Contd.

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\sqrt{2}\, w - \frac{z}{\sqrt{2}}\right)^2\right] dw$$

Let $u = \sqrt{2}\, w - z/\sqrt{2}$. Then

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

Since the integrand is the pdf of $N(0;1)$, the integral is equal to unity, & we obtain

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{2}} \cdot e^{-z^2/4} = \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} \cdot e^{-z^2/\left[2(\sqrt{2})^2\right]}$$

which is the pdf of $N(0;\sqrt{2})$

Thus, Z is a normal random variable with zero mean & variance $\sqrt{2}$.

Q: Consider the transformation

$$R = \sqrt{X^2+Y^2}, \quad \theta = \tan^{-1}\frac{Y}{X}$$

Find $f_{R\theta}(r,\theta)$ $\quad R \longrightarrow r$, $\theta \longrightarrow \theta$

Solution :-

From above it is clear that $r \geq 0$ & $0 \leq \theta \leq 2\pi$. With this assumption, it is clearly a rectangular to polar cartesian system, & have a single solution :

$$x = r\cos\theta \; ; \; y = r\sin\theta$$

$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

where $Z \Rightarrow R$ & $W \Rightarrow \theta$

$$R^2 = x^2 + y^2$$

$$2r\frac{dr}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r.\cos\theta}{r} = \cos\theta$$

Similarly $2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

$$\theta = \tan^{-1}\frac{y}{x} \Rightarrow \frac{\partial \theta}{\partial x} = \frac{y}{1+y^2/x^2} \cdot \frac{-1}{x^2} = \frac{-y}{(x^2+y^2)\cancel{x^2}}\cancel{x^2}$$

$$= \frac{-y}{r^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\theta = \tan^{-1}\cancel{\frac{y}{x}} \Rightarrow \frac{\partial \theta}{\partial y} = \frac{1}{x} \cdot \frac{1}{\frac{(x^2+y^2)}{x^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$J(x,y) = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = \frac{\cos^2\theta}{r} + \frac{\sin^2\theta}{r}$$

$$J(x,y) = \frac{1}{r}$$

$$f_{R\theta}(r,\theta) = r\, f_{XY}(x,y)$$

$$f_{R\theta}(r,\theta) = r\, f_{XY}(r\cos\theta, r\sin\theta)$$

Answer

The joint density function of two continuous random variables is

$$f(x,y) = \begin{cases} C\,xy & , 0<x<2,\ 1<y<3 \\ 0 & , \text{otherwise} \end{cases}$$

Find a) C , b) $P(0<X<1,\ 1<Y<2)$

c) $P(X<1, Y>2)$, d) Marginal distribution function of X & Y , e) JOINT DISTRIBUTION Functions of X & Y & f) $P(X+Y<3)$

Solution :–

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{x=0}^{2}\int_{y=1}^{3} C\,xy\,dx\,dy = 1 \Rightarrow C = \frac{1}{8}$$

(b) $P(0<X<1,\ 1<Y<2)$

$$= \int_{0}^{1}\int_{1}^{2} \frac{xy}{8}\,dx\,dy = \frac{3}{32}$$

c) $P(X<1, Y>2)$

$$= \int_{0}^{1}\int_{2}^{3} \frac{xy}{8}\,dx\,dy = \frac{5}{32}$$

d) Marginal distribution

$$F_{XY}(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(u,v)\,du\,dv$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{\infty} \frac{xy}{8}\,dx\,dy$$

Now there will be three cases

1) $x<0 \Rightarrow F_{XY}(x) = 0$

2) $0<x<2 \Rightarrow F_{XY}(x) = \int_{u=0}^{x}\int_{v=1}^{3} \frac{1}{8}xy\,dx\,dy$

3) $2<x \Rightarrow F_{XY}(x) = 1$ { Note Total SUM OF ALL PROBABILITY WILL Be 1 }

Similarly $F_{XY}(y)=0$ for $-\infty<y<1$

for $1<y<3$

$$F_{XY}(y) = \int_{-\infty}^{\infty}\int_{1}^{y} \frac{xy}{8}\,dx\,dy$$

$$= \frac{1}{8}\int_{0}^{2}\int_{1}^{y} \frac{xy}{8}\,dx\,dy = \frac{1}{8}\left[\int_{1}^{y} y\int_{0}^{2} x\,dx\right]dy$$

$$= \frac{1}{8}\int_{1}^{y} y\left[\frac{4}{2}-\frac{0}{2}\right]dy = \frac{1}{8}\times 2\left[\frac{y^2}{2}\right]_{1}^{y}$$

$$= \frac{1}{4}\times\frac{1}{2}\left\{\left[y^2\right]-1\right\} = \frac{1}{8}(y^2-1)$$

& For $y>3$

$$F_{XY}(y) = 1$$

$$F_2(y) = \begin{cases} 0 & ; \quad y<1 \\ (y^2-1)/8 & ; \quad 1<y<3 \\ 1 & ; \quad y>3 \end{cases}$$

e) Joint distribution Function $F_{XY}(x,y) = ?$

$$F_{XY}(x,y) = P(X\le x,\ Y\le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

⇒ Now For different ranges of x & y, F_{XY} will be different.

i.e. there will be three cases of Y for three cases of X.

Case (i)

Now for $x<0$

(1) $y<1 \Rightarrow F_{XY}(x,y) = 0$

(2) $1<y<3 \Rightarrow F_{XY}(x,y) = 0$

(3) $3<y \Rightarrow F_{XY}(x,y) = 0$

Case (2) $0 < x < 2$

(1) $y < 1 \Rightarrow F_{XY}(x,y) = 0$

(2) $1 < y < 3 \Rightarrow F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$

$$= \frac{1}{8}\int_{0}^{x}\int_{1}^{y} xy\,dx\,dy = \frac{1}{8}\left[\frac{x^2(y^2-1)}{4}\right] = \frac{1}{32}\left[x^2(y^2-1)\right]$$

Case (iii) $2 < x$ → In this case

(1) $y < 0$ $F_{XY}(x,y) = 0$

(2) $1 < y < 3$, $F_{XY}(x,y) = \int_{-\infty}^{\infty}\int_{1}^{y} \frac{xy}{8}\,dx\,dy$

For all x

$$= \int_{1}^{y}\int_{-\infty}^{\infty} \frac{xy}{8}\,dx\,dy = \frac{1}{8}\int_{-\infty}^{\infty} x \int_{1}^{y} y\,dy\,dx$$

for all x

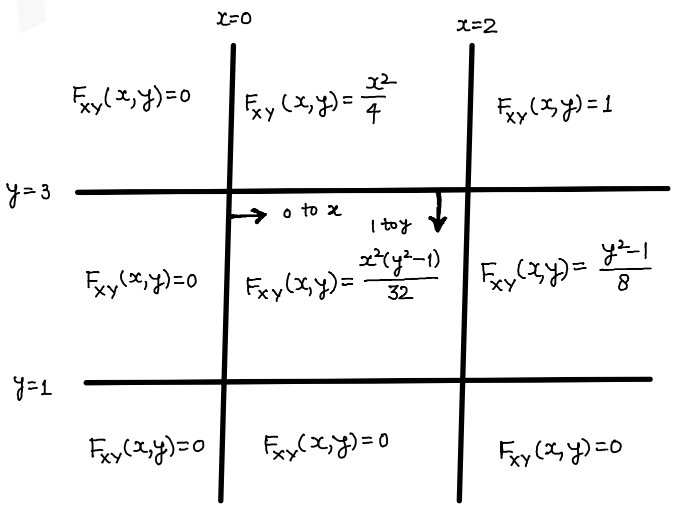
$$= \frac{1}{8}\int_{-\infty}^{\infty} x\,\frac{(y^2-1)}{2}\,dx = \frac{1}{16}\int_{-\infty}^{\infty} x\,dx\,(y^2-1)$$

$$= \frac{1(y^2-1)}{8\cdot 2}\int_{-\infty}^{\infty} x\,dx = \frac{1}{8}\frac{(y^2-1)}{2}\left\{\int_{-\infty}^{0} x\,dx + \int_{0}^{2} x\,dx + \int_{2}^{\infty} x\,dx\right\}$$

$$= \frac{y^2-1}{8\cdot 2}\left[0 + \frac{x^2}{2}\Big|_0^2 + 0\right] = \frac{(y^2-1)}{8\cdot 2}\times\frac{4-0}{2} = \frac{y^2-1}{8}$$

3) for $y > 3$

$F_{XY}(x,y) = 0$



$P(X+Y<3) = ?$

$\Rightarrow P(X+Y<3) = P(X < 3-Y)$

or $P(X+Y<3) = P(Y < 3-X)$

i.e. Probability that the value of sum of two random variable should be less than 3.

Given $0 < X < 2$

& $1 < Y < 3$

upper limit of x

upper limit of y

$$P[X+Y<3] = \int_{0}^{2}\int_{1}^{3-x} f_{XY}(x,y)\,dx\,dy$$

lower limit of x

lower limit of y

$$= \frac{1}{8}\int_{0}^{2} x\left(\int_{1}^{3-x} y\,dy\right)dx = \frac{1}{8}\int_{0}^{2}\frac{x}{2}\left\{(3-x)^2 - 1\right\}dx$$

$$= \frac{1}{8}\int_{0}^{2}\frac{x}{2}(9+x^2-6x-1)\,dx = \frac{1}{8}\int_{0}^{2}\frac{x}{2}(8+x^2-6x)\,dx$$

$$= \frac{1}{8}\left[\frac{8x^2}{2} + \frac{x^4}{4} - \frac{6x^3}{3}\right]_0^2 = \frac{1}{8}\left[4\times 4 + \frac{16}{4} - \frac{6\times 8}{3}\right] - 0$$

$$= \frac{1}{8}\left[\frac{\cancel{16}+4-\cancel{16}}{2}\right] = \frac{1}{2\cdot 2} = \frac{1}{4} \quad \text{Ans}$$

Similarly:-

$$P[(X+Y<3)] = P[X < 3-Y]$$

$$= \int_{1}^{3}\int_{0}^{3-y} \frac{xy}{8}\,dx\,dy = \frac{1}{8}\left[\int_{1}^{3} y \int_{0}^{3-y} x\,dx\,dy\right]$$

for X for Y

$$= \frac{1}{8}\times\frac{1}{2}\int_{1}^{3} y\,x^2\Big|_0^{3-y} dy = \frac{1}{16}\int_{1}^{3} y(3-y)^2\,dy$$

$$= \frac{1}{16}\int_{1}^{3} y(9+y^2-6y)\,dy = \frac{1}{16}\int_{1}^{3}(9y+y^3-6y^2)\,dy = \frac{1}{16}\left[\frac{9y^2}{2}+\frac{y^4}{4}-\frac{6y^3}{3}\right]_1^3$$

$$= \frac{1}{16}\left[\frac{9y^2}{2}+\frac{y^4}{4}-\frac{6y^3}{3}\right]_1^3 = \frac{1}{16}\left[\frac{9\times(9)}{2}+\frac{81}{4}-\frac{6\times 27}{3}\right] - \frac{1}{16}\left[\frac{9\times 1}{2}+\frac{1}{4}-\frac{6}{3}\right]$$

$$= \frac{1}{16}\left[\frac{81}{4}\{3\} - 54\right] - \frac{1}{16}\left[\frac{19}{4} - 2\right]$$

$$= \frac{243}{64} - \frac{54}{16} - \frac{19}{64} + \frac{2}{16} = \frac{224}{64} - \frac{52\times 4}{16\times 4} = \frac{224-208}{64}$$

$$= \frac{16}{64} = \frac{1}{8}\times 2 = \frac{1}{4} \quad \text{Ans}$$

Characteristics of Random Variables

1. Expectation

Let X be a discrete random variable such that

$$[X] = [x_1, x_2, x_3, x_4, \ldots\ldots x_m]$$

The expectation or expected value or mean of X is defined as

$$\mu_x = E(x) = \sum_{j=1}^{m} x_j f(x_j) \longrightarrow (16)$$

where $f(x_j)$ is the probability function.

If all the probabilities are equal, then

$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = \ldots\ldots = f(x_m) = \frac{1}{m}$$

Hence

$$\mu_x = E(x) = \frac{\sum_{j=1}^{m} x_j}{m} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_m}{m} \longrightarrow (16)$$

Thus in this case

Expectation = Arithmatic Mean

For a continuous r.v. X having a density function $f(x)$, the expectation is defined as

$$\boxed{\mu_x = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx} \longrightarrow (17)$$

Functions of Random Variable

If X is a R.V., then $Y = g(x)$ is also a R.V. Hence

$$E(Y) = E[g(x)] = \sum_{x} g(x) \cdot f(x) \rightarrow (18)$$

Discrete Case

&

$$E[Y] = E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \rightarrow (19)$$

Continuous Case

The above result can be generalized for functions of two, or more variables. Thus

$$E[g(x,y)] = \sum_{x} \sum_{y} g(x,y) f(x,y) \rightarrow (20)$$

Discrete Case

$$E[g(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy \rightarrow (21)$$

Continuous Case

Theorems on Expectation

1. $E(cX) = c\,E(X) \longrightarrow (22)$
2. $E(X+Y) = E(X) + E(Y) \longrightarrow (23)$
3. If two R.V. X & Y independent to each other then

$$E(XY) = E(X).E(Y) \longrightarrow (24)$$

Theorem [Equation 23]

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\, dx\, dy$$

Let $A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\, dx\, dy$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\right] dx$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = E[X]$$

$f_X(x) \longrightarrow$ Marginal PDF

In a similar, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y \left\{\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\right\} dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$f_Y(y)$: Marginal PDF

Thus,

$$\boxed{E[X+Y] = E[X] + E[Y]}$$ Proved

Theorem equ. (24)

$$E[XY] = E[X]E[Y]$$

Proof:-

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\, f_{XY}(x,y)\, dx\, dy$$

But as we know X & Y are independent

$$f_{XY}(x,y) = f_X(x)\cdot f_Y(y)$$

$$E_{XY}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\, f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E[X]\, E[Y]$$

Hence $\boxed{E[XY] = E[X]\,E[Y]}$

Similarly we can also prove that

$$E[g_1(X)\, g_2(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)\, g_2(y)\, f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} g_1(x)\, f_X(x)\, dx \int_{-\infty}^{\infty} g_2(y) f_Y(y)\, dy$$

$$= E[g_1(X)\, g_2(Y)]$$

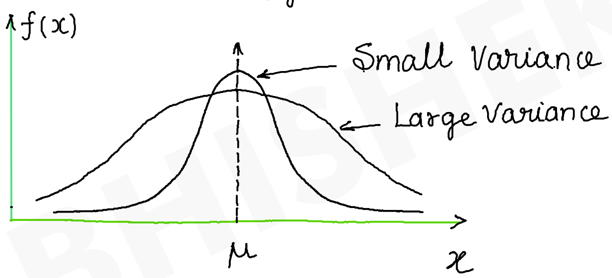
$$\boxed{E[g_1(X)\cdot g_2(Y)] = E[g_1(X)]\cdot E[g_2(Y)]}$$

Variance and Standard Deviation

To characterise a random variable mean 'μ' & standard deviation from its mean, play a dominant role.

$E(X-\mu)$ is used to indicate the deviation, or dispersion about its mean.

Now let us consider a curve as shown in figure below



Random Variables of Same Mean & Different Variance

Let us consider the curves I & II, to be symmetric about μ. Then $(X-\mu)$ is positive for $X>\mu$, & negative for $X<\mu$. Because of the symmetry of the curve about μ, resulting

$E(X-\mu)=0$ in both the cases. Thus $\longrightarrow$ (25)

$E(X-\mu)$ is not a proper term to characterize a r.v.

Now $E[(X-\mu)^2]$ is more useful function in comparison of $E[(X-\mu)^2]$

$E[(X-\mu)^2]$ is known as variance of X, which have square unit.

And square root of this variance will give standard deviation of X.

Variance of a random variable X is defined as

$$Var(X) = \sigma_x^2 = E[(X-\mu)^2] \longrightarrow (26)$$

where μ is the mean of X

$$\mu = E(X)$$

Standard deviation of a random variable X is defined as

$$\sigma_x = \sqrt{Var(X)} = \sqrt{E[(X-\mu)]^2}$$

$$\mu = E(X) \longrightarrow (27)$$

Theorem 1

$$\sigma^2 = E[(X-\mu)^2] = E(X^2) - \mu^2 \longrightarrow (28)$$

Proof:-

$$\sigma^2 = E[X^2 + \mu^2 - 2X\mu]$$

$$= E[X^2] + E[\mu^2] - 2E[\mu X]$$

Contd.

$$\sigma^2 = E(X^2) - 2\mu E(X) + E(\mu^2)$$

$$= E(X^2) - 2\mu\cdot\mu + \mu^2 E[1]$$

$$= E[X^2] - \mu^2$$

Note $E(1) = 1$
$E(X) = \mu$

Theorem 2 : If C is a constant, then

$$\mathrm{Var}(CX) = C^2\,\mathrm{Var}(X) \longrightarrow (29)$$

Proof :- $\mathrm{Var}(CX) = E[(Cx - C\mu)^2]$

$$= E[C^2(X-\mu)^2]$$

$$= C^2 E[(X-\mu)^2]$$

$$= C^2\,\mathrm{Var}(X)$$

Theorem 3

If X and Y are independent random variables then

a) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \longrightarrow (30)$

b) $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \longrightarrow (31)$

Proof :- we know that

$$\mathrm{var}(X) = E[(X-\mu)^2],$$

where

$\mu = E[X]$, Mean of X

Similarly $\mathrm{var}[Y] = E[(Y-\mu_y)^2]$

where $\mu_y = E[Y]$

$$\mathrm{Var}(x+y) = E[\{(x+y) - (\mu_x + \mu_y)\}^2]$$

$$= E[\{(X-\mu_x) + (Y-\mu_y)\}^2] = E[(X-\mu_x)^2 + (Y-\mu_y)^2 + 2(X-\mu_x)(Y-\mu_y)]$$

$$= E[(X-\mu_x)^2] + E[(Y-\mu_y)^2] + 2E[(X-\mu_x)(Y-\mu_y)]$$

Given X & Y are independent R.V.'s.

$$\Rightarrow E[(X-\mu_x)\cdot(Y-\mu_y)]$$

$$= E[(X-\mu_x)]\cdot E[(Y-\mu_y)]$$

Now, $E[(X-\mu_x)] = 0$,

$E[(Y-\mu_y)] = 0$.

So, we can write $\boxed{\mathrm{Var}(x+y) = \mathrm{Var}(X) + \mathrm{Var}(Y)}$

UNIFORM DISTRIBUTION

Normal Approximation to Binomial Distribution

If n is large & if neither p nor q is too close to zero, then the binomial distribution can be closely approximated by a normal distribution with standarized R.V. is given by

$$Z = \frac{X - np}{\sqrt{npq}}$$

In practice $np \geq 5$ and $nq \geq 5$ give satisfactory approximation.

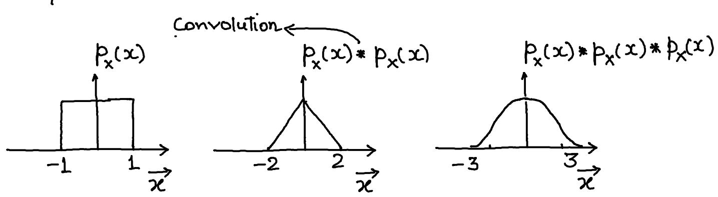
$\boxed{\text{Central Limit Theorem}}$

"The probability density of a sum of N independent random variables tend to approach a normal density as the number N increases."

The mean & variance of this normal density are the sum of means and variance of N independent random variables.

$$\lim_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma} \leq x\right] = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}\,dv \longrightarrow (50)$$

i.e. if we add a lot of indendent random variable their joint PDF tend to become Gaussian.



or equivalently

$$\lim_{n\to\infty} P\left[\frac{\tilde{x}_n-\mu}{\sigma/\sqrt{n}} > x\right] = Q(x)$$

where $\tilde{x}_n = \frac{x_1+x_2+x_3+\cdots+x_n}{n}$

$\longrightarrow$ Note For a gaussian random variable y,

$$P(y>a) = Q(a)$$

Uniform & other distributions :-

A random variable x is said to have a uniform distribution in the region $a \leq x \leq b$ if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & , \ a \leq x \leq b \\ 0 & , \ \text{otherwise} \end{cases} \longrightarrow (51)$$

$$\text{Mean} = \frac{a+b}{2}, \quad \text{variance} = \frac{(b-a)^2}{12} \longrightarrow (52)$$

Rayleigh Distribution

PDF for a R.V. is given by

$$f(x) = \begin{cases} \frac{x}{a^2} e^{-x^2/2a^2} & ; \ 0 \leq x \leq \infty \\ 0 & ; \ a>0 \end{cases} \longrightarrow (53)$$

Attains Max. at $r=a$

$$\text{Mean} \rightarrow \mu = a\sqrt{\frac{\pi}{2}}, \quad \underset{\text{variance}}{\sigma^2} = \alpha\beta^2 \longrightarrow (54)$$

Similarly

$$Var(X-Y) = E\left[\{(X-Y)-(\mu_x-\mu_y)\}^2\right]$$
$$= E\left[\{(X-\mu_x)-(Y-\mu_y)\}^2\right]$$
$$= E\left[(X-\mu_x)^2\right] + E\left[(Y-\mu_y)^2\right]$$
$$-2E\left[(X-\mu_x)(Y-\mu_y)\right]$$

Now $E[(X-\mu_x).(Y-\mu_y)] = E[X-\mu_x].E[Y-\mu_y]$

As X & Y are independent

Now $E[X-\mu_x]=0$ & $E[Y-\mu_y]=0$

Hence

$$Var(X-Y) = E[(X-\mu_x)^2] + E[(Y-\mu_y)^2]$$
$$-2E[(X-\mu_x)].E[(Y-\mu_y)]$$
$$= Var(X) + Var(Y) - 0$$

$$\Rightarrow \boxed{Var(X-Y) = Var(X) + Var(Y)}$$

Second Method :- To prove above equ.

$$Var(X-Y) = Var(X+(-Y))$$
$$= Var(X) + Var(-Y)$$
$$= Var(X) + (-1)^2 \, Var(Y)$$
$$\Rightarrow Var(X-Y) = Var(X) + Var(Y)$$

Proved

Standardized Random Variable :-

If μ & σ $(\sigma>0)$ are the respective mean & deviation of R.V. X, then an associated Standardized Random Variable (or Normalized R.V.) is defined as

$$X^* = \frac{X-\mu}{\sigma} \longrightarrow (32)$$

Properties of X^*

a) $E(X^*) = 0$

b) $Var(X^*) = 1$

$\longrightarrow (33)$

Moments

The r^{th} moment of a random variable X about the origin is defined as

$$\mu_r' = E[(X)^r] \quad , r=0,1,2,3,.. \longrightarrow (34)$$

The r^{th} moment of a random variable X about the mean μ is defined as

$$\mu_r = E[(X-\mu)^r] \quad , \quad r=0,1,2, \longrightarrow (35)$$

Note $\mu_r' \longrightarrow$ r^{th} Moment about origin

$\mu_r \longrightarrow$ r^{th} Moment about Mean μ

Equ. (35) $\mu_r = E[(X-\mu)^r] \longrightarrow (36)$

$$= \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx \; ; \text{ If } X \text{ is a continuous R.V.}$$

$$\mu_r = \sum_m (x_m-\mu)^r f(x_m) \; ; \text{ If } X \text{ is a discrete R.V.} \longrightarrow (37)$$

Joint Moment of (X,Y)

about origin $\mu_{kn}' = E[X^k Y^n] \longrightarrow (38)$

$$= \begin{cases} \sum_i \sum_j x_i^k y_j^n \cdot f(x_i, y_j) & \text{for discrete R.V.} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^n f_{XY}(x,y)\,dx\,dy & \text{for cont. R.V.} \end{cases}$$

COVARIANCE

$$\boxed{\begin{aligned} Cov(X,Y) &= \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] \\ \text{or} \quad Cov(X,Y) &= \sigma_{XY} = E[XY] - E(X)\,E(Y) \end{aligned}}$$

→ (39)

Note :-

① X & Y are said to orthogonal iff

$$E[XY] = 0$$

② X & Y are said to be uncorrelated iff (or independent)

$$Cov(X,Y) = 0$$

or $E[XY] = E[X] \cdot E[Y]$

Q. Find the covariance of X & Y if (a) they are independent, (b) Y is related to X by $Y = aX + b$.

Solution

$$Cov[XY] = \sigma_{XY} = E[(X-\mu_x)(Y-\mu_y)]$$

$$= E[XY] - E[X]E[Y]$$

Now if X & Y are independent then

$$E[XY] = E[X]\,E[Y]$$

Hence

$$\sigma_{XY} = E[X]E[Y] - E[X]E[Y] = 0$$

(b) $E[XY] = E[X(aX+b)]$

$$= E[aX^2 + bX]$$

$$= aE[X^2] + bE[X] = aE[X^2] + b\mu_X$$

Thus,

$$\sigma_{XY} = E[XY] - E[X]\,E[Y]$$

$$= aE[X^2] + b\mu_X - \mu_X \cdot (a\mu_X + b)$$

Note

$$\sigma_{XY} = aE[X^2] + b\mu_X - \mu_X(a\mu_X + b)$$

$$= aE[X^2] + \cancel{b\mu_X} - a\mu_X^2 - \cancel{b\mu_X}$$

$$= aE[X^2] - a\mu_X^2$$

$$= a\left[E[X^2] - \mu_X^2\right]$$

Now consider equ. (28) $\quad variance(X) = E[X^2] - \mu_X^2$

$$\sigma_{XY} = a\,\sigma_X^2$$

Note :-

If X & Y are independent, then they are uncorrelated. But the converse is not true.

Q. let $Z = aX + bY$, where a & b are arbitrary constants. Show that X & Y are independent, then

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2$$

Solution:-

$$\sigma_X^2 = Var(X) = E[X^2] - \mu_X^2$$

$$\sigma_Y^2 = Var(Y) = E[Y^2] - \mu_Y^2$$

Now $\mu_Z = E[Z] = E[aX + bY] = aE[X] + bE[Y]$

$\Rightarrow \mu_Z = a\mu_X + b\mu_y$

Now $\sigma_Z^2 = E[(Z-\mu_Z)^2]$

$$= E[\{(aX+bY) - (a\mu_X + b\mu_y)\}^2]$$

$$= E[\{a(X-\mu_X) + b(Y-\mu_y)\}^2]$$

$$= E[a^2(X-\mu_X)^2] + E[b^2(Y-\mu_y)^2] + 2E[a(X-\mu_X) \cdot b(Y-\mu_y)]$$

Now if X & Y are independent then

$$2E[a(X-\mu_X)\,b(Y-\mu_y)] = 2aE[(X-\mu_X)]\,E[(Y-\mu_y)]$$

$$= 2ab \cdot 0 \cdot 0 = 0$$

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab \cdot 0$$

$$\Rightarrow \boxed{\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2}$$

Proved

Let X be a discrete R.V. that can assume values 0, 1, 2, 3, 4, 5, ... Then the probability function of X is given by

Poission Distribution :

$$\boxed{f(x) = P(X = x) = \frac{\lambda^{x} \cdot e^{-\lambda}}{(x)!}} \rightarrow (43)$$

$x = 0, 1, 2, 3, 4, \ldots\ldots$

λ is a positive constant

The probilities of poission distribution are

$$\boxed{\begin{array}{l} \lambda \equiv \text{mean} \\ \lambda \equiv \text{variance} \end{array}} \quad (44)$$

Poission Approximation to Binomial Distribution

In Binomial distribution if n is large & p is close to zero, then it can be approximated by Poission distribution with $\mu = np$.

In practice, $n \geqslant 50$ & $np \leq 5$ gives satisfactory approximation.

Imp

Normal Distribution :→

A R.V. X is called Normal (or Gaussian) R.V. if its PDF is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma_x^2}\right\} \rightarrow (45)$$

Mean $\equiv \mu$, variance $\equiv \sigma^2$

Corresponding Distribution Function

$$F(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-(v-\mu)^2/2\sigma_x^2}\, dv$$

Let $\frac{v-\mu}{\sigma_x} = q$, $\frac{1}{\sigma_x} dv = dq$, when $v \to x$, $q = (x-\mu)/\sigma_x$

$v \to -\infty$, $q \to -\infty$

$$F_X\left(\frac{x-\mu}{\sigma}\right) = P\left(X \leq \frac{x-\mu}{\sigma}\right) = \frac{1}{\cancel{\sigma_x}\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} \cancel{\sigma_x} \cdot e^{-q^2/2}\, dq$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-q^2/2}\, dq \rightarrow (46)$$

Imp
Note :-

Q-function $\quad Q(z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-s^2/2}\, ds \rightarrow (47)$

Q(z) is also known as complementary error function erfc(z).

There is a function known as error function or erf(z), defined as (on next page)

Contd.

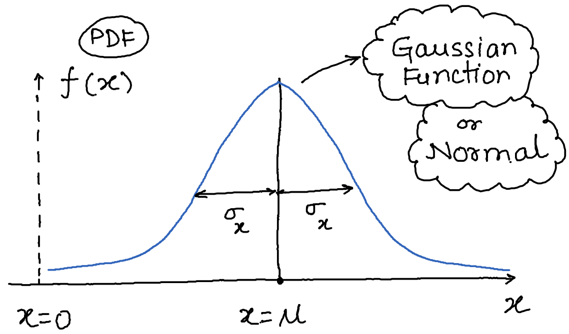
From eq. (47),

$$F_X\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x-\mu/\sigma} e^{-\xi^2/2}\, d\xi$$

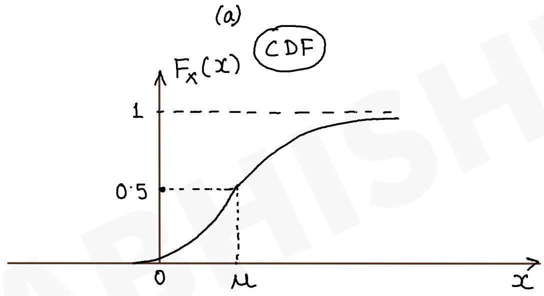
$$= 1 - Q\left(\frac{x-\mu}{\sigma}\right) \longrightarrow (48)$$

or $\underline{F_X(z) = 1 - Q(z)}$

The values of $Q(z)$ function is Tabulated for different values of z



(a)



(b)

Note:— Tips for Q function and error function

1) $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}\, du$

2) $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}\, du$

3) $Q(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-u^2/2}\, du$ (49)

4) $\operatorname{erf}(0) = 0$ and $\operatorname{erf}(\infty) = 1$

5) $1 - Q(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}\, du$

Let $Z = \frac{X-\mu}{\sigma}$, equ. (47) can be rewritten as

$$F(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\xi^2/2}\, d\xi + \frac{1}{\sqrt{2\pi}}\int_{0}^{z} e^{-\xi^2/2}\, d\xi$$

Note $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-x^2/2}\, dx = \frac{1}{2}$

& $\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x^2/2}\, dx = \frac{1}{2}$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^z e^{-\xi^2/2}\, d\xi$$

Let $\xi/\sqrt{2} = u \Rightarrow d\xi = \sqrt{2}\, du$

if $\xi \to 0$, $u \to 0$

$\xi \to z$, $u \to \frac{z}{\sqrt{2}}$

$$= \frac{1}{2} + \frac{2\sqrt{2}}{2\sqrt{2\pi}}\int_0^{z/\sqrt{2}} e^{-u^2}\, du$$

$$= \frac{1}{2} + \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{2}} e^{-u^2}\, du = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right] = \frac{1}{2}\left[1 + 1 - \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)\right]$$

$$= 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)$$

$$= 1 - Q(z) = 1 - Q\left(\frac{X-\mu}{\sigma}\right)$$