Tut - 4

Ques 1 The height of a hill is expressed as $h(x, y) = 5(2xy - 3x^2 - 4y^2 - 18x + 28y + 6)$. Find location & height peak?

Sol^n:- Height of Hill $\Rightarrow$ $h(x, y) = 5(2xy - 3x^2 - 4y^2 - 18x + 28y + 6)$

Now $\because$ we know that

at peak $\vec{\nabla} h(x, y) = 0$

$\Rightarrow \vec{\nabla} h(x, y) = 0\hat{i} + 0\hat{j}$

$\frac{\partial h}{\partial x} \Rightarrow \frac{\partial}{\partial x} 5(2xy - 3x^2 - 4y^2 - 18x + 28y + 6)$

$= 5(2y - 6x - 18)$ — (1)

$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}(5(2xy - 3x^2 - 4y^2 - 18x + 28y + 6))$

$= 5(2x - 8y + 28)$ — (2)

$\Rightarrow 5(2y - 6x - 18)\hat{i} + 5(2x - 8y + 28)\hat{j} = 0\hat{i} + 0\hat{j}$

On comparing, we get

$2y - 6x - 18 = 0$
$2x - 8y + 28 = 0$

$y - 3x - 9 = 0$ — (3)
$x - 4y + 14 = 0$ — (4)

After common

from (3) & (4)

$y - 3x = 9$ — (3)
$-4y + x = -14$ — (4)

Multiply (3) by 4

$4y - 12x = +36$
$-4y + x = -14$

$-11x = 22$

$\boxed{x = -2}$

put $x = -2$ in eqn (3)

$$y - 3x - 9 = 0$$
$$y - 3\times(-2) - 9 = 0$$
$$y + 6 - 9 = 0$$
$$y - 3 = 0$$
$$y = 3$$

location of Peak is $(-2, 3)$

Height of peak

$$= 5\left(2\times(-2\times3) - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 6\right)$$
$$= 5\left((2\times-6) - 3(4) - 4(9) + 36 + 84 + 6\right)$$
$$= 5\left(-12 - 12 - \cancel{36} + \cancel{36} + 84 + 6\right)$$
$$= 5(-24 + 90)$$
$$= 5(66)$$
$$= 330 \text{ units.}$$

Ques 2:- find the divergence of function $f = 2k$. Comment on the result.

Soln:- Given :- $f = 2\hat{k}$

$$\text{div. } f = \vec{\nabla}\cdot\vec{f} = \vec{\nabla}\cdot 2\hat{k}$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(f_x, f_y, f_z)$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(0, 0, 2)$$
$$\text{div } f = 0 = \vec{\nabla}\cdot\vec{f} \quad \text{— (1)}$$

we can say that $\vec{f}$ is not diverging.

Q4 3:- Plot the function $f(x, y) = x\hat{j} - y\hat{i}$. Also find the curl of this function.

Ans:- Given $f(x, y) = x\hat{j} - y\hat{i}$ —(1)

Now, curl f = $\vec{\nabla} \times \vec{f}$

$\Rightarrow \vec{\nabla} \times (x\hat{j} - y\hat{i}) \Rightarrow \vec{\nabla} \times (x\hat{j} - y\hat{i} + 0\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \vec{\nabla} \times \vec{f}$$

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial(x)}{\partial z}\right) - \hat{j}\left(\frac{\partial(0)}{\partial x} - \frac{\partial(-y)}{\partial z}\right) + \hat{k}\left(\underbrace{\frac{\partial(x)}{\partial x}}_{1} + \underbrace{\frac{\partial(y)}{\partial y}}_{1}\right)$$

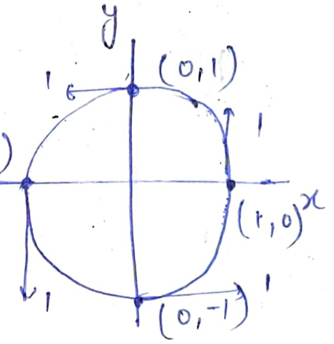
$\Rightarrow 2\hat{k}$

$\vec{\nabla} \times \vec{f} = 2\hat{k}$

Plotting (1)

$(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$

$\hat{j}$ $-\hat{i}$ $-\hat{j}$ $\hat{i}$



Ques 4 A parallel plate capacitor is filled with a material having permittivity = 80, permeability = $\mu = \mu_0$ & resistivity 0.25 Ω m. An alternating signal $V = V_0 \sin(2\pi\nu t)$ is applied across the plate of the capacitor. Calculate the ratio of conduction current density to displacement current density.

Sol^n:- $\Rightarrow \dfrac{(J_d)\text{ Conduction Current density}}{(J_c)\text{ Displacement current density}} = ??$

We know that

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} \quad -(1)$$

and

displacement vector $\vec{D} = \varepsilon \cdot \vec{E}$

$|\vec{D}| = \varepsilon E = \varepsilon \cdot \frac{V}{d}$ $[\because V = E \cdot d]$

$= |\vec{D}| = \frac{\varepsilon}{d} \cdot V_0 \sin(2\pi \nu t)$ — (2)

using (2) into (1)

$$\vec{J}_d = \frac{\partial}{\partial t}\left(\frac{\varepsilon}{d} V_0 \sin(2\pi \nu t)\right)$$

$$\boxed{J_d = 2\pi \nu \cdot \frac{\varepsilon}{d} \cdot \cos(2\pi \nu t)} \quad -(3)$$

Also

$$J_c = \sigma \cdot E = \frac{1}{\rho} \cdot \frac{V}{d} = \frac{1}{\rho \cdot d} V_0 \sin 2\pi \nu t \quad -(4)$$

from (3) & (4) we get

$$\frac{J_c}{J_d} = \frac{\frac{1}{\rho \cdot d} V_0 \sin 2\pi \nu t}{2\pi \nu \cdot \frac{\varepsilon}{d} \cos(2\pi \nu t)}$$

$$\frac{J_c}{J_d} = \frac{1}{2\pi \nu \rho \varepsilon} \cdot \tan(2\pi \nu t) \quad -(5)$$

Ques 5.:- An alternating signal $V = V_0 \sin(2\pi \nu t)$ is applied across a piece of copper. Calculate the ratio of conduction current to displacement current if $\nu = 50$ Hz. The resistivity & permitivity of copper are given to be $1.68 \times 10^{-8}\ \Omega m$, $5.4 \times 10^{-11}\ C^2/N m^2$.

Sol^n: using eq^n (5)

$$\frac{J_c}{J_d} = \frac{1}{2 \times 3.14 \times 50 \times 1.68 \times 10^{-8} \times 5.4 \times 10^{-11}} \times \tan(2\pi \times 50 \times t)$$

$\Rightarrow \frac{1 \times \tan(2\pi \times 50 \times t)}{2848.608 \times 10^{-19}}$

$$\frac{J_c}{J_d} = 3.5 \times 10^{15} \tan(100\pi t)$$

Qus 6: A square loop of wire of side 5 Cm is placed in a uniform magnetic field B, such that the normal to the plane of the loop subtends an angle of 60° with direction of B. If the magnetic field str. is given $(0.5 - 0.002t^3)$ tesla. then find the induced emf in the loop at $t = 2$ s.

Solⁿ:

Side of Sq. loop (a^2) = 5 Cm = 25×10^{-4} m

field Strength = $(0.5 - 0.002t^3)$ T

$t = 2$ sec.

$\theta = 60°$

Now, we know that

induced emf $= -\frac{\partial \phi}{\partial t}$ —(1)

Also $\phi = \vec{B}\cdot\vec{A} = \vec{B}\cdot\vec{A}\cos\theta$

$\phi = B\cdot A\cdot \cos 60° = (0.5)\, B\cdot A$

induced EMF $= -\frac{d}{\partial t}(0.05\, BA)$

at $t = 2$ sec.

induced Emf $= -0.5 \times A \times \frac{\partial}{\partial t}(0.5 - 0.002t^3)$

$= 3 \times 10^5$ T.

Que 7: Justify that $E(x,t) = E_0 e^{i(kx-\omega t)}\hat{j}$ and $B(x,t) = B_0 e^{i(kx-\omega t)} k$ represent electric & magnetic field vectors of an EM wave propagating in free space.

Solⁿ: Given $\vec{E}(x,t) = E_0 e^{i(kx-\omega t)}\hat{j}$ —(1)

$\vec{B}(x,t) = B_0 e^{i(kx-\omega t)}\hat{k}$ —(2)

As we can see from (1) & (2) $\vec{E}$ & $\vec{B}$ are ⊥ to each other.

Also ~~$\vec{E}$ &~~ $\vec{E}\times\vec{B}$ comes out to be ⊥ to the direction of propagation.

∴ we can state that these are field vectors of an em wave propagating in free space.

Hence Proved

Ques 8: The conductivity & Relative Permittivity for for given materials are 5 S/m and 1 respectively. An electric field $E = E_0 \sin(2\pi \nu t)$ is applied across the material. Calculate the value of frequency ν at which peak value of conduction & displacement current density becomes equal.

Sol^n :-

$$\left.\begin{aligned} \text{Conductivity} &= 5\ S/m \\ \text{Permitivity}(\varepsilon) &= 1 \\ E &= E_0 \sin(2\pi\nu t) \end{aligned}\right]$$

$$(J_d)_{max} = (J_c)_{max}$$

$$2\pi\nu \cdot \frac{V_0 \varepsilon}{d} = \frac{V_0}{\rho d}$$

$$\nu = \frac{1}{2\pi \varepsilon \rho}$$

$$\because \frac{1}{\rho} = \sigma$$

Conductivity

$$\nu = \frac{\sigma}{2\pi\varepsilon} \quad \text{— (A)}$$

using (*) in eq^n (A)

$$\nu = \frac{5}{2 \times 3.14 \times 8.85 \times 10^{-12}} = 8.99 \times 10^{10}\ Hz.$$

Q 9. Calculate the phase diff. b/w electric field and magnetic field inside a good conductor.

Sol^n: For a good conductor, we know that k_- & k_+ are equal. (electric & magnetic field → phase b/w them (=))

OR

If an EM wave travels in a good conductor than the values of α & β are almost equal i.e.

$$\tan\phi = \frac{k_1}{k_2} = \frac{\beta}{\alpha} \simeq 1$$

(k_1 → wave vector & imag; β → imaginary; α Real → Real)

ϕ → phase angle b/w electric & magnetic field.

free space → Real
media → Real
Metal → Real & imaginary
Decay amplitude decay.

$$\phi = \tan^{-1}(1)$$

$$\phi = 45°$$

So, the phase difference b/w electric and magnetic field inside a good conductor is 45°.

Que 10:- The resistivity, permittivity & permeability of copper are $1.68 \times 10^{-8}\ \Omega m$, $5.4 \times 10^{-11}\ C^2/N m^2$ & $1.26 \times 10^{-6}\ N/A^2$. Calculate the skin depth for copper at optical frequencies ($\sim 10^{15}$ Hz.) Also comment on Result.

Sol^n:- Skin depth $= d = \frac{1}{k_-} = \sqrt{\frac{2}{\omega\sigma\mu}}$ $\quad \sigma = 1/\rho$

$$d = \sqrt{\frac{2\rho}{2\pi f\cdot\mu}} = \sqrt{\frac{2\rho}{2\pi f\cdot\mu}} \quad -(2)$$

$$\Rightarrow \sqrt{\frac{2\times1.68\times10^{-8}}{2\times3.14\times10^{15}\times1.26\times10^{-6}}}$$

$$\Rightarrow \sqrt{\frac{3.36\times10^{-8}}{7.9128\times10^{9}}} \Rightarrow \sqrt{0.4246\times10^{-17}} \Rightarrow \sqrt{4.246\times10^{-18}}$$

$$d \Rightarrow 2.04\times10^{-9}\,m$$

i.e. metals are opaque for such frequency.