ANTENNA AND WAVE PROPAGATION

UNIT: 1 VECTOR ANALYSIS

Vector Algebra (in Cartesian coordinates only)*:

- Vector analysis is a mathematical tool with which electromagnetic concepts are most conveniently expressed and best comprehended.
- A quantity can be either a scalar or a vector:
  - → Scalar: is a quantity that has only magnitude. (time, mass, distance, temperature, entropy, electric potential, population etc).
  - → ~~Veto~~ Vectors: is a quantity that has both magnitude and direction. (velocity, force, displacement, electric field intensity).
- There is another class of physical quantities called tensors, of which scalars and vectors are special cases.
- To distinguish between scalars and vectors, vectors are represented with an arrow on top of the letter eg. $\vec{A}$ and $\vec{B}$.
- EM theory is essentially a study of some particular fields.
  - → Field: is a function that specifies a particular quantity everywhere in a region.
    - scalar fields: temperature distribution in a building, sound intensity in a theater, electric potential in a region, refractive index ~~a~~ of a ~~sto~~ stratified medium.
    - vector fields: gravitational force on a body in space, the velocity of raindrops in the atmosphere are examples of vector fields.

~~Vector~~

- Unit Vector:
  - A vector $\vec{A}$ has both magnitude and direction.
  - The magnitude of $\vec{A}$ is a scalar written as A or $|\vec{A}|$.
  - A unit vector $\hat{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity ~~cts~~ and its direction is along $\vec{A}$.

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|}, \quad |\hat{a}_A| = 1$$

  - A vector $\vec{A}$ may be represented as:

$$\vec{A} = (A_x, A_y, A_z) = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$

where, A_x, A_y, A_z are ~~cop~~ components of $\vec{A}$ and $\hat{a}_x, \hat{a}_y, \hat{a}_z$ are unit vectors in x-, y- and z-directions respectively.

  - The magnitude of vector $\vec{A}$ is given by:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and the unit vector $\hat{a}_A$ along $\vec{A}$ is given by:

$$\hat{a}_A = \frac{A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

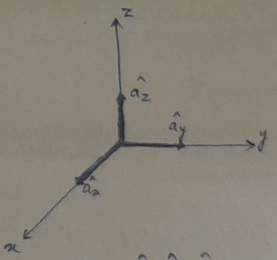


fig: unit vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$

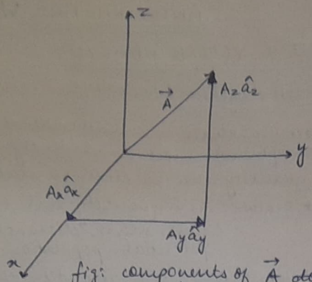


fig: components of $\vec{A}$ ~~along~~ along $\hat{a}_x, \hat{a}_y, \hat{a}_z$

- **Vector addition and subtraction:**

- Addition: $\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\hat{a}_x + (A_y + B_y)\hat{a}_y + (A_z + B_z)\hat{a}_z$

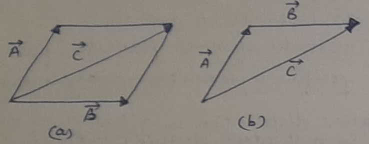


fig: vector addition
(a) parallelogram rule
(b) head to tail rule

- subtraction: $\vec{D} = \vec{A} - \vec{B} = (A_x - B_x)\hat{a}_x + (A_y - B_y)\hat{a}_y + (A_z - B_z)\hat{a}_z$

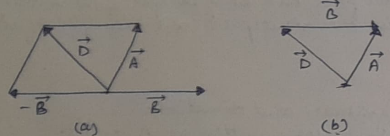


fig: vector subtraction
(a) parallelogram rule
(b) head to tail rule

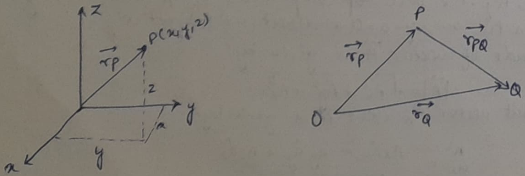
- **Position and Distance vectors:**

- Position vector: (radius vector) $\vec{r}_P$ of point P is defined as the directed distance from origin O to P.

$$\vec{r}_P = OP = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \qquad [P = (x, y, z)]$$

- Distance vector: is the displacement from one point to another.

$$\vec{r}_{PQ} = \vec{r}_Q - \vec{r}_P = (x_Q - x_P)\hat{a}_x + (y_Q - y_P)\hat{a}_y + (z_Q - z_P)\hat{a}_z$$



- **Vector Multiplication:**

- Dot product: (scalar product) is defined geometrically as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of smaller angle between them, when they are drawn tail to tail.

$$\vec{A} \cdot \vec{B} = A \cdot B \cdot \cos\theta_{AB}$$

If $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ then

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

scalar product or dot product results into a scalar quantity. Two vectors are said to be orthogonal or perpendicular with each other if $\vec{A} \cdot \vec{B} = 0$.

Dot product obeys following:

(i) commutative : $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

(ii) Distributive : $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

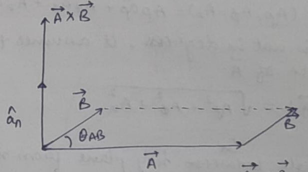
(iii) $\vec{A} \cdot \vec{A} = |\vec{A}|^2 = A^2$

Also note that:

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$
$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

- Cross product: is a vector quantity whose magnitude is the area of the parallelogram formed by the two vectors and is in the direction of advance of a right-handed screw as $\vec{A}$ is turned into $\vec{B}$.

$$\vec{A} \times \vec{B} = AB \sin\theta_{AB}\, \hat{a}_n$$



If $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + (A_xB_y - A_yB_x)\hat{a}_z$$

It is obtained by "crossing" terms in cyclic ~~manner~~ permutation, hence name "cross product".

Cross product obeys following:

(i) Anticommutative : $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

(ii) Not Associative : $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

(iii) Distributive : $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

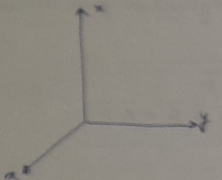
(iv) $\vec{A} \times \vec{A} = 0$

Also note that

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z, \quad \hat{a}_y \times \hat{a}_z = \hat{a}_x, \quad \hat{a}_z \times \hat{a}_x = \hat{a}_y.$$
$$\hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$$

Coordinate Systems and Transformation

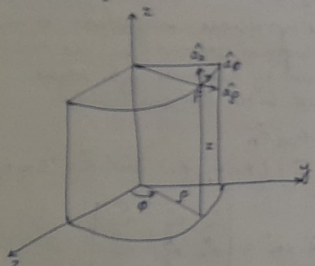
- A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or non-orthogonal.
- Nonorthogonal systems are hard to work with, and they are of little or no practical use.
- Orthogonal system is one in which the coordinate surfaces are mutually perpendicular.
  e.g. cartesian, circular cylindrical, spherical, elliptical cylindrical, parabolic cylindrical, conical, prolate spheroidal, oblate spheroidal, ellipsoidal.
- Cartesian coordinates (x, y, z)



$-\infty < x < \infty$
$-\infty < y < \infty$
$-\infty < z < \infty$

$\vec{A} = A_x \hat{a_x} + A_y \hat{a_y} + A_z \hat{a_z} = (A_x, A_y, A_z)$

- Circular cylindrical coordinates (ρ, ϕ, z)



$0 \le \rho < \infty$
$0 \le \phi < 2\pi$, $\cancel{360°}$
$-\infty < z < \infty$

2π is excluded in the range for ϕ since $\phi = 2\pi$ is equivalent to $\phi = 0$.

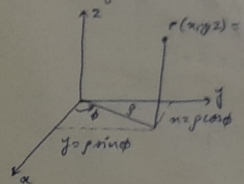
$\vec{A} = (A_\rho, A_\phi, A_z) = A_\rho \hat{a_\rho} + A_\phi \hat{a_\phi} + A_z \hat{a_z}$

$\hat{a_\phi}$ is not in degrees, it assumes the units of $\vec{A}$.

$|\vec{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

$\rho \to$ radius of the cylinder
$\phi \to$ azimuthal angle / angle ~~between~~ in x-y plane from x-axis
$z \to$ height of the cylinder.

$\hat{a_\rho} \cdot \hat{a_\rho} = \hat{a_\phi} \cdot \hat{a_\phi} = \hat{a_z} \cdot \hat{a_z} = 1$
$\hat{a_\rho} \cdot \hat{a_\phi} = \hat{a_\phi} \cdot \hat{a_z} = \hat{a_z} \cdot \hat{a_\rho} = 0$
$\hat{a_\rho} \times \hat{a_\phi} = \hat{a_z}$, $\hat{a_\phi} \times \hat{a_z} = \hat{a_\rho}$, $\hat{a_z} \times \hat{a_\rho} = \hat{a_\phi}$



Relationship between cartesian and cylindrical coordinates:

$\rho = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}\frac{y}{x}$, $z = z$ — ①

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ — ②

$\hat{a_x} = \cos\phi\, \hat{a_\rho} - \sin\phi\, \hat{a_\phi}$
$\hat{a_y} = \sin\phi\, \hat{a_\rho} + \cos\phi\, \hat{a_\phi}$
$\hat{a_z} = \hat{a_z}$

$\hat{a_\rho} = \cos\phi\, \hat{a_x} + \sin\phi\, \hat{a_y}$
$\hat{a_\phi} = -\sin\phi\, \hat{a_x} + \cos\phi\, \hat{a_y}$
$a_z = a_z$

Note that $|\hat{a_x}| = |\hat{a_y}| = |\hat{a_\rho}| = |\hat{a_\phi}| = 1$ since $\sqrt{\sin^2\phi + \cos^2\phi} = 1$

In matrix form, we write the transformation of vector $\vec{A}$ from (A_x, A_y, A_z) cartesian to (A_ρ, A_ϕ, A_z) cylindrical

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{— ③}$$

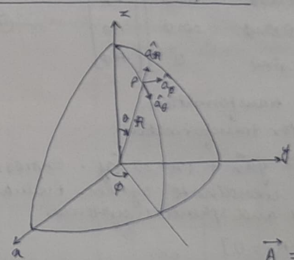
The inverse of the transformation $(A_\rho, A_\phi, A_z) \to (A_x, A_y, A_z)$ is obtained as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad \text{— ④}$$

① and ② are for point-to-point transformation.
③ and ④ are for vector-to-vector transformation.

- Spherical coordinates (r, θ, ϕ)



$0 \le r < \infty$
$0 \le \theta \le \pi$
$0 \le \phi < 2\pi$

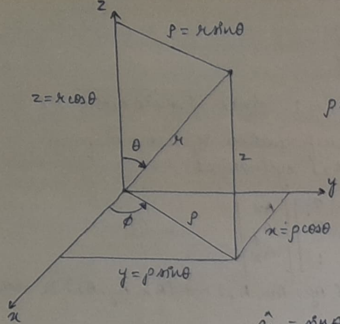
When ϕ changes from 0 to 2π, θ needs to be varied from 0 to π only to map the spherical volume within the distance r from the origin.

$\vec{A} = A_r \hat{a_r} + A_\theta \hat{a_\theta} + A_\phi \hat{a_\phi} = (A_r, A_\theta, A_\phi)$

$|\vec{A}| = \sqrt{(A_r^2 + A_\theta^2 + A_\phi^2)}$

$r \to$ radius of the sphere
$\theta \to$ ~~collat~~ colatitude / angle between z-axis and position vector r.
$\phi \to$ azimuthal angle / angle in x-y plane from x-axis.

$\hat{a_r} \cdot \hat{a_r} = \hat{a_\theta} \cdot \hat{a_\theta} = \hat{a_\phi} \cdot \hat{a_\phi} = 1$
$\hat{a_r} \cdot \hat{a_\theta} = \hat{a_\theta} \cdot \hat{a_\phi} = \hat{a_\phi} \cdot \hat{a_r} = 0$
$\hat{a_r} \times \hat{a_\theta} = \hat{a_\phi}$, $\hat{a_\theta} \times \hat{a_\phi} = \hat{a_r}$, $\hat{a_\phi} \times \hat{a_r} = \hat{a_\theta}$



Relationship between cartesian and cylindrical coordinates and spherical coordinates

$\rho = \sqrt{x^2+y^2}$, $r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$,

$\phi = \tan^{-1}\frac{y}{x}$ —①

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ —②

$\hat{a}_x = \sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi$

$\hat{a}_y = \sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi$

$\hat{a}_z = \cos\theta\,\hat{a}_r - \sin\theta\,\hat{a}_\theta$

$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$

$\hat{a}_\theta = \cos\theta\cos\phi\,\hat{a}_x + \cos\theta\sin\phi\,\hat{a}_y - \sin\theta\,\hat{a}_z$

$\hat{a}_\phi = -\sin\phi\, a_x + \cos\phi\,\hat{a}_y$.

In matrix form, the $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{—③}$$

The inverse transformation $(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$ is similarly obtained ~~are~~ according to:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \quad \text{—④}$$

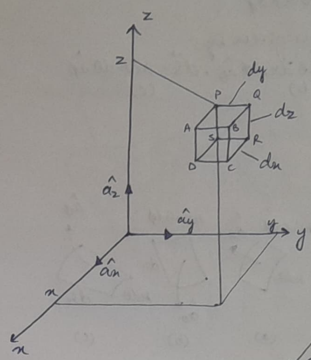
① and ② are for point-to-point transformation.
③ and ④ are for vector-to-vector transformation.

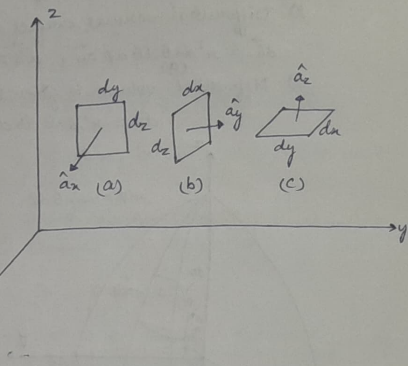
Q! Given that $P(-2,6,3)$ and $\vec{A} = y\hat{a}_x + (x+z)\hat{a}_y$. Express P and $\vec{A}$ in cylindrical and spherical coordinate system. Evaluate $\vec{A}$ at P in the cartesian, cylindrical and spherical systems.
[Hint: for $\vec{A}$, $A_x = y$, $A_y = x+z$, $A_z = 0$]

Ans! Example 2.1 pg 37 [sadiku]

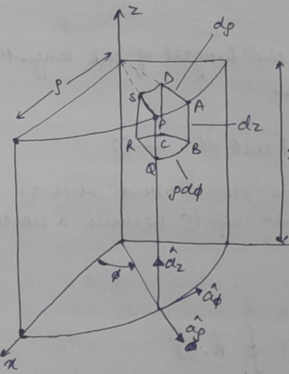
Vector Calculus

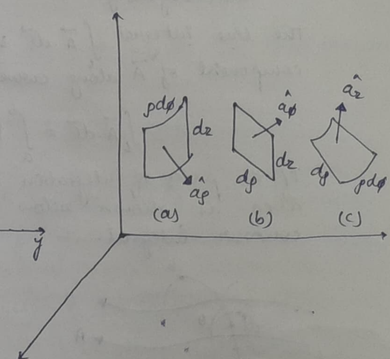
- Differential length, area and volume
  - Cartesian coordinate systems:
    1) Differential displacement is given by:
    $$d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$
    2) Differential normal surface area:
    $$d\vec{S} = dx\,dy\,\hat{a}_z \;(c), \quad d\vec{S} = dy\,dz\,\hat{a}_x \;(a), \quad d\vec{S} = dx\,dz\,\hat{a}_y \;(b)$$
    3) Differential volume is given by:
    $$dv = dx\,dy\,dz$$



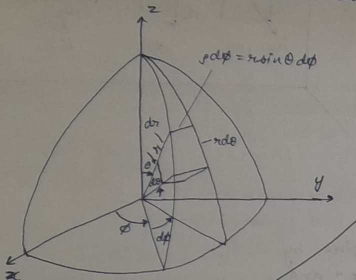


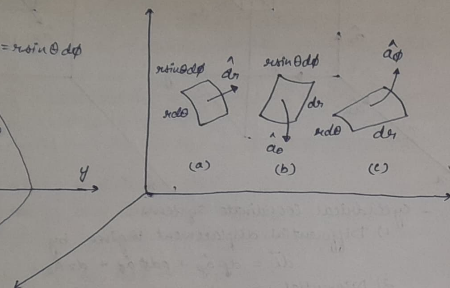
  - Cylindrical coordinate systems:
    1) Differential displacement is given by
    $$d\vec{l} = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$
    2) Differential normal surface area is given by
    $$d\vec{S} = \rho\, d\phi\, dz\,\hat{a}_\rho \;(a), \quad d\vec{S} = d\rho\, dz\,\hat{a}_\phi \;(b), \quad d\vec{S} = \rho\, d\rho\, d\phi\,\hat{a}_z \;(c)$$
    3) Differential volume is given by
    $$dv = \rho\, d\rho\, d\phi\, dz$$





- Spherical coordinate System:
  1) Differential displacement is given by
  $$d\vec{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$
  2) Differential normal surface area is given by
  $$\underset{(a)}{d\vec{S} = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r}, \quad \underset{(b)}{d\vec{S} = r\sin\theta\,dr\,d\phi\,\hat{a}_\theta}, \quad \underset{(c)}{d\vec{S} = r\,dr\,d\theta\,\hat{a}_\phi}$$
  3) Differential volume is given by
  $$dv = r^2\sin\theta\,dr\,d\theta\,d\phi$$



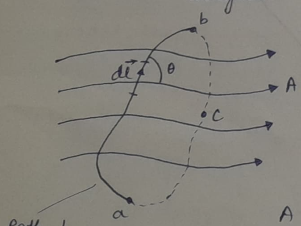


- Line, surface and volume integrals
  - Line: By "line" we mean the path along a curve in space. Terms such as line, curve and contour are used interchangeably.

The line integral $\int_L \vec{A}\cdot d\vec{l}$ is the integral of the tangential component of $\vec{A}$ along curve L.

$$\int_L \vec{A}\cdot d\vec{l} = \int_a^b |\vec{A}|\cos\theta\, dl \quad —①$$

If the path of integration is a closed curve such as abca as shown below then eq. ① becomes a closed contour integral.



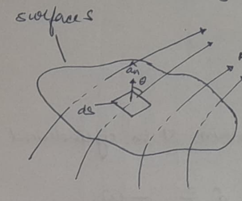
$$\oint_L \vec{A}\cdot d\vec{l}$$

which is called the circulation of $\vec{A}$ around L.

A common example of line integral is the work done on a particle.

- ~~Surface~~ Surface: Given a vector field $\vec{A}$, continuous in a region containing the smooth surface S, we define the surface integral or the flux of $\vec{A}$ through S as

$$\Psi = \int_S |\vec{A}|\cos\theta\, dS = \int_S \vec{A}\cdot\hat{a}_n\, dS$$



or simply $\Psi = \int_S \vec{A}\cdot d\vec{S}$ —②

where, at any point on S, $\hat{a}_n$ is the unit normal to S.

For a closed surface (defining a volume) eq. ② becomes

$$\Psi = \oint_S \vec{A}\cdot d\vec{S}$$

which is referred to as the net outward flux of $\vec{A}$ ~~through~~ from S.

- Volume: Notice that a closed path defines an open ~~surfa~~ surface, whereas a closed surface defines a volume.

We define the volume integral of the scalar ρ_v over the volume v, as

$$\int_v \rho_v\, dv \quad —③$$

- Del operator

The del operator, written as ∇, is the vector differential operator. In Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \quad —①$$

This operator, otherwise known as the gradient operator, is not a vector itself, but it operates on a scalar function.

The operator is useful in defining:
- The gradient of scalar V, ∇V
- The divergence of a vector $\vec{A}$, $\nabla\cdot\vec{A}$
- The curl of a vector $\vec{A}$, $\nabla\times\vec{A}$
- The Laplacian of a scalar V, $\nabla^2 V$

In cylindrical coordinates,

To obtain ∇ in terms of ρ, ϕ, z we use

$$\rho = \sqrt{x^2+y^2}, \quad \tan\phi = \frac{y}{x}$$

~~Hence~~, and,

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}$$

Substituting these in eq. ① we obtain ∇ in cylindrical coordinates as

$$\nabla = \hat{a}_\rho \frac{\partial}{\partial \rho} + \hat{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{a}_z \frac{\partial}{\partial z} \quad \text{—②}$$

In spherical coordinates,

To obtain ∇ in terms of r, θ, ϕ we use

$$r = \sqrt{x^2+y^2+z^2}, \quad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}, \quad \tan\phi = \frac{y}{x}$$

and,

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$dy = \sin\theta\sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Substituting these in eq. ① we obtain ∇ in spherical coordinates as

$$\nabla = \hat{a}_r \frac{\partial}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \quad \text{—③}$$

Gradient of a ~~sa~~ scalar

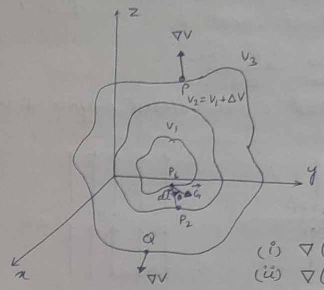
The gradient of a scalar field at any point is the maximum rate of change of the field at that point.

The gradient of a scalar field V is a vector that represents both the magnitude and direction of the maximum space rate of increase of V.

$$\nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial x}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) V$$

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \quad \text{—①}$$

The gradient of V can be expressed in cartesian coordinates as in eq. ①.



For cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \quad \text{—②}$$

For spherical coordinates:

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi \quad \text{—③}$$

Formulas on gradient:

(i) $\nabla(V+U) = \nabla V + \nabla U$

(ii) $\nabla(VU) = V\nabla U + U\nabla V$

(iii) $\nabla(V/U) = \dfrac{U\nabla V - V\nabla U}{U^2}$

(iv) $\nabla V^n = nV^{n-1}\nabla V$

where U and V are scalars and n is an integer.

Fundamental properties of the gradient of a scalar field V:

1) The magnitude of ∇V equals the maximum rate of change in V per unit distance.
2) ∇V points in the direction of maximum rate of change of V.
3) ∇V at any point is perpendicular to the constant V surface that passes through that point.
4) The projection of ∇V in the direction of a unit vector $\hat{a}$ ~~is $\nabla V \cdot \hat{a}$ in the direction~~ and is called the directional derivative of V along $\hat{a}$. This is the rate of change of V in the direction of $\hat{a}$.

Directional derivative : $\Delta V = \dfrac{\vec{a}}{|a|}$

5) If $\vec{A} = \nabla V$, V is said to be the scalar potential of $\vec{A}$.

Q: Find the gradient of the following scalar fields:

(a) $V = e^{-z}\sin 2x \cosh y$ (b) $U = \rho^2 z\cos 2\phi$ (c) $W = 10r\sin^2\theta\cos\phi$

Ans: (a) $2e^{-z}\cos 2x\cosh y\,\hat{a}_x + e^{-z}\sin 2x\sinh y\,\hat{a}_y - e^{-z}\sin 2x\cosh y\,\hat{a}_z$

(b) $2\rho z\cos 2\phi\,\hat{a}_\rho - 2\rho z\sin 2\phi\,\hat{a}_\phi + \rho^2\cos 2\phi\,\hat{a}_z$

(c) $10\sin 2\theta\cos\phi\,\hat{a}_r + 10\sin 2\theta\cos\phi\,\hat{a}_\theta - 10\sin\theta\sin\phi\,\hat{a}_\phi$.

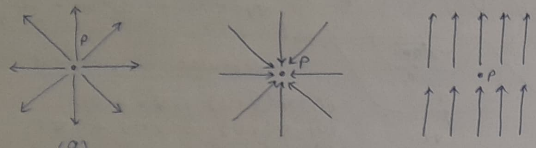
Divergence of a Vector and Divergence theorem:

The divergence of $\vec{A}$ as the net outward flow of flux per unit volume over a closed incremental surface.

The divergence of $\vec{A}$ at a given point P is the outward flux per unit volume as the volume shrinks about P.

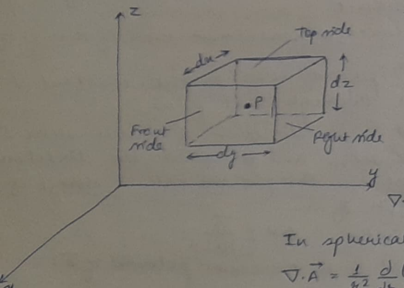
$$\text{div } \vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta v}$$

where Δv is the volume enclosed by the closed surface S in which P is located.



(a) positive divergence (b) negative divergence (c) zero divergence.

The divergence of a vector field can also be viewed as simply the limit of the field's source strength per unit volume (or source density); it is positive at a source point in the field, and negative at a sink point, or zero where there is neither sink nor source.



The divergence of $\vec{A}$ at point $P(x_0, y_0, z_0)$ in a Cartesian system is given by

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{--- (1)}$$

In cylindrical coordinates

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad \text{--- (2)}$$

In spherical coordinates

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \quad \text{--- (3)}$$

The following properties of the divergence of a vector field

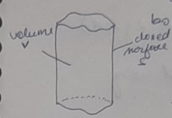
1) It provides a scalar field (because scalar product is involved).
2) $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$
3) $\nabla\cdot(V\vec{A}) = V\nabla\cdot\vec{A} + \vec{A}\cdot\nabla V$

From the definition of divergence of $\vec{A}$, it is not difficult to expect that

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot\vec{A}\, dv$$

This is called the divergence theorem, otherwise known as Gauss-Otrogradsky theorem.

The divergence theorem states that the total outward flux of a vector field $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of $\vec{A}$.



Proof: Subdivide volume v into a large number of small cells. If the k^{th} cell has volume Δv_k and is bounded by surface S_k.

$$\oint_S \vec{A}\cdot d\vec{S} = \sum_k \oint_{S_k} \vec{A}\cdot d\vec{S} = \sum_k \frac{\oint_{S_k} \vec{A}\cdot d\vec{S}}{\Delta v_k}\Delta v_k \quad \text{--- (1)}$$

Since the outward flux to one cell is inward to some neighbouring cells, there is cancellation on every interior surface, so the sum of the surface integrals over the S_k's is the same as the surface integral over the surface S.

Taking the limit of right-hand side of eq. (1) gives

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot\vec{A}\, dv$$

Q: Determine the divergence of these vector fields:

(a) $\vec{P} = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

(b) $\vec{Q} = \rho\sin\phi\,\hat{a}_\rho + \rho^2 z\,\hat{a}_\phi + z\cos\phi\,\hat{a}_z$

(c) $\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

Ans: (a) $\nabla\cdot\vec{P} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz) = 2xyz + x$

(b) $\nabla\cdot\vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$

(c) $\nabla\cdot\vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$

$= 2\cos\theta\cos\phi$

- **Curl of a vector and Stoke's theorem**

The curl of $\vec{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

That is,

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{l}}{\Delta S} \right)_{max} \hat{a}_n \longrightarrow *$$

where the area ΔS is bounded by the curve L and $\hat{a}_n$ is the unit vector ~~mo~~ normal to the surface ΔS and is determined by using the right-hand rule.

This is independent of the coordinate system (in eq. *)

In cartesian coordinates the curl of $\vec{A}$ is found using

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad \text{— } 1.a$$

$$\nabla \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] a_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] \hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \hat{a}_z \quad \text{— } 1.b$$

Using point-to-point vector transformation techniques, we obtain the curl of $\vec{A}$ in cylindrical coordinates as

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \quad \text{— } 2.a$$

$$\nabla \times \vec{A} = \frac{1}{\rho} \left[\frac{\partial A_z}{\partial \phi} - \rho \frac{\partial A_\phi}{\partial z} \right] \hat{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \hat{a}_\phi + \frac{1}{\rho} \left[\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right] \hat{a}_z \quad \text{— } 2.b$$

In spherical coordinates as

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta \, \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta \, A_\phi \end{vmatrix} \quad \text{— } 3.a$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta} \left[\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] \hat{a}_r + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right] \hat{a}_\theta + \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{a}_\phi \quad \text{— } 3.b$$

The following are the properties of the curl:

1) The curl of vector field is another ~~a~~ vector field.
2) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
3) $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$
4) $\nabla \times (V\vec{A}) = V \nabla \times \vec{A} + \nabla V \times \vec{A}$
5) The divergence of the curl of a vector field vanishes; i.e., $\nabla \cdot (\nabla \times \vec{A}) = 0$.
6) The curls of the gradient of a scalar field vanishes; i.e., $\nabla \times \nabla V = 0$ or $\nabla \times \nabla = 0$.

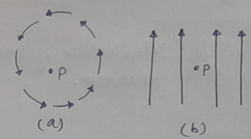


fig (a): curl of a vector field around P is directed out of the page.

fig (b): vector field with curl = 0.

Also, from the definition of the curl of $\vec{A}$ in eq. — (*), we may expect that

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S}$$

This is called Stoke's theorem.

Stoke's theorem states that the circulation of vector field $\vec{A}$ around a (closed) path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by L, provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on S.

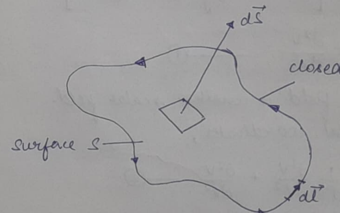


fig: determining the sense of $d\vec{l}$ and $d\vec{S}$ involved in Stoke's theorem.

Proof: The surface S is subdivided into a large number of cells. If the k^{th} cell has surface are ΔS_k and is bounded by path L_k

$$\oint_L \vec{A} \cdot d\vec{l} = \sum_k \oint_{L_k} \vec{A} \cdot d\vec{l} = \sum_k \frac{\oint_{L_k} \vec{A} \cdot d\vec{l}}{\Delta S_k} \Delta S_k$$

There is cancellation on every interior path, so the sum of the line integrals around the L_k's is the same as the line integral around the bounding curve L.

Therefore, taking the limit of the right-hand side of the above equation as $\Delta S_k \to 0$ and incorporating eq. — ⊛, we get

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S (\nabla \times \vec{A})\cdot d\vec{S}.$$

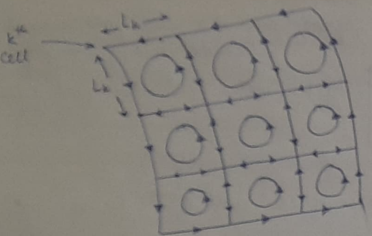


fig: Illustration of Stoke's theorem.

- **Laplacian of a Scalar**

The Laplacian of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

In Cartesian coordinates,

$$\text{Laplacian } V = \nabla\cdot\nabla V = \nabla^2 V$$

$$= \left[\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right]\cdot\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{—— ①}$$

Notice that: Laplacian of a scalar field is another scalar field.

Using transformation, In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{—— ②}$$

and in spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{—— ③}$$

A scalar field V is said to be harmonic in a given region if its Laplacian vanishes in that region.

In other words, if

$$\nabla^2 V = 0 \quad \text{—— *}$$

is satisfied in the region, the solution for V in eq. — ⊛ is harmonic (it is of the form of sine or cosine).

eq — ⊛ is called Laplace's equation.

Laplacian of a vector $\vec{A}$, $\nabla^2\vec{A}$ is defined as the gradient of the divergence of $\vec{A}$ minus the curl of the curl of $\vec{A}$.

$$\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A}) \quad \text{—— ④}$$

Eq. — ④ can be applied in any coordinate system.

In cartesian coordinates only eq. ④ becomes,

$$\nabla^2\vec{A} = \nabla^2 A_x\hat{a}_x + \nabla^2 A_y\hat{a}_y + \nabla^2 A_z\hat{a}_z \quad \text{— ⑤}.$$

Q: Find the Laplacian of the scalar fields

(a) $V = e^{-z}\sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10 r \sin^2\theta \cos\phi$.

Ans: Example 3.11 [Sadiku]

Q: Prove $\nabla\cdot(\nabla\times\vec{A}) = 0$.