Second Order linear D.E

Note The general second order linear D.E is given by →

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\cdot y = R(x) \quad \text{— (A)}$$

or $$y'' + P(x)y' + Q(x)y = R(x)$$

or $$y'' + Py' + Qy = R \quad \text{— (1)}$$

where $P(x)$, $Q(x)$, $R(x)$ are fns of x only or constants.

→ If $R(x) = 0$, (1) is called homogeneous D.E (HDE) or Reduced D.E

If $R(x) \neq 0$, (1) is called nonhomogeneous (NHDE) or Complete D.E

So $$y'' + P(x)y' + Q(x)y = R(x) \quad \text{— (1)}$$

is a nonhomogeneous D.E (NHDE)

& $$y'' + P(x)y' + Q(x)y = 0 \quad \text{— (2)}$$

is a homogeneous D.E (HDE)

Note → Existence & Uniqueness thm →

Thm (1) Consider NHDE $y'' + Py' + Qy = R$ — (1). let $P(x)$, $Q(x)$ & $R(x)$ be ~~continuous~~ continuous on $[a, b]$. If $x_0 \in [a, b]$ & y_0, y_0' are any numbers then equation (1) has one and only one solution $y(x)$ on $[a, b]$

Optional

s.t $y(x_0) = y_0$ & $y'(x_0) = y_0'$.

→ It means that (under given assumptions) if $x_0 \in [a, b]$ and if we have the value of $y(x)$ & $y'(x)$ at pt. x_0 (Say $y(x_0) = y_0$ / $y'(x_0) = y_0'$) then there will always exist a unique solution of D.E (1) that assumes these values.

eg → Consider D.E → $y''+y=0$ ① , $\left.\begin{matrix} y(0)=0 \\ y'(0)=1 \end{matrix}\right\}$ → Initial value problem.

(2)

→ $\boxed{\text{Here } x_0=0,\ y_0=0,\ y_0'=1}$

Optional

→ We know that → $y = c_1\cos x + c_2\sin x$ — ②

is the G.S of D.E ①

Now as the initial conditions given are $y(0)=0$, $y'(0)=1$.

we can see that

→ $y(0)= c_1(1)+c_2(0) \Rightarrow \underline{c_1=0}$

Also $y'(x) = -c_1\sin x + c_2\cos x$

$y'(0) = -c_1(0)+c_2(1) \Rightarrow \boxed{c_2=1}$

So $\boxed{y = \sin x}$ is the Particular soln that satisfies these initial conditions

→ So corresponding to $x_0=0$ $\exists$ unique soln

$y=\sin x$ that satisfies D.E ① + Initial conditions

$y(0)=0$ + $y'(0)=1$

Note If D.E is → $y''+y=0$ ① but initial conditions are $\left.\begin{matrix} y(0)=1 \\ y'(0)=0 \end{matrix}\right\}$ — ②

Optional

→ Then try yourself

Check → $[y=\cos x]$ is Particular soln

that satisfies conditions ②

Thm 2 If y_g is general soln (G.S) of HDE

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{--- (1)}$$

And y_p is particular soln (P.S) of NHDE

$$y'' + P(x)y' + Q(y)y = R(x) \quad \text{--- (2)}$$

Then G. Soln of NHDE is →
$$= y_g + y_p$$

[HDE → Homogeneous DE
NHDE → Nonhomogeneous D.E]

(*) So to find the G.S of NHDE (2) we need to

1) Step (1) → 1st find G.S (y_g) of HDE
2) Step (2) → 2ndly find P.S (y_p) of NHDE

So that $= y_g + y_p$ is G.S of (2).]

Thm 3 If y_1, y_2 are any two particular soln of HDE (1) then $c_1 y_1 + c_2 y_2$ is also soln of HDE (1)

OR

Linear combination of any two particular solns of HDE is also soln of HDE (1)

Note → $\boxed{\text{Step (1)}}$ Finding General Soln of Homogeneous D.E (HDE)

$$\left[y'' + P(x)y' + Q(x)y = 0 \right] \text{--- (1)}$$

Note Linearly dependent + independent fns

→ Two functions $f(x)$ + $g(x)$, defined on $[a, b]$ are said to be L.D (Linearly dependent) if one is scalar or constant multiple of other.

that is, $f(x) = k\,g(x)$, $\frac{f(x)}{g(x)} = k$, k is Constant.

(*) More general def of LI/LD will be done in Linear Algebra)

→ Otherwise they are called L.I (Linearly independent) functions.

(*) eg → ① $f(x) = x$, $g(x) = 3x$

They are L.D. $\because \left[\begin{aligned} g(x) &= 3x \\ &= 3 \cdot f(x) \end{aligned} \right]$

② $f(x) = 1$, $g(x) = x$.

They are L.I. $\because f(x) \neq k\,g(x)$

or $\frac{f(x)}{g(x)} \neq k$ (constant)

③ In case $f(x) = 0$ + $g(x)$ is any nonzero fn.

Then $f(x)$ + $g(x)$ will be ______ ??

Thm ① → If $\boxed{y_1 + y_2}$ are any two Particular Solns of HDE ① and $\boxed{\text{are L.I.}}$ (that is, y_1 + y_2 are two L.I Soln of HDE ①) then

$\boxed{c_1 y_1 + c_2 y_2}$ is $\boxed{\text{General Soln}}$ of HDE ①

Lemma ① If y_1 & y_2 are any two solns of HDE ① on $[a, b]$ then their Wronskian ($W(y_1, y_2)$) is either identically zero or never zero on $[a, b]$.

→ Wronskian of fns $f(x)$ & $g(x)$ is $= W(f(x), g(x))$

$$= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \rightarrow \text{Determinant}$$

→ If we have three functions, say y_1, y_2, y_3 → Then their Wronskian

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Lemma ② If y_1 & y_2 are any two solns of HDE ① on $[a, b]$ then they are L.D on $[a, b]$ iff their Wronskian is identically zero on $[a, b]$

(*) So to check whether two solutions y_1 & y_2 (or two fns in general) are L.D or L.I using Wronskian we get following result →

→ L.D, Wronskian $(W) = 0$

→ L.I, Wronskian $(W) \neq 0$

So we will use this concept of Wronskian to check whether two (or more) fns are L.I or L.D.

eg ① Show that $y = c_1 x + c_2 x^2$ — ① is G. Soln of D.E → $x^2 y'' - 2xy' + 2y = 0$ — ② in any interval not containing zero. (Optional)

Soln $\boxed{\text{Method ①}}$ → Diff y in eqn ① to get y' & y''. Put values of y, y' & y'' in ② to check L.H.S = R.H.S.

$\boxed{\text{Method ②}}$ → Let us use the theory we did before

Consider $y_1 = x$, $y_2 = x^2$

Clearly $y_1' = 1$, $y_1'' = 0$. Putting y_1', y_1'' & y_1 in ② we get $x^2(0) - 2x(1) + 2(x) = 0$

$0 = 0$

$\Rightarrow \underline{y_1 = x}$ is soln of D.E ②

Similarily $y_2' = 2x$, $y_2'' = 2$. Put y_2', y_2'' & y_2 in ② we get $x^2(2) - 2x(2x) + 2(x^2) = 0$

$\underline{0 = 0}$

$\Rightarrow y_2 = x^2$ is also soln of D.E ②

So from Thm → $c_1 y_1 + c_2 y_2 = c_1 x + c_2 x^2$ is also soln of D.E ②

(But still it is not G.S since we have to check whether y_1 & y_2 are L.I)

So $W(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$

$= 2x^2 - x^2$

$= x^2 \neq 0 \ \forall x - \{0\}$

So $W \neq 0 \ \forall x - \{0\}$

$\Rightarrow y_1 = x$ & $y_2 = x^2$ are L.I $\forall x - \{0\}$

So by thm → $c_1 x + c_2 x^2$ is G.S of ② $\forall x - \{0\}$.