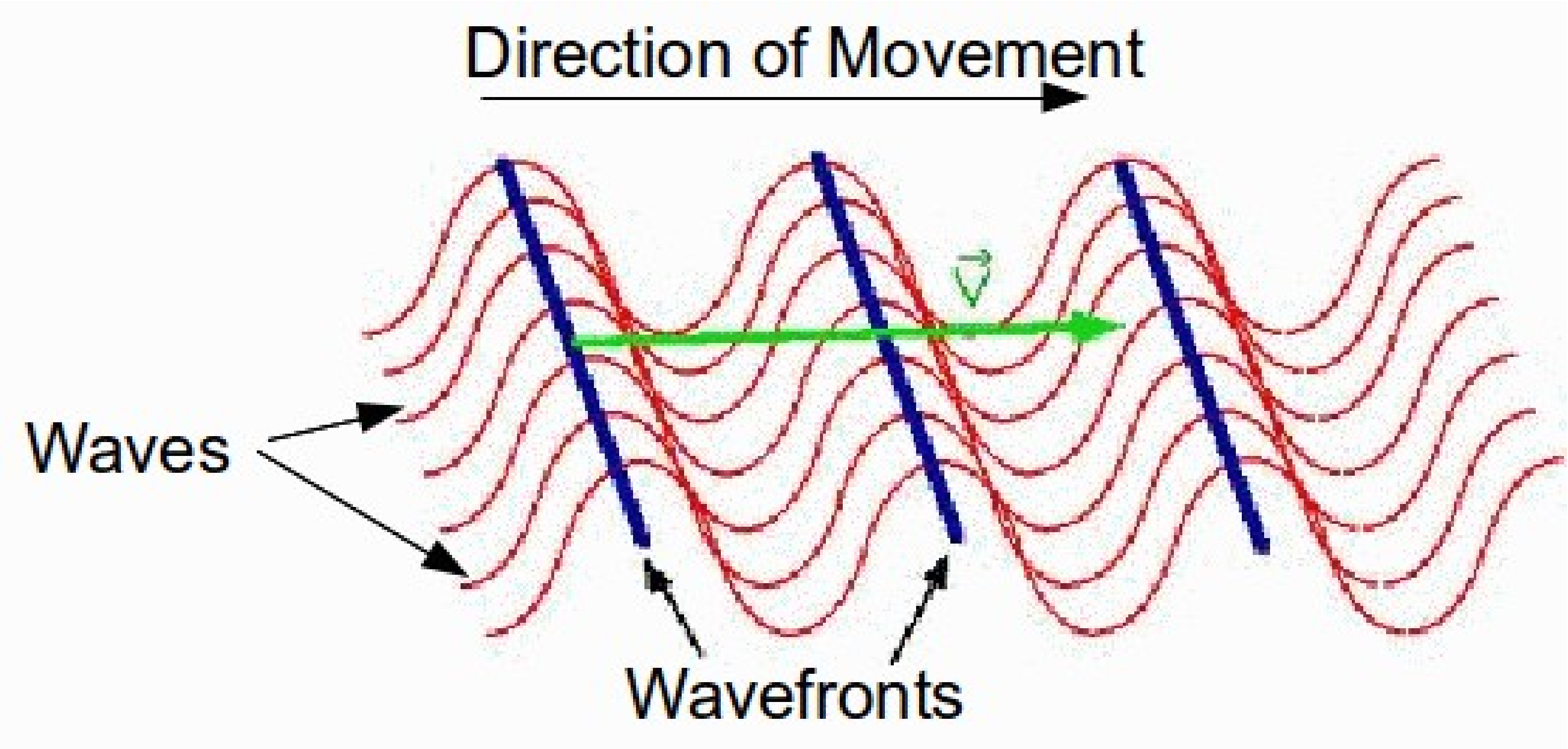
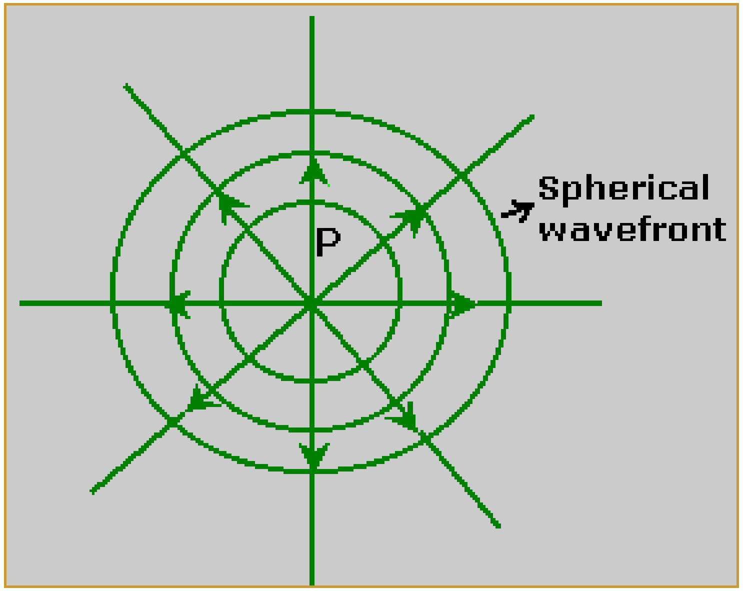
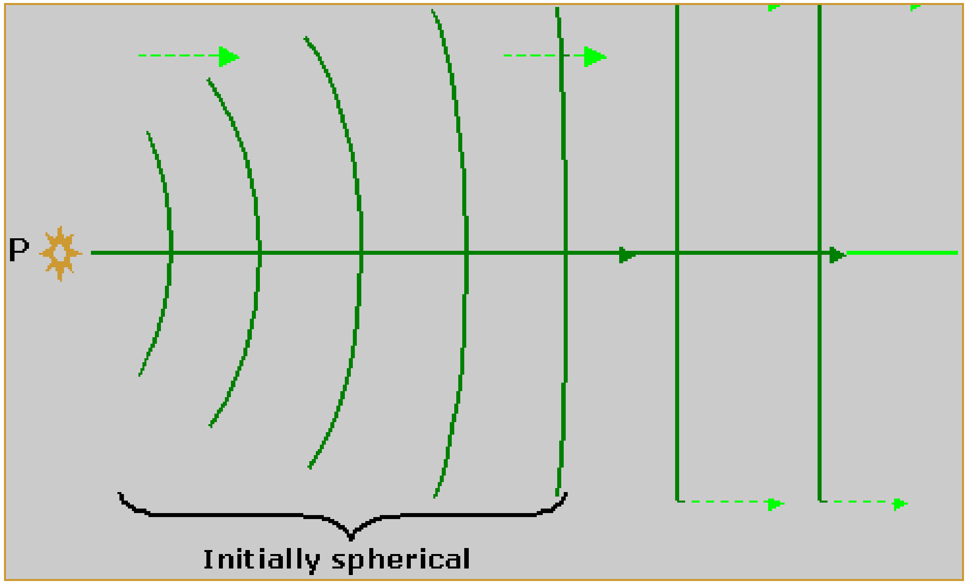
Diffraction

Wavefront:


Direction of Movement
$\vec{v}$
Waves
Wavefronts


P
Spherical
wavefront

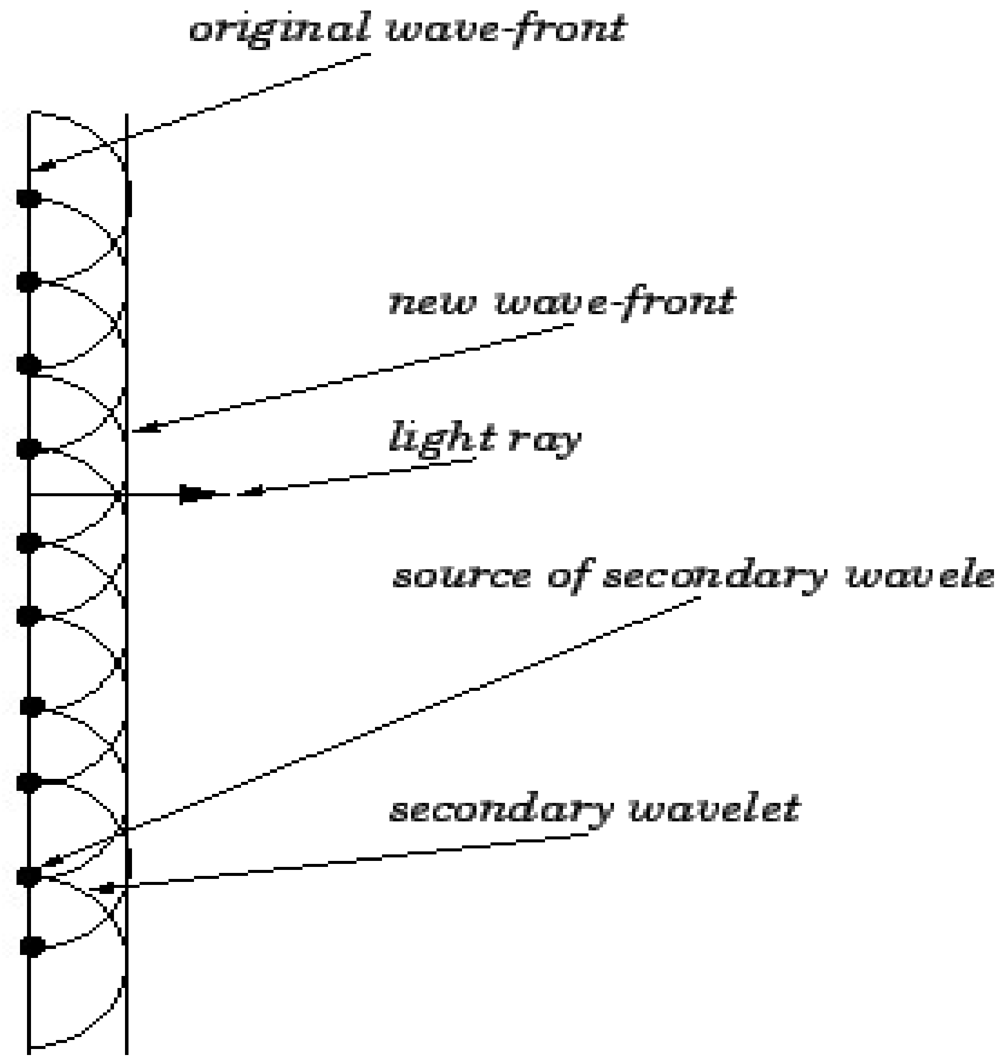

P
Initially spherical

Diffraction definition:

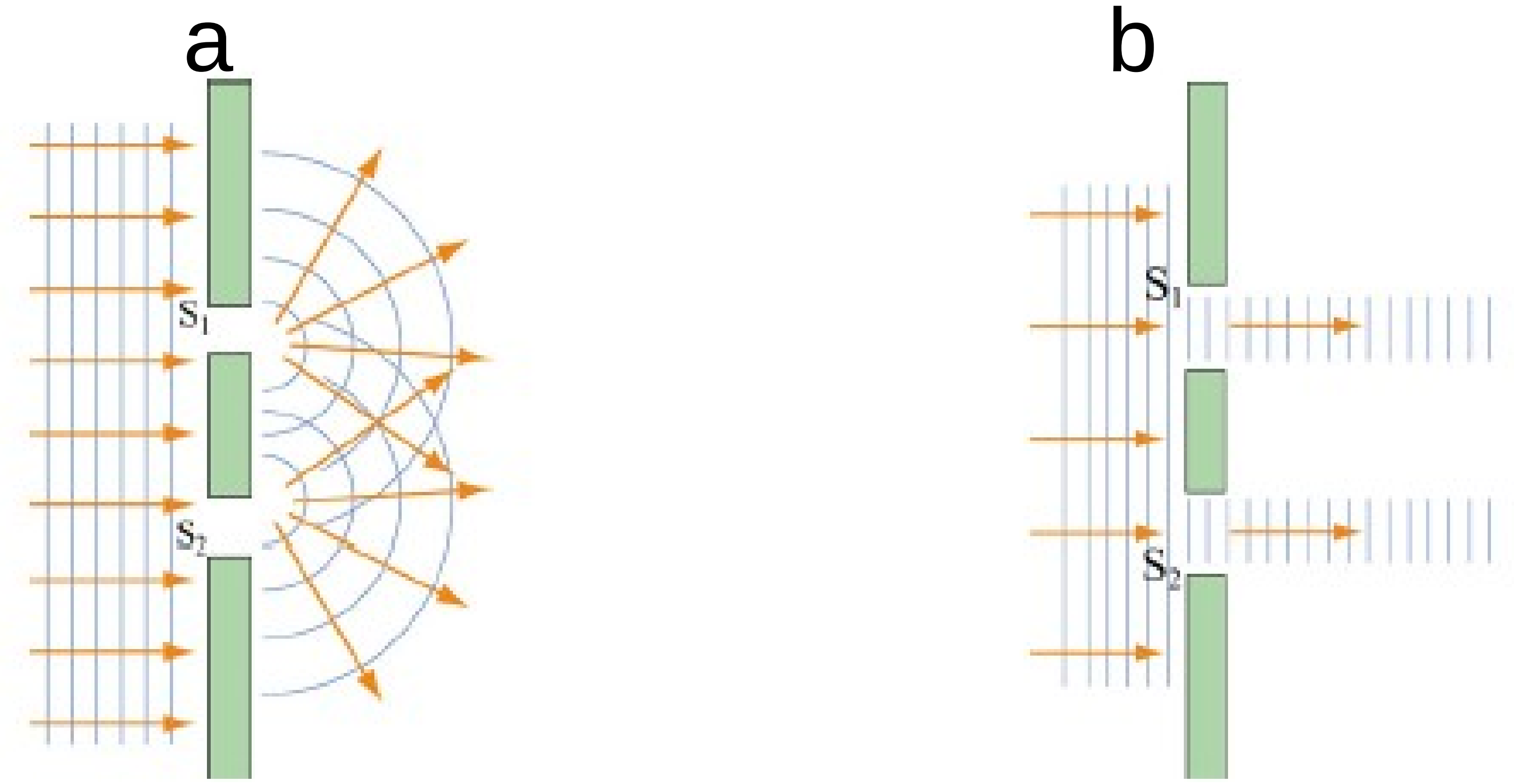
Bending/spreading out of waves as they pass by some objects or through a finite-width aperture.

Huygens's principle:

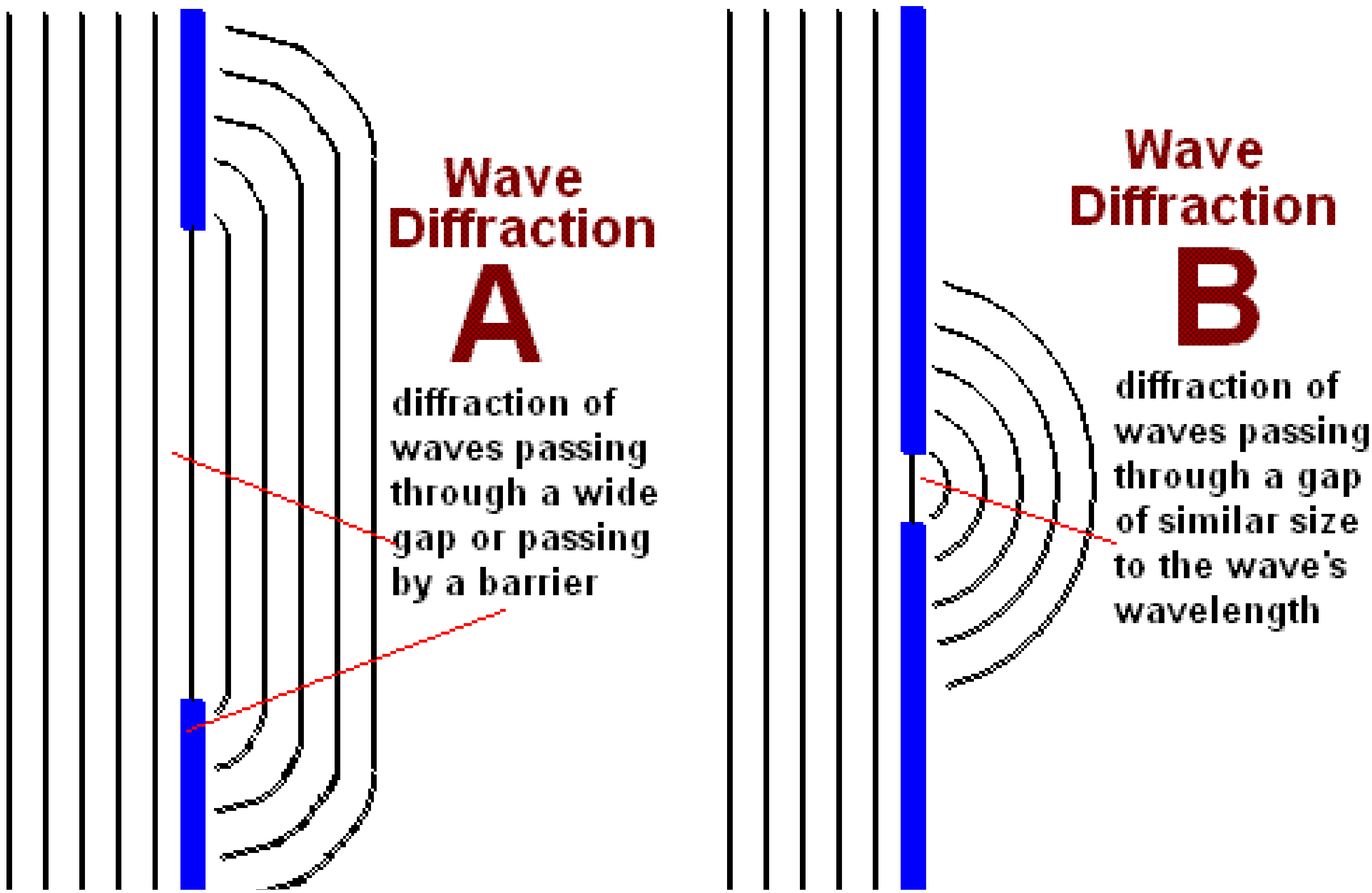
Every unobstructed point on a wavefront will act a source of secondary spherical waves which spread out in the forward direction. The new wavefront is the surface tangent to all the secondary spherical waves.



If a portion of the wavefront enters a different medium (enters glass from air, for example), then the wavelets generated by each portion of the wavefront travel with the velocity that is appropriate for the medium that the wavefront is in.



(a) Spreading of light producing diffraction pattern.
(b) Absence of diffraction patterns if paths of light waves are straight lines.



Wave Diffraction
A
diffraction of waves passing through a wide gap or passing by a barrier
Wave Diffraction
B
diffraction of waves passing through a gap of similar size to the wave's wavelength

Difference between interference and diffraction:

- In Interference, minima are usually perfectly dark while this is not the case for diffraction.

- In interference, all maxima are of same intensity but they have varying intensity in diffraction.

- Fringe width could be equal in some cases in interference while they are never equal in diffraction.

- In interference, interaction takes place between two separate wavefronts originating from two coherent sources while in diffraction, interaction takes place between secondary wavelets originating from same wavefront.

Difference between interference and diffraction:

If you have two infinitely-narrow double slits, there will be just interference, but for finite-width slits there can be both interference and diffraction effects.

Finite width slit: the width of slit is comparable with the wavelength λ.

One continuous wide slit is equivalent to the $N \to \infty$ limit of the N-slit result of interference.

Diffraction is simply the $N \to \infty$ limit of interference, there is technically no need to introduce a new term for it. But on the other hand, a specific kind of pattern arises, so it makes sense to give it its own name.

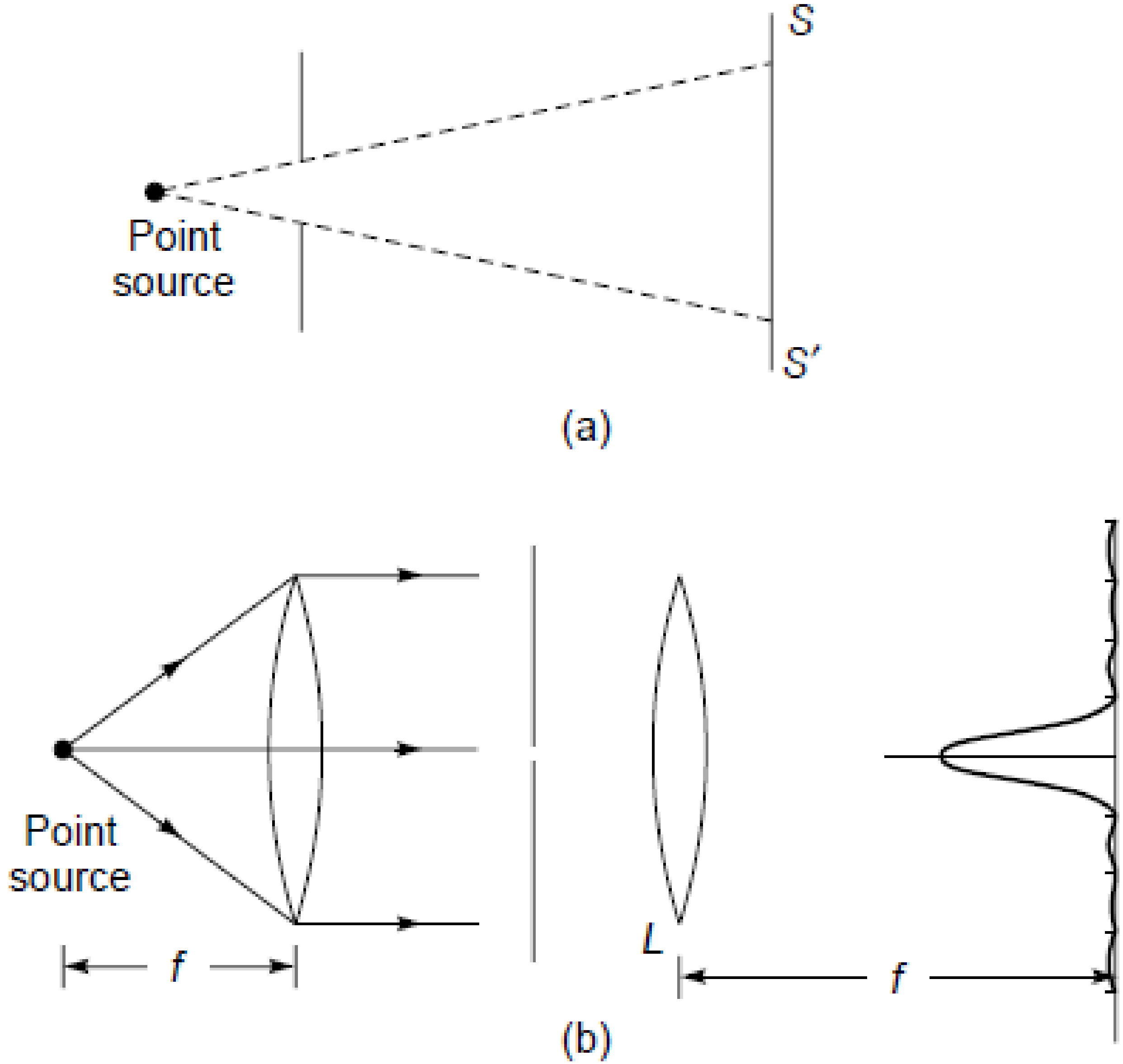
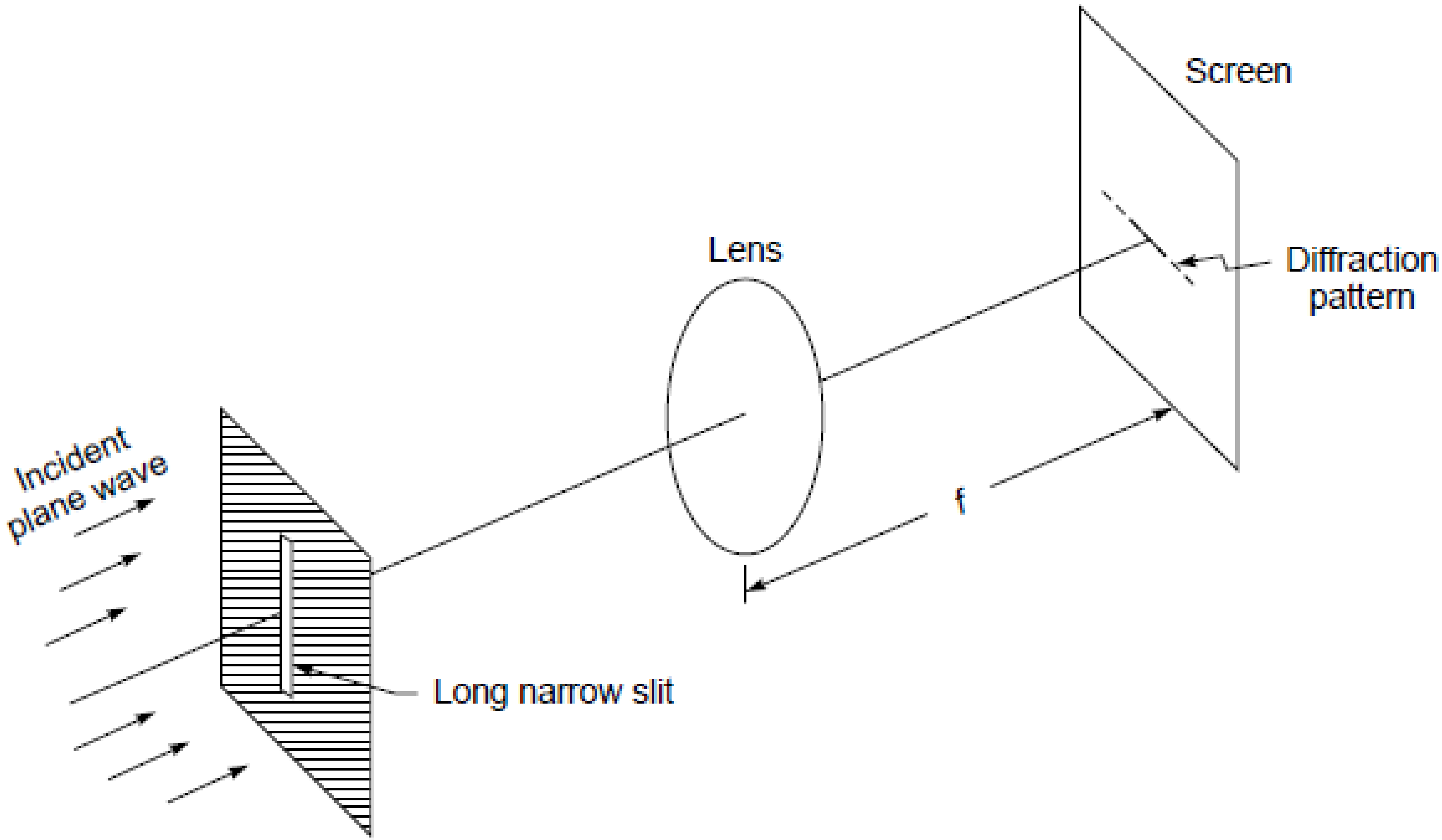
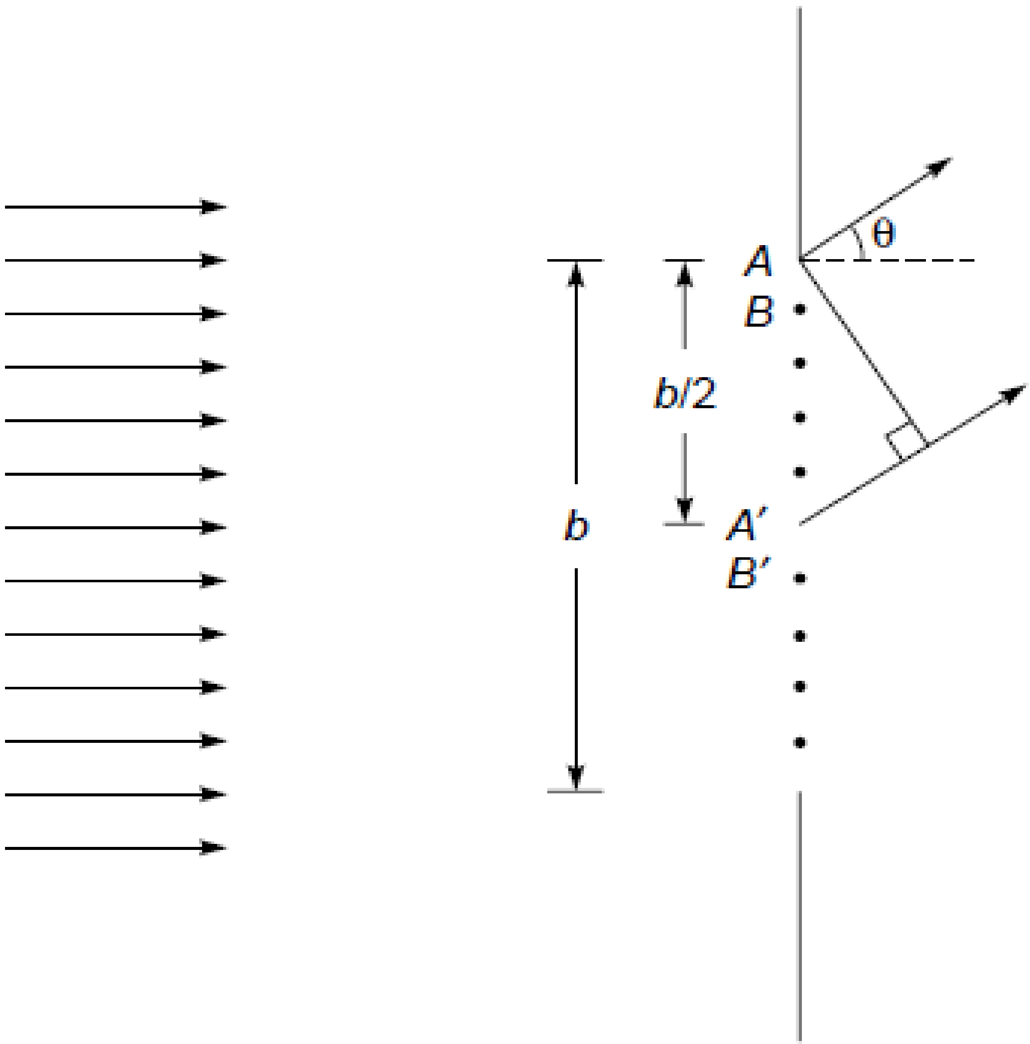


Fig. 18.2 (a) When either the source or the screen (or both) is at a finite distance from the aperture, the diffraction pattern corresponds to the Fresnel class. (b) In the Fraunhofer class both the source and the screen are at infinity.

Diffraction at single slit:



Simplistic way of finding Minima:



Condition for minima if slit is divided in 2 parts:

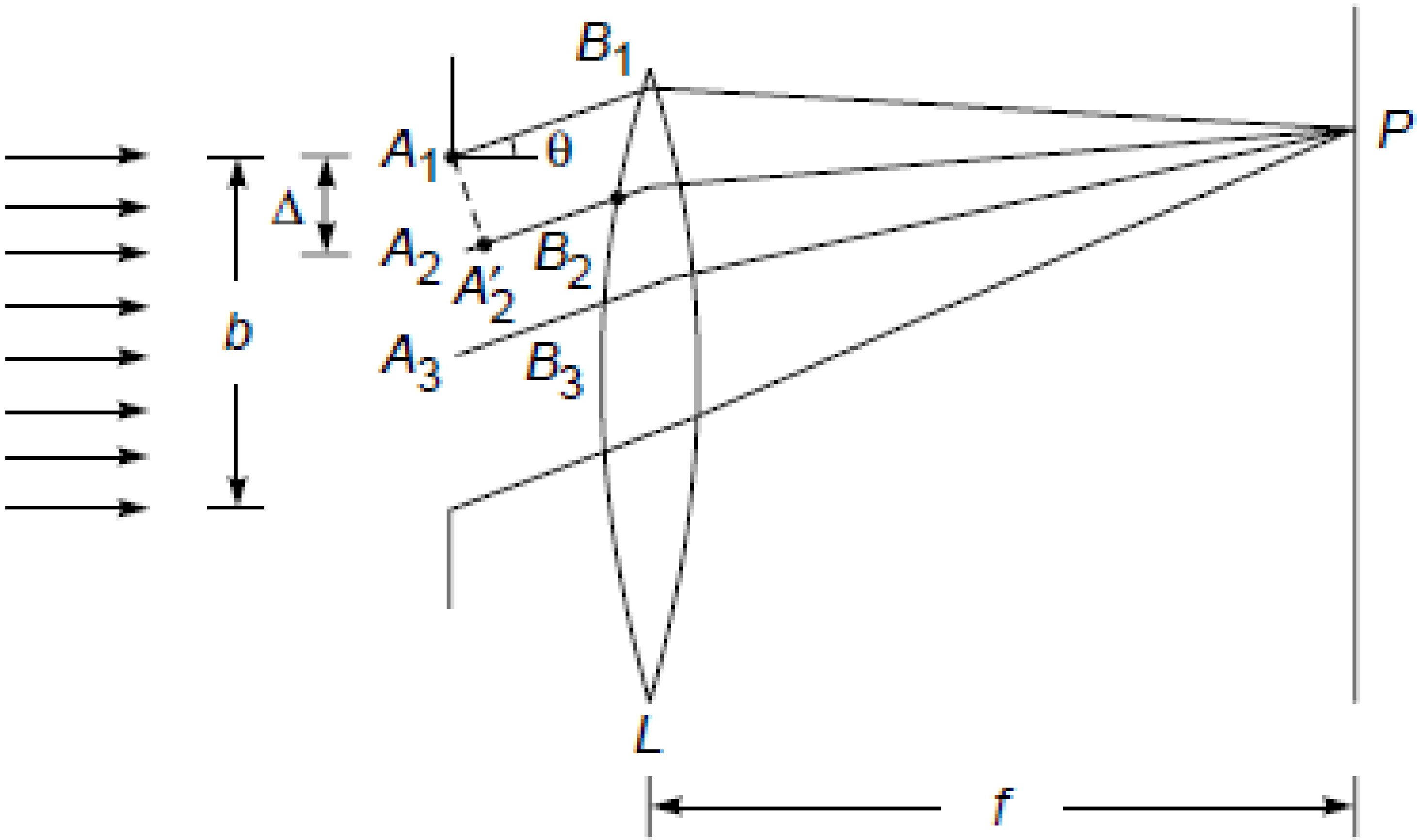
(b/2) sin(θ) = λ/2 or

b sin(θ) = λ

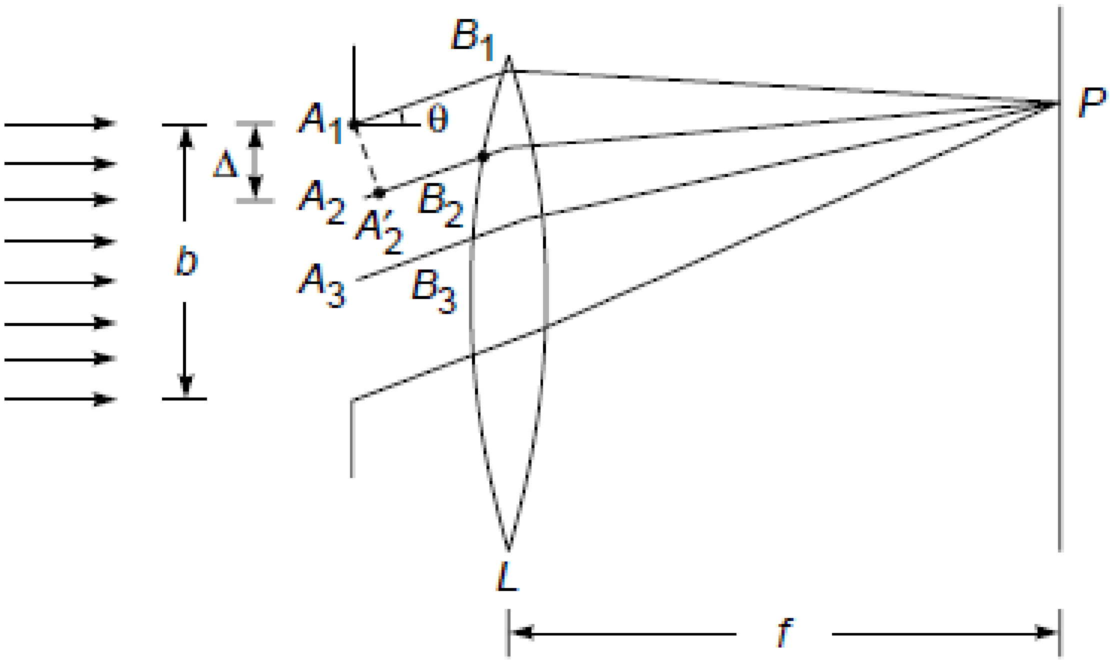
And if slit is divided in 4 parts :
(b/4) sin(θ) = λ/2

And so on …….

Diffraction at single slit:



If the distance between two consecutive points be Δ. Thus, if the number of point sources is n, then $b = (n - 1)\Delta$.



The additional path traversed by the disturbance emanating from point A_2 will be $A_2A'_2$, where

$A_2A'_2 = \Delta.\sin(\theta)$.

Corresponding phase difference, $\varphi = (2\pi/\lambda).\Delta.\sin(\theta)$

if the field at point P due to the disturbance emanating from point A1 is "a cos(ωt)", then the field due to the disturbance emanating from A2 is "a cos (ωt – φ)

Field due to the disturbance emanating from A_3 is "a cos (ωt – 2φ)" and so on. For A_n it will be "a cos (ωt – (n-1)φ)".

Resultant field at point P will be,

$E = a\cos(\omega t) + a\cos(\omega t - \varphi) + a\cos(\omega t - 2\varphi) \ldots\ldots\ldots + a\cos(\omega t - (n-1)\varphi)$

Where $\varphi = (2\pi/\lambda).\Delta.\sin(\theta)$

Mathematically,

$a\cos(\omega t) + a\cos(\omega t - \varphi) + a\cos(\omega t - 2\varphi) \ldots\ldots\ldots + a\cos(\omega t - (n-1)\varphi)$

$$= a\frac{\sin(n\varphi/2)}{\sin(\varphi/2)}\cos\left(\omega t - \frac{1}{2}(n-1)\varphi\right)$$

$$E = a\frac{\sin(n\varphi/2)}{\sin(\varphi/2)}\cos\left(\omega t - \frac{1}{2}(n-1)\varphi\right) = E_\theta \cos\left(\omega t - \frac{1}{2}(n-1)\varphi\right)$$

where the amplitude E_θ of the resultant field is given by :

$$E_\theta = a\frac{\sin(n\varphi/2)}{\sin(\varphi/2)}$$

And $\varphi = (2\pi/\lambda).\Delta.\sin(\theta)$

In the limit of $n \to \infty$ and $\Delta \to 0$ in such a way that $n\Delta \to b$,

$$\frac{n\varphi}{2} = \frac{\pi}{\lambda} n\Delta \sin(\theta) \rightarrow \frac{\pi}{\lambda} b \sin(\theta)$$

$$\varphi = \frac{2\pi}{\lambda}\frac{b\sin(\theta)}{n}$$

If $n \to \infty$ then $\varphi \to 0$; $\sin(\varphi/2)=\varphi/2$

$$E_\theta \approx a\frac{\sin(n\varphi/2)}{\varphi/2} = na\frac{\sin(\pi b \sin(\theta)/\lambda)}{\pi b \sin(\theta)/\lambda}$$

$$E_\theta \approx a\frac{\sin(n\varphi/2)}{\varphi/2} = na\frac{\sin(\pi b\sin(\theta)/\lambda)}{\pi b\sin(\theta)/\lambda}$$

$$= A\frac{\sin\beta}{\beta}$$

$$where, A = na; \beta = \frac{\pi b\sin(\theta)}{\lambda} = n\frac{\varphi}{2}$$

$$Thus, E = A\frac{\sin(\beta)}{\beta}\cos\left(\omega t - \frac{1}{2}(n-1)\varphi\right)$$

$$E = A\frac{\sin(\beta)}{\beta}\cos(\omega t - \beta), because\, n \rightarrow \infty$$

Intensity distribution for this will be given by :

$$I = I_0\frac{\sin^2\beta}{\beta^2}$$

Where I_0 ($I_0 = A^2$) gives intensity at θ=0.

$$E_\theta \approx a\frac{\sin(n\varphi/2)}{\varphi/2} = na\frac{\sin(\pi b\sin(\theta)/\lambda)}{\pi b\sin(\theta)/\lambda}$$

$$= A\frac{\sin\beta}{\beta}$$

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Intensity distribution for this will be given by :

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Where I_0 ($I_0 = A^2$) gives intensity at θ=0.

Positions of Maxima and Minima:

The intensity is zero when $\beta = m\pi$ $\quad m \neq 0$

When $\beta = 0$, $(\sin\beta)/\beta = 1$ and $I = I_0$, which corresponds to the maximum of the intensity.

$$but\ \beta = \frac{\pi b \sin(\theta)}{\lambda}$$

$$\frac{\pi b \sin(\theta)}{\lambda} = m\pi$$

$$Hence, b\sin\theta = m\lambda;\ where\ m = \pm 1, \pm 2, \pm 3, \ldots\ldots (minima)$$

First minimum occurs at $\theta = \pm \sin^{-1}(\lambda/b)$, second minimum occurs at $\theta = \pm \sin^{-1}(2\lambda/b)$ and so on. Can be used for fringe width calculations.

Upper limit of $\sin(\theta)$ is 1 so max. value of m is integer closest to b/λ.

Positions of Maxima and Minima:

Intensity distribution is given by

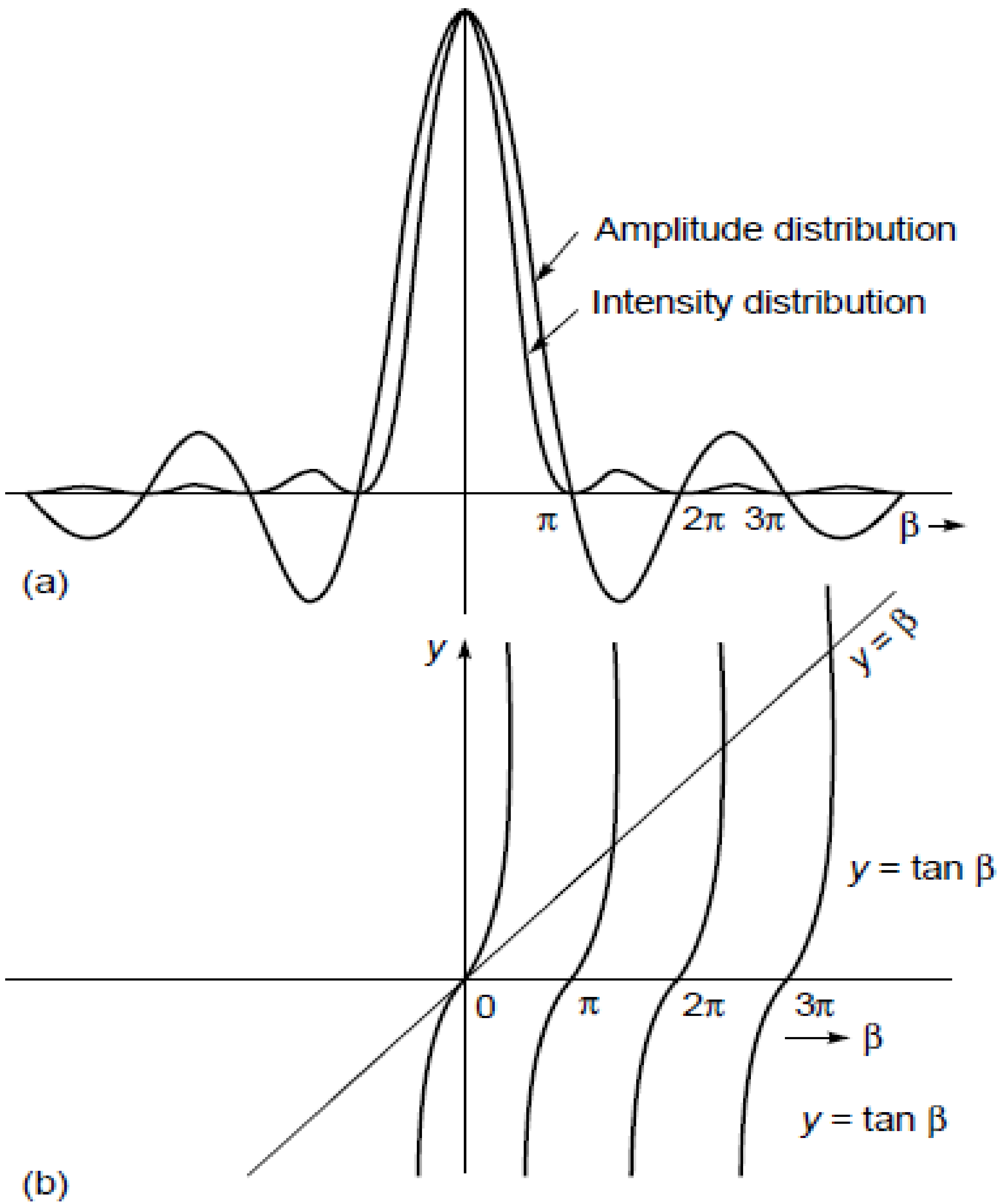
$$I = I_0 \frac{\sin^2\beta}{\beta^2}$$

$$\frac{dI}{d\beta} = I_0\left(\frac{2\sin\beta\cos\beta}{\beta^2} - \frac{2\sin^2\beta}{\beta^3}\right) = 0$$

$$\Rightarrow \sin\beta\left(\beta - \tan\beta\right) = 0$$

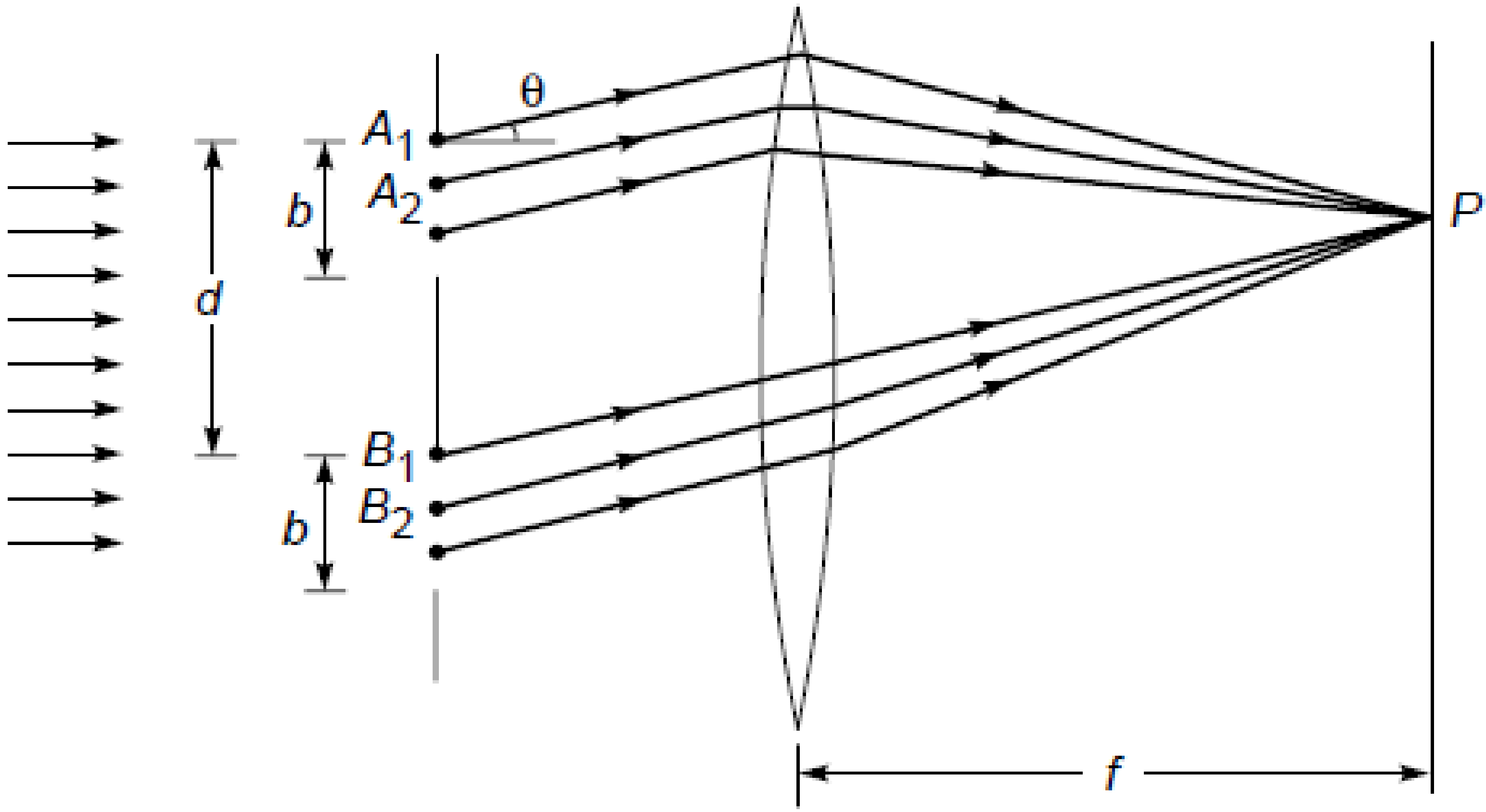
Condition sin(β) = 0 or β = mπ (m≠0) gives minima.

Condition tan(β) = β gives maxima. β=0 gives central Maxima. Rest of roots are found by intersection of curves y = β and y = tan(β)

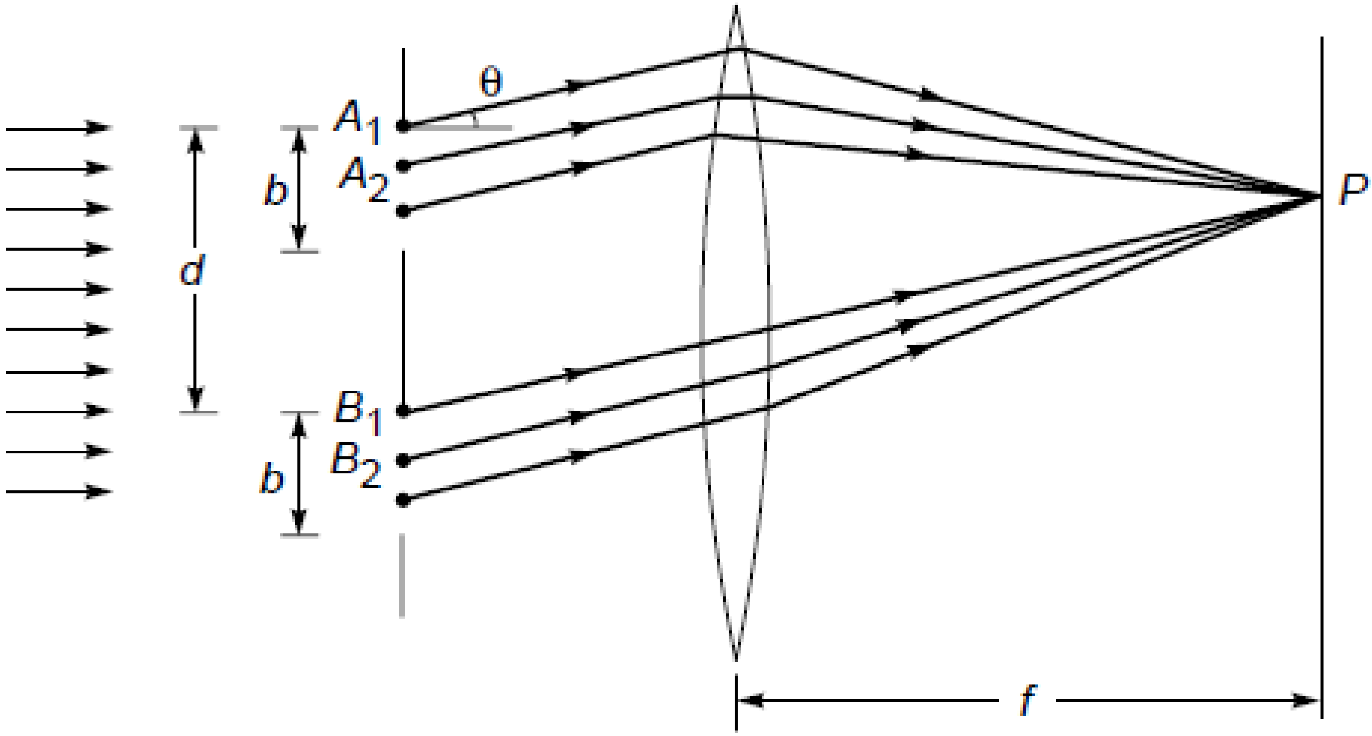


(a) Intensity distribution.
(b) roots of equation $\tan(\beta) = \beta$. Roots are: $\beta=1.43\pi$, 2.86π and so on.

Two slit diffraction pattern :



Resultant intensity distribution will be product of the single-slit diffraction pattern and the interference pattern produced by two point sources separated by distance “d”.



$$\textit{The field produced by first slit at point P will be}: E_1 = A\frac{\sin(\beta)}{\beta}\cos(\omega t - \beta)$$

$$\textit{The field produced by second slit at point P will be}:$$

$$E_2 = A\frac{\sin(\beta)}{\beta}\cos(\omega t - \beta - \Phi_1)\ \textit{where}\ \Phi_1 = \frac{2\pi}{\lambda} d\sin(\theta)$$

Φ_1 gives phase difference between disturbances reaching point P corresponding points such as (A_1, B_1), (A_2, B_2)… at distance *d*.

The resultant field will be given by:

$$E = E_1 + E_2$$

$$= A\frac{\sin(\beta)}{\beta}\cos(\omega t-\beta) + A\frac{\sin(\beta)}{\beta}\cos(\omega t-\beta-\Phi_1)$$

$$= A\frac{\sin(\beta)}{\beta}(\cos(\omega t-\beta)+\cos(\omega t-\beta-\Phi_1))$$

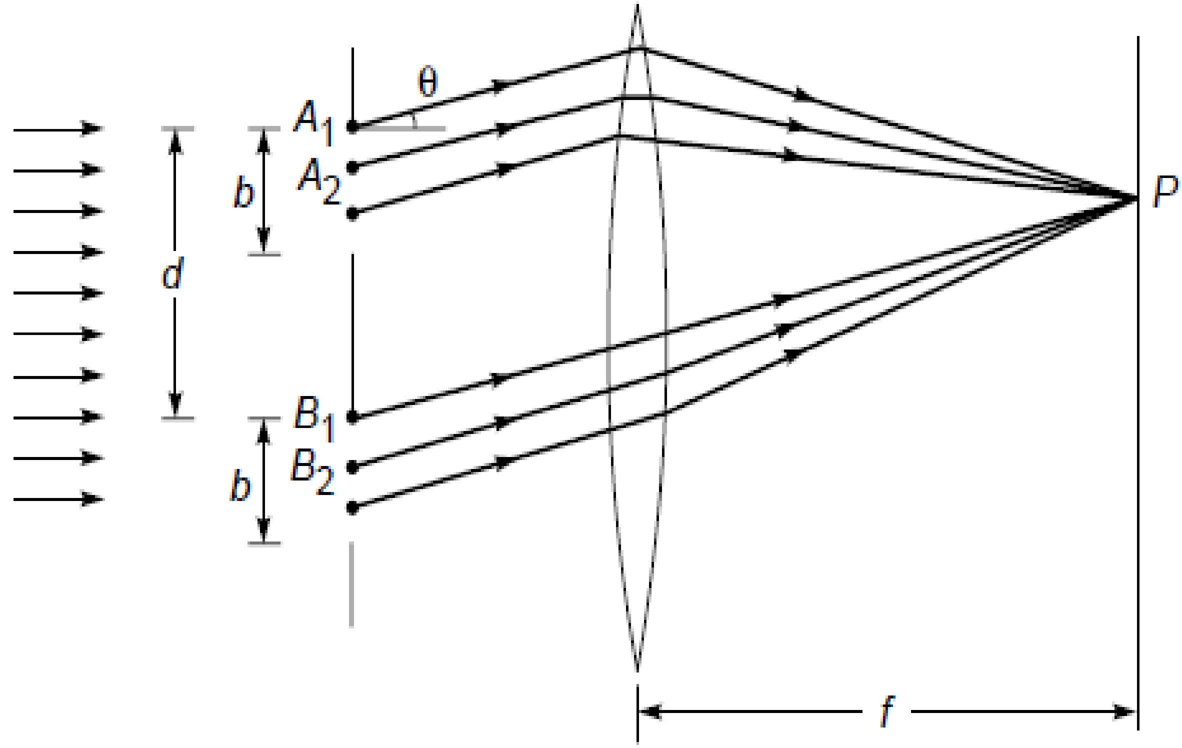
$$= 2A\frac{\sin(\beta)}{\beta}\cos\left(\frac{\Phi_1}{2}\right)\cos\left(\omega t-\beta-\frac{\Phi_1}{2}\right); where\,\Phi_1=\frac{2\pi}{\lambda}d\sin(\theta)$$

The intensity distribution will be given by

$$\boxed{I=4I_0\frac{\sin^2\beta}{\beta^2}\cos^2\left(\frac{\Phi_1}{2}\right); where\, I_0=A^2}$$

In this expression, $(\sin^2\beta)/\beta^2$ represents diffraction pattern produced by single slit of width *b*.

Second term $\cos^2(\Phi_1/2)$ represents interference produced by two point sources separated by distance d.



$$I = 4 I_0 \frac{\sin^2 \beta}{\beta^2} \cos^2\left(\frac{\Phi_1}{2}\right)$$

$$\beta = \frac{\pi b \sin(\theta)}{\lambda}$$

If slit widths are very small (b → 0, sin(β) → β) so that there is no variation of of the term $(\sin^2\beta)/\beta^2$ with θ, then one simply obtains Young's interference pattern.

Positions of minima :

$$I = 4 I_0 \frac{\sin^2\beta}{\beta^2} \cos^2\left(\frac{\Phi_1}{2}\right); where\, \beta = \frac{\pi b \sin(\theta)}{\lambda}; \Phi_1 = \frac{2\pi}{\lambda} d \sin(\theta)$$

Intensity is zero wherever

$\beta = \pi, 2\pi, 3\pi$ …..

Which means $b \sin(\theta) = m\lambda$ $\quad$ $m = 1,2,3$…….

Or when $\Phi_1 = \pi, 3\pi, 5\pi$ …..

Which means $d \sin(\theta) = (n-1/2)\lambda$ $\quad$ $n = 1,2,3$…….

Positions of maxima :

$$I = 4 I_0 \frac{\sin^2\beta}{\beta^2} \cos^2\left(\frac{\Phi_1}{2}\right); \textit{where}\, \beta = \frac{\pi b \sin(\theta)}{\lambda}; \Phi_1 = \frac{2\pi}{\lambda} d \sin(\theta)$$

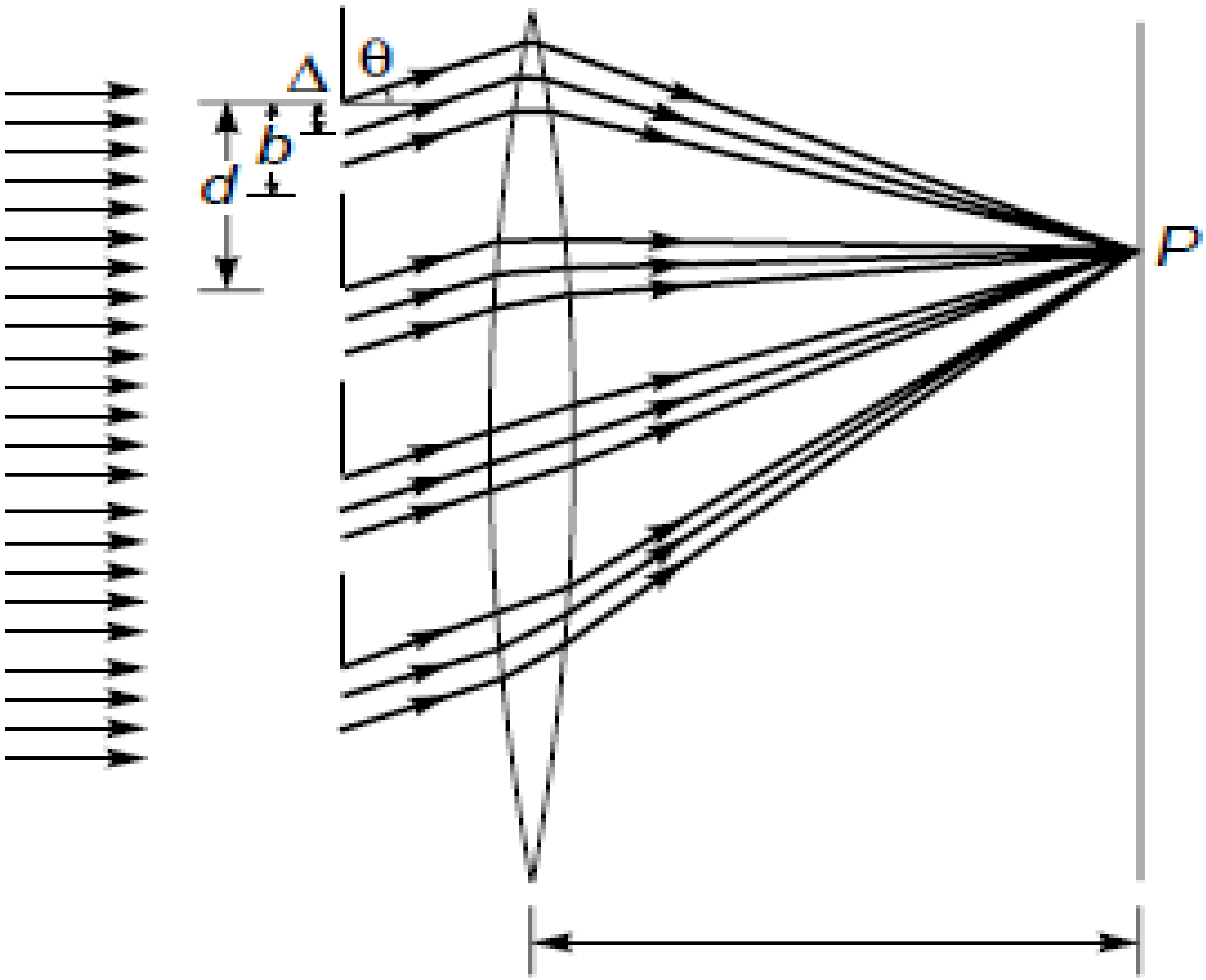
Interference maxima wherever
when $\Phi_1/2 = 0, \pi, 2\pi$ ….. or $(\pi/\lambda)d\sin(\theta) = 0, \pi, 2\pi$……
Which means $d \sin(\theta) = 0, \lambda, 2\lambda, 3\lambda$…….

This can be used to calculate fringe width for double slit interference maxima.

This will be approximate positions of maxima provided variation of diffraction pattern is not too rapid.

A maximum may not occur at all if θ corresponds to diffraction minimum ($b \sin(\theta) = \lambda, 2\lambda, 3\lambda$…..). **These are called missing orders.**

N-slit diffraction pattern:



The resultant field at any arbitrary point P will be :

$$E = A\frac{\sin(\beta)}{\beta}\cos(\omega t-\beta)+A\frac{\sin(\beta)}{\beta}\cos(\omega t-\beta-\Phi_1)+A\frac{\sin(\beta)}{\beta}$$

$$\cos(\omega t-\beta-2\Phi_1)+......+A\frac{\sin(\beta)}{\beta}\cos(\omega t-\beta-(N-1)\Phi_1)$$

$$=A\frac{\sin(\beta)}{\beta}(\cos(\omega t-\beta)+\cos(\omega t-\beta-\Phi_1)+\cos(\omega t-\beta-2\Phi_1)+......+\cos(\omega t-\beta-(N-1)\Phi_1))$$

Using same trignometric relation as used in single slit

$$=A\frac{\sin(\beta)}{\beta}\frac{\sin(\frac{N\Phi_1}{2})}{\sin(\frac{\Phi_1}{2})}\cos\left(\omega t-\beta-\frac{(N-1)}{2}\Phi_1\right)$$

$$where\,\beta=\frac{\pi b\sin(\theta)}{\lambda};\Phi_1=\frac{2\pi}{\lambda}d\sin(\theta)$$

The corresponding intensity distribution will be :

$$I = I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2\left(\frac{N\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$

In this expression, $(\sin^2\beta)/\beta^2$ represents diffraction pattern produced by single slit of width *b*.

Second term $(\sin^2(N\Phi_1/2)/\sin^2(\Phi_1/2))$ represents interference produced by N equally spaced point sources separated by distance d.

$$I = I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2\left(\frac{N\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$

$$When\, N = 1 : I = I_0 \frac{\sin^2(\beta)}{\beta^2}$$

Same as we obtained in single slit

$$When\, N = 2 : I = I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2\left(\frac{2\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$

$$= I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\left(2\sin\left(\frac{\Phi_1}{2}\right)\cos\left(\frac{\Phi_1}{2}\right)\right)^2}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$

$$= 4 I_0 \frac{\sin^2(\beta)}{\beta^2} \cos^2\left(\frac{\Phi_1}{2}\right)$$

Same as we obtained in double slit

Position of principal maxima :

$$I = I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2\left(\frac{N\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$

The condition for principal maxima is :

$$\frac{\Phi_1}{2} = m\pi \,; m = 0,1,2,3\ldots$$

$$\Phi_1 = \frac{2\pi}{\lambda} d \sin(\theta)$$

$$\Rightarrow \frac{\pi}{\lambda} d \sin(\theta) = m\pi$$

$$\Rightarrow d \sin(\theta) = m\lambda \,; m = 0,1,2,3\ldots$$

Position of principal maxima :

$$I = I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2\left(\frac{N\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$

The condition for principal maxima is :

$$\frac{\Phi_1}{2} = m\pi \,;\, m = 0,1,2,3\ldots$$

$$\Phi_1 = \frac{2\pi}{\lambda} d \sin(\theta)$$

Something doesn't make Sense here!

$$\Rightarrow \frac{\pi}{\lambda} d \sin(\theta) = m\pi$$

$$\Rightarrow d \sin(\theta) = m\lambda \,;\, m = 0,1,2,3\ldots$$

Position of principal maxima :

L' Hôpital's rule or L' hospital's rule

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty,$$

If $\displaystyle \lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

$$E = A\frac{\sin(\beta)}{\beta}\frac{\sin\left(\frac{N\Phi_1}{2}\right)}{\sin\left(\frac{\Phi_1}{2}\right)}\cos\left(\omega t - \beta - \frac{(N-1)}{2}\Phi_1\right)$$

Apply L' Hôpital's rule on

$$\lim_{\frac{\Phi_1}{2}\to m\pi}\frac{\sin\left(\frac{N\Phi_1}{2}\right)}{\sin\left(\frac{\Phi_1}{2}\right)} = \lim_{\frac{\Phi_1}{2}\to m\pi}\frac{N\cos\left(\frac{N\Phi_1}{2}\right)}{\cos\left(\frac{\Phi_1}{2}\right)} = \pm N$$

$$Thus, E = N\frac{Asin(\beta)}{\beta}\cos\left(\omega t - \beta - \frac{(N-1)}{2}\Phi_1\right)$$

Physically, at these maxima fields produced by each of the slits are in phase and hence resultant field (E) is N times of field produced by single slit

Intensity will be given in this case by

$$I = I_0 N^2 \frac{\sin^2(\beta)}{\beta^2}$$

$$where\,\beta = \frac{\pi b \sin(\theta)}{\lambda} = \frac{\pi b}{\lambda}\frac{m\lambda}{d} = \frac{\pi b m}{d}\,;\, m = 0,1,2,3\ldots..$$

- Intensity has large value unless $(\sin^2\beta)/\beta^2$ itself is small.
- Since $\sin(\theta) \leq 1$, m can not be greater than d/λ.
- It means more is the number of slit (N), intensity of maxima will be more.
- This concept is used in diffraction grating where you have 15000 or so slits per inch!

Position of minima :

$$I = I_0 \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2\left(\frac{N\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)} ; \textit{where}\, \beta = \frac{\pi b \sin(\theta)}{\lambda} ; \Phi_1 = \frac{2\pi}{\lambda} d \sin(\theta)$$

Intensity is zero wherever

$\beta = \pi, 2\pi, 3\pi$ ….. (minima for single slit!)

Which means b sin(θ) = nλ n = 1,2,3…….

Or when $N\Phi_1/2 = p\pi$ $p \neq N, 2N, 3N$……

Because Intensity "***I***" become indeterminate at these points.

$$\frac{N\Phi_1}{2} = p\pi \Rightarrow \frac{N \frac{2\pi}{\lambda} d \sin(\theta)}{2} = p\pi$$

$$\Rightarrow d \sin(\theta) = p \frac{\lambda}{N}$$

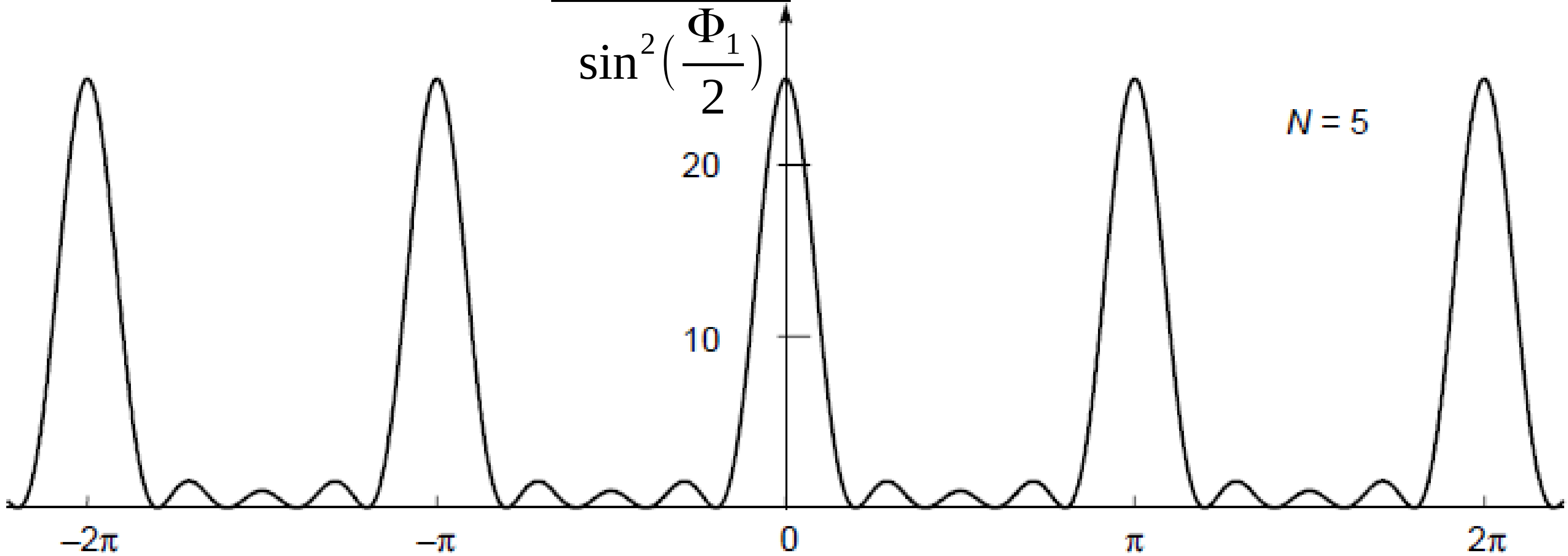
$$d\sin(\theta)=\frac{p\lambda}{N};\, p\neq N, 2N, 3N\,.....$$

$$d\sin(\theta)=\frac{\lambda}{N},\frac{2\lambda}{N},\frac{3\lambda}{N}....\frac{(N-1)\lambda}{N},\frac{(N+1)\lambda}{N},\frac{(N+2)\lambda}{N}.....$$
$$\frac{(2N-1)\lambda}{N},\frac{(2N+1)\lambda}{N},\frac{(2N+2)\lambda}{N}$$

Notice the missing terms! They correspond to principal maxima. There are (N-1) minima between 2 principal maxima.

Between two consecutive minima, intensity has to have a maximum; these maxima are called secondary maxima. There will be (N-2) secondary maxima between two principal maxima.

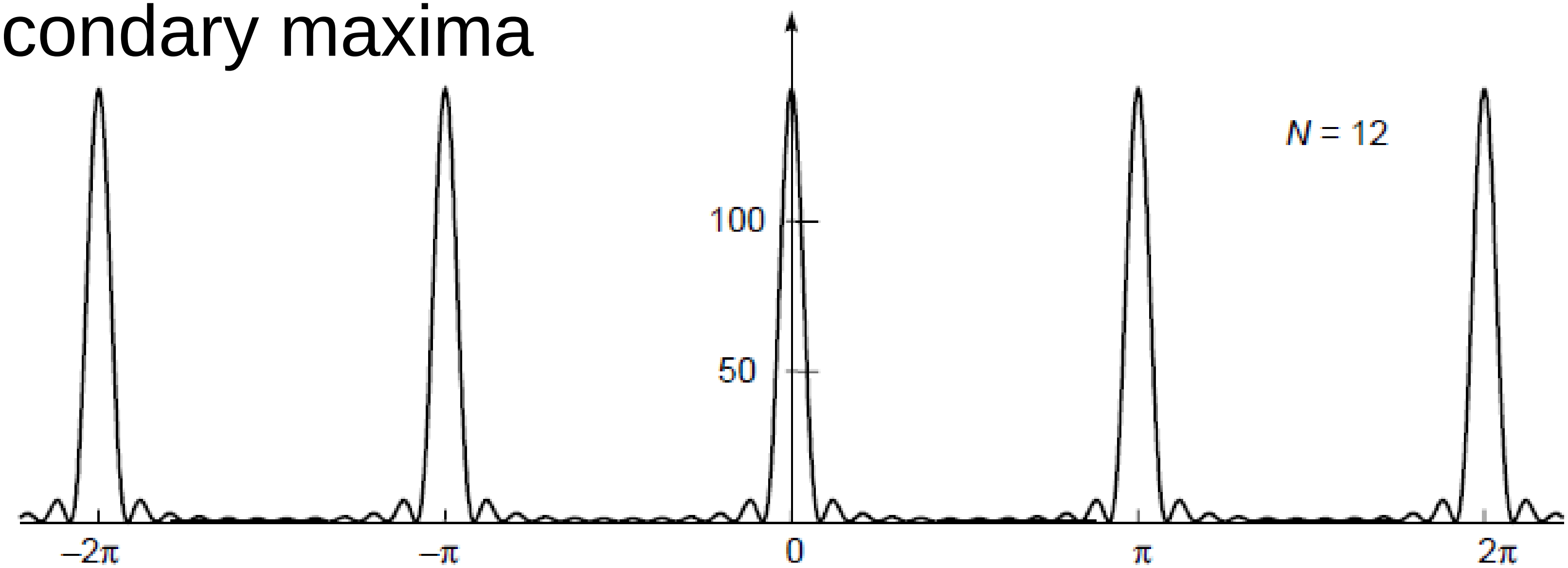
$$\frac{\sin^2\left(\frac{N\,\Phi_1}{2}\right)}{\sin^2\left(\frac{\Phi_1}{2}\right)}$$



(a)

N-1 minima

N-2 secondary maxima



$$\rightarrow \frac{\Phi_1}{2}$$

What if angle of principal maxima is same as diffraction minima?

This will happen when these conditions are satisfied simultaneously

$\mathbf{d \sin(\theta) = m\lambda}$ m=0,1,2,……. (Principal maxima)

And $\mathbf{b \sin(\theta) = \lambda, 2\lambda, 3\lambda}$**…..** (diffraction minima)

These are referred as **missing order.**

Diffraction grating:

A very large number of equidistant slits is called diffraction grating.

Corresponding diffraction pattern is called as the grating spectrum.

Principal maxima: $\mathbf{d \sin(\theta) = m\lambda}$ $m=0,1,2,\ldots\ldots.$
As it depends on wavelength, so principal maxima ($m \neq 0$) for different λ will give different θ.
Can be used for measurement of λ.

More is number of slits, narrower will be principal maxima.
Usually 15,000 per inch slits are there.

Lines should be as equally spaced as possible.

Grating Spectrum :

Principal maxima: $\mathbf{d \sin(\theta) = m\lambda}$ m=0,1,2,……..
This equation is also called the **grating equation**.

The zeroth order principal maxima occurs at $\theta=0$ irrespective of wavelength.
Thus for white light, central maximum will be white.

For $m\neq0$, θ are different for different λ, various spectral components appear at different locations.

Dispersive power of grating :

Principal maxima: **d sin(θ) = mλ** m=0,1,2,……..

Differentiating this equation:

$$d\cos(\theta)\Delta\theta = m\Delta\lambda$$

$$\Rightarrow \frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d\cos(\theta)}$$

$\Delta\theta/\Delta\lambda$ is called **dispersive power.**

Conclusions:

1. Dispersive power is proportional to "*m*" (order of principal maximum). Higher is m, well separated will be maxima corresponding to 2 close wavelengths like sodium doublet. Zeroth order principal maxima will overlap.

Conclusions contd:

2. Dispersive power is inversely proportional to “*d*” (the grating element). Smaller is “*d*”, larger will be angular dispersion.
3. Dispersive power is inversely proportional to cos(θ). if θ is very small then cos(θ)$\simeq$1,

$$\frac{d\theta}{d\lambda}=\frac{m}{d\cos(\theta)}\ becomes\ \frac{d\theta}{d\lambda}=\frac{m}{d}$$

Such spectrum is known as **normal spectrum**. For this dθ is directly proportional to dλ.

Resolving power of grating :

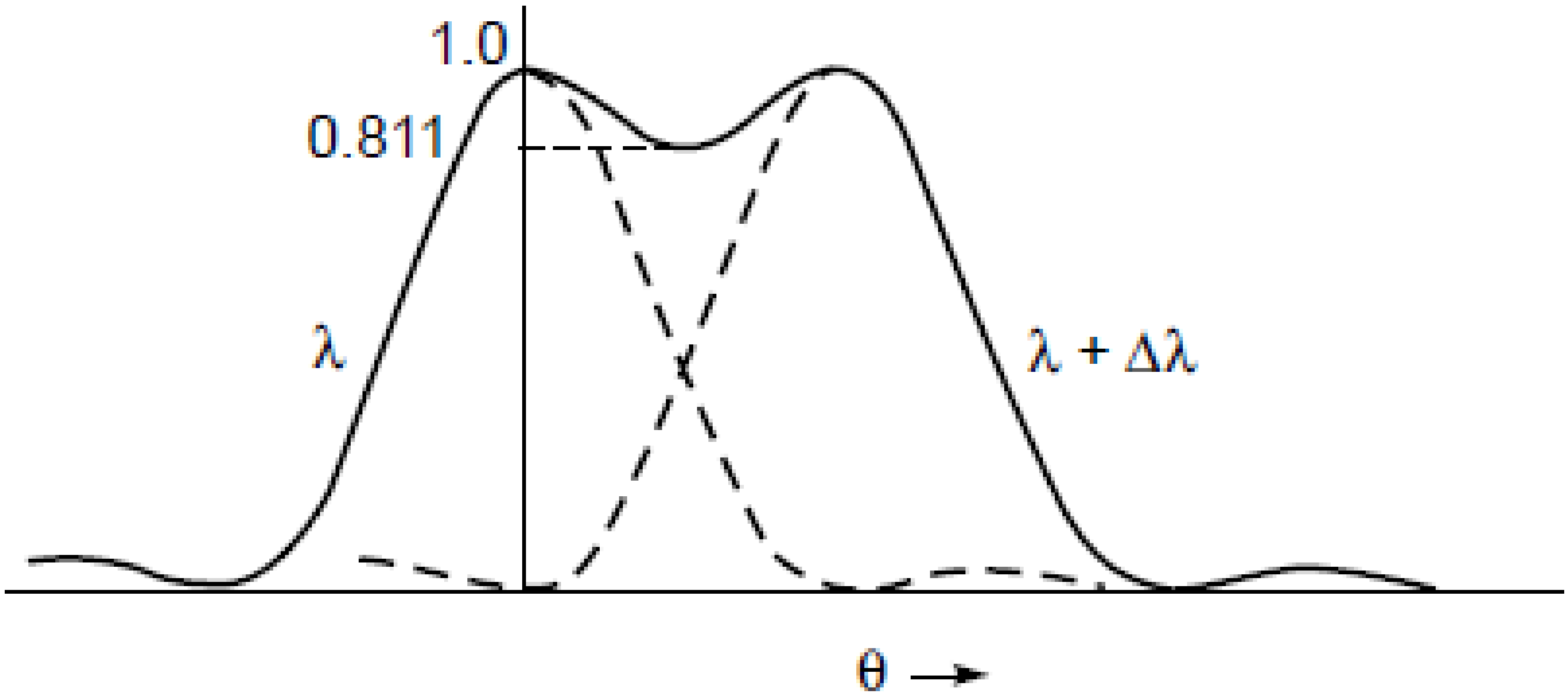
Minimum separation at which two objects look separate is Called “limit of resolution”.

Smaller is separation between 2 objects an instrument can Resolve, higher is its resolving power and better is the Instrument.

In case of diffraction grating, resolving power is power of distinguishing two nearby spectral lines.

Rayleigh's Criterion :

If the principal maximum corresponding to wavelength λ + Δλ falls on first minimum (on either side) of the wavelength λ, then the two wavelengths λ and λ + Δλ are said to be just resolved.



Mathematically this means that

If angle θ is the angle corresponding to m^{th} order spectrum then these conditions are satisfied simultaneously :

principal maximum for wavelength $\lambda+\Delta\lambda$:

$$d \sin(\theta) = m(\lambda+\Delta\lambda)$$

minimum for wavelength λ:

$$d \sin(\theta) = m\lambda+\lambda/N$$

Equating both sides: $m(\lambda+\Delta\lambda) = m\lambda+\lambda/N$

Or $m\,\Delta\lambda = \lambda/N$

Or $\lambda/\Delta\lambda = m\,N$

$\lambda/\Delta\lambda$ is called the **resolving power of a grating**.

Resolving power; $\lambda/\Delta\lambda = m N$

- Resolving power depends on total number of lines in grating exposed to incident light (N).

- Resolving power is proportional to “order of spectrum”.

dθ = Angular separation between the principal maxima of 2 patterns
Δθ = half angular width of principal maxima of each pattern