e^{-k_- z}}\,e^{i(k_+ z-\omega t)}\hat{x}$$

Damping term

EM wave equation in conducting medium :

$$\vec{E} = E_{0y}\ \underbrace{e^{-k_- z}}_{\text{Damping term}}\ e^{i(k_+ z-\omega t)}\hat{y} \qquad \vec{B} = B_{0x}\ \underbrace{e^{-k_- z}}_{\text{Damping term}}\ e^{i(k_+ z-\omega t)}\hat{x}$$

Skin depth (δ) is defined as the distance at which $\vec{E}$ and $\vec{B}$ are reduced to 1/e of its initial value.

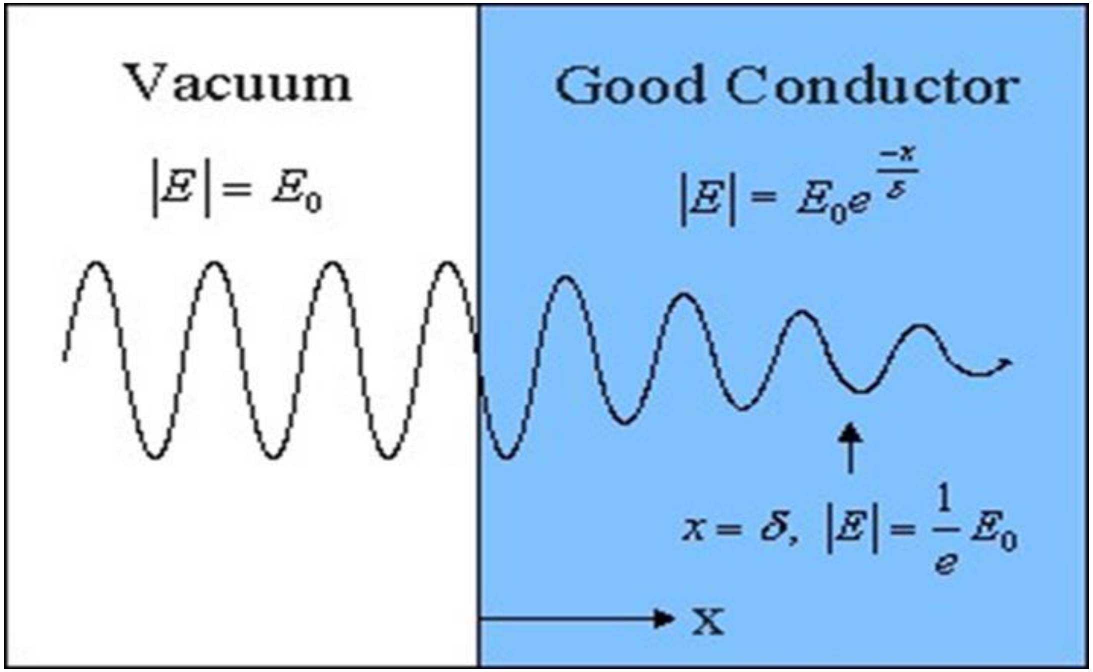
When $z = \frac{1}{k_-}$, $\left|\vec{E}\right| = \frac{E_{0y}}{e}$ and $\left|\vec{B}\right| = \frac{B_{0x}}{e}$, this value of z will be skin depth (**δ**).

Real part of wave vector k_+ determines wavelength (λ) and propagation speed (v).

$$\lambda = \frac{2\,\pi}{k_+} \qquad \text{and} \qquad v = \frac{\omega}{k_+}$$

EM wave equation in conducting medium :

Schematic representation of skin depth (Wave is travelling in x-direction):



EM wave equation in conducting medium :

For poor conductor ($\sigma \ll \epsilon\omega$ or $\frac{\sigma}{\epsilon\omega} \ll 1$) :

$$k_+ = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\;\right]^{1/2}$$

Since $\frac{\sigma}{\epsilon\omega} \ll 1$, so it can be neglected

$$\Rightarrow k_+ \cong \omega\sqrt{\frac{\mu\epsilon}{2}}\left[1+\sqrt{1}\;\right]^{1/2}$$

$$\Rightarrow k_+ \cong \omega\sqrt{\frac{\mu\epsilon}{2}}\left[2\;\right]^{1/2}$$

$$\Rightarrow k_+ \cong \omega\sqrt{\mu\epsilon}$$

EM wave equation in conducting medium :

For poor conductor ($\boldsymbol{\sigma \ll \epsilon\omega}$ or $\boldsymbol{\frac{\sigma}{\epsilon\omega} \ll 1}$) :

$$k_- = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[-1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\,\right]^{1/2}$$

Uisng binomial theorem when $x < 1$, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 \ldots\ldots$

$$\Rightarrow k_- = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[-1+\left(1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2+\cdots\right)\right]^{1/2}$$

$$\Rightarrow k_- = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/2}$$

$$\Rightarrow k_- = \omega\sqrt{\frac{\mu\epsilon}{2}}\,\frac{1}{\sqrt{2}}\left(\frac{\sigma}{\epsilon\omega}\right)$$

$$\boldsymbol{\Rightarrow k_- = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}}$$

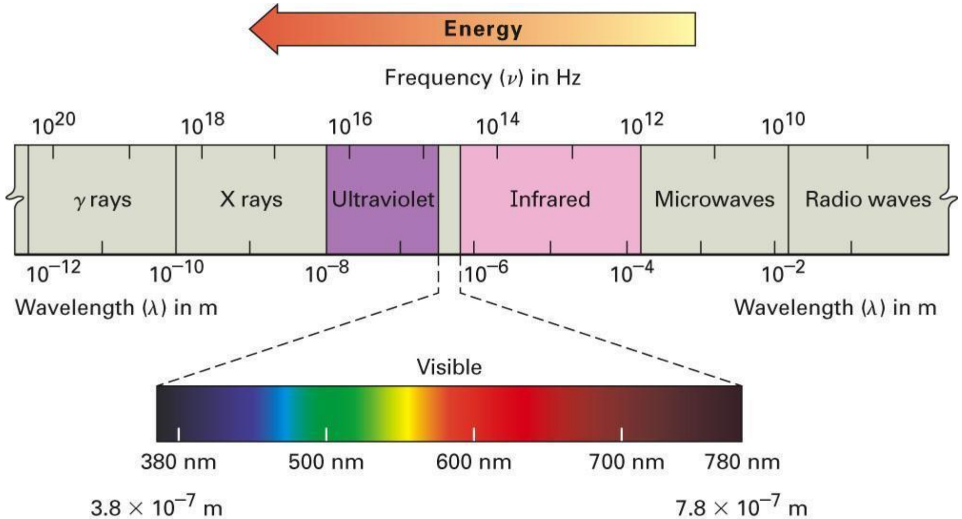
EM wave equation in conducting medium :

For poor conductor ($\sigma \ll \epsilon\omega$ or $\frac{\sigma}{\epsilon\omega} \ll 1$) :

$$\Rightarrow k_{-} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$$\textbf{Skin depth } (\delta) = \frac{1}{k_{-}} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

Notice that skin depth (δ) is independent of frequency (ω) for a poor conductor. That means out of spectrum shown below whatever is incident wave, the penetration into medium will be same as long as $\boldsymbol{\sigma \ll \epsilon\omega}$ *is true.*



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EM wave equation in conducting medium :

For good conductor ($\sigma \gg \epsilon\omega$ or $\frac{\sigma}{\epsilon\omega} \gg 1$) :

$$k_+ = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]^{1/2}$$

Since $\frac{\boldsymbol{\sigma}}{\boldsymbol{\epsilon\omega}} \gg \mathbf{1}$, so we can ignore 1 in comparison to $\frac{\boldsymbol{\sigma}}{\boldsymbol{\epsilon\omega}}$ and k_+ will be

$$k_+ \cong \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\left(\frac{\sigma}{\epsilon\omega}\right)\right]^{1/2}$$

$$\Longrightarrow k_+ \cong \sqrt{\frac{\sigma\omega\mu}{2}}$$

EM wave equation in conducting medium :

For good conductor ($\sigma \gg \epsilon\omega$ or $\frac{\sigma}{\epsilon\omega} \gg 1$) :

$$k_{-} = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[-1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\;\right]^{1/2}$$

Since $\frac{\sigma}{\epsilon\omega} \gg \mathbf{1}$, so we can ignore 1 in comparison to $\frac{\sigma}{\epsilon\omega}$ and k_{-} will be

$$k_{-} \cong \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\left(\frac{\sigma}{\epsilon\omega}\right)\right]^{1/2}$$

$$\Rightarrow k_{-} \cong \sqrt{\frac{\sigma\omega\mu}{2}} \qquad \textit{we just derived} \quad k_{+} \cong \sqrt{\frac{\sigma\omega\mu}{2}}$$

Hence, for good conductor, $k_{+} = k_{-}$

EM wave equation in conducting medium :

For good conductor ($\boldsymbol{\sigma \gg \epsilon\omega}$ or $\boldsymbol{\frac{\sigma}{\epsilon\omega} \gg 1}$) :

$$k_- \cong \sqrt{\frac{\sigma\omega\mu}{2}}$$

$$\textbf{Skin depth } (\boldsymbol{\delta}) = \frac{\mathbf{1}}{\boldsymbol{k_-}} = \sqrt{\frac{\mathbf{2}}{\boldsymbol{\sigma\omega\mu}}}$$

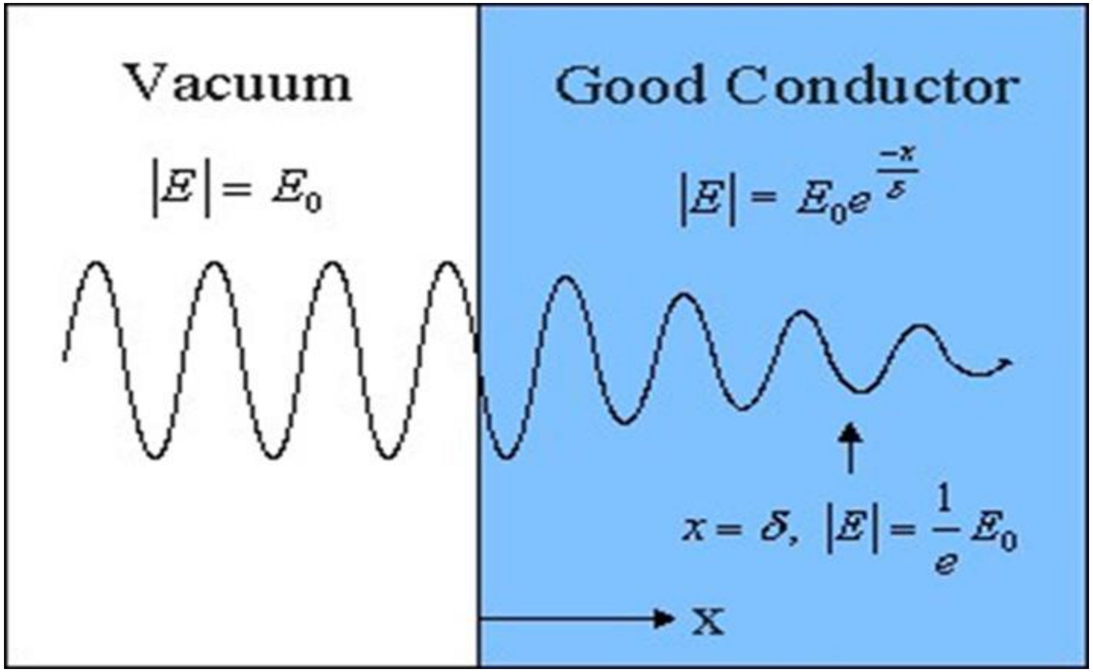
Notice that skin depth (δ) is dependent on frequency (ω) for a good conductor. Higher is frequency (ω), less will be skin depth (δ).

For copper, with $\mu \approx \mu_0$ 0 and $\sigma = 5.8 \times 10^7\ S/m$ at a frequency of 60 Hz, $\delta \approx$ 9 mm;
at 1 MHz, $\delta \approx 6.6 \times 10^{-5}$ m
and at 30,000 MHz (radar wavelength of 1 cm), $\delta \approx 3.8 \times 10^{-7}$ m.

We see also why a conductor can act to 'shield' a region from electromagnetic waves.

EM wave equation in conducting medium :

Schematic representation of skin depth :



For ex. Silver has conductivity $\sigma = 6.30 \times 10^7\, S/m$ and permittivity $\epsilon_0 \approx \epsilon \approx 8.85 \times 10^{-12}\, F/m$. For frequency 10^{10} Hz, condition of good conductor is satisfied, skin depth is approx. 0.6 μm.

Wavelength of radiation of frequency 10^{10} Hz is about 3 cm; but in silver wavelength is $\lambda = \frac{2\pi}{k_+} = \frac{2\pi}{k_-} = 2\pi\delta \approx 4\mu m$. (for good conductor $k_+ = k_-$)

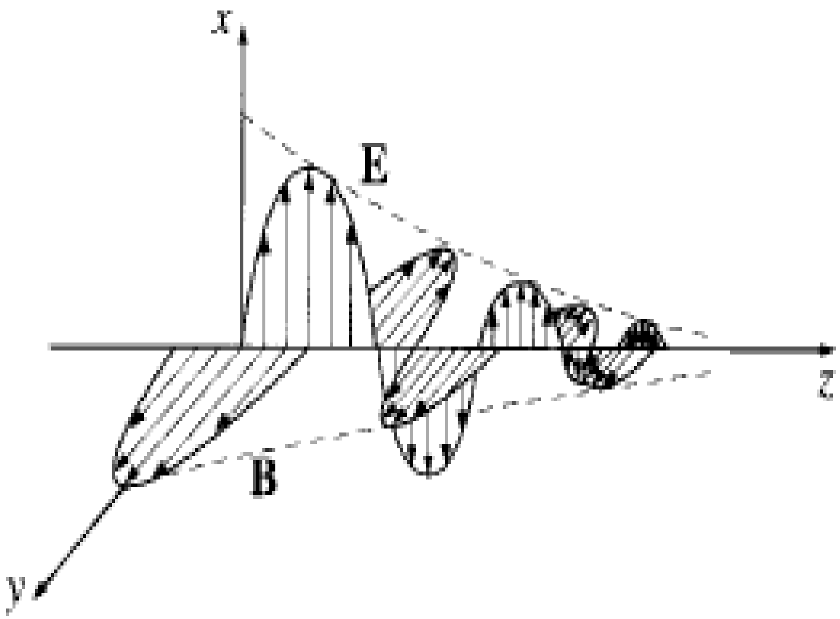
EM wave equation in conducting medium :

For good conductor ($\sigma \gg \epsilon\omega$ or $\frac{\sigma}{\epsilon\omega} \gg 1$) :

The phase difference between electric filed and magnetic field in a good conductor is given by

$$\tan\phi = \frac{k_-}{k_+} = 1 \qquad \text{(for good conductor } k_+ = k_-\text{)}$$

$$\Rightarrow \phi = 45°$$



A schematic representation of phase difference and exponential decaying amplitude in a good conductor for a wave travelling in z–direction, electric field oscillating along x-axis and magnetic field oscillating along y-axis.

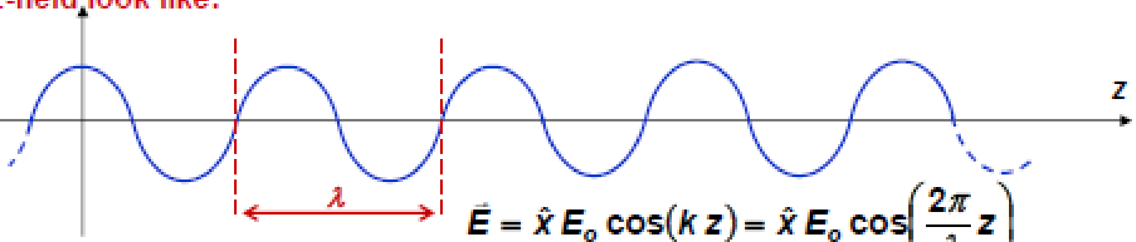
A pictorial representation of travelling wave :

Sinusoidal Solutions of Electromagnetic Wave Equation - V

Consider the plane wave:

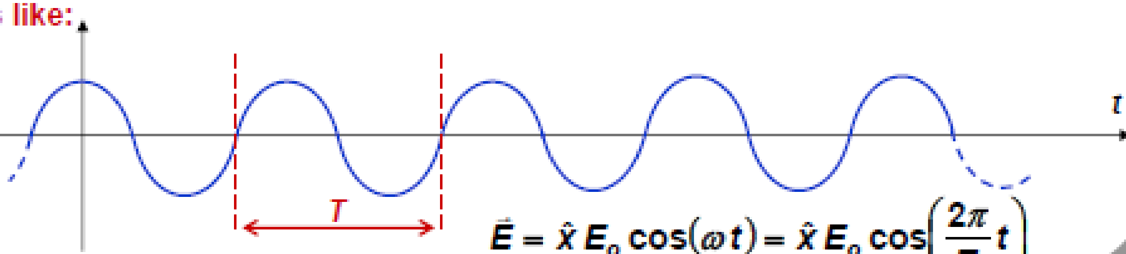
$$\vec{E} = \hat{x}\, E_o \cos(\omega t - k z) \qquad \vec{H} = \hat{y}\, \frac{E_o}{\eta_o} \cos(\omega t - k z)$$

If a person takes a snapshot of the wave in space at any time, say at $t = 0$, he will see E-field look like:



$$\vec{E} = \hat{x}\, E_o \cos(k z) = \hat{x}\, E_o \cos\left(\frac{2\pi}{\lambda} z\right)$$

If a person sits at one location, say $z = 0$, he will see an oscillating E-field in time that looks like:



$$\vec{E} = \hat{x}\, E_o \cos(\omega t) = \hat{x}\, E_o \cos\left(\frac{2\pi}{T} t\right)$$