Random Variable

Random Experiments

An experiment whose outcome can not be predicted exactly, hence is random, is called a random experiment (ex. toss of a coin, drawing of a card from a deck of playing cards, etc.)

Sample Space

The collective outcomes of a random experiment form a sample space.

Sample Point

A particular outcome from a sample space is called a sample point.

Events

Collection of outcomes is called an event. An event is a subset of Sample Space.

Random Variable :-

Random Variable is a real valued function defined over the sample space of random experiment.

It is also known as stochastic variable, or random function, or stochastic function.

For Example :-

For a three tosses a fair coin, there are only eight (2^3) possible outcomes of this experiments.

$$S \equiv [\ HHH \quad HHT \quad HTH \quad THH \quad HTT \quad THT \quad TTH \quad TTL\ ]$$

$$X \equiv [\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8\ ]$$

Sample Space

Let X is a random variable, represents number of heads.

$$X = \left\{ \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \end{array} \right\}$$

Discrete Probability Distribution :→

Let X is Discrete Random Variable & $x_1, x_2, x_3, \ldots$ be the values that X can assume in increasing order of magnitude. Then probability that random variable X has a value one of possible values x_j $\{j = 1, 2, 3, \ldots\}$ will be given by $\boxed{P(X = x_j) \Rightarrow f(x_j)}$

Now properties of $f(x)$

① $f(x) \geq 0 \rightarrow (1)$

② $\sum_x f(x) = 1 \rightarrow (2)$

If $f(x)$ follows the properties shown in equ (1) & equ (2), then $f(x)$ is said to be a probability distribution function.

$f(x) \rightarrow$ Probability Function or Probability Density Function or Probability Mass Function.

Ex:- Find $f(x)$ probability function for three fair coin toss.

Soln:- Since coin is fair probability each of the eight outcomes $= \frac{1}{8}$.

then probability that outcome have no heads

$$P(X=0) = P(X = x_8) = \frac{1}{8}$$

the probability that outcome have one head

$$P(X=1) = P(X = x_5) + P(X = x_6) + P(X = x_7)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

The probability that outcome have two heads

$$P(X=2) = P(x_2) + P(x_4) + P(x_3)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Similarly, probability function that outcome have all three heads

$$P(X=3) = P(x_1) = \frac{1}{8}$$

Now Probability Distribution among with random variable X will be given by

$X \equiv$	0	1	2	3
$f(x) \equiv$	1/8	3/8	3/8	1/8

Note: $\sum_x f(x) = 1$

Cumulative Distribution Function :→

The CDF $F_X(x)$ of a discrete random variable X may be written as

$$F_X(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

Properties of CDF

1. $F_X(x) \geq 0$
2. $F_X(\infty) = 1$
3. $F_X(-\infty) = 0$
4. $F_X(x)$ is a nondecreasing function, that is $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$

if $x_1 < x_2$

$F_X(x_2) = P(X \leq x_2)$

$= P[(X \leq x_1) \cup (x_1 < x \leq x_2)]$

$F_X(x_2) = P(X \leq x_1) + P(x_1 < X < x_2)$

$= F_X(x_1) + P(x_1 < X < x_2)$

Ex:- For an event in which three fair coin are tossed, find the CDF. (RV will be head count)

Soln;- $X = \{0 \quad 1 \quad 2 \quad 3\}$

$\quad\quad x_1 \quad x_2 \quad x_3 \quad x_4$

$f(x) = \{\frac{1}{8} \quad \frac{3}{8} \quad \frac{3}{8} \quad \frac{1}{8}\}$

$F_X(x) = P(X \leq x)$

For $-\infty < x \leq 0$

$F_X(0) = P(X \leq 0) = \frac{1}{8}$

$0 < x \leq 1$

$F_X(1) = P(X \leq 1) = P[(X \leq 0) \cup (0 < X \leq 1)]$

$= P[X \leq 0] + P[0 < X \leq 1] = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$

$= \frac{1}{2}$

$F_X(2) = P(X \leq 2) = P[(X \leq 1) \cup (1 < X \leq 2)]$

$= P[(X \leq 1)] + P[1 < X \leq 2] = \frac{1}{2} + \frac{3}{8}$

$= \frac{7}{8}$

$F_X(3) = P(X \leq 3) = P[(X \leq 2) \cup (2 < X \leq 3)]$

$= P(X \leq 2) + P(2 < X \leq 3)$

$= \frac{7}{8} + \frac{1}{8} = 1$

Continuous Random Variable :-

A continuous RV X can assume any value in a certain interval.

or

"A random variable that takes on an infinite number of values is known as a continuous Random variable."

Continuous Probability Distribution :-

Let there be a function $f(x)$, such that

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here the function $f(x)$ is known as a probability function or probability distribution for a continuous Random Variable

Probability of X lying between a & b is

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Now, for a continuous case, the probability of X being equal to any particular value is zero.

Hence,

$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$

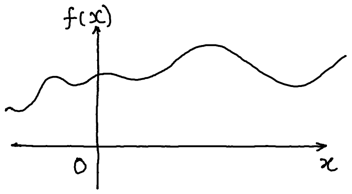
$= P(a \leq X \leq b)$

Random Variable

CDF for continuous R.V.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

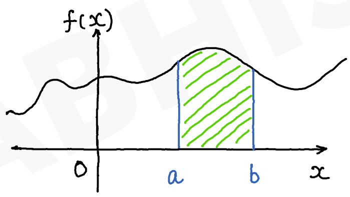
Note:- From property of $f(x)$ {PDF}, $f(x) \geq 0$, $f(x)$ will be a curve, which cannot fall below x-axis



Probability Density Function of continuous R.V.

Now, the probability that $P(a<X<b)$ will be

$$P(a<X<b) = \int_{a}^{b} f(x)\,dx = \begin{bmatrix}\text{Area under} \\ \text{the curve } f(x) \\ \text{between } a \,\&\, b\end{bmatrix}$$



For a continuous case, the probability of X being equal to any particular value is zero. Hence either or both the signs "<" can be replaced by the sign "≤". Thus

$$P(a<x<b) = P(a \leq x \leq b) = P(a<x\leq b) = P(a \leq x < b)$$

Q. ① Find c for pdf

$$f(x) = \begin{cases} c(x-1) \; ; & 1<x<4 \\ 0 \; ; & \text{otherwise} \end{cases}$$

② Find $P(2<X<3)$

③ Find CDF

Solution :- To fulfill the given function as a probility density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{1}^{4} c(x-1)\,dx = 1 \Rightarrow c = \frac{2}{9}$$

② $$P(2<X<3) = \int_{2}^{3} \frac{2}{9}(x-1)\,dx = \frac{2}{9}\left(\frac{x^2}{2} - x\right)\Bigg|_{2}^{3}$$

$$= \frac{1}{3}$$

③ CDF ≡ Distribution Function

$$F_X(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$F_X(x) = F_X(1) = P(X<1)$$

$$= \int_{-\infty}^{1} f(x)\,dx = 0$$

$$F_X(4) = P(X<4) = P(X<1) + P(X<4) = \int_{-\infty}^{x} f(u)\,du =$$

$$= \int_{-\infty}^{1} f(u)\,du + \int_{1}^{x} f(u)\,du$$

$$= \int_{1}^{x} \frac{2(u-1)}{9}\,du = \frac{(x-1)^2}{9}$$

Last for $4<X$

$$F_X(x) = \int_{1}^{4} \frac{2(u-1)}{9}\,du + \int_{4}^{x} (0)\,du$$

$$= 1$$

Random Variable

JOINT DISTRIBUTION :-

CONTINUOUS CASE

Let X & Y be two continuous random variable defined over $f(x,y)$.

$f(x,y)$ be the JOINT Probability Distribution follows following

1) $f(x,y) \geq 0$
2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Marginal PDF →

$$f_{XY}(x) = \int_{v=-\infty}^{\infty} f(u,v)\,dv \quad \left\{ \begin{array}{l} \text{Note: for fixed } x \\ y \text{ is varying} \end{array} \right.$$

$$f_{XY}(y) = \int_{u=-\infty}^{\infty} f(u,v)\,du \quad \left\{ \begin{array}{l} \text{Note: for fixed } y \\ x \text{ is varying} \end{array} \right.$$

JOINT CDF

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

Marginal CDF

$$F_{XY}(x) = P(X \leq x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(u,v)\,du\,dv$$

$$F_{XY}(y) = P(Y \leq y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

Independent Random Variable :-

Let X & Y be discrete random variables.

X & Y is said to be independent if

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

$$\Rightarrow \quad f(x,y) = \underbrace{f_{XY}(x) \cdot f_{XY}(y)}_{\text{Marginal PDF}}$$

Marginal CDF if X & Y are independent

$$P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

$$F_{XY}(x,y) = F_{XY}(x) \cdot F_{XY}(y)$$

CONDITIONAL Distributions

If X & Y are discrete random variables, then the conditional probability function of Y given X is defined as

$$f(y/x) = P(Y=y/X=x) = \frac{f(x,y)}{f_{XY}(x)}$$

$$f(x/y) = P(X=x/Y=y) = \frac{f(x,y)}{f_{XY}(y)}$$

The above equations are valid for both continuous & discrete random variables.

Ex:- The Joint density function of two continuous Random Variable X & Y is given by

$$f(x,y) = \begin{cases} 2 & ; \text{ for } 0<x<1,\ 0<y<x \\ 0 & ; \text{ otherwise} \end{cases}$$

(1) Find Marginal density function, &
(2) The conditional density functions.

Solution:-

$$f_{XY}(x) = \int_{-\infty}^{\infty} f(u,v)\,dv = \int_{0}^{x} 2\,dv = 2x \ ; 0<x<1$$

for v

$f_{XY}(x) = 0$ otherwise

$$f_2(y) = \int_{-\infty}^{\infty} f(u,v)\,du\,dv = \int_{y}^{1} 2\,du = 2(1-y), \text{ for } 0<y<1$$

for u

Random Variable

JOINT DISTRIBUTION :→

Discrete Case :→

Let X & Y, are two discrete random variables The JOINT PROBABILITY function of X & Y is given by

$$f(x,y) = P(X=x, Y=y)$$

where $f(x,y)$ satisfy the following properties

1) $f(x,y) \geq 0$

2) $\sum_x \sum_y f(x,y) = 1$

Now let,

$$[X] = [x_1 \quad x_2 \quad x_3 \cdots\cdots x_m]$$

$$[Y] = [y_1 \quad y_2 \quad y_3 \cdots\cdots y_n]$$

$$\& \quad [XY] = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & x_2y_3 & \cdots & x_2y_n \\ x_3y_1 & x_3y_2 & x_3y_3 & \cdots & x_3y_n \\ \vdots & \vdots & \vdots & & \\ x_my_1 & x_my_2 & x_my_3 & \cdots & \end{bmatrix}$$

Marginal Probability Density Function

For X given "x_j"

$$f(x_j) = \sum_{K=1}^{n} f(x_j, y_K) \quad , \; K = 1,2,3,\ldots m$$

For Y given "y_K"

$$f(y_K) = \sum_{j=1}^{m} f(x_j, y_K) \quad , \; j = 1,2,3,\ldots n$$

CDF (JOINT)

$$F(X,Y) = \sum_{X \leq x_j} \sum_{Y \leq y_K} f(x_j, y_K)$$

Marginal CDF

$$F_X(x) = P(X \leq x)$$

$$= \sum_{u \leq x} \sum_{v} f(u,v)$$

$$F_Y(y) = P(Y \leq y) = \sum_{u} \sum_{v \leq y} f(u,v)$$

X \ Y	y_1	y_2	y_3	⋯	⋯	⋯	y_n
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	$f(x_1,y_3)$	⋯	⋯	⋯	$f(x_1,y_n)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	$f(x_2,y_3)$	⋯	⋯	⋯	$f(x_2,y_n)$
x_3	⋮	⋮	⋯	⋯	⋯	⋯	⋮
⋮	⋮	⋮	⋮	⋮	⋮		⋮
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	$f(x_m,y_3)$	⋯	⋯	⋯	$f(x_m,y_n)$

$\sum_{K=1}^{n} f(x_1, y_K)$

$\sum_{K=1}^{n} f(x_2, y_K)$

$\sum_{j=1}^{m} f(x_j, y_K)$

$\sum_{j=1}^{m} f(x_j, y_2)$

$\sum_{j=1}^{m} \sum_{K=1}^{n} f(x_j, y_K) = 1$

$$\boxed{\begin{array}{l} \text{Marginal Probability of } X \Rightarrow \underset{P(X=x_j)}{f(x_j)} = \sum_{K=1}^{n} f(x_j, y_K) \\ \text{Marginal PDF for } Y \Rightarrow \underset{P(Y=y_K)}{f(y_K)} = \sum_{j=1}^{m} f(x_j, y_K) \\ \boxed{\text{Note; } f(x_m, y_n) = P(X=x_m, Y=y_n)} \end{array}}$$

Marginal CDF

$$P(X \leq x) = F_{XY}(x) = \sum_{u \leq x} \sum_{v} f(u,v)$$

For given x & all values of y

$$P(Y \leq y) = F_{XY}(y) = \sum_{u} \sum_{v \leq y} f(u,v)$$

For given y & all values of X

Random Variable

Ex:- Joint PDF of two random variables X & Y is given by

$$f(x) = \begin{cases} c(x^2+2y) \; ; & x = 0, 1, 2 \;, \; y = 1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

Find 1) $c = ?$, 2) $P(X=2, Y=3)$,

3) $P(X \leq 1, Y > 2)$ & 4) Marginal PDF

5) Find whether X & Y are independent or not

6) Find $f(y/1)$ & $f(x/2)$

7) $P(Y=3/X=2)$, 8) $P(X=0/Y=4)$

9) $f(2/y)$, 10) $f(3/x)$

Solution :→

X \ y	1	2	3	4	
0	2C	4C	6C	8C	20C
1	3C	5C	7C	9C	24C
2	6C	8C	10C	12C	36C
	11C	17C	23C	29C	80C

a) Table gives $f(x,y)$. The grand total $80C$ must be 1

$$80C = 1 \Rightarrow c = \frac{1}{80}$$

b) The desired probability is given by the entry corresponding to $X=2$ & $Y=3$

$$P(X=2, Y=3) = 10C = \frac{1}{8}$$

c) The desired probability is given by adding the entries in the dotted rectangle of the Table.

Thus, $P(X \leq 1, Y > 2) = (6C + 8C + 7C + 9C) = 30C = \frac{3}{8}$

d) Marginal Probability functions of X & Y are obtained by adding the entries of the rows, and the columns, respectively

$$P(X=x) = f_X(x) = \begin{cases} 20C = \frac{1}{4} \;, & x=0 \\ 24C = \frac{3}{10} \;, & x=1 \\ 36C = \frac{9}{20} \;, & x=2 \end{cases}$$

$$P(Y=y) = f_y(y) = \begin{cases} 11C = \frac{11}{80} \;, & y=1 \\ 17C = \frac{17}{80} \;, & y=2 \\ 23C = \frac{23}{80} \;, & y=3 \\ 29C = \frac{29}{80} \;, & y=4 \end{cases}$$

Similarly $P(X \leq 1, Y > 2) = (6C + 8C + 7C + 9C) = 30C = \frac{3}{8}$

5) For X & Y to be independent, we must have

$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$ for all x & y.

$P(X=2, Y=3) = \frac{1}{8}$

$P(X=2) = \frac{9}{20}$, $P(Y=3) = \frac{23}{80}$

From above it is clear that, the given random variable are not independent

6) $f(y/1) = ?$

$$f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{(x^2+2y)/80}{f_X(x)}$$

for $x=1$ $\quad f(y/1) = \frac{1+2y}{80 \cdot f_X(1)} = \frac{1+2y}{80 \times \frac{3}{10}} = \frac{1+2y}{24}$

$\Rightarrow f(x/2) = ?$

$$f(x/y) = \frac{f(x,y)}{f_y(y)} = \frac{(x^2+2y)/80}{f_y(y)}$$

for $y=2$ $\quad f(x/y) = \frac{x^2+4}{80 \cdot f_y(2)} \Rightarrow f(x/2) = \frac{x^2+4}{80 \times 17/80}$

$$= \frac{x^2+4}{17}$$

7) $P(Y=3/X=2)$,

$$f(3/2) = \frac{(4+6)/80}{9/20} = \frac{5}{18}$$

8) $P(X=0/Y=4) = f(0/4) = \frac{(0+8)/80}{29/80} = \frac{8}{29}$

9) $f(2/y) = f(x/y)\big|_{x=2} = \frac{(4+2y)/80}{f_y(y)}$

$$f(2/y) = \begin{cases} \dfrac{(4+2y)}{80}\Big/\dfrac{11}{80} = \dfrac{6}{11} & \text{for } y=1 \\ \dfrac{(4+2y)}{80}\Big/\dfrac{17}{80} = \dfrac{4+2y}{17} = \dfrac{8}{17} ; & y=2 \\ \dfrac{4+2y}{23} = \dfrac{10}{23} , & y=3 \\ \dfrac{(4+2y)/80}{29/80} = \dfrac{4+2y}{29} = \dfrac{12}{29} & , y=4 \end{cases}$$

(f) $f(3/x) = f(y/x)\Big|_{y=3} = \dfrac{(x^2+6)}{80}\Big/ f_x(x)$

Hence

$$f(3/x) = \begin{cases} \dfrac{x^2+6}{80}\Big/\dfrac{1}{4} = \dfrac{x^2+6}{20} = \dfrac{3}{10}, & x=0 \\ \dfrac{x^2+6/80}{3/10} = \dfrac{x^2+6}{24} = \dfrac{7}{24}, & \text{for } x=1 \\ \dfrac{(x^2+6)/80}{9/20} = \dfrac{x^2+6}{36} = \dfrac{5}{18}, & \text{for } x=2 \end{cases}$$

FUNCTIONS OF RANDOM VARIABLE

A. Random Variable $g(x)$

Given a r.v. X & a function $g(x)$, the expression

$$Y = g(X)$$

defines a new r.v. Y with a given number,

we denote D_y the subset of range (range of X) such that $g(x) \le y$ Then

$$F_Y(y) = \int_{D_y} f_x(x)\,dx \rightarrow (1)$$

P.D.F. of R.V. x.

Now Let X be a continuous R.V. with PDF $f_x(x)$. If the transformation $y = g(x)$ is one to one & has the inverse transformation

$$x = g^{-1}(y) = h(y) \rightarrow (2)$$

then the PDF of Y is given by

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X[h(y)]\left|\frac{d\,h(y)}{dy}\right| \rightarrow (3)$$

Note that if $g(x)$ is a continuous monotonic increasing or decreasing function, then the transformation $y = g(x)$ is one to one.

If the transformation is not one to one $f_Y(y)$ is obtained as follows

Denoting the real roots of $y = g(x)$ by x_K, that is

$$y = g(x_1) = \cdots\cdots = g(x_K) = \cdots\cdots$$

then

$$f_Y(y) = \sum_K \frac{f_X(x_K)}{|g'(x_K)|} \rightarrow (4)$$

where $g'(x_K)$ is the derivative of $g(x)$.

One Function of two random variable

Given two random variable X & Y with a function $g(x,y)$. the expression

$$Z = g(X, Y) \rightarrow (5)$$

is a New R.V..

With z a given number, we denote by D_z the region of the xy plane such that $g(x,y) \le z$. Then

$$[Z \le g] = \{g(x,y) \le z\} = \{(x,y) \in D_z\}$$

where $\{(x,y) \in D_z\}$ is the event consisting of all outcomes λ such that the point $\{X(\lambda), Y(\lambda)\}$ is in D_z.

$$F_Z(z) = \iint_{D_z} f_{XY}(x,y)\,dx\,dy \rightarrow (6)$$

Two Function of Two Random Variables

Given two R.V. X and Y and two functions $g(x,y)$ and $h(x,y)$, the expression

$$Z = g(X,Y) \quad , \quad W = h(X,Y) \rightarrow (7)$$

defines two new R.V.s Z and W.

With z & w two given numbers we denote D_{ZW} the subset of R_{XY} [range of (X,Y)] such that $g(x,y) \leq z$ & $h(x,y) \leq w$. Then

$$(Z \leq z, W \leq w) = [g(x,y) \leq z, h(x,y) \leq w]$$
$$= \{(X,Y) \in D_{ZW}\} \rightarrow (8)$$

where $\{X,Y) \in D_{ZW}\}$ is the event consisting of all outcomes λ such that the point $\{X(\lambda), Y(\lambda)\} \in D_{ZW}$.

Hence,

$$F_{ZW}(z,w) = P(Z \leq z, W \leq w)$$
$$= P\{(X,Y) \in D_{ZW}\} \rightarrow (9)$$

In continuous case we have

$$F_{ZW}(z,w) = \iint_{D_{ZW}} f_{XY}(x,y)\, dx\, dy \rightarrow (10)$$

Determination of $f_{ZW}(z,w)$ from $f_{XY}(x,y)$ will be obtained as

Let X and Y be two continuous R.V. with joint PDF $f_{XY}(x,y)$. If the transformation

$$Z = g(x,y) \quad , \quad W = h(x,y) \rightarrow (11)$$

is one to one & has the inverse transformation

$$x = q(z,w) \quad , \quad y = r(z,w) \rightarrow (12)$$

then the joint pdf of Z and W is given by

$$f_{ZW}(z,w) = f_{XY}(x,y) \left| J(x,y) \right|^{-1} \rightarrow (13)$$

where

$x = q(z,w)$, $y = r(z,w)$ and

$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} \rightarrow (14)$$

which is the Jacobian of the transformation.

Questions:-

Let $Y = 2X+3$. If a random variable X is uniformly distributed over $[-1,2]$. Find $f_Y(y)$.

Given $f_X(x) = \begin{cases} \frac{1}{3} & ; -1 \leq x \leq 2 \\ 0 & : \text{otherwise} \end{cases}$

Soln.:- Eq. $y = g(x) = 2x+3$, has a single solution

$\Rightarrow x = \frac{y-3}{2}$, range $y \in [1, 7]$. $g'(x) = 2$

for $x = -1$
$x = 2$

Contd..

Thus, $-1 \le x_1 \le 2$ and by equ (4)

$$f_Y(y) = \sum_K \frac{f_X(x_K)}{|g'(x_K)|}$$

$$f_Y(y) = \frac{1}{2} f_X(x_1) = \begin{cases} 1/6 & ; 1 \le y \le 7 \\ 0 & ; \text{elsewhere} \end{cases}$$

Q: Given $Y = aX + b$, show that if

$X = N(\mu : \sigma^2)$, then

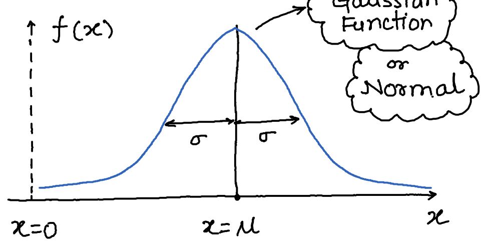
$Y = N(a\mu + b ; a^2\sigma^2)$

Note :- $N(\mu, \sigma)$ is known as Normalize Distribution (or we can say a function), the mean of this function is μ & σ^2 is variance where σ is standard deviation.

Normal Distribution $\equiv$ Gaussian Distribution

Given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \longrightarrow (15)$$



Solution :- $y = g(x) = ax + b$ has a single solution $x = \frac{y-b}{a}$ & $g'(x) = a$. The range of y is $(-\infty, \infty)$

Hence $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$ ✓

Since $X = N(\mu ; \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Now $$f_Y(y) = \frac{1}{\sqrt{2\pi.\sigma^2}} \cdot \frac{1}{(a)} \exp\left[-\frac{1}{2\sigma^2}\left(\frac{y-b}{a} - \mu\right)^2\right]$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 a^2}} \exp\left[-\frac{1}{2\sigma^2 a^2}\left(y - b - a\mu\right)^2\right]$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 a^2}} \exp\left[\frac{\{y - (b + a\mu)\}^2}{2\sigma^2 a^2}\right] \longrightarrow (2)$$

Compare equ.(2) with equ(1), we find that

$$f_Y(y) = N(b + a\mu ; a^2\sigma^2)$$

So if $X = N(\mu ; \sigma^2)$ then

$Y = N(a\mu + b ; a^2\sigma^2)$

Q. If $Y = X^2$, find $f_Y(y)$ if $X = N(0;1)$

Soln:- If $y < 0$, then the equation $y = x^2$ do't have real solutions $\Longrightarrow$ hence $f_Y(y) = 0$ for $y < 0$

If $y > 0$, then $y = x^2$ has two solutions

$x_1 = \sqrt{y}$, $x_2 = -\sqrt{y}$

Now, $y = g(x) = x^2$ & $g'(x) = 2x$.

Hence from the equation $f_Y(y) = \frac{1}{g'(x)} f_X(x)$

$$f_Y(y) = \frac{1}{2x} \frac{1}{\sqrt{2\pi}} \cdot e^{-(x^2/2)} \quad (\because \mu = 0 \ \& \ \sigma = 1) \text{ Given}$$

$$f_Y(y) = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot u(y)$$

Note $u(y)$ is here, as there is no real solution for $y<0$, hence $f_Y(y) = 0$ for $y < 0$

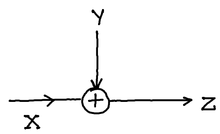
Ans

Question :-

The input to a noisy communication channel is a binary random variable X with $P(X=0) = P(X=1) = \frac{1}{2}$. The output of channel Z is given by $X+Y$, where Y is the additive noise introduced by the channel.

Assuming that X & Y are independent & $Y = N(0;1)$, find the density function of Z.

Solution :→



Note in the question density function of Y is given by $Y = N(0;1)$.

$$P(X=0) = \frac{1}{2}, \quad P(X=1) = \frac{1}{2}$$

$$F_Z(z) = P(Z \le z) = P(Z \le z | X=0) \cdot P(X=0) + P(Z \le z | X=1) \cdot P(X=1)$$

Now

$$\Rightarrow P(Z \le z | X=0) = P(X+Y \le z | X=0)$$

$$= P(0+Y \le z) = P(Y \le z)$$

$$= F_Y(z)$$

Again $P(Z \le z | X=1) = P(X+Y \le z | X=1)$

$$= P(Y \le z-1) = F_Y(z-1)$$

$$\Rightarrow \quad F_Z(z) = \frac{1}{2} F_Y(z) + \frac{1}{2} F_Y(z-1)$$

Now, $Y = N(0;1) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)$

Using relationship of PDF & CDF

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{2} f_Y(z) + \frac{1}{2} f_Y(z-1)$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} e^{-z^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(z-1)^2/2}\right]$$

Ans

Question :—

Consider the Transformation

$$Z = aX + bY, \quad \& \quad W = cX + dY$$

Find the joint density function $f_{ZW}(z, w)$ in terms of $f_{XY}(x, y)$.

Solution :— Assuming that the equ.

$$ax + by = z$$

& $\quad cx + dy = w \quad$ has one & only one solution iff $ad - bc \neq 0$

then

$$x = \alpha z + \beta w, \quad y = \gamma z + \eta w$$

where $\alpha = \frac{d}{ad-bc}$, $\beta = \frac{-b}{ad-bc}$,

$$\gamma = \frac{c}{ad-bc}, \quad \eta = \frac{a}{ad-bc}$$

Now

$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

So, using equ. (13) & (14)

$$f_{ZW}(z, w) = \frac{1}{|ad-bc|} f_{XY}(\alpha z + \beta w, \gamma z + \eta w)$$

(Answer)

Q_{n} :- X & Y are independent normalized R.V.'s

Find the PDF of $Z = X + Y$.

Solution :- The PDF of X & Y are

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

In this case, we have to first find out $f_Z(z)$.

As there are only three R.V.'s, we have to assume another R.V. 'W' in such a manner

$Y = W$ has a unique solution with $Z = X + Y$.

In this case $X = Z - W$, $Y = W$

Since

$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Now $f_{ZW}(z,w) = f_{XY}(z-w, w)$

Now Marginal PDF $f_Z(z)$ will be obtained as

$$f_{ZW}(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw$$

$$= \int_{-\infty}^{\infty} f_{XY}\{(z-w), w\}\,dw$$

As X & Y are independent, then $\{P(X=x, Y=y) = P(X=x)\,P(Y=y)\}$

$$\boxed{f_{ZW}(z) = \int_{-\infty}^{\infty} f_X(z-w)\cdot f_Y(w)\,dw}$$

V.Imp Note :- If two or more independent random variable are adding, then their PDF (Density Function) will be convoluted.

i.e $Z = X + Y \Rightarrow f_{ZW}(z,w) = \int_{-\infty}^{\infty} f_X(z-w)\cdot f_Y(w)\,dw$

Generally, if $Z = x_1 + x_2 + x_3 + x_4 + \cdots\cdots + x_n$

then PDF $f_Z(x) = f_{x_1}(x) * f_{x_2}(x) * f_{x_3}(x) \cdots\cdots f_{x_n}(x)$

Joint Marginal Probability Functions

Now in question it is given that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{and} \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Now, as we have developed an equation.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-w)\cdot f_Y(w)\,dw$$

$$= \int_{-\infty}^{\infty} f_X(z-w)\, f_Z(w)\,dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-w)^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-z^2/2} \cdot e^{+zw} \cdot e^{-w^2/2} \cdot e^{-w^2/2}\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-z^2/2} \cdot e^{zw} \cdot e^{-w^2}\,dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}+zw-w^2}\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2}+w^2-zw\right)}\,dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2}+w^2+\frac{z^2}{4}-\frac{z^2}{4}-zw\right)}\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\left\{\left(\frac{z^2}{2}-\frac{z^2}{4}\right) + \left(w^2+\frac{z^2}{4}-zw\right)\right\}\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\frac{z^2}{2} + \left(\sqrt{2}\,w - \frac{z}{\sqrt{2}}\right)^2\right]\right\}dw$$