Unit-I
(Ordinary Differential EQNs)

→ In general most of the laws in nature, physics, chemistry or any other field are mostly expressed in form of differential equations.

Also, while solving various physical problems in science & engineering, when we convert them to mathematical form or equation (Mathematical modeling) then they usually comes out to be in the form of differential equations. So it is necessary that we should know how to solve differential equations.

(Differential equations are tools that are used to study change in physical world)

eg → Newton's Second law motion →

$$F = ma = m\frac{d^2x}{dt^2}$$

→ Current (I) flowing through LCR-circuit

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E$$

→ ordinary differential equation →

A relation between dependent variable ~~(xyz)~~ independent variable ~~(xyz)~~ and derivative of dependent variable w.r.t one or more independent variable is called D.Eqn.

eg → $\frac{dy}{dx} + y = x$ $\quad y \to$ dep, $x \to$ Ind.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = y \quad y = f(x)$$

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial z} = x \quad \to y = f(x, z)$$

x & $z \to$ independent variables

Ordinary D.E (ODE) → The differential equation that contain ordinary derivatives (that is, here dependent variable depends only on one independent variable)

eg → $\frac{dy}{dx} + y = x$] $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 3.$]

Here $y = f(x)$ → [LCR - circuit / ODE] → Motion of mass attached to spring.

→ **Partial differential equation** → In case dependent variable depends upon more than one independent variable then we get partial derivatives & differential equation is called PDE.

eg → let $y = f(x, z)$ then $\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial z} = 0$

$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} + \frac{\partial y}{\partial x} = 2$

are partial diff. eqn

[Heat eqn / Wave eqn → PDE]

→ **Order of a D.E** → is the order of the highest derivative term occuring in the D.E

eg → 1) $y'' \cdot y + y^2 = x^2$ ⟶ Order = 2.

2) $[1 + (y')^2]^{1/2} = 5y$ → Order = 1.

3) $[(y'')^2] = 1 + (y')^3$ ⟶ Order = 2.

Note Linear ODE → A D·E of order (n) is said to be linear if it is / or it can be expressed in the form →

$$a_0(x) + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^n(x) = b(x) \quad \text{--- (1)}$$

where $a_n \neq 0 \ \forall x$.

→ y' ~~y_1~~ → 1st derivative, y'' → 2nd derivative
— y^n or $y^n(x)$ → nth derivative

→ $a_0, a_1, \ldots, a_n$ are fns of x only or constant.

So from above def it is clear that in a linear ODE

1) Dependent variable (y) & its various derivatives occur in degree (1)

2) ∄ (there does not exist) any term containing product of dependent variable (y) and/or its various derivatives

3) ∄ any term containing transcendental fn of dependent variable (y) and/or its derivatives

→ D·E that is not linear is called nonlinear D.E.

Note From above points it is clear that →

1 Point (1) → Linear ODE will not contain terms like
$y^2, y^3, y^4, \ldots$ or $(y')^2, (y'')^2, \ldots$

2 Point (2) → Linear ODE will not contain terms like
$yy', yy'', y'y'', y''y''', \ldots$

3 Point (3) → Linear ODE will not contain terms of type →
$\sin y, \cos y, e^y, \ln y, \sin^{-1}y, \cosh y, \ldots$

(*) Transcendental fn → A fn that does not satisfy the Polynomial eqn, that is, it can not be expressed as finite combination/sequence of algebraic operations of $\oplus, \ominus, /, \times, \wedge$.

Check whether given ODEs are linear or not

1) $y' + x^2 y = x^3$ (L)

2) $(y')^2 + xy = y$ (NL)

3) $y' + xy^2 = x$ (NL)

4) $xy' + \sin y = 1$ (NL)

5) $y' + \sin x = x$ (L)

6) $[1+y']^{1/2} = y''$ (???)

Note → Solution of a D.E → A relation between dependent variable + independent variable that does not contain any derivative term + satisfies the D.E is called soln of a D.E.

eg → If $y' = y$ is a D.E then
$y = ce^x$ is soln of this D.E.

Types of Solution → 1) General Soln (Complete/Primitive)
2) Particular Soln 3) Singular Soln.

1 General Soln → A soln that satisfies D.E and contains as many constants as the order of a D.E. is called G.S.

eg → $y = ce^{x}$ is G.S of $y' = y$ (One arbitrary constant in soln)

$y = c_1 \cos x + c_2 \sin x$ is G.S of $y'' + y = 0$. (Two arbitrary constants in soln)

(*) Geometrically G.S of a D.E represents family of curves.

2 Particular Soln → If we give value to constants in the G.S we get Particular soln.

eg → $y = 2e^{x}$ is P.S of $y' = y$

$y = \cos x + 2\sin x$ is P.S of $y'' + y = 0$

3 Singular Soln → A solution that cannot be obtained from general soln of a D.E but still satisfies the D.E.

eg (1) $(y')^2 + xy' = y$ ⟶ G.S → $y = cx + c^2$

Singular Soln → $y = -x^2/4$.

(2) $(y')^2 - 4y = 0$

G.S → $y = (x+c)^2$ | Singular Soln → $y = 0$

(*) Geometrically singular soln represents a curve that is ~~repres~~ formed from the envelope of family of curves of G.S.

(*) In eg(1) & eg(2), we can see that singular solution of the D.E cannot be obtained from the general solution of D.E by giving any value to constants but they still satisfy D.E.

Note Initial value Problem (IVP) → A given D.E along with given conditions is said to form initial value problem (IVP) if all the given conditions (to find constants) are prescribed at only one single point.

eg → $y'' + y' = x+1$, $\left.\begin{array}{l} y(0) = 1 \\ y'(0) = -1 \end{array}\right\}$

→ $y''' + xy' = y$, $\left.\begin{array}{l} y(0) = 1 \\ y'(0) = 0,\ y''(0) = -1 \end{array}\right\}$

→ **Boundary value problem (BVP)** → A D.E along with given conditions is said to form BVP if the conditions are prescribed at more than one pt.

eg → 1) $y'' + y = x$, $\left.\begin{array}{l} y(0) = 0 \\ y(1) = 1 \end{array}\right\}$

2) $y''' + xy = y'$, $\left.\begin{array}{l} y(0) = 0 \\ y'(0) = 1 \\ y(1) = -1 \end{array}\right\}$

Note Formation of a D.E → (Optional)

Working method to form a D.E from relation

1) Count No. of arbitrary constants in a given relation.
2) Differentiate the ~~Dx~~ relation or equation as many times as the No. of constants
3) Eliminate constants from relations & get D.E.

eg ①. Form a D.E from relation →

$$y = cx^2 \quad -① \qquad \text{So} \quad y' = 2cx$$

$$\Rightarrow c = y'/2x \quad -②$$

Put ② in ①
we get

$$y = \left(y'/2x\right)(x^2) = y'/2 \, x$$

$$\Rightarrow \boxed{xy' - 2y = 0}$$

eg ②. Show that relation $y^2 = m(a^2 - x^2)$ ① satisfies D.E → $(xy)y'' + x(y')^2 - yy' = 0$ ②

Soln → ∵ $y^2 = m(a^2 - x^2)$, m & a constants.

$$2yy' = m(-2x)$$

$$\Rightarrow yy' = -mx \quad ③ \qquad \Rightarrow m = -yy'/x \quad ④$$

Again diff w.r.t (x)

$$yy'' + (y')^2 = -m \quad -⑤$$

Put ④ in ⑤ $\Rightarrow yy'' + (y')^2 = yy'/x$

$$\Rightarrow \left[(xy)y'' + x(y')^2 - yy' = 0\right]$$

Proved.

Note Methods to solve first order ODE of form $\rightarrow$ $y' = f(x, y)$ | $F(x, y, y') = 0$.

$\rightarrow$ **Variable Separable method** $\rightarrow$

eg① Solve the D.E $\rightarrow$ $(x \ln x) y' = y$. — ①

Here we take the terms of same variable on one side & then integrate to get soln.

$$\rightarrow \quad x \ln x \frac{dy}{dx} = y \quad \Rightarrow \frac{1}{y} dy = \frac{1}{x \ln x} dx$$

$$\int \frac{1}{y} dy = \int \frac{1/x}{\ln x} dx \Rightarrow \ln y = \ln(\ln x) + c$$

$$\Rightarrow \ln y - \ln(\ln x) = c \quad \Rightarrow \ln\left(\frac{y}{\ln x}\right) = c$$

$$\Rightarrow y = e^{c} \ln x$$

$$\boxed{y = c_1 \ln x} \text{ — ② Soln}$$

Eg② (see bottom of Page 8)

$\rightarrow$ **Equations reducible to variable separable form (By Substitution)**

Eqns of form $\rightarrow$ $\frac{dy}{dx} = f(ax + by + c)$ | Put $ax + by + c = z$.

eg① Find General Solution of $\rightarrow$ $\frac{dy}{dx} = (x + y + 1)^2$ — ①

∵ not in variable separable form in the present case.

So Put $x + y + 1 = z$ — ② $\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \text{— ③}$$

Put ② + ③ in ①

$$\frac{dz}{dx} - 1 = z^2$$

$$\Rightarrow \frac{1}{1 + z^2} dz = dx \Rightarrow \int \frac{1}{z^2 + 1} dz = \int dx \Rightarrow \tan^{-1}(z) = x + c$$

$$\boxed{\tan^{-1}(x + y + 1) = x + c}$$

Note →

It is not always necessary that we will always get D·E where substitution will be of the form $(ax+by+z)=z$. It depends on the D·E given

eg② Find the G·S of given D·E

$$xy' = xe^{-y/x} + y \quad — ①$$

Soln

∵ from ① we have $y' = e^{-y/x} + y/x$ — ②

Put $y/x = t$ — ③ $\Rightarrow y = xt$

So that $\frac{dy}{dx} = t + x\frac{dt}{dx}$ — ④

Put ③ & ④ in ② we get

$$t + x\frac{dt}{dx} = e^{-t} + t \Rightarrow x\frac{dt}{dx} = e^{-t}$$

$$\Rightarrow e^{t}\,dt = \frac{1}{x}dx \quad \text{Integrating we get}$$

$$\int e^{t}dt = \int \frac{1}{x}dx$$

$$e^{t} = \ln(x) + C.$$

$$\Rightarrow \boxed{e^{y/x} = \ln(x) + C} \quad \text{Required G. Soln}$$

eg② of Variable Separable method

Find G.S of →

$(1+x)y - (1+y)x\frac{dy}{dx} = 0$

$x>0, y>0$

↓

$\frac{dy}{dx}x(1+y) = (1+x)y$

Soln → Rearranging terms we get

$(1+1/y)dy = (1+1/x)dx$

Integrating →

$y + \ln y = x + \ln x + c$

$\Rightarrow (y-x) + \ln(y/x) = c$

eg **Try yourself** →

1) $\frac{dy}{dx} = 1 + x + y + xy.$ $\left[y = c e^{x^2/2 + x} - 1 \right].$

2) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ $\left[e^{y} = e^{x} + x^3/3 + c \right]$

3) $x\frac{dy}{dx} = e^{-xy} - y$ $\left[e^{xy} = x + c \right]$

4) $(x+1)\left(\frac{dy}{dx} - 1\right) = 2(y-x)$ $\left[y = x + c\,(x+1)^2 \right]$