UES 009
Mechanics

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

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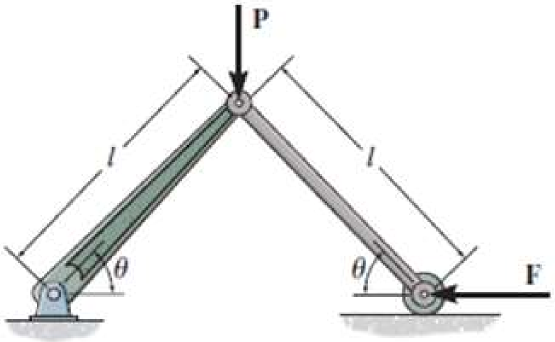
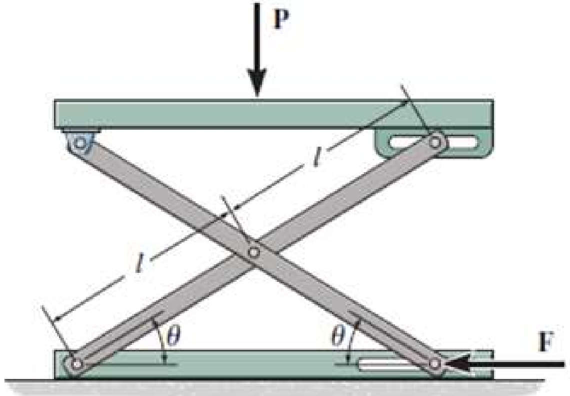
Method of Virtual Work

Scissors Lift Platform

Method of Virtual Work





Scissors Lift Platform

Rigid Body Equilibrium

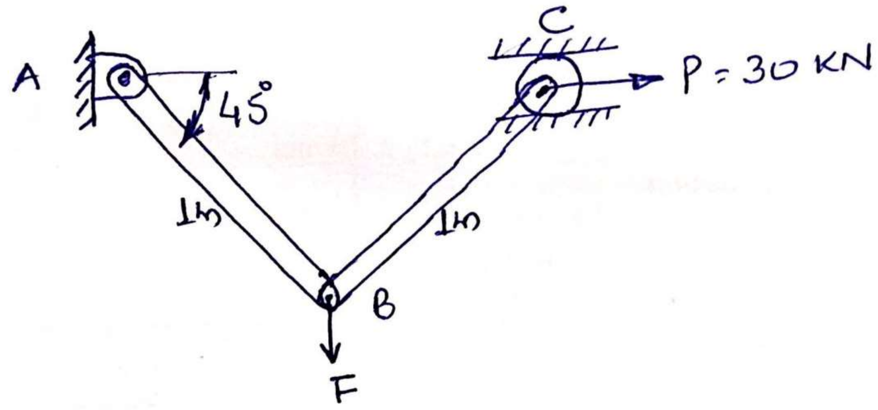
Steps to analyze a system:

- Draw FBD(contains external forces, constraints/ reactions)
- Write equations of equilibrium
- Solve equations of equilibrium to find unknowns.
- The other important parameter kept in mind is **DOF**

(DOF – parameter used to specify a system)

Rigid Body Equilibrium

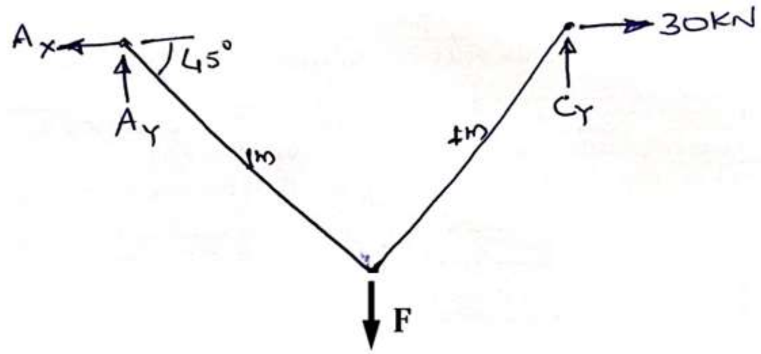
Ex. Find the magnitude of force **F** required to keep the system in equilibrium.



Solution: Draw FBD of the system

In this diagram, there are four unknowns but we can write only three equations of equilibrium i.e.

$\sum F_x = 0, \quad \sum F_y = 0$ and $\quad \sum M = 0$



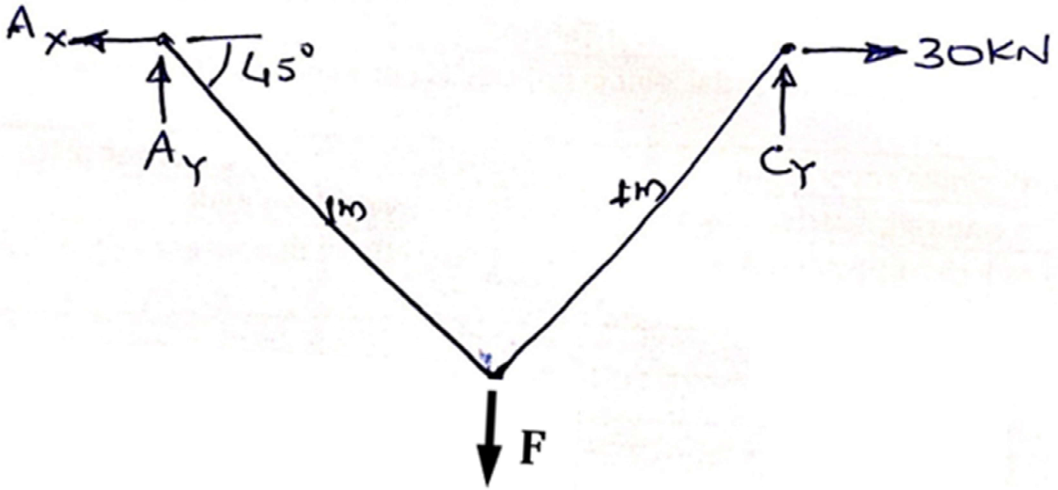
Rigid Body Equilibrium

$\sum F_x = 0 \Rightarrow A_x = 30\ kN$

$\sum F_y = 0 \Rightarrow A_y + C_y = \mathrm{F}$ ……………(1)

$\sum M_A = 0 \Rightarrow F \times l\ cos\theta - C_y \times 2l cos\theta = 0$

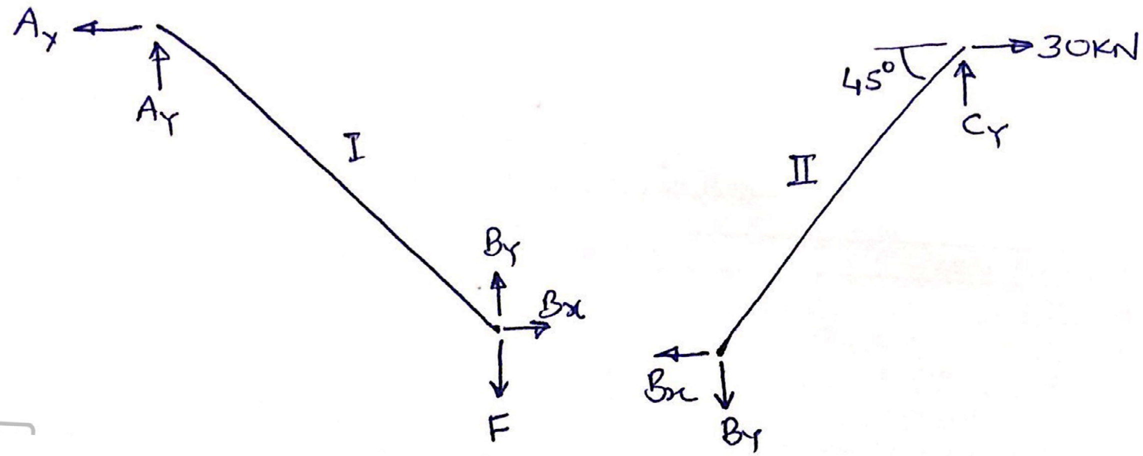
$F = 2\ C_y$ ……….(2)



To further solve this we need to draw separate FBDs of both the bars .

Rigid Body Equilibrium

From FBD-II



$B_x = 30\ kN$
$B_y = C_y$,

$\sum M_B = 0$
$-C_y \times l\ cos45 + 30 \times lsin45 = 0$
$C_y = 30\ kN$
From eq. (2) $F = 2\ C_y$
$\boldsymbol{F = 2 \times 30 = 60\ kN}$

Previous methods (FBD, $\sum F$, $\sum M$) are generally employed for a body whose equilibrium position is known or specified.

Virtual Work Method

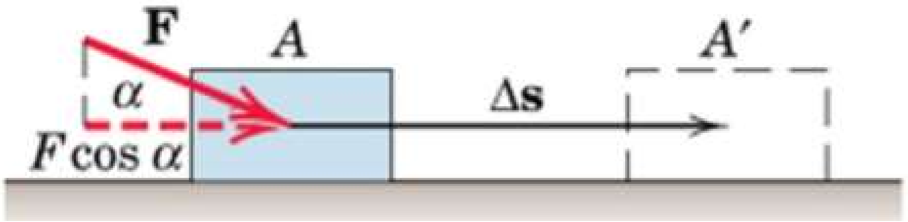
- For problems in which bodies are composed of interconnected members that can move relative to each other.
- various equilibrium configurations are possible and must be examined.
- Previous methods can still be used but are not the direct and convenient.

Virtual Work

Method of Virtual Work:

- is suitable for analysis of multi-link structures (pin-jointed members) which change configuration.
- Effective when a simple relation can be found among the displacement of the points of application of various forces involved.
- based on the concept of work done by a force.
- enables us to examine stability of systems in equilibrium.

Work done by a Force



Work = Force X corresponding displacement

Work done by the component of the force in the direction of the displacement,

$$\mathbf{W = \mathit{F}\ cos\alpha \times \Delta s}$$

Virtual Work

Virtual = Unreal or Imaginary

Virtual displacement: Displacement does not really exist but only is assumed to exist so that we may compare various possible equilibrium positions to determine the correct one.

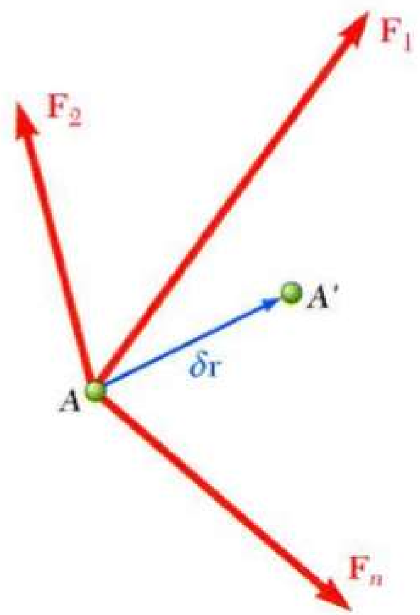
It is imaginary displacement of a system under equilibrium and is consistent with the system (according to DOF).

Virtual Work

A particle A is acted upon by various forces, imagine a small virtual displacement ***δr*** of the particle.

<u>The corresponding work done</u>,

$$\begin{aligned}\delta U &= \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + \vec{F}_3 \cdot \delta\vec{r} \\ &= \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot \delta\vec{r} \\ &= \vec{R} \cdot \delta\vec{r}\end{aligned}$$



Virtual Work

Principle of Virtual Work:

- If a particle is in equilibrium, the total virtual work of forces acting on the particle is **zero** for any virtual displacement.
- If a rigid body is in equilibrium, the total virtual work of external forces acting on the body is **zero** for any virtual displacement of the body.

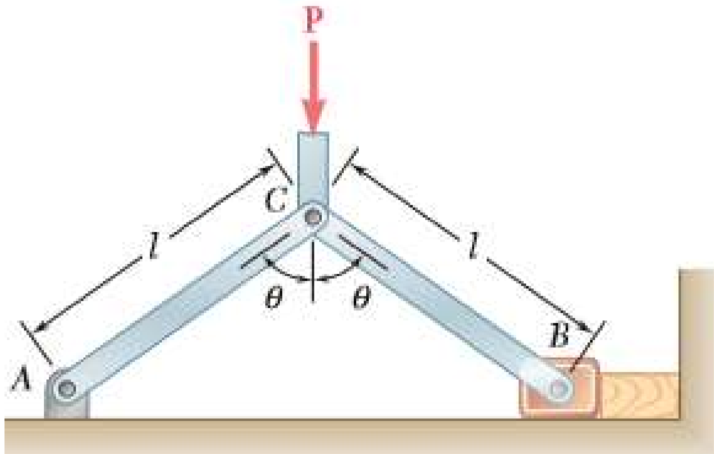
If a system of connected rigid bodies remains connected during the virtual displacement, only the work of the **external forces** needs to be considered since, work done by internal forces (equal, opposite, and collinear) cancels each other.

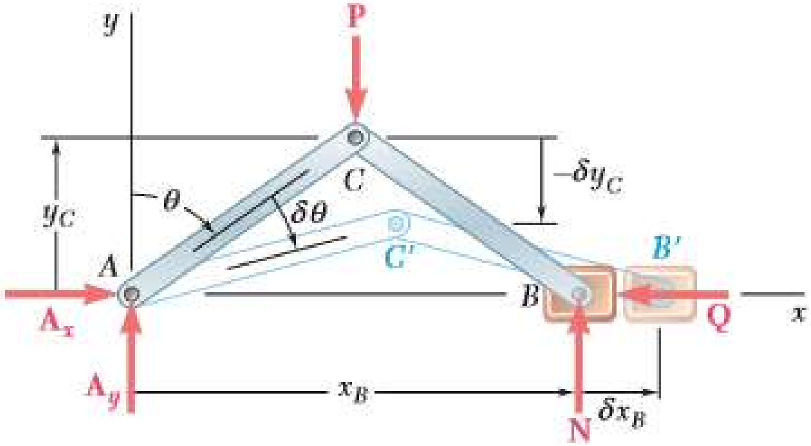
Virtual Work

- ✓ To solve the problem using **Principle of Virtual Work**, a small virtual displacement is assumed according to the **DOF** of the system.
- ✓ The net work done is calculated **(W = F X d)** by calculating work done due to each external force.
- ✓ Since the system is in equilibrium, no displacement is possible, so the net work done shall have to be equal to **zero**.
- ✓ So, the work done is put equal to **zero** and equation is solved to find unknowns.

Ex. Determine the force exerted by the vice on the block when a given force **P** is applied at ***C***. Assume that there is no friction.

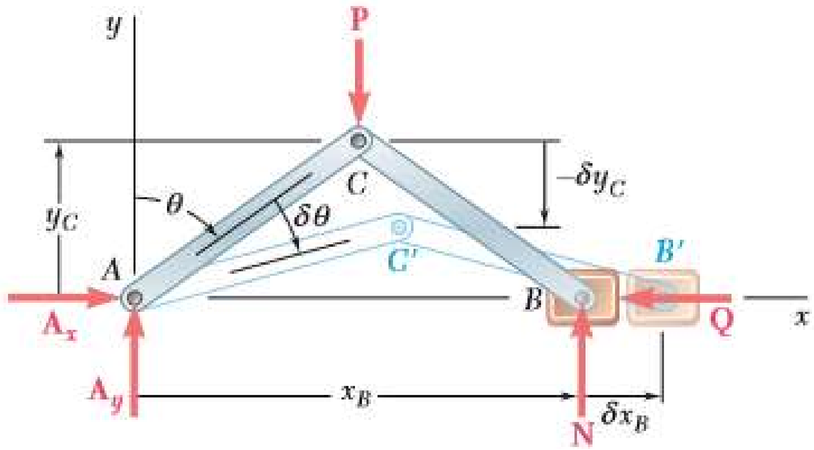
Under the action of force **P**, the points '***A***' and '***B***' will tend to go away. Since ***A*** is a hinge joint shall not move but the block at **B** will move towards right. This will cause a reactive force **Q** that will be acting at ***B*** in the opposite direction.





Express x_B and y_C ⟹ $x_B = 2l\sin\theta$ $\qquad y_C = l\cos\theta$

• $\boldsymbol{x_B}$ **increases while** $\boldsymbol{y_C}$ **decreases**



Expressing x_B and y_C in terms of θ and differentiating w.r.t. θ

$$x_B = 2l\sin\theta \qquad y_C = l\cos\theta$$

$$\delta x_B = 2l\cos\theta\,\delta\theta \quad \delta y_C = -l\sin\theta\,\delta\theta$$

Work due to forces **P** is positive and work done due to force **Q** is negative.

So, $\boldsymbol{\delta W = -Q\delta x_B + P.\delta y_C = 0}$

$$0 = -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta$$

$$Q = \frac{1}{2}P\tan\theta$$

SIGN CONVENTIONS:

$$x_B = 2l\sin\theta \qquad y_C = l\cos\theta$$

$$\delta x_B = 2l\cos\theta\,\delta\theta \quad \delta y_C = -l\sin\theta\,\delta\theta$$

$$\boldsymbol{\delta y_C = -|l\sin\theta\delta\theta|}$$

$$\boldsymbol{\delta W = -Q\delta x_B + P.\delta y_C = 0}$$

NOTE: The **minus (−) sign** obtained on differentiating is **not to be used** in the equation for work done. Only the **absolute** value of the displacement is to be used.

$$0 = -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta$$

$$Q = \frac{1}{2}P\tan\theta$$

SIGN CONVENTIONS:

1. When movements in the system (**translation** or **rotation)** are in the direction of **Force** or **Moment,** then work done is **positive (+)**

2. In writing the **final equation of work done** on the system we will only consider the sign convention as explained in **Point 1.**

As we are providing an infinitesimal displacement in the system (translation or rotation) we put the differentiated values **(absolute)** in final equation as already explained.

Virtual Work

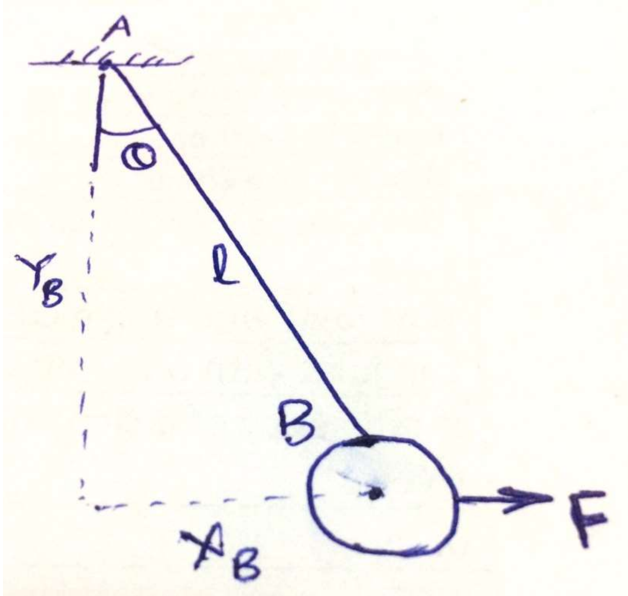
Steps to solve the problem :

- Identify external forces
- Fix the origin (a point which does not move)
- Find distances of the points where external forces are acting w.r.t. the origin
- Imagine a small virtual displacement according to DOF
- Take derivative of all the distances noted in previous step (it is actually the displacement of each point)
- Calculate virtual work done by external forces.

(*Note*: Reactions / constraint forces do not do any work)

Virtual Work

Ex. Find the magnitude of force **F** required to keep the pendulum in the given equilibrium position. The bob of pendulum has a mass '**m**'.



Solution:

▪Fix point A as the origin.

▪Write distance of point B from the origin.

$$Y_B = l\cos\theta$$

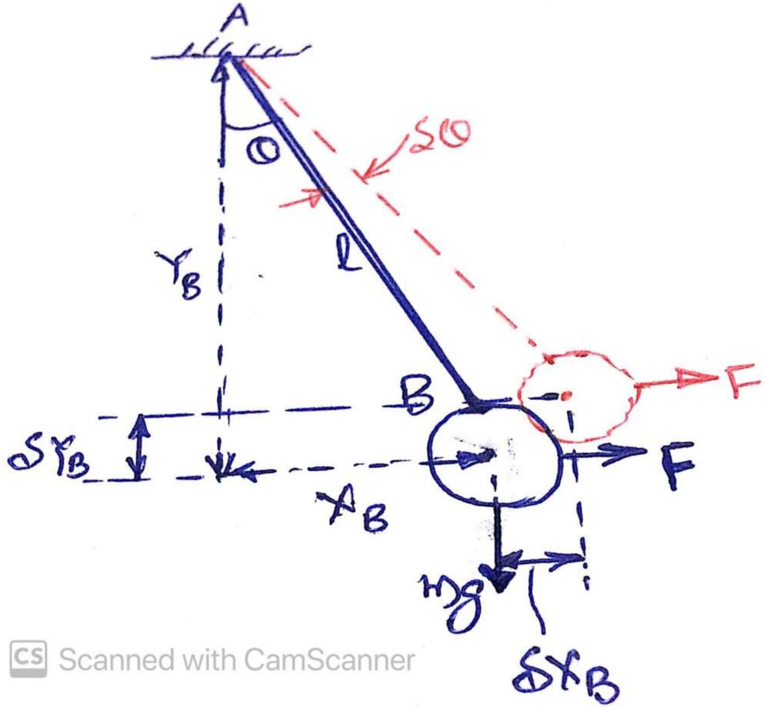
$$X_B = l\sin\theta$$

▪Assume a small virtual displacement under the action of force F (marked in red).

Virtual Work

Point B has been shifted to a new location.

Differentiate X_B and Y_B w.r.t. θ *to* find δX_B and δY_B (displacement of point B).



$\delta Y_B = l(-\sin\theta.\delta\theta) = -l\sin\theta.\delta\theta$

$\delta X_B = l\cos\theta.\delta\theta$

Calculate virtual work done

δW = F.δX_B $-$ mg. δY_B=0

δW = F($l\cos\theta.\delta\theta$) −mg($l\sin\theta.\delta\theta$) = 0

F$\cos\theta$ = mg $\sin\theta$

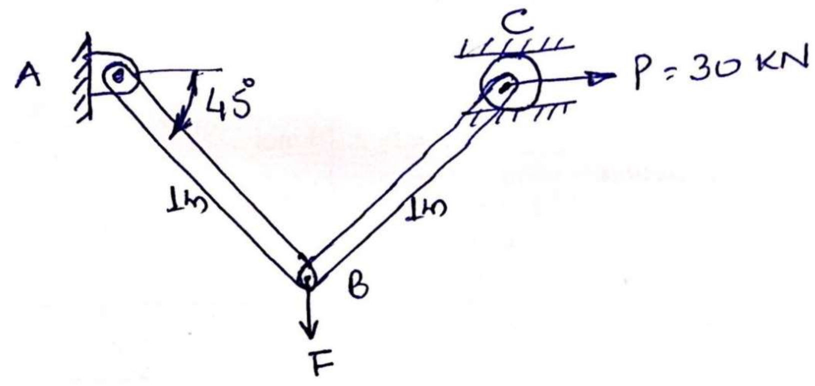
F = mg tan θ

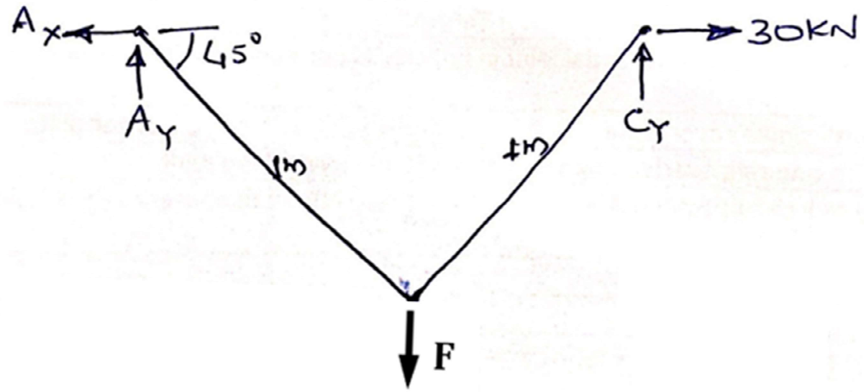
Ex. Find the magnitude of force **F** required to keep the system in equilibrium.

Solution:

- Fix point A as the origin.
- Write distances of points B and C from the origin.

$$Y_B = l \sin \theta$$
$$X_C = 2l \cos \theta$$



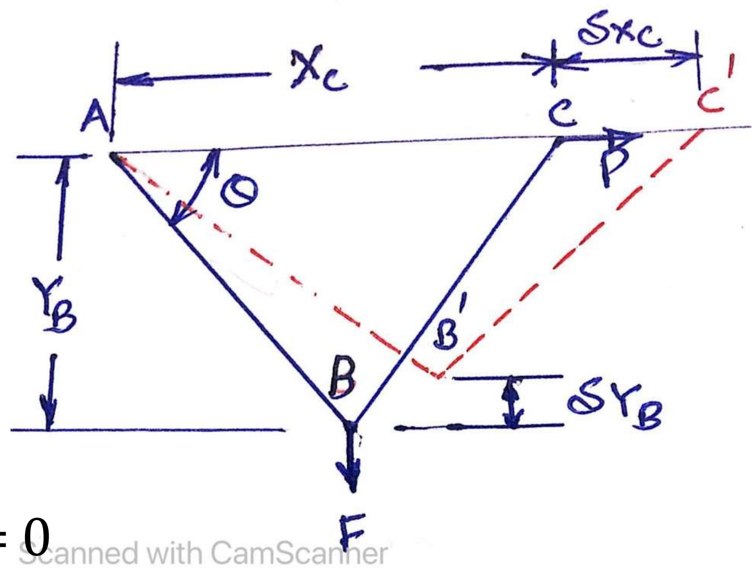


- Assume a small virtual displacement under the action of force P (=30 kN) (marked in red).

Points B and C have been shifted to a new locations.
Take derivatives of X_C and Y_B to find δX_C and δY_B (displacement of points B and C).

$\delta Y_B = l\cos\theta . \delta\theta$ and

$\delta X_C = 2l(-\sin\theta . \delta\theta) = -2l\sin\theta . \delta\theta$



Calculate virtual work done due to force F and force P and put it equal to zero.
Solve the equation and find out F.

$\delta W = -F.\delta Y_B + P.\delta X_C = 0$

$\delta W = -F(l\cos\theta . \delta\theta) + P(2l \sin\theta . \delta\theta) = 0$

$F\cos\theta = 2P \sin\theta$

$\mathbf{F = 2P \tan\theta = 2 \text{ X } 30 \text{ X } 1 = 60 \text{ kN}}$

Virtual Work

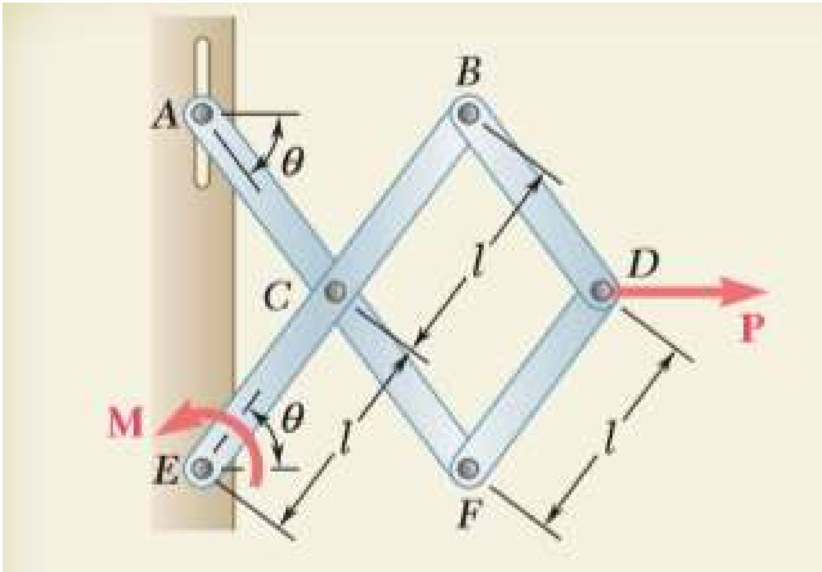
Major Advantages of the Virtual Work Method

- It is not necessary to dismember the systems in order to establish relations between the active forces.
- Relations between active forces can be determined directly without reference to the reactive forces.
- The method is particularly useful in determining the position of equilibrium of a system under known loads (This is in contrast to determining the forces acting on a body whose equilibrium position is known –studied earlier).
- The method requires that internal frictional forces do negligible work during any virtual displacement.
- If internal friction is appreciable, work done by internal frictional forces must be included in the analysis.

Ex. Using the method of virtual work, determine the magnitude of the couple **M** required to maintain the equilibrium of the mechanism shown.

Solution:

- Choosing a coordinate system with origin at E, we write



$x_D = 3l\cos\theta$

$\delta x_D = -3l\sin\theta . \delta\theta$

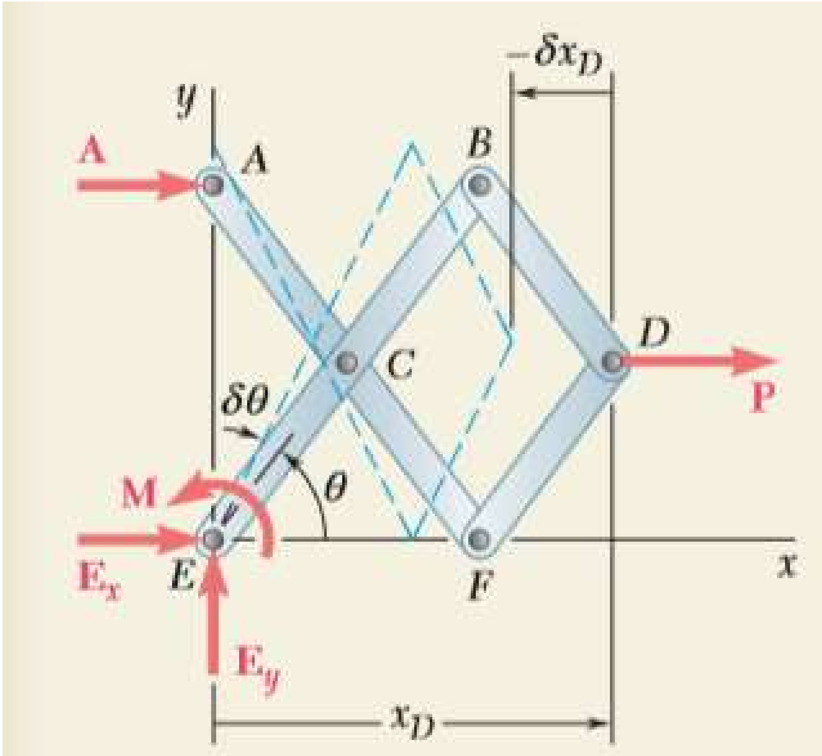
Since the reactions A, E_x, and E_y will do no work and the total work done by **M** and **P** must be zero.

$\delta W = 0$:

$+ M.\delta\theta - P.\delta x_D = 0$

$+ M.\delta\theta - P(3l\sin\theta.\delta\theta) = 0$

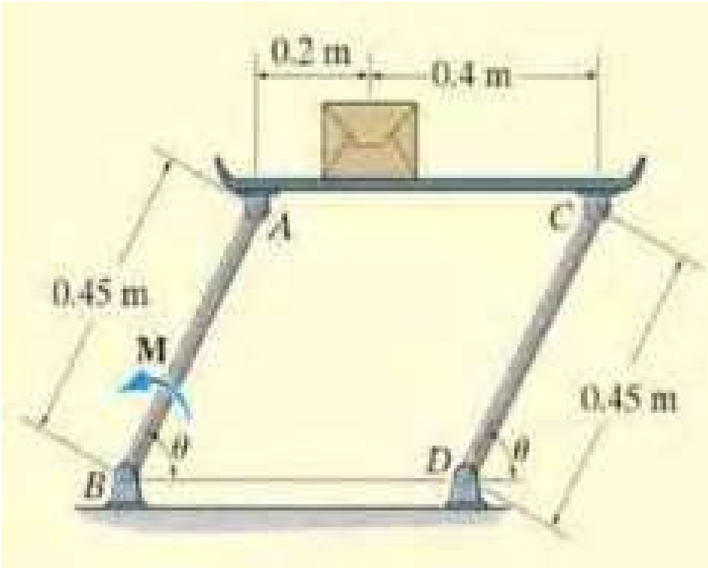
$\mathbf{M = 3P}\boldsymbol{l}\ \mathbf{sin}\ \boldsymbol{\theta}$



Ex. If the box lying on the frame has a mass of **10 kg**, determine the couple moment **M** needed to maintain equilibrium when θ = 60°. Neglect the mass of the members.

Solution:

Virtual Displacements: When θ undergoes a virtual displacement $\delta\theta$. Only the couple moment M, and the weight of the box (= 10 × 9.81 N) do work.



The position coordinate y_E measured from the fixed point B.

$y_E = 0.45 \sin\theta$

Ex. If the box lying on the frame has a mass of 10 kg, determine the couple moment **M** needed to maintain equilibrium when θ = 60°. Neglect the mass of the members.

The position coordinate y_E measured from the fixed point B.

$y_E = 0.45 \sin\theta$

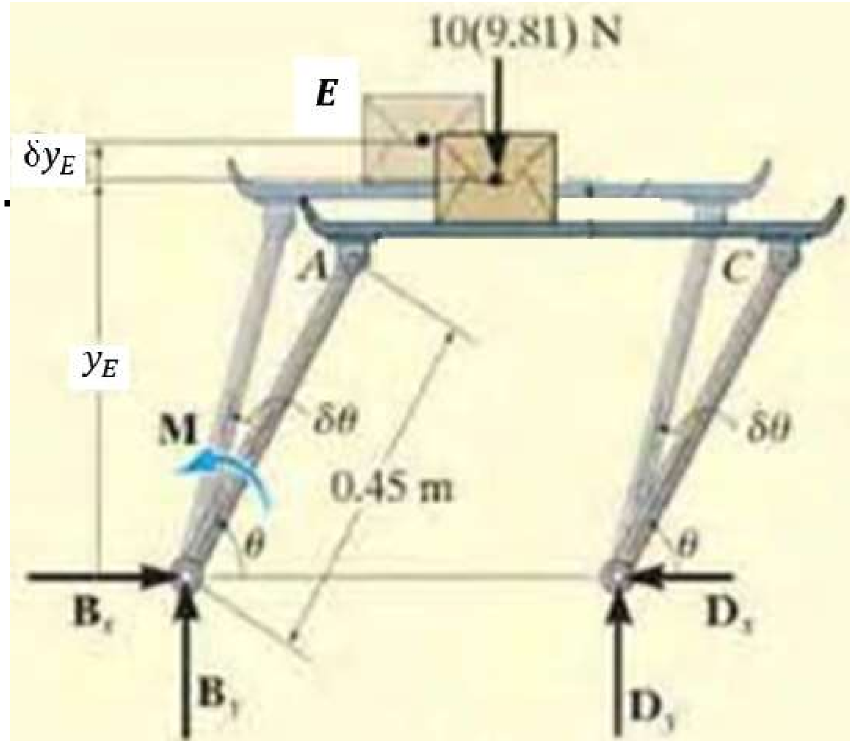
$\delta y_E = 0.45 \cos\theta . \delta\theta$

The force **98.1N** does positive work and the moment ***M*** does negative work

δW = 0: $98.1 . \delta y_E - M . \delta\theta = 0$

$98.1(0.45 \cos 60 . \delta\theta) - M . \delta\theta = 0$

***M*= 22.07 N-m**



Ex. Determine the required force $\boldsymbol{P}$ needed to maintain equilibrium of the scissors linkage when θ = 60°. The spring is unstretched when θ = 30°. Neglect the mass of the links.

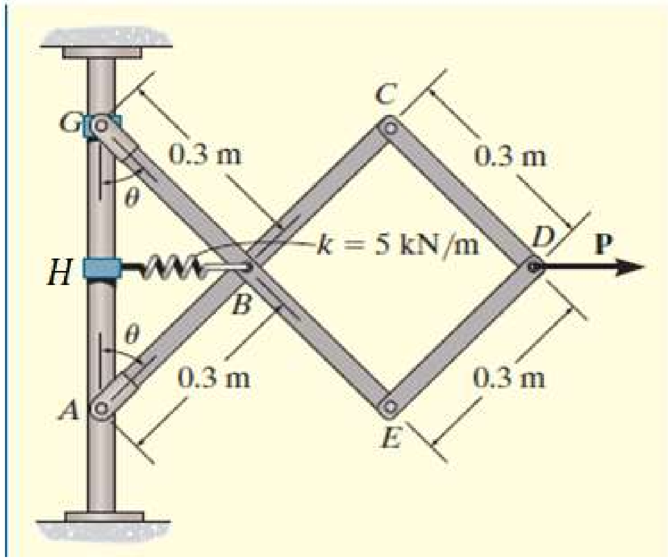
The spring is unstretched when θ=30° and at equilibrium θ=60°,

so, stretch of spring, $s = HB - HB' = 0.3\ sin60 - 0.3\ sin30$= 0.11

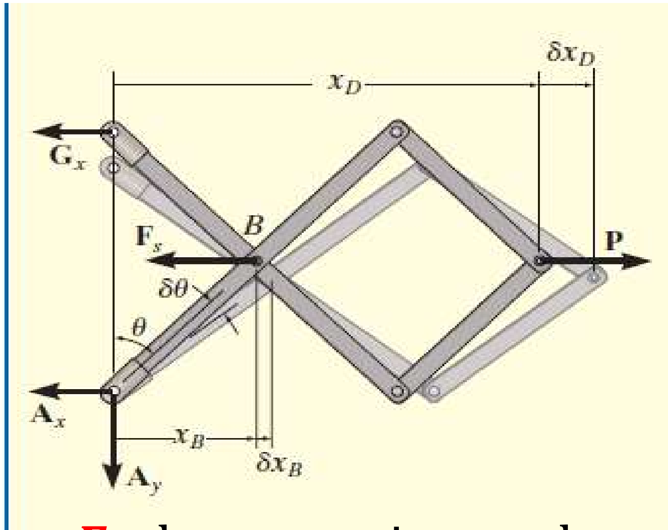
The magnitude of the force

exerted by the spring at B,

$\boldsymbol{F_s} = ks = 5 \times 10^3 \times 0.11 = 550\ N$



A small virtual displacement in θ, by $\delta\theta$ is given under the action of force $\boldsymbol{P}$



$x_B = 0.3 \sin\theta,$

$x_D = 3(0.3 \sin\theta) = 0.9 \sin\theta$

$\delta x_B = 0.3\cos\theta.\delta\theta$

$\delta x_D = 0.9\cos\theta.\delta\theta$

The force $\boldsymbol{P}$, does positive work and force $\boldsymbol{F_s}$ does negative work

δW = 0: $P.\delta x_D - F_s.\delta x_B = 0$

$P(0.9\cos\theta.\delta\theta) - 550(0.3\cos\theta.\delta\theta) = 0$

$\boldsymbol{P} = \mathbf{183.33\ N}$

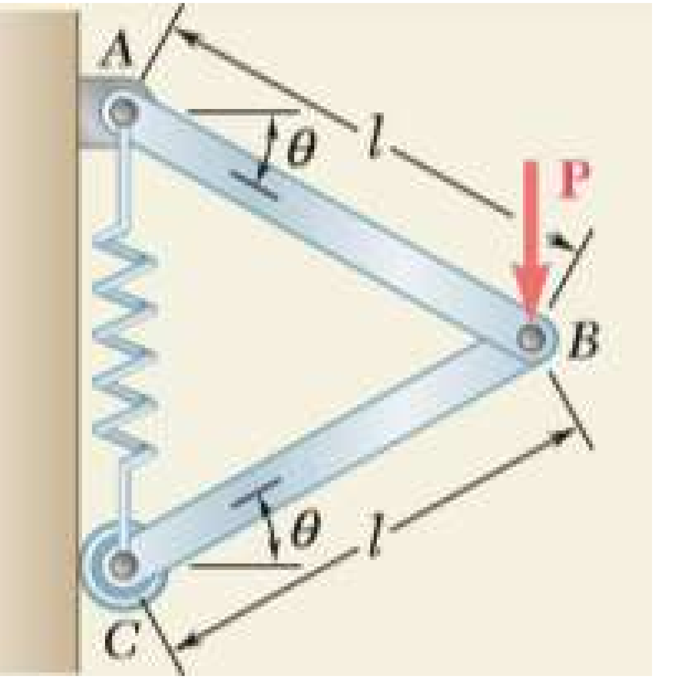
Ex. Determine the expressions for $\boldsymbol{\theta}$ and for the tension in the spring which correspond to the equilibrium position of the mechanism. The unstretched length of the spring is $\boldsymbol{h}$, and the constant of the spring is $\boldsymbol{k}$. Also find tension in the spring. Neglect the weight of the bars.

Solution:

With the coordinate system shown:

$$y_B = l\sin\theta$$

$$y_C = 2l\sin\theta$$



$y_B = l \sin\theta$

$y_C = 2l \sin\theta$

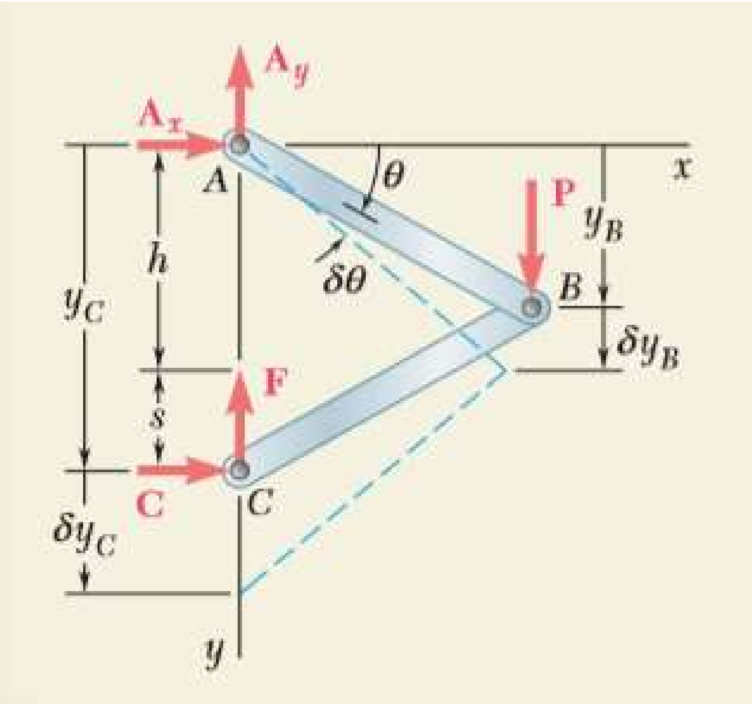
The elongation of the spring is

$s = y_C - h = 2l \sin\theta - h$

The magnitude of the force

exerted at C by the spring is

$F_s = k.s = k(2l \sin\theta - h)$ ....(1)



A small virtual displacement in θ, by $\delta\theta$ is assumed as shown (in the direction of force **P**)

$\delta y_B = l\cos\theta.\delta\theta \qquad \delta y_C = 2l\cos\theta.\delta\theta$

Principle of Virtual Work: Since the reactions Ax, Ay, and C do no work, the total virtual work done by **P** and **F** must be zero.

$\delta W = 0$: ⇒ $P.\delta y_B - F_s.\delta y_C = 0$

$P(l\cos\theta.\delta\theta) - ks(2l\cos\theta.\delta\theta) = 0$

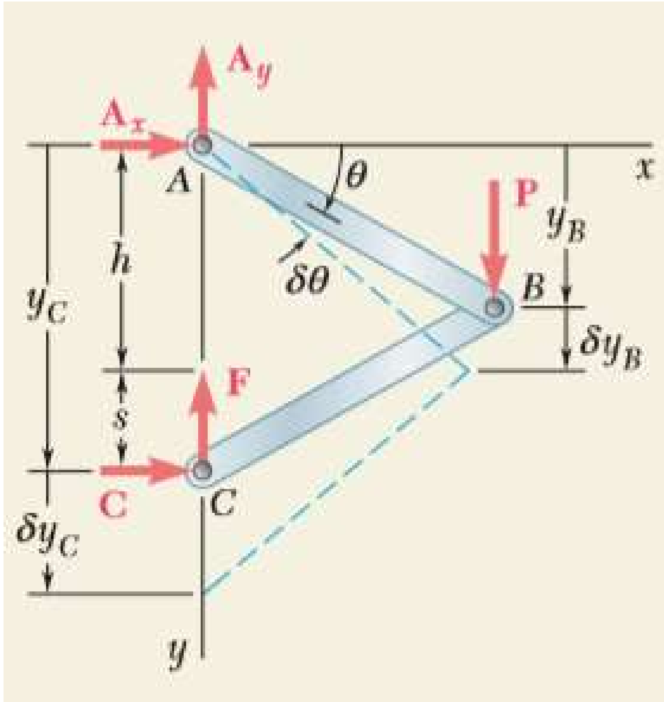
$P - 2ks = 0,$

$P - 2k\,(2l\sin\theta\; - h) = 0$

$P - 4kl\sin\theta\; + 2kh) = 0$

$P + 2kh = 4kl\sin\theta$

$\sin\theta\; = \frac{P+2\,kh}{4kl}$ ⇒ $\boldsymbol{\theta = sin^{-1}\left(\frac{P+2\,kh}{4kl}\right)}$



Substitute $\sin\theta$ in eq. (1), to find tension in the spring,

we obtain $\boldsymbol{F_s = \frac{1}{2}P}$

THANK YOU