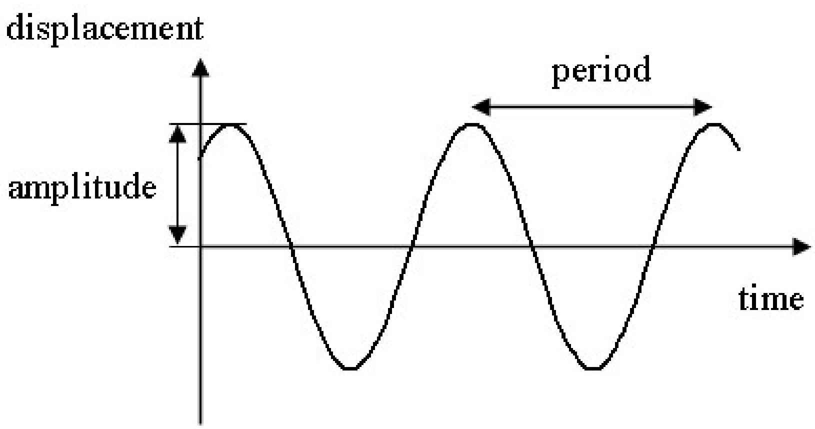
Oscillations and Waves



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- Simple Harmonic Motion (SHM)—Spring Oscillations
- Mathematical Representation of Simple Harmonic Motion
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- The Period and Sinusoidal Nature of SHM
- The Simple Pendulum
- Damped Oscillations
- Forced Oscillations

Periodic Motion

A special type of motion called periodic motion. It is a repeating motion of an object in which the object continues to return to a given position after a fixed time interval.

Examples: Pendulum and a beach ball floating on the waves

The back and forth movements of an object called oscillations.

We will focus our attention on a special case of periodic motion called **simple harmonic motion (SHM)**

We shall find that all periodic motions can be modeled as combinations of simple harmonic motions. Thus, simple harmonic motion forms a basic building block for more complicated periodic motion.

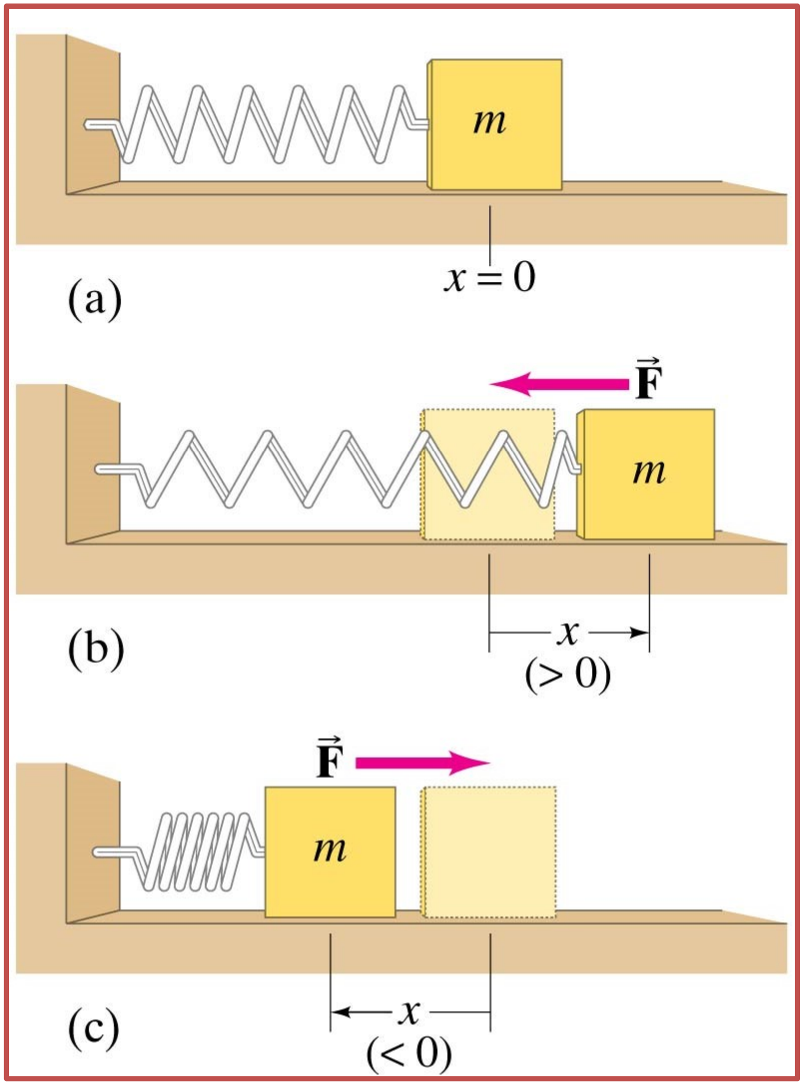
Oscillations and Waves

- Simple harmonic motion also forms the basis for our understanding of *mechanical waves.* Sound waves, seismic waves, waves on stretched strings, and water waves are all produced by some source of oscillation.

- As a sound wave travels through the air, elements of the air oscillate back and forth; as a water wave travels across a pond, elements of the water oscillate up and down and backward and forward.

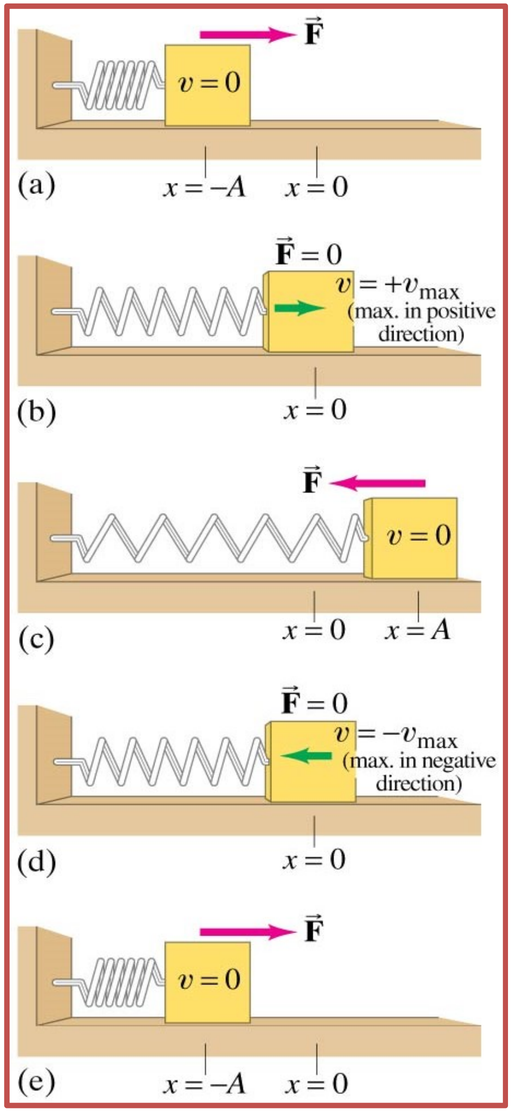
- To explain many other phenomena in nature, we must understand the concepts of oscillations and waves.

Simple Harmonic Motion-Spring Oscillations



If an object vibrates or oscillates back and forth over the same path, each cycle taking the same amount of time, the motion is called periodic. The mass and spring system is a useful model for a periodic system.

Simple Harmonic Motion-Spring Oscillations



- Displacement is measured from the equilibrium point
- Amplitude is the maximum displacement
- A cycle is a full to-and-fro motion; this figure shows half a cycle
- Period is the time required to complete one cycle
- Frequency is the number of cycles completed per second

Simple Harmonic Motion-Spring Oscillations

- We assume that the surface is frictionless.
- There is a point where the spring is neither stretched nor compressed; this is the equilibrium position.
- We measure displacement from that point ($x = 0$)

The force exerted by the spring depends on the displacement:

$$F = -kx. \qquad \text{[force exerted by spring]}$$

- The minus sign on the force indicates that it is a restoring force—it is directed to restore the mass to its equilibrium position.
- k is the spring constant

Any vibrating system where the restoring force is proportional to the negative of the displacement is in simple harmonic motion (SHM), and is often called a simple harmonic oscillator.

Simple Harmonic Motion-Spring Oscillations

Applying Newton’s second law:

$F = ma$

So, $-kx = ma$;

$a = -kx/m$

So, an object moves with simple harmonic motion whenever its acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium.

Mathematical Representation of SHM

$a = dv/dt = d^2x/dt^2$, and so we can express Equation 15.2 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{15.3}$$

If we denote the ratio k/m with the symbol ω^2 (we choose ω^2 rather than ω in order to make the solution that we develop below simpler in form), then

$$\omega^2 = \frac{k}{m} \tag{15.4}$$

and Equation 15.3 can be written in the form

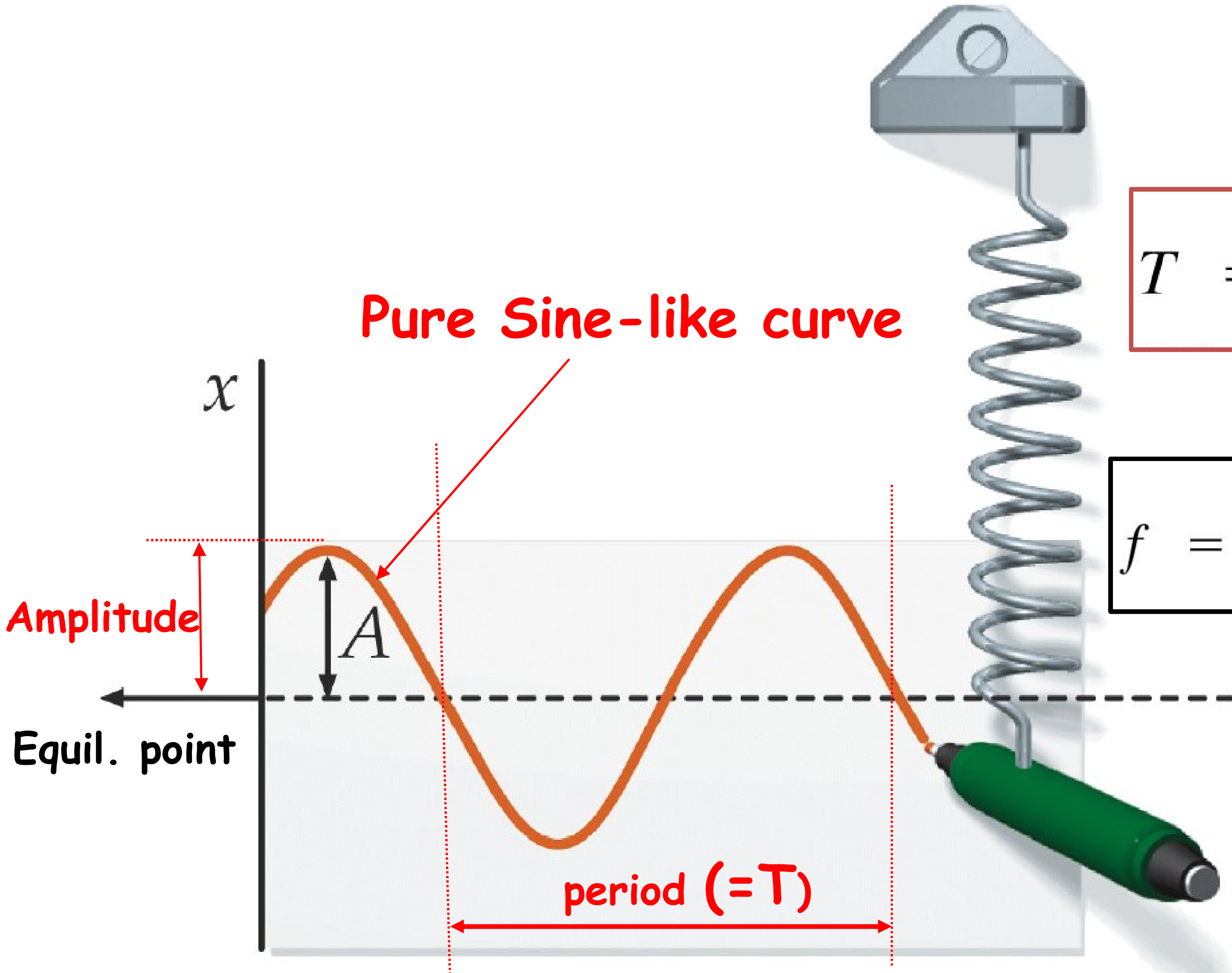
$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \omega = \sqrt{\frac{k}{m}} \tag{15.5}$$

Differential equation of SHM

The mathematical solution to Equation 15.5 is a function x(t) that satisfies this second-order differential equation.

$\boxed{x = A\cos(\omega t + \Phi)}$ where A, ω and Φ are constants

Displacement v/s time



$$T = 2\pi\sqrt{\frac{m}{k}}.$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Displacement, velocity and acceleration v/s time

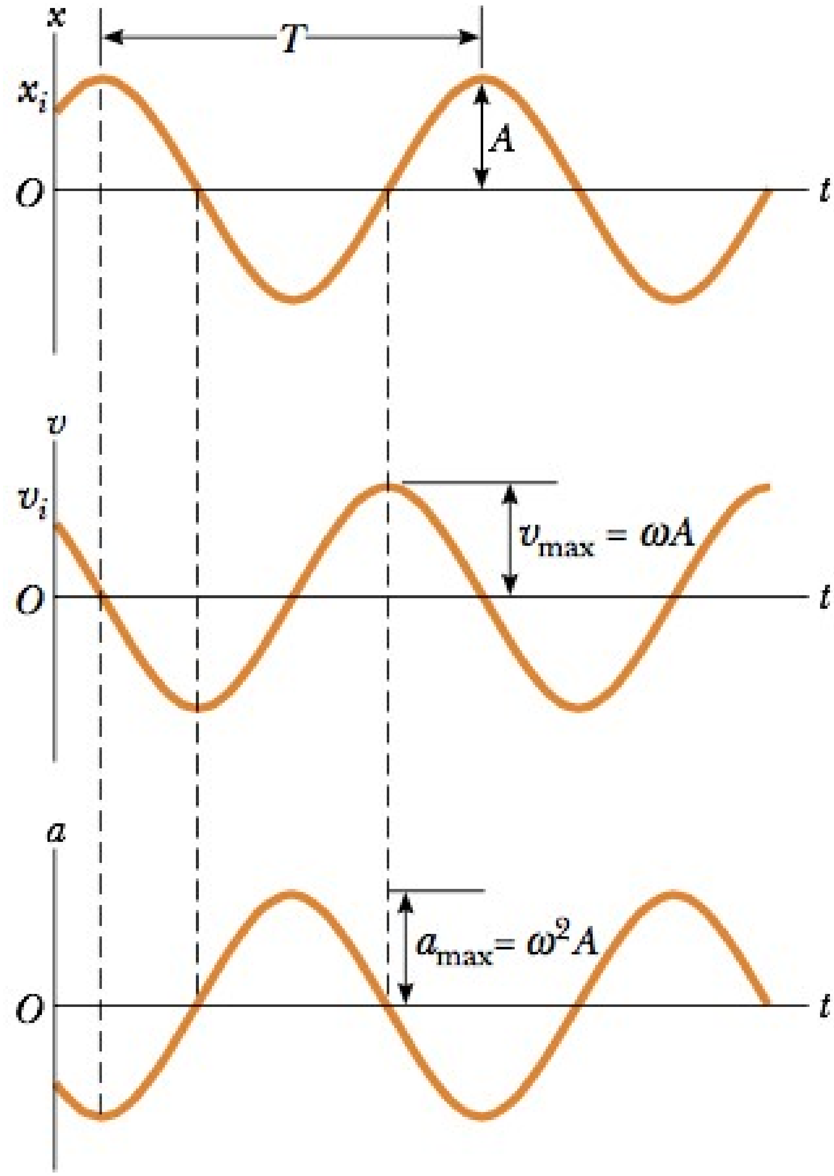
$$x(t) = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}}\,A$$

$$a_{\max} = \omega^2 A = \frac{k}{m}\,A$$



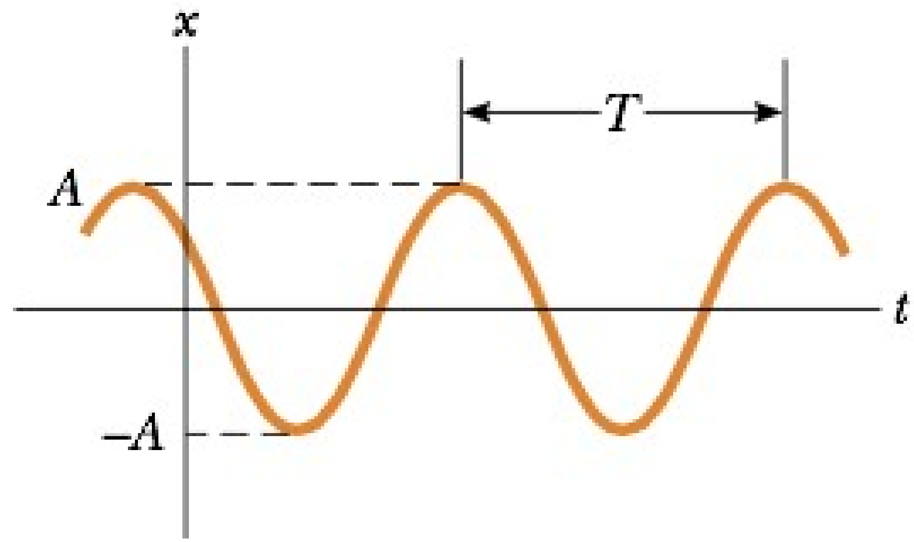
Summary: SHM (Spring Oscillator)

Following Equations form the basis of the mathematical representation of simple harmonic motion.

$$F = -kx$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$x = A\cos(\omega t + \Phi)$$

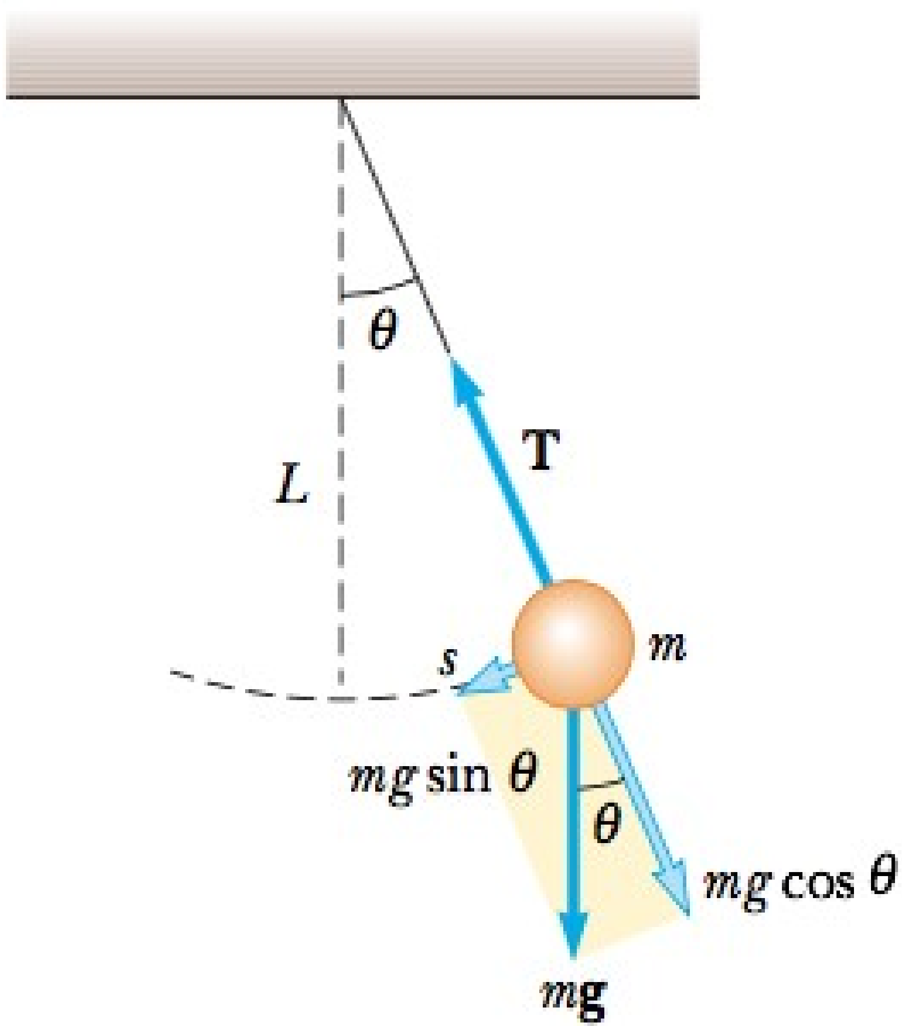


$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

$$\boxed{f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

The Simple Pendulum Case

A simple pendulum consists of a mass at the end of a lightweight cord. We assume that the cord does not stretch, and that its mass is negligible.



In order to be in SHM, the restoring force must be proportional to the negative of the displacement,

$F = -mg \sin \theta$

Apply Newton's second law for motion in the tangential direction:

$$F_t = -mg \sin \theta = m \frac{d^2 s}{dt^2}$$

Because $s = L\theta$ and L is constant, this equation reduces to

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin \theta$$

The Simple Pendulum case

Therefore, for small angles, the force is approximately proportional to the angular displacement.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \qquad \omega = \sqrt{\frac{g}{L}}$$

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

Spring Oscillator case

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

The period is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}.}$$

Energy Conservation in Oscillatory Motion

In an ideal system with no non-conservative forces, the total mechanical energy is conserved.

$$\boxed{E = K + U}$$

We already know that the PE and KE of a spring is given by:

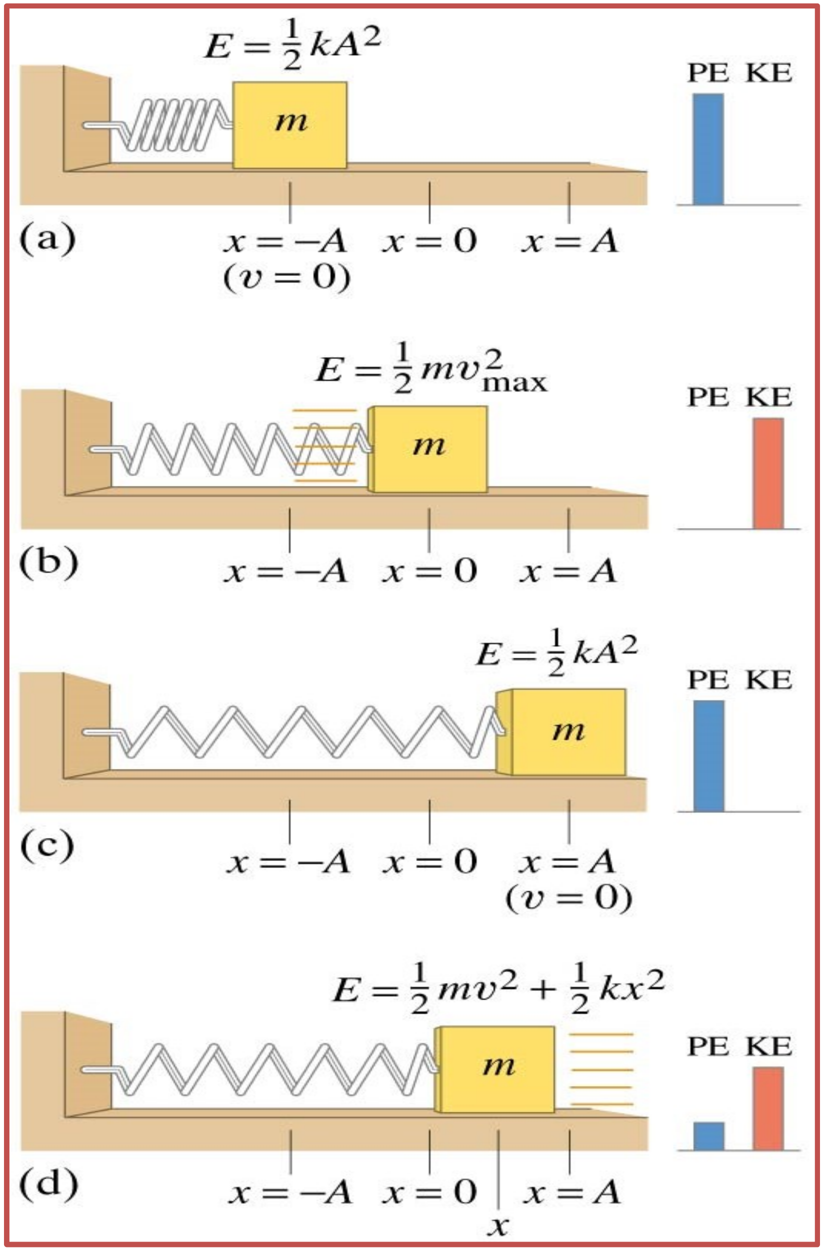
$$U = ½\ kx^2$$

$$K = ½\ mv^2$$

$$\boxed{E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2}$$

The total mechanical energy will be conserved, as we are assuming the system is frictionless.

Energy in Simple Harmonic Motion



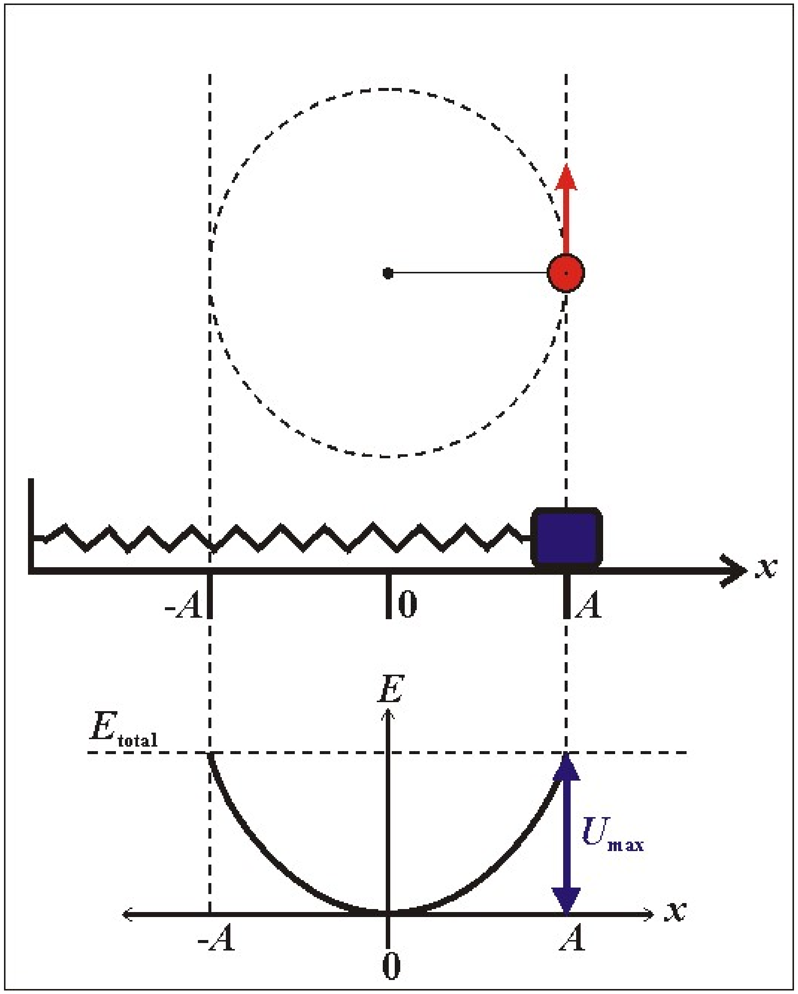
- If the mass is at the limits of its motion, the energy is all potential.
- If the mass is at the equilibrium point, the energy is all kinetic.

$$K_{max} = \frac{1}{2}mA^2\omega^2 = \frac{1}{2}mA^2(k/m)$$

$$K_{max} = \frac{1}{2}kA^2$$

$$E = \frac{1}{2}m(0)^2 + \frac{1}{2}kA^2 = \frac{1}{2}kA^2.$$

Energy in Simple Harmonic Motion



The total energy is, therefore

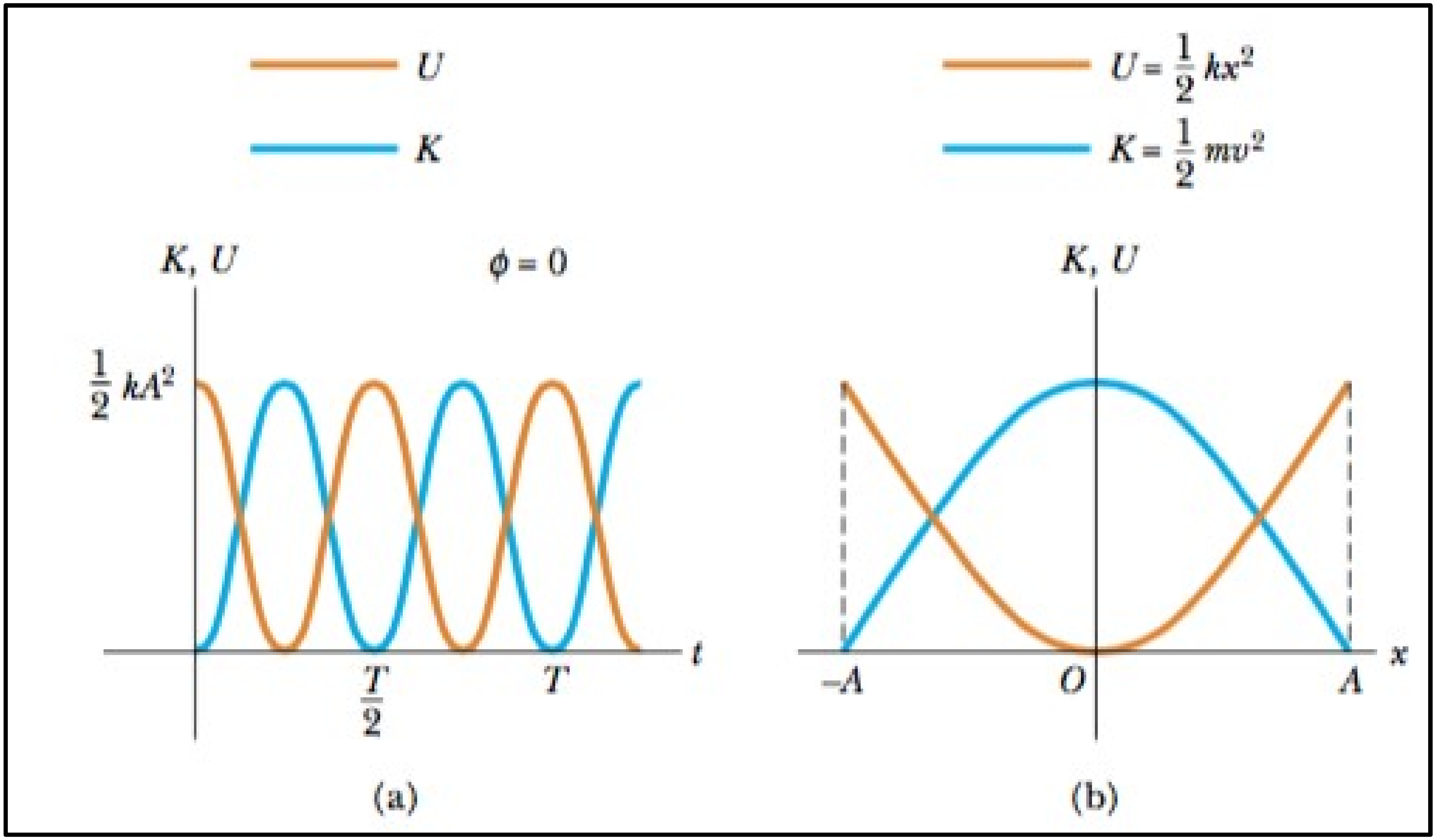
$$E = \tfrac{1}{2} kA^2$$

And we can write:

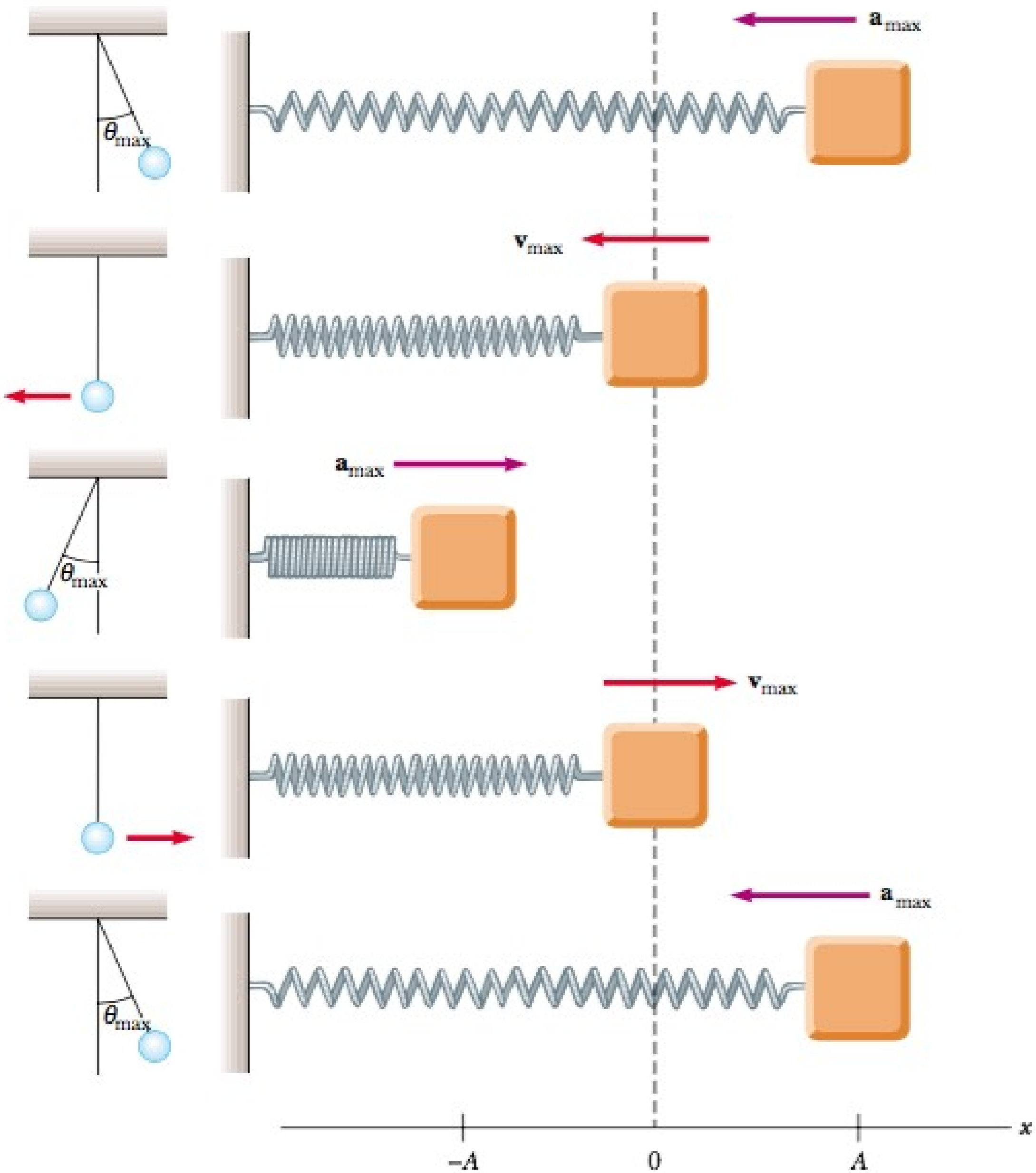
$$\boxed{\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.}$$

Energy Conservation in Oscillatory Motion

This diagram shows how the energy transforms from potential to kinetic and back, while the total energy remains the same.



Energy in Simple Harmonic Motion

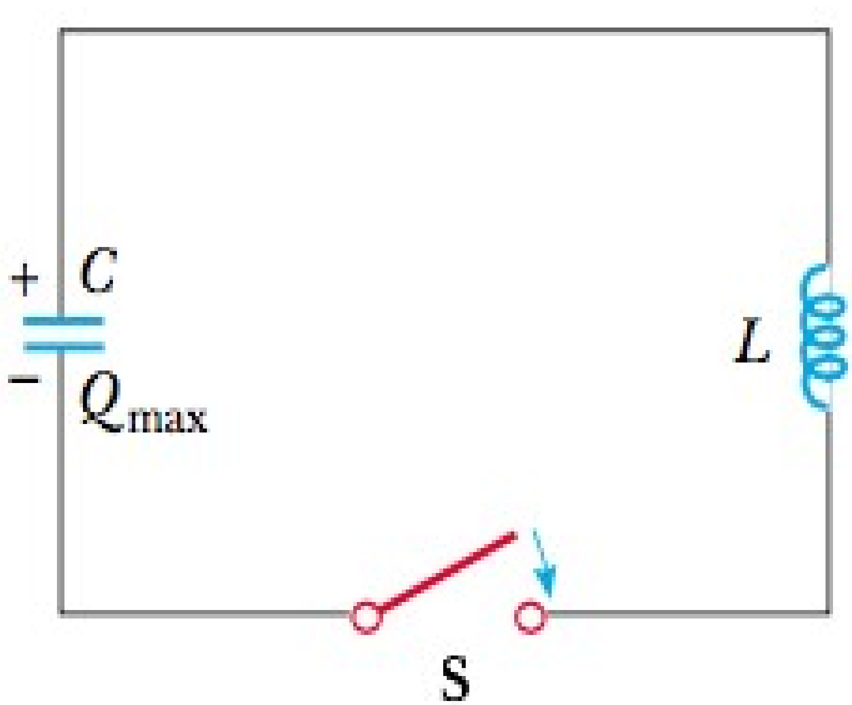


t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Electrical Oscillator: LC circuit

Electrical Oscillator: LC circuit

When a capacitor is connected to an inductor the combination is an LC circuit.



The Eqn. of voltage due to the Instantaneous current *I* can be written as;

$$\frac{Q}{C} + L\,\frac{d^2Q}{dt^2} = 0 \qquad \{I = dQ/dt.\}$$

$$\boxed{\frac{d^2Q}{dt^2} = -\frac{1}{LC}\,Q}$$

Solution is :

$$Q = Q_{max}\cos(\omega t + \phi) \qquad \omega = \frac{1}{\sqrt{LC}}$$

If the capacitor is initially charged and the switch is then closed, we find that both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values. If the resistance of the circuit is zero, no energy is transformed to internal energy.

Energy in Electrical Oscillator: LC circuit

The potential energy ½ kx^2 stored in a stretched spring is analogous to the electric potential energy ½ $C(\Delta V_{max})^2$ stored in the capacitor.
The kinetic energy ½ mv^2 of the moving block is analogous to the magnetic energy ½ LI^2 stored in the inductor, which requires the presence of moving charges.

When the capacitor is fully charged, the energy U in the circuit is stored in the electric field of the capacitor and is equal to

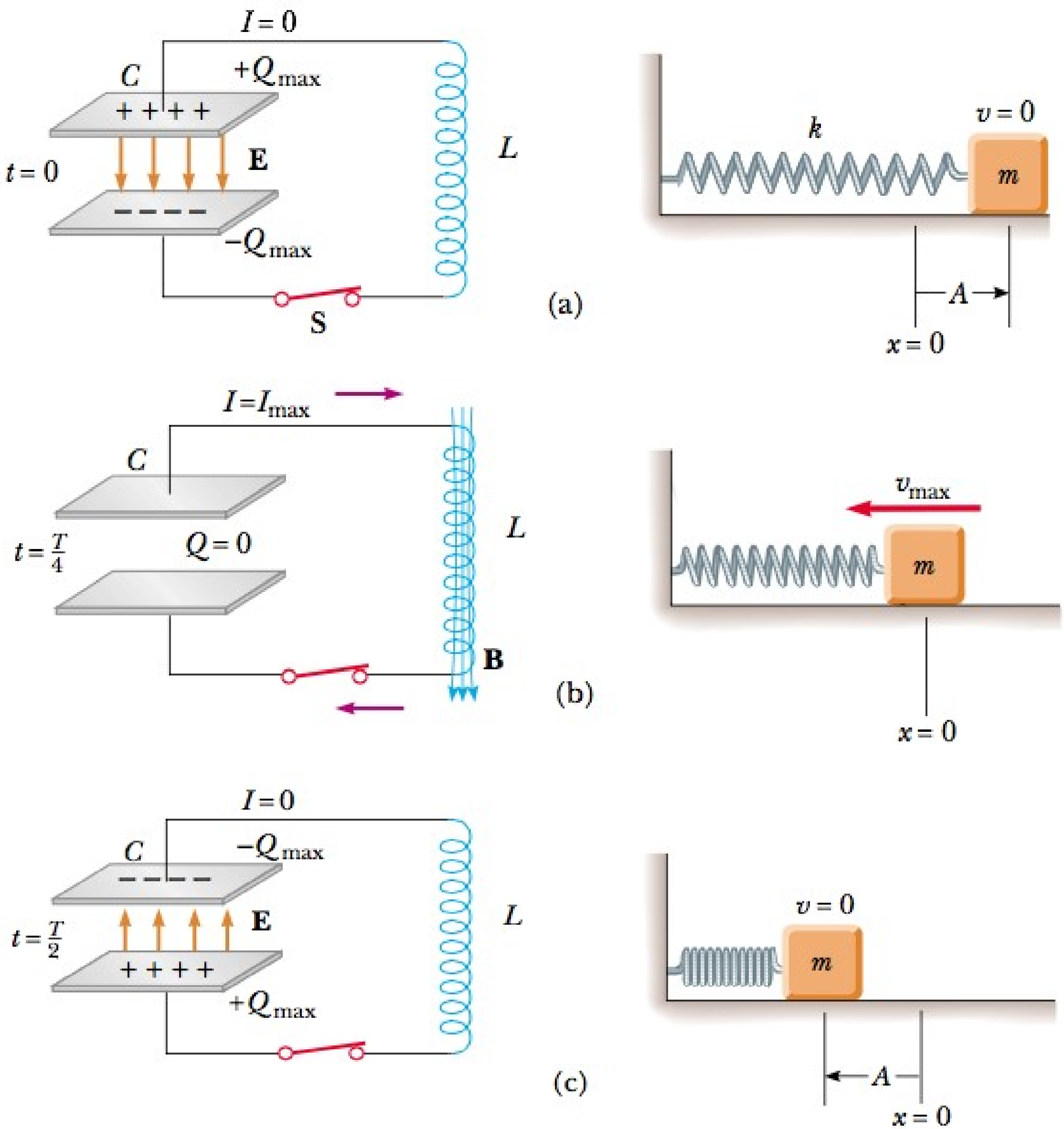
$$U = Q^2_{max}/2C$$

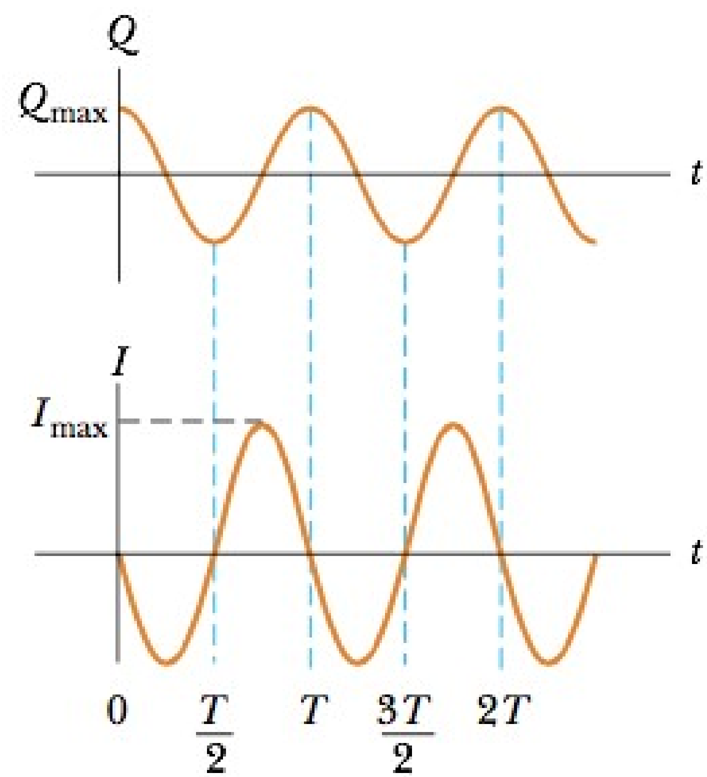
$$U = U_C + U_L = \frac{Q^2}{2C} + \tfrac{1}{2}LI^2$$

Assumed the circuit resistance to be zero, so no energy is transformed to internal energy and none is transferred out of the system of the circuit. Therefore, the total energy of the system must remain constant in time.

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \tfrac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

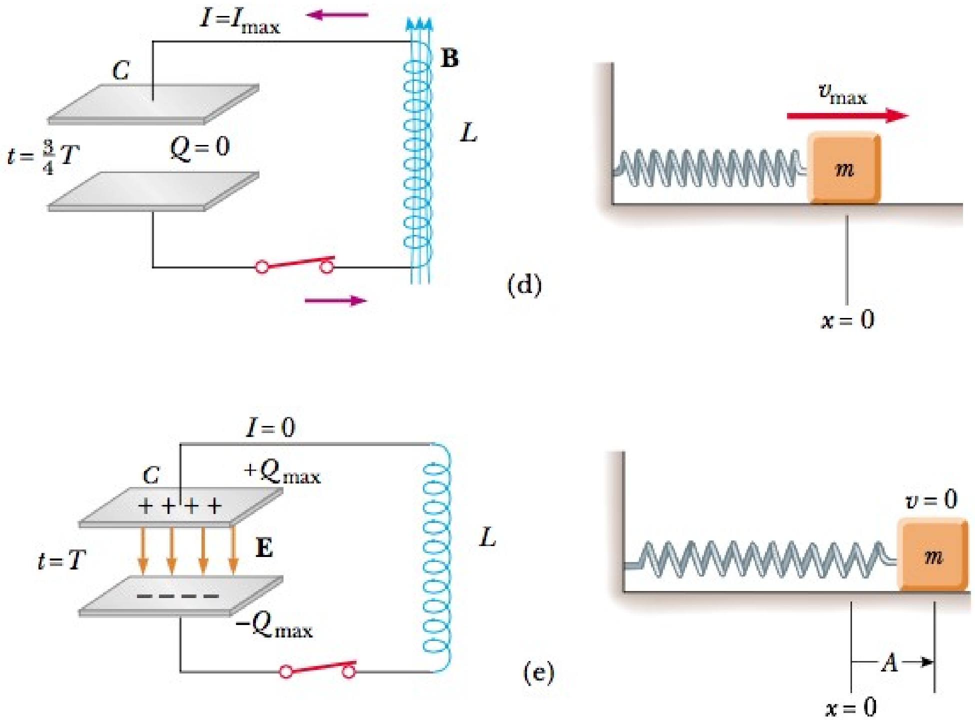
Energy in Electrical Oscillator: LC circuit





$$\frac{Q_{max}^2}{2C} = \frac{LI_{max}^2}{2}$$

Energy in Electrical Oscillator: LC circuit



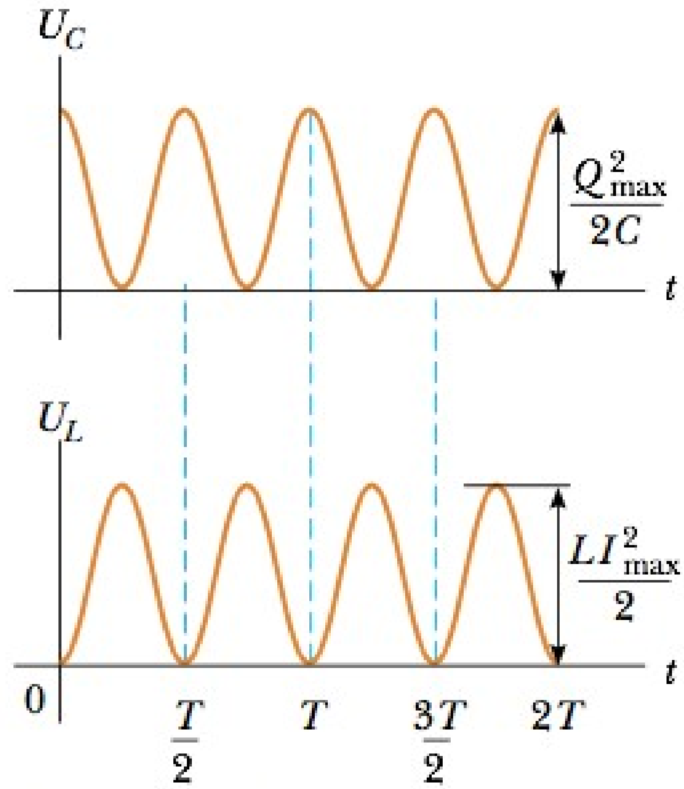


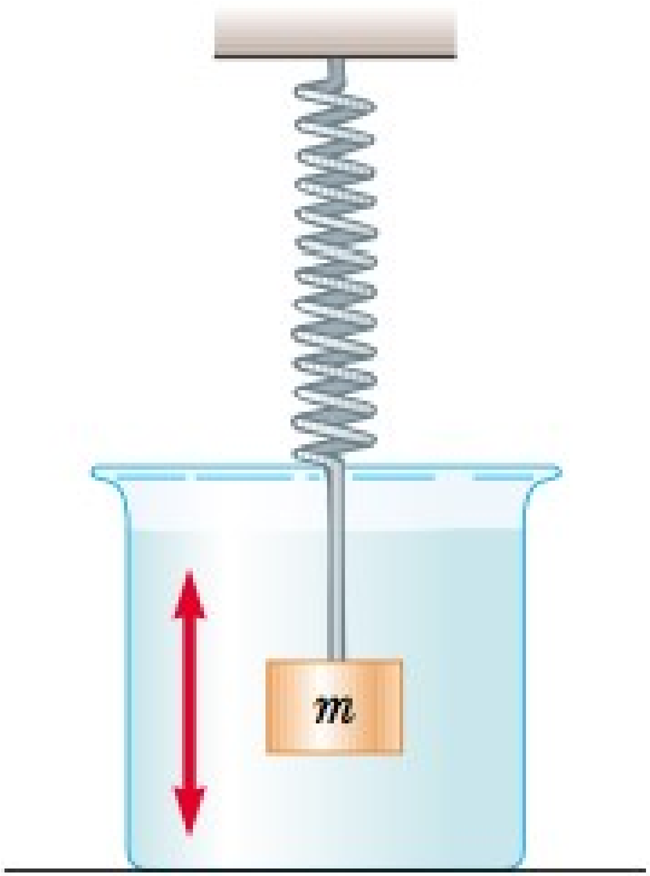
Figure 32.19 Plots of U_C versus t and U_L versus t for a resistanceless, nonradiating LC circuit. The sum of the two curves is a constant and equal to the total energy stored in the circuit.

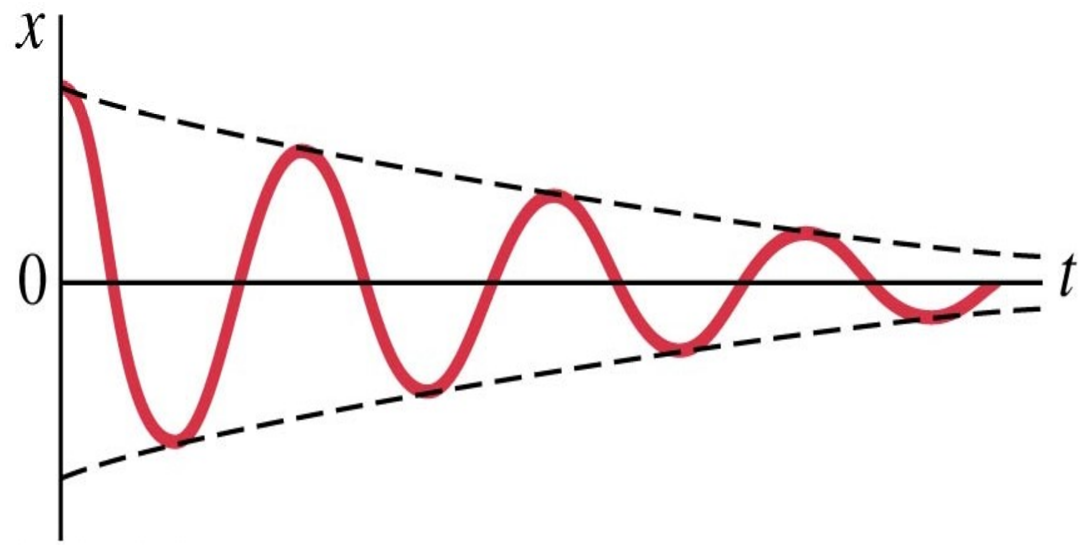
$$U = \frac{Q^2_{max}}{2C}(\cos^2 \omega t + \sin^2 \omega t) = \frac{Q^2_{max}}{2C}$$

because $\cos^2 \omega t + \sin^2 \omega t = 1$

Damped Harmonic Motion

Damped harmonic motion is harmonic motion with a frictional or drag force. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be damped.





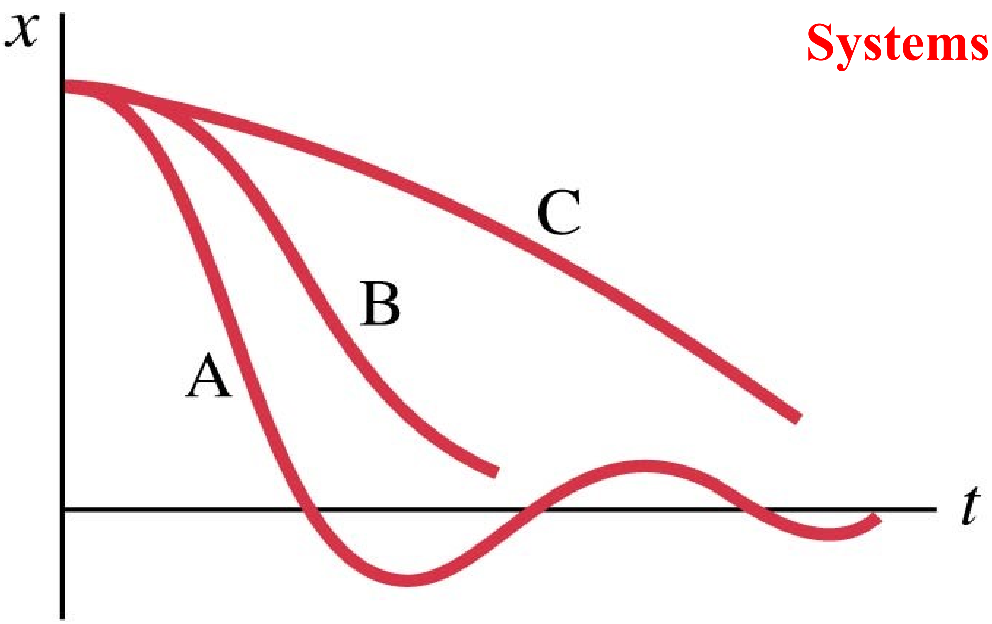
Damped Harmonic Motion

If the damping is large, it no longer resembles SHM

A: Underdamping: There are a few small oscillations before the oscillator comes to rest.

B: Critical damping: this is the fastest way to get to equilibrium.

C: Overdamping: the system is slowed so much that it takes a long time to get to equilibrium.



Systems where damping is wanted/unwanted

Damping is wanted:

Automobile, shock absorbers

Damping is unwanted

Clocks and watches

Damped Harmonic Motion

The retarding force is often observed when an object moves through air, for instance. Because the can be expressed as
Rretarding force : R = -b**v** (where b is a damping coefficient)
Restoring force : F = -kx (where k is a spring constant)
Using Newton's second law as ;

$$\sum F_x = -kx - bv_x = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

When the retarding force is small compared with the maximum restoring force that is, when b is small, the solution to above equation is;

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

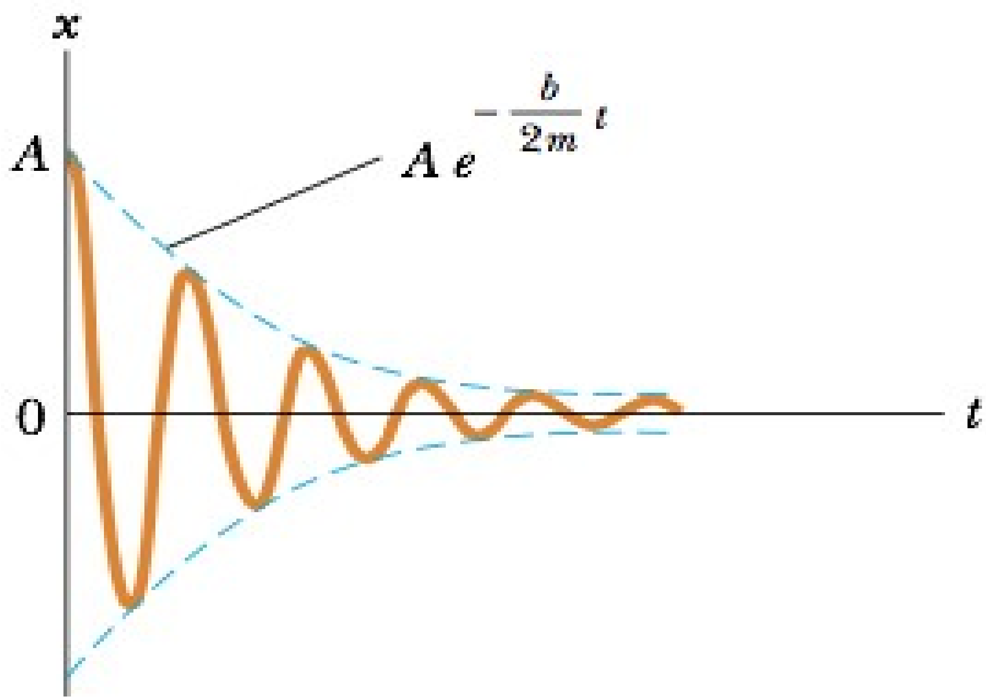
Damped Harmonic Motion

when the retarding force is small, the oscillatory character of the motion is preserved but the amplitude decreases in time, with the result that the motion ultimately ceases. Any system that behaves in this way is known as a **damped oscillator.**

$$\boxed{\omega = \sqrt{\omega_0^{\,2} - \left(\frac{b}{2m}\right)^2}}$$

$$\omega_0 = \sqrt{k/m}$$

represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the natural frequency of the system



Amplitude decays exponentially with time

Forced Oscillations; Resonance

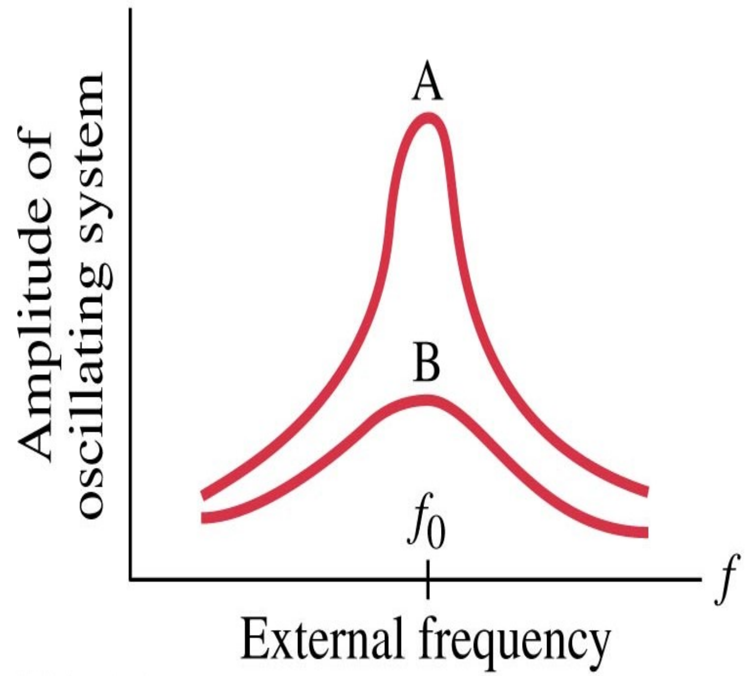
Forced vibrations occur when there is a periodic driving force. This force may or may not have the same period as the natural frequency of the system. If the frequency is the same as the natural frequency, the amplitude becomes quite large. This is called resonance.

The sharpness of the resonant peak depends on the damping.

If the damping is small, (A) it can be quite sharp;

if the damping is larger, (B) It is less sharp.

Like damping, resonance can be wanted or unwanted. Musical instruments and TV/radio receivers depend on it.



Forced Oscillations; Resonance

UPI / Bettmann Newsphotos

(a) (b)

Figure 15.26 (a) In 1940 turbulent winds set up torsional vibrations in the Tacoma Narrows Bridge, causing it to oscillate at a frequency near one of the natural frequencies of the bridge structure. (b) Once established, this resonance condition led to the bridge's collapse.

Electrical Oscillator: LCR circuit

$$LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2R \qquad \left\{ I = dQ/dt. \right\}$$

$$LI\frac{d^2Q}{dt^2} + I^2R + \frac{Q}{C}I = 0$$

Now divide by I

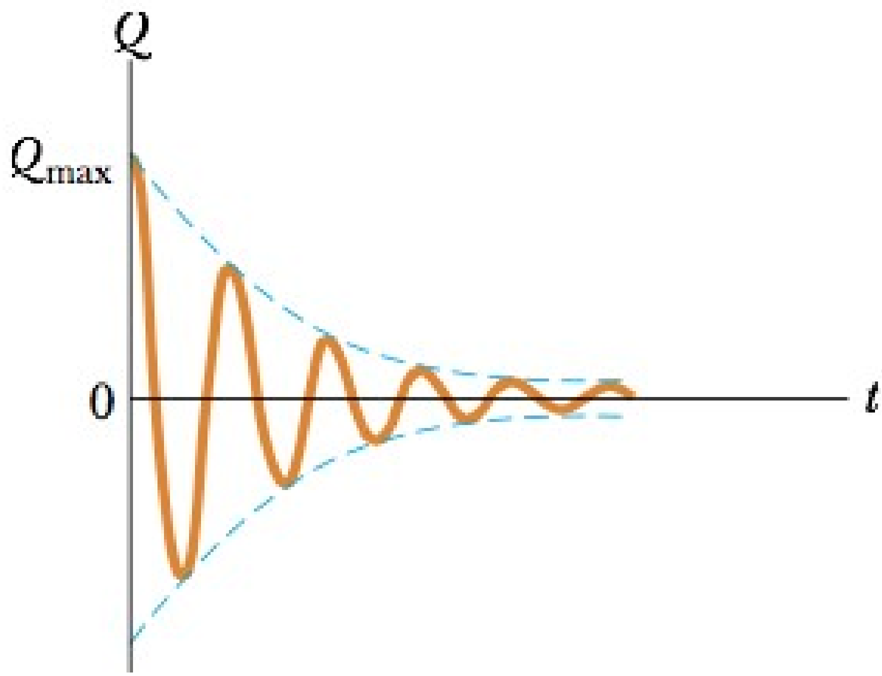
$$L\frac{d^2Q}{dt^2} + IR + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$Q = Q_{\max}\, e^{-Rt/2L} \cos \omega_d t$$

ncy at which the circuit oscillate

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2}$$



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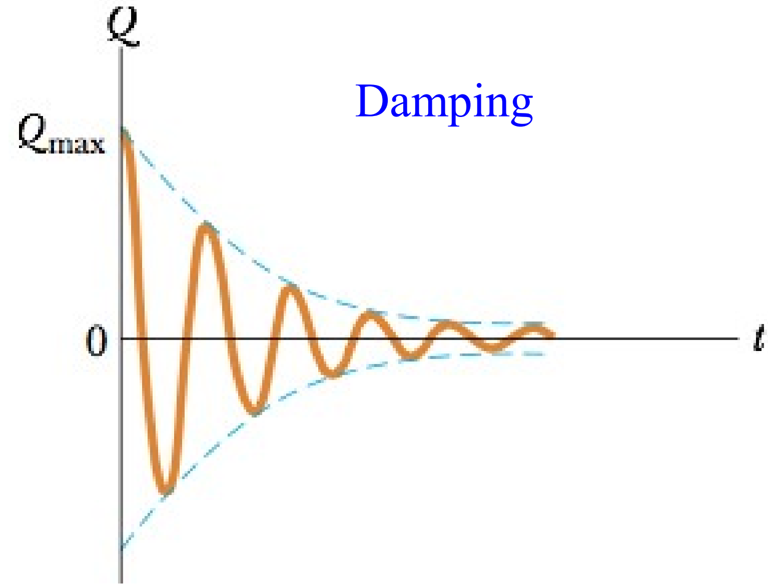
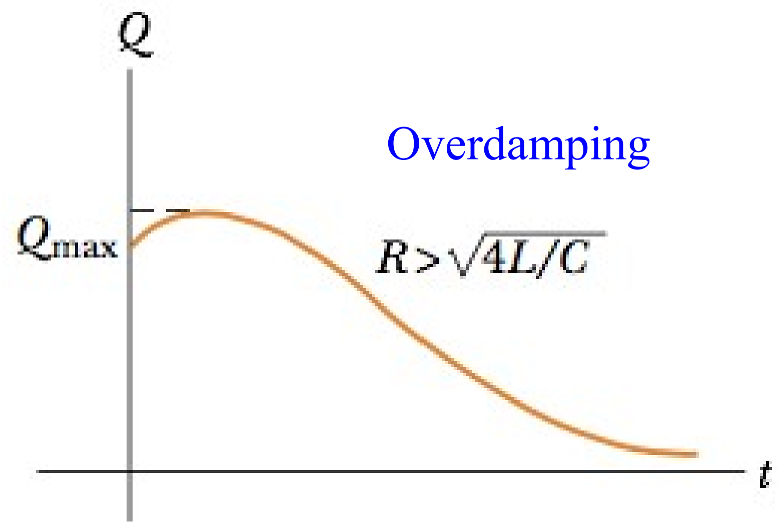


There exists a critical resistance value

$$R_c = \sqrt{4L/C}$$

above which no oscillations occur.
A system with $R = Rc$ is said to be critically damped.
When R exceeds Rc, the system is said to be overdamped



Electrical Oscillator: Mechanical Oscillator

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \dfrac{dQ}{dt} \leftrightarrow v_x = \dfrac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\dfrac{dI}{dt} = \dfrac{d^2Q}{dt^2} \leftrightarrow a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\dfrac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0 \leftrightarrow m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0$	Damped object on a spring

Logarithmic Decrement (δ)

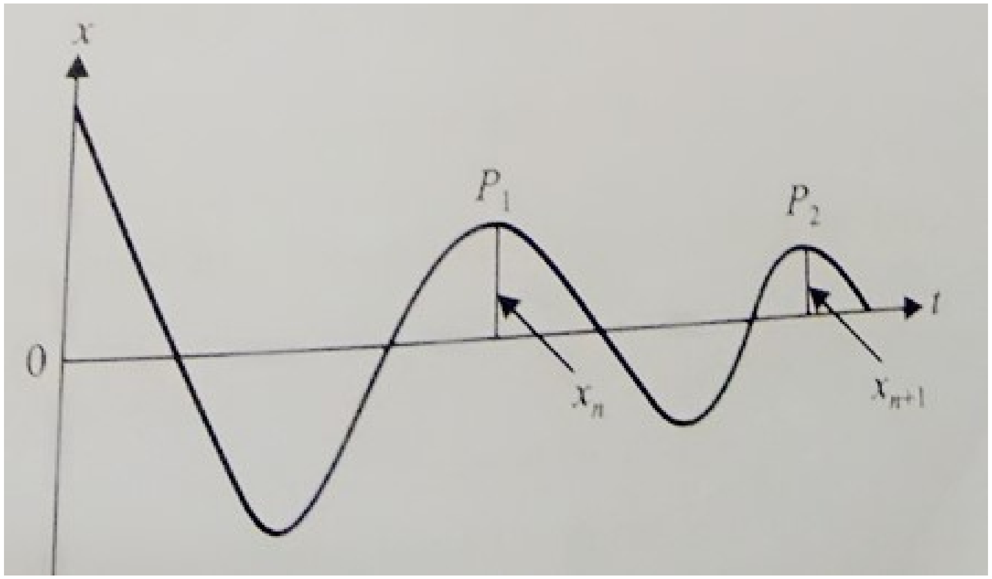
Logarithmic decrement measures the rate at which the amplitude of the oscillatory motion decay.

Let P_1 & P_2 be the two successive maxima corresponding to amplitudes, say x_n and x_{n+1}, and separated by a time period T.
If the maximum P_1 is occurring at $t_1 = t$, then P_2 will occur at $t_2 = t+2\pi/\omega$

$$\delta = -\log_e\{x_{n+1}/x_n\}$$

$$x = Ae^{-\frac{b}{2m}t}$$

$\delta = rT$
here $r = b/2m$



Relaxation Time (τ)

The relaxation time is a measures of time (τ_a) during which the amplitude of an oscillatory motion decay to $1/e$ of its initial value.

Or

The time (τ_e) during which the energy of an oscillatory motion decays to to $1/e$ of its initial value.

$$x = Ae^{-\frac{b}{2m}t}$$

By definition, when $t = \tau_a$, $x = A/e$

$$\tau_a = 1/r$$

here $r = b/2m$

$$\boxed{\delta = rT = T/\tau_a}$$

Quality Factor (Q)

The quality factor measures the quality of the oscillator;
less the losses (due to damping), more the quality.
It is also called the figure of merit and is
defined as:
2π the energy stored in the damped harmonic oscillator to the energy loss per cycle.

$$Q = 2\pi(E / -dE)$$

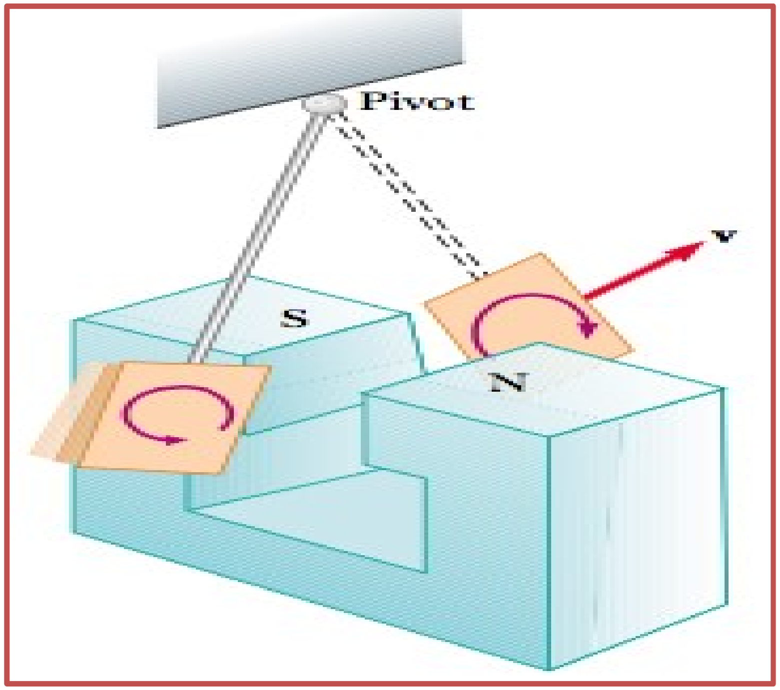
Using $E = E_0 e^{-2rt}$
And

$$-dE = 2rE_0 e^{-2rt} dt$$

$$Q = \omega / 2r$$

Examples of damping: Eddy Currents

As we know that, an emf and a current are induced in a circuit by a changing magnetic flux. In the same manner, circulating currents called eddy currents are induced in bulk pieces of metal moving through a magnetic field.



Demonstrated by allowing a flat copper plate attached at the end of a rigid bar to swing back and forth through a magnetic field. As the plate enters the field, the changing magnetic flux induces an emf in the plate, which in turn causes the free electrons in the plate to move, producing the swirling eddy currents.

Application: Eddy Currents

The braking systems on many subway and rapid-transit cars make use of electro-magnetic induction and eddy currents. An electromagnet attached to the train is positioned near the steel rails. The braking action occurs when a large current is passed through the electro- magnet. The relative motion of the magnet and rails induces eddy currents in the rails, and the direction of these currents produces a drag force on the moving train.
Also, as a safety measure, some power tools use eddy currents to stop rapidly spinning blades once the device is turned off.

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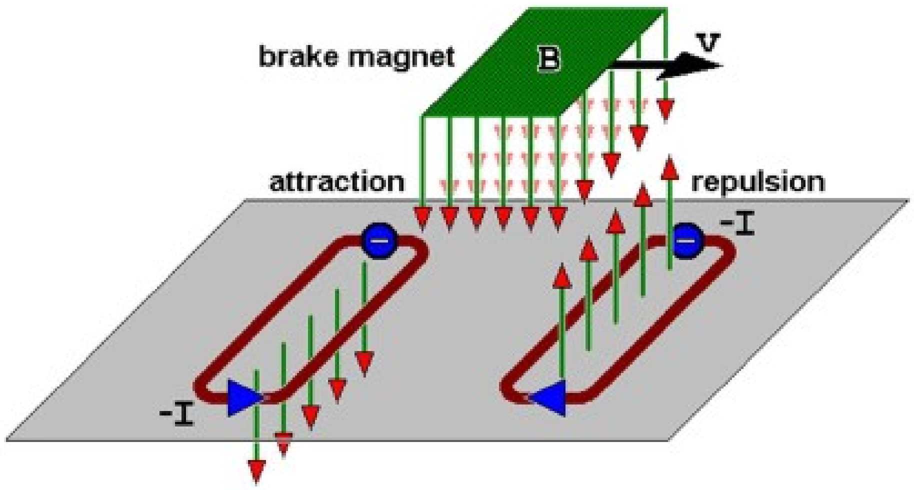
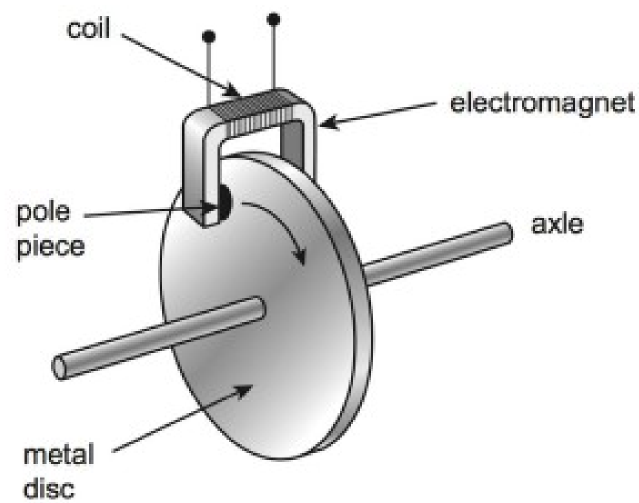


Linear braking system



circular braking system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