Quantum Mechanics

Tut - 9

Ques 1/: Kinetic energy of an e^- and photon is 4.55×10^{-25} J. Calculate the velocity, momentum and wavelength of electron and photon?

Solution:-

(*) $E_k = \frac{1}{2} m v_0^2$ — (1)

$K.E. = 4.55 \times 10^{-25}$ J

$M_e = 9.1 \times 10^{-31}$ Kg

in eq. (1)

$$4.55 \times 10^{-25} = \frac{1}{2} \times 9.1 \times 10^{-31} \times v^2$$

$$\frac{4.55 \times 10^{-25} \times 2}{9.1 \times 10^{-31}} = v^2$$

$v^2 = 1.00 \times 10^6$

$v = 10^3$ m/s

(*) Momentum of electron is given as

$P = m_0 v$ — (2)

$= 9.1 \times 10^{-31} \times 10^3 = 9.1 \times 10^{-28}$ kg m/s

(*) wavelength of e^-, $\lambda = h/p$ — (3)

$$= \frac{6.62 \times 10^{-34} \text{ Js}}{9.1 \times 10^{-31} \times 10^3} = 7.27 \times 10^{-7} \text{ m}$$

(*) For Photon

→ $E = \frac{hc}{\lambda}$

$E = 4.55 \times 10^{-25}$ J, $C = 3 \times 10^{8}$ m/s $h = 6.62 \times 10^{-34}$ Js

$\lambda = \frac{hc}{E}$

$= \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{4.55 \times 10^{-25}} = 4.365 \times 10^{-1}$ m

→ Velocity $v = 3 \times 10^{8}$ m/s

$P = h/\lambda$

→ momentum $= P = \frac{6.62 \times 10^{-34}}{4.365 \times 10^{-1}} = 1.517 \times 10^{-33}$ kg m/s

Ques 2 :- Write down the conditions for the acceptable wave function & prove that $\psi = Ae^{-x^2}$ $(-\infty \le x \le \infty)$ is an acceptable wave function.

Ans:- (i) wavefunction must be finite everywhere.

(ii) must be single value.

iii) It must be continuous.

iv) It's derivative must be continuous.

i) finite [$\psi = Ae^{-x^2}$

$\lim_{x \to \pm\infty} \psi(x) = Ae^{-x^2}$ ——— (1)

$= 0$, this function is finite everywhere.

$\frac{A}{e^{x^2}}$

$\frac{A}{e^{\infty}} - \frac{A}{e^{-\infty}}$

$\frac{A}{\infty} - \frac{A}{e^{-\infty}}$

$0 - \infty$

0

(ii) check for some values $x = 1, 2, 3 - - -$

$\psi(1) = Ae^{-1}$; $\psi(-1) = Ae^{-1}$] Single valued

$\psi(2) = Ae^{-4}$; $\psi(-2) = Ae^{-4}$

——— (2)

Que 3:- The wave function of free particle in normalized state is represented by

$$\psi = N e^{-\left(\frac{x^2}{2a^2}\right) + ik}$$

Calculate the normalization factor N & the max. probability of finding the particle.

Sol^n:- The normalization condition is

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \quad \text{---} (1)$$

Putting the value of ψ & ψ^* in above eq^n we get

$$\Rightarrow \int_{-\infty}^{\infty} N e^{-(x^2/2a^2) - ikx} \cdot N e^{-(x^2/2a^2) + ikx} \, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} N^2 \underline{e^{-2x^2/2a^2} \, dx} = 1$$

$$\left[\because \int_{-\infty}^{\infty} e^{-x^2/a^2} dx = \sqrt{\pi}\, a \right.$$

$$\Rightarrow N^2 \cdot a\sqrt{\pi} = 1$$

$$\Rightarrow \boxed{N^2 = \frac{1}{a\sqrt{\pi}} \Rightarrow N = \frac{1}{a^{1/2}\pi^{1/4}}}$$

The max. probability $P(x)$ can be given as

$$P(x) = |\psi^*(x)\,\psi(x)|$$

$$= N^2 e^{-x^2/a^2}$$

$$= \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$

Que 4:- which of the following are eigenfunctions of operator $\frac{\partial^2}{\partial x^2}$? Find out appropriate eigenvalues of following function.

Sol^n:- Given that $f(x) = \sin x$

(i) operating $\frac{\partial^2}{\partial x^2}$ on $f(x)$, we get

$$\frac{\partial^2}{\partial x^2}(\sin x) = -\sin x = -f(x)$$

$\sin x$ is an eigenfunction having eigen value -1.

(ii) $f(x) = \sin^2 x \Rightarrow \frac{1}{2}\frac{\partial}{\partial x^2}(2\sin^2 x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(1 - \cos 2x)$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial x}(0 + 2\sin 2x) = \frac{1}{2}(4\cos 2x) = 2\cos 2x.$$

Hence it is not an eigen function for $f(x) = \sin^2 x$

Continued

iii) $\lim_{x\to\pm\infty} \frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(Ae^{-x^2}\right)$

$= Ae^{-x^2} \cdot -2x$

or $= -2A \frac{x}{e^{x^2}}$

Put limits $\infty, -\infty$

$= -2A \frac{\infty}{e^{\infty^2}}$ & $-2A \frac{(-\infty)}{(e^{-\infty})}$

Apply L-Hospital Rule

$\left[\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \right.$

$\lim_{x\to\pm\infty} \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) = \frac{-2A}{2x e^{x^2}}$

$\lim_{x\to\pm\infty} \frac{-A}{x e^{x^2}} = -\frac{A}{\infty} = 0$ — ③

$\frac{-A}{\infty} + \frac{A}{-\infty} = 0$

iv) $\lim_{x\to\pm\infty} \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(-2Axe^{-x^2}\right)$

$= -2A\left[xe^{-x^2}\cdot(-2x) + e^{-x^2}\right]$

$= -2A\left[-2x^2e^{-x^2} + e^{-x^2}\right]$

$= -2A\left[\frac{-2x^2}{e^{x^2}} + e^{-x^2}\right]$

$= -2A\left[\frac{-2x^2+1}{e^{x^2}}\right]$

Apply L-Hospital Rule.

$\lim_{x\to\pm\infty} \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) = \lim_{x\to\pm\infty} -2A\left[\frac{-4x}{e^{x^2}\cdot 2x}\right] = \lim_{x\to\pm\infty} \frac{4A}{2x^2e^{x^2}+e^{x^2}} = \frac{4A}{\infty} = 0$ — ④

from eqn ①, ②, ③ & ④ the above fun is Allowed

Ques 5 :- A particle limited to x axis has wave function $\psi = ax$ between $x=0$ & $x=1$; $\psi = 0$ elsewhere. (a) find probability that the particle can be found b/w $x = 0.45$ & $x = 0.55$.
(b) find expectation value $\langle x \rangle$ of particle's position.

Soln :- $\psi = ax \quad 0 \le x \le 1$

$\psi = 0 \quad$ elsewhere

Probability of particle b/w 0.45 & 0.55

$$\text{probability} = \int_{x_1}^{x_2} \psi^* \psi \, dx = \int_{0.45}^{0.55} a^2 x^2 \, dx$$

$$= a^2 \int_{0.45}^{0.55} x^2 \, dx$$

$$= \frac{a^2}{3} \left| x^3 \right|_{0.45}^{0.55}$$

$$= \frac{a^2}{3} \left| (0.55)^3 - (0.45)^3 \right|$$

$$= 0.0251 \, a^2.$$

(b) find the expectation value of particle's position is

$$\langle x \rangle = \int_0^1 \psi^* \hat{x} \psi \, dx$$

$$= \int_0^1 ax \cdot x \cdot ax \, dx = \int_0^1 a^2 x^3 dx = a^2 \int_0^1 x^3 dx$$

$$= \frac{a^2}{4} \left| x^4 \right|_0^1$$

$$= a^2/4 \quad \text{(Ans)}$$

Ques 6 :- In a region of space, a particle with zero energy has a wave fuction $\psi = A e^{-(x^2/L^2)}$. Determine the steady state pot. energy as a function of x.

Soln :- Steady state potential

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - U)\psi = 0 \quad -(1)$$

$$E = 0; \quad \psi = A e^{-x^2/L^2} \quad (*)$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(A e^{-x^2/L^2}) = A e^{-x^2/L^2} \cdot \left(\frac{-2x}{L^2}\right)$$

$$= -\frac{2A}{L^2} x \cdot e^{-x^2/L^2}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2}{L^2} A x e^{-x^2/L^2}\right)$$

$$= -\frac{2A}{L^2}\left[x \cdot e^{-x^2/L^2}\left(\frac{-2x}{L^2}\right) + e^{-x^2/L^2}\right]$$

$$= -\frac{2A}{L^2}\left[-\frac{2x^2}{L^2} e^{-x^2/L^2} + e^{-x^2/L^2}\right]$$

$$= \frac{-2}{L^2}\left[-\frac{2x^2}{L^2} + 1\right]\psi$$

$$= \left(\frac{4x^2}{L^4} - \frac{2}{L^2}\right)\psi \qquad -(2)$$

$$\left[\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi + \frac{2m}{\hbar^2}[E-U]\psi = 0$$

$$\frac{4x^2}{L^4} - \frac{2}{L^2} + \frac{2m}{\hbar^2}(-U) = 0$$

$$\frac{4x^2}{L^4} - \frac{2}{L^2} = \frac{2m}{\hbar^2} U$$

$$\boxed{U = \frac{\hbar^2}{2m}\left(\frac{4x^2}{L^4} - \frac{2}{L^2}\right)}$$

Ques 7/ A proton is confined in an ~~a~~ infinite square well of ~~width~~ length 10 fm. Calculate the energy & wavelength of photon emitted when proton undergoes a transition from the first excited state ($n=2$) to ground state ($n=1$).

Ans:- Energy of particle in an infinite square well is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The energy E and wavelength λ of a photon emitted as the particle makes a transition from the $n=2$ state to the $n=1$ state are

$$E = E_2 - E_1 = \frac{2^2\pi^2\hbar^2}{2mL^2} - \frac{1^2\pi^2\hbar^2}{2mL^2} \Rightarrow (2^2-1^2)\frac{\pi^2\hbar^2}{2mL^2} = \frac{3\pi^2\hbar^2}{2mL^2}$$

or $\frac{3h^2}{8mL^2} = \frac{3\times(6.62\times10^{-34})^2}{8\times1.672\times10^{-27}\times(10\times10^{-15})^2} \Rightarrow 9.84\times10^{-13} J$

wavelength $\lambda = \frac{hc}{E} = \frac{6.627\times10^{-34}\times 3\times10^{8}}{9.849\times10^{-13}} = 2.01\times10^{-13}\,m$

$= 201\ fm$

Ques 8:- Electron with energies of 1.0 eV & 2.0 eV are incident on a barrier 10.0 eV high & 0.5 nm wide. (a) Find their respective transmission probabilities.
(b) How are these affected if barrier is doubled in width?

Ans :- Transmission probability is

$T = e^{-2kL}$

k is the wave no. inside barrier & is given by

$k = \frac{\sqrt{2m(U-E)}}{\hbar}$

for e^- 1.0 eV energy

$k_1 = \frac{\sqrt{2m(10-1)\times 1.6\times10^{-19}}}{1.054\times10^{-34}}$

$= \frac{\sqrt{2\times(9.1\times10^{-31}\,kg)\times 9\times 1.6\times10^{-19}}}{1.054\times10^{-34}}$

$k_1 = 1.542\times10^{10}\,m^{-1}$

$T_1 = e^{-2k_1L}$

$= e^{-2\times1.54\times10^{10}\times0.50\times10^{-9}}$

$T_1 = e^{-15.42} = 2\times10^{-7}$

for 2.0 eV

$k_2 = \frac{\sqrt{2\times9.1\times10^{-31}\times8\times1.6\times10^{-19}}}{1.054\times10^{-34}}$

Ans $\frac{\sqrt{15.26\times10^{-25}}}{1.054\times10^{-34}}$

$14.48\times10^{+9}$

$1.44\times10^{+10}$

$= 1.44\times10^{10}\,m^{-1}$

$T_2 = e^{-2k_2L} = e^{-2\times1.44\times10^{10}\times0.50\times10^{-9}}$

$= e^{-14.48} = 5.14\times10^{-7}$

2ND

$T_1 = e^{-30.84} = 4.039\times10^{-14}$

$T_2 = e^{-28.96} = 2.64\times10^{-13}$