UNIT : 2 Electrostatic fields and magnetostatic fields

Electrostatic fields

• Coulomb's law and field intensity.

Coulomb's law deals with the force a point charge exerts on another point charge.

Point charge : charge located on a body whose dimensions are much smaller than other relevant dimensions.
eg. a collection of charges on a pinhead.
e^-'s are regarded as point charges.

The polarity may be positive or negative.
charges are generally measured in coulombs (C).

1 C = 6×10^{18} electrons.

It is a very large unit of charge because,

1e = -1.6019×10^{-19} C.

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is :

1) Along the line joining them
2) Directly proportional to the product $Q_1 Q_2$ of the charges.
3) Inversely proportional to the square of the distance R between them.

Mathematically,

$$F = \frac{k\, Q_1 Q_2}{R^2}$$

where k is the proportionality constant. ~~whose~~

In SI units Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N), so that $k = 1/4\pi\varepsilon_0$. The constant ε_0 is known as the permittivity of free space. (In farads per meter (F/m)).

$$\varepsilon_0 = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi} \text{ F/m.}$$

and

$$k = \frac{1}{4\pi\varepsilon_0} \simeq 9 \times 10^9 \text{ m/F.}$$

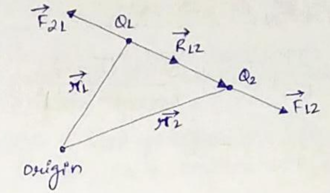


fig: Coulomb vector force on point charges Q_1 and Q_2.

If Q_1 and Q_2 are located at points having position vectors $\vec{r_1}$ and $\vec{r_2}$, then the force $\vec{F_{12}}$ on Q_2 due to Q_1 is given by

$$\vec{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\, \hat{a}_{R_{12}} \quad \text{——①}$$

where, $\vec{R_{12}} = \vec{r_2} - \vec{r_1}$, $R = |\vec{R_{12}}|$ and $\hat{a}_{R_{12}} = \frac{\vec{R_{12}}}{R}$

substituting these in eq. ①, we get

$$\vec{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^3}\, \vec{R_{12}}$$

or

$$\vec{F_{12}} = \frac{Q_1 Q_2 (\vec{r_2} - \vec{r_1})}{4\pi\varepsilon_0 |\vec{r_2} - \vec{r_1}|^3} \quad \text{——②}$$

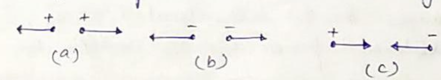
Note that :

1) From figure above, the force $\vec{F_{21}}$ on Q_1 due to Q_2 is given by

$$\vec{F_{21}} = |\vec{F_{12}}|\, \hat{a}_{R_{21}} = |\vec{F_{12}}|\, (-\hat{a}_{R_{12}})$$

or $\vec{F_{21}} = -\vec{F_{12}}$ $\quad [\because \hat{a}_{R_{21}} = -\hat{a}_{R_{12}}]$

2) Like charges repel while unlike charges attract.



3) The distance R between the charged bodies Q_1 and Q_2 must be large compared ~~to~~ with the linear dimensions of the ~~di~~ bodies. i.e. Q_1 and Q_2 must be point charges.
4) Q_1 and Q_2 must be static (at rest).
5) The signs of Q_1 and Q_2 must be taken into account. For like charges $Q_1 Q_2 > 0$, for unlike charges $Q_1 Q_2 < 0$.
6) The quantity of total charge remains constant (conservation)

If we have more than two point charges, we can use the principle of superposition.
It states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located at points with position vectors $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_N}$, the resultant force $\vec{F}$ on a charge Q located at point $\vec{r}$ is the vector sum of the forces exerted on Q by each of the charges.

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_N$$

$$\vec{F} = \frac{QQ_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|^3} + \frac{QQ_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^3} + \dots \frac{QQ_N(\vec{r}-\vec{r}_N)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_N|^3}$$

or $$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{K=1}^{N}\frac{Q_K(\vec{r}-\vec{r}_K)}{|\vec{r}-\vec{r}_K|^3} \quad \text{— (3)}$$

The electric field intensity (or electric field strength) $\vec{E}$ is the force that a unit positive charge experiences when placed in an electric field.

$$\vec{E} = \lim_{Q\to 0}\frac{\vec{F}}{Q}$$

or $$\vec{E} = \cancel{\lim}\ \frac{\vec{F}}{Q} \quad \text{— (2)}$$

For $Q>0$, $\vec{E}$ is in the direction of $\vec{F}$ and is measured in newtons per coulomb (N/C) or volts per meter (V/m).

$\vec{E}$ at point $\vec{r}$ due to a point charge located at $\vec{r}'$ is obtained from eq. (2).

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{Q(\vec{r}-\vec{r}')}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|^3} \quad \text{— (**)}$$

or $$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \quad \text{— (4)}$$

For N point charges $Q_1, Q_2 \dots Q_N$ located at $\vec{r}_1, \vec{r}_2 \dots \vec{r}_N$, $\vec{E}$ at point $\vec{r}$ is obtained from eqs. (3) and (4), as

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots + \vec{E}_N$$

$$\vec{E} = \frac{Q_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^3} + \dots + \frac{Q_N(\vec{r}-\vec{r}_N)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_N|^3}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{K=1}^{N}\frac{Q_K(\vec{r}-\vec{r}_K)}{|\vec{r}-\vec{r}_K|^3} \quad \text{— (5)}$$

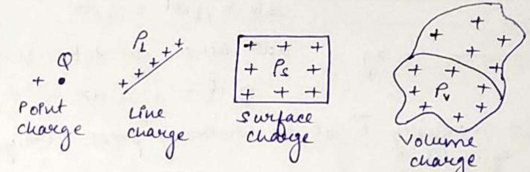
Q: Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4) respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

Ans: Example 4.1 [Sadiku]

- Electric fields due to continuous charge distributions

So far we have considered only forces and electric fields due to point charges, which are essentially charges occupying very small physical space.

It is also possible to have continuous charge distributions along a line, on a surface or in a volume, as illustrated below.



ρ_L (C/m) → Line charge density

ρ_S (C/m²) → surface charge density

ρ_v (C/m³) → Volume charge density

[ρ (without subscript) ↳ radial distance in cylindrical coordinates]

The charge element (dQ) and the total charge (Q) due to these charge distributions are obtained as

$$dQ = \rho_L\,dl \longrightarrow Q = \int_L \rho_L\,dl \quad \text{(line charge)}$$

$$dQ = \rho_S\,dS \longrightarrow Q = \int_S \rho_S\,dS \quad \text{(surface charge)}$$

$$dQ = \rho_v\,dv \longrightarrow Q = \int_v \rho_v\,dv \quad \text{(volume charge)}$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_S, ρ_v may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution.

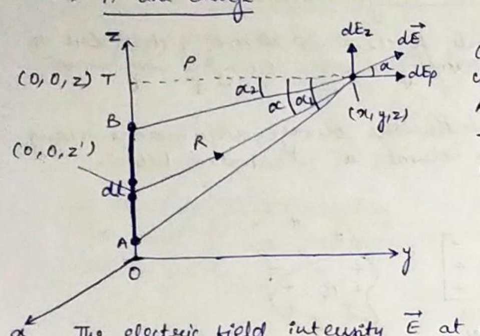
We treat dQ as a point charge. Thus be replacing Q in eq. (**) with charge element $dQ = \rho_L\,dl$, $\rho_S\,dS$ or $\rho_v\,dv$ and integrating we get

$$\vec{E} = \int_L \frac{\rho_L\,dl}{4\pi\varepsilon_0 R^2}\hat{a}_R \quad \text{(line charge) — (1)}$$

$$\vec{E} = \int_S \frac{\rho_S\,dS}{4\pi\varepsilon_0 R^2}\hat{a}_R \quad \text{(surface charge) (sheet charge) — (2)}$$

$$\vec{E} = \int_v \frac{\rho_v\,dv}{4\pi\varepsilon_0 R^2}\hat{a}_R \quad \text{(volume charge) — (3)}$$

→ A line charge



Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis.

The charge element dQ associated with element $dl = dz$ of line is

$$dQ = \rho_L dl = \rho_L dz$$

and hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L dz$$

The electric field intensity $\vec{E}$ at an arbitrary point $P(x,y,z)$ can be found using eq. – ①.

Field point (x,y,z) → The field point is the point at which the field is to be evaluated.

source point (x', y', z')

$$dl = dz'$$

$$\vec{R} = (x,y,z) - (0,0,z') = x\hat{a}_x + y\hat{a}_y + (z-z')\hat{a}_z$$

or

$$\vec{R} = \rho\hat{a}_\rho + (z-z')\hat{a}_z$$

$$R^2 = |\vec{R}|^2 = x^2+y^2+(z-z')^2 = \rho^2 + (z-z')^2$$

$$\frac{\hat{a}_R}{R^2} = \frac{\vec{R}}{|\vec{R}|^3} = \frac{\rho\hat{a}_\rho + (z-z')a_z}{[\rho^2+(z-z')^2]^{3/2}}$$

substituting all this into eq. ①, we get

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho\hat{a}_\rho + (z-z')\hat{a}_z}{[\rho^2+(z-z')^2]^{3/2}} dz' \quad \text{—④}$$

To evaluate this, it is convenient that we define α, α_1 and α_2

$$R = [\rho^2 + (z-z')^2]^{1/2} = \rho\sec\alpha$$

$$z' = OT - \rho\tan\alpha, \quad dz' = -\rho\sec^2\alpha\, d\alpha$$

Hence eq. ④ becomes,

$$\vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\,[\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z]\,d\alpha}{\rho^2\sec^2\alpha}$$

$$\vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2}[\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z]\,d\alpha$$

Thus for a finite line charge,

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\hat{a}_z\right] \quad \text{—⊛}$$

Special case: for an infinite line charge,

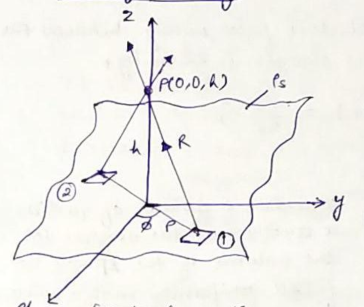
point B is at $(0,0,\infty)$ and
point A is at $(0,0,-\infty)$

so that $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = -\frac{\pi}{2}$; the z-component vanishes and eq. ⊛ becomes

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

If line is not along z-axis, ρ is perpendicular distance from the line to the point of interest and $\hat{a}_\rho$ is the unit vector along that distance directed from the line charge to the field point.

→ A surface charge



Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_s.

The charge associated with an elemental area dS is

$$dQ = \rho_s dS$$

From eq. ②, the contribution to the $\vec{E}$ field at point $P(0,0,h)$ by the charge dQ on the elemental surface 1 is

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\hat{a}_R \quad \text{—⑤}$$

~~from~~ from figure,

$$\vec{R} = \rho(-\hat{a}_\rho) + h\hat{a}_z, \quad R = |\vec{R}| = [\rho^2 + h^2]^{1/2}$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|}, \quad dQ = \rho_s dS = \rho_s\rho\, d\phi\, d\rho.$$

Substitution of these terms into eq. ⑤ gives

$$d\vec{E} = \frac{\rho_s\rho\, d\phi\, d\rho\,[-\rho\hat{a}_\rho + h\hat{a}_z]}{4\pi\varepsilon_0[\rho^2 - h^2]^{3/2}}$$

For every element 1, there is a corresponding element 2 whose contribution along $\hat{a}_\rho$ cancels that of element 1
→ symmetry of the charge distribution.

Thus the contributions to E_ρ add up to zero so that $\vec{E}$ has only z-component.

Mathematically, it can be shown as, by replacing $\hat{a}_\rho$ with $\cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0<\phi<2\pi$ gives zero.

Therefore,

$$\vec{E} = \int_S d\vec{E}_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{[\rho^2+h^2]^{3/2}}\hat{a}_z$$

$$\vec{E} = \frac{\rho_s h}{4\pi\varepsilon_0}2\pi\int_0^\infty [\rho^2+h^2]^{-3/2}\frac{1}{2}d(\rho^2)\,\hat{a}_z$$

$$\vec{E} = \frac{\rho_s h}{2\varepsilon_0}\left\{-[\rho^2+h^2]^{-1/2}\right\}_0^\infty\hat{a}_z = \frac{\rho_s}{2\varepsilon_0}\hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_z \quad \text{—— ⑥}$$

that is, $\vec{E}$ has only z-component if the charge is in the xy-plane. Eq. ⑥ is valid for $h>0$; for $h<0$, we would need to replace $\hat{a}_z$ with $-\hat{a}_z$. In general for a infinite sheet of charge

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_n \quad \text{—— (**)}$$

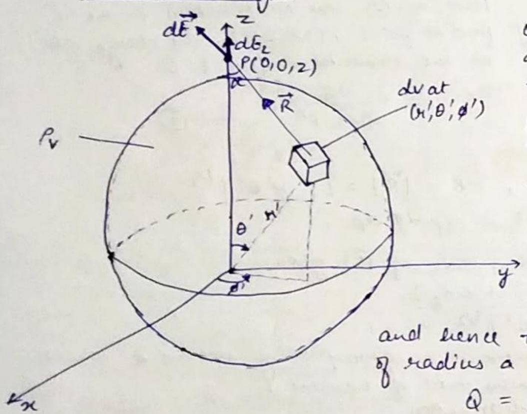
where $\hat{a}_n$ is the unit vector normal to the sheet.

Note: Electric field is normal to the sheet and it is independent of the distance between the sheet and the point of observation P.

In a parallel plate capacitor, the electric field existing between the ~~to~~ two plates having equal and opposites charges is given by,

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_n + \frac{-\rho_s}{2\varepsilon_0}(-\hat{a}_n) = \frac{\rho_s}{\varepsilon_0}\hat{a}_n$$

→ A Volume Charge



consider a sphere of radius a centered at the origin. Let the volume of the sphere be filled uniformly with a volume-charge density ρ_v (in C/m^3).

The charge dQ associated with the elemental volume dv chosen at (ρ', θ', ϕ') is

$$dQ = \rho_v\, dv$$

and hence the total charge in a sphere of radius a is

$$Q = \int_v \rho_v\, dv = \rho_v \int_v dv$$

$$Q = \rho_v \frac{4\pi a^3}{3}$$

The electric field $d\vec{E}$ outside the sphere at $P(0,0,z)$ due to the elementary volume charge is

$$d\vec{E} = \frac{\rho_v\, dv}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

where $\hat{a}_R = \cos\alpha\,\hat{a}_z + \sin\alpha\,\hat{a}_\rho$

By symmetry of the charge distributions, the contributions to E_x or E_y add up to zero. We are left with only E_z, given by

$$E_z = \vec{E}\cdot\hat{a}_z = \int_v dE\cos\alpha = \frac{\rho_v}{4\pi\varepsilon_0}\int_v \frac{dv\cos\alpha}{R^2} \quad \text{—— ⑦}$$

Expressions for dv, R^2 and $\cos\alpha$

$$dv = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi' \quad \text{—— ⑧}$$

applying the cosine rule, we have

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

It is convenient to evaluate the integral in terms of R and r'. Hence we express $\cos\theta'$, $\cos\alpha$, and $\sin\theta' d\theta'$ in terms of R and r', that is,

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR}$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'} \quad \text{—— ⑨}$$

Differentiating eq. ⑨ with respect to θ' and keeping z and r' fixed, we obtain

$$\sin\theta'\, d\theta' = \frac{R\,dR}{zr'} \quad \text{—— ⑩}$$

As θ' varies from 0 to π, R varies from $(z-r')$ to $(z+r')$ if P is outside the sphere.

Substituting ⑧ and ⑩ in ⑦ yields

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2 \frac{R\,dR}{zr'}\,dr'\,\frac{z^2+R^2-r'^2}{2zR}\frac{1}{R^2}$$

$$E_z = \frac{\rho_v 2\pi}{8\pi\varepsilon_0 z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left[1 + \frac{z^2-r'^2}{R^2}\right]dR\,dr'$$

$$E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2}\int_0^a r'\left[R - \frac{z^2-r'^2}{R}\right]_{z-r'}^{z+r'} dr'$$

$$E_z = \frac{\rho_v\pi}{4\pi\varepsilon_0 z^2}\int_0^a 4r'^2\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{1}{z^2}\left(\frac{4}{3}\pi a^3\rho_v\right)$$

or $$\vec{E} = \frac{Q}{4\pi\varepsilon_0 z^2}\hat{a}_z \quad \text{—— (11)}$$

This result is obtained for $\vec{E}$ at $P(0,0,z)$.

By symmetry of the charge distribution, the electric field at $P(r, \theta, \phi)$ is readily obtained from eq. (11) as

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r \quad \text{——} (***)$$

which is identical to the electric field at the same point due to a point charge Q located at the origin or the center of the spherical charge distribution.

- Electric Flux Density

The flux due to the electric field $\vec{E}$ can be calculated by using the general definition of flux in eq. below:

$$\Psi = \int_S \vec{A}\cdot d\vec{s} \quad \text{——} (1)$$

For practical reasons, this quantity is not usually considered to be the most useful flux in electrostatics.

Also the equations of Electric field ($\vec{E}$) due to various continuous charge distributions and free space, show that the electric field intensity is dependent on the medium in which charge is placed.

Suppose a new vector field $\vec{D}$ is defined by,

$$\vec{D} = \varepsilon_0 \vec{E} \quad \text{——} (2)$$

We use eq. (1) to define electric flux Ψ in terms of $\vec{D}$ as,

$$\Psi = \int_S \vec{D}\cdot d\vec{s}$$

In SI units, one line of electric flux emanates from +1C and terminates on −1C. Therefore, the electric flux is measured in coulombs. Hence the vector field $\vec{D}$ is called the electric flux density, and is measured in coulombs per square meter. (C/m²).

$\vec{D}$ is also called electric displacement.

From (2), it is apparent that all the formulas derived for $\vec{E}$ from coulomb's law can be used in calculating $\vec{D}$, we only have to multiply those formulas by ε_0.

Example: for an infinite sheet of charge, we have

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_n, \quad \text{then}, \quad \vec{D} = \frac{\rho_s}{2}\hat{a}_n$$

and for a volume charge distribution, we have

$$\vec{E} = \int_v \frac{\rho_v\, dv}{4\pi\varepsilon_0 R^2}\hat{a}_R, \quad \text{then} \quad \vec{D} = \int_v \frac{\rho_v\, dv}{4\pi R^2}\hat{a}_R$$

from these equations it can be noted that $\vec{D}$ is a function of charge and position only, it is independent of the medium.

Q: Determine $\vec{D}$ at (4,0,3) if there is a point charge -5π mC at (4,0,0) and a line charge 3π mC/m along the y-axis.

Ans: Example 4.7 [sadiku] page. 131

- Gauss's Law - Maxwell's Equation

Gauss's law states that the electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface.

Thus, $\Psi = Q_{enc}$

that is, $\Psi = \oint d\Psi = \oint_S \vec{D}\cdot d\vec{s}$ = total charge enclosed Q

$$= \int_v \rho_v\, dv$$

or

$$Q = \underbrace{\oint_S \vec{D}\cdot d\vec{s}}\; = \int_v \rho_v\, dv \quad \text{——} (1)$$

Applying divergence theorems to this term, we have

$$\oint_S \vec{D}\cdot d\vec{s} = \int_v \nabla\cdot\vec{D}\, dv \quad \text{——} (2)$$

Comparing the two volume integrals in eq. (1) and (2) we get,

$$\rho_v = \nabla\cdot\vec{D} \quad \text{——} (3)$$

Eq. (3) is the first Maxwell's equation, states that the volume charge density is same as the divergence of the electric flux density. Sometimes called as source equation. It is equivalent to Coulomb's law of force between point charges.

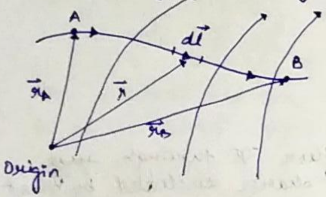
Q: Given that $\vec{D} = z\rho\cos^2\phi\,\hat{a}_z$ C/m², calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \le z \le 2$m.

Ans: Example 4.8 [Sadiku] page 138.

- Relationship between $\vec{E}$ and V - Maxwell's Equation

→ Electric potential:

Suppose we wish to move a point charge Q from point A to point B in an electric field $\vec{E}$. From coulomb's law, the force on Q is $\vec{F} = Q\vec{E}$ so that the work done in displacing the charge by $d\vec{l}$ is



$$dW = -\vec{F}\cdot d\vec{l} = -Q\vec{E}\cdot d\vec{l}$$

The negative sign indicates that the work is being done by an external agent. Thus total work done or the potential energy required, in moving Q from A to B is,

$$W = -Q\int_A^B \vec{E}\cdot d\vec{l}$$

Dividing W by Q in this equation, gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the potential difference between points A and B. Thus,

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}\cdot d\vec{l} \quad \text{J/C [joules / coulomb] or volts (V)}$$

- A → initial point, B → final point
- V_{AB} → negative, then loss in potential energy
  V_{AB} → positive, then gain in potential energy in moving Q from A to B.
- V_{AB} is independent of the path taken.

If $\vec{E}$ due to point charge Q located at origin, then

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

$$\text{So, } V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\,\hat{a}_r$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

or

$$V_{AB} = V_B - V_A$$

where V_B and V_A are the potentials or absolute potentials at B and A.

The potential at any point is the potential difference between that point and a chosen point (or reference point) at which the potential is zero.

\# Vectors whose line integral does not depend on the path of integration are called conservative. Thus $\vec{E}$ is conservative.

→ Relationship b/w $\vec{E}$ & V:

We know that, the potential difference between points A and B is independent of the path taken. Hence,

$$V_{BA} = -V_{AB}$$

that is, $V_{BA} + V_{AB} = \oint_L \vec{E}\cdot d\vec{l} = 0$

or

$$\oint_L \vec{E}\cdot d\vec{l} = 0 \quad \text{—① [integral form]}$$

This shows that the line integral of $\vec{E}$ along a closed path must be zero. Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying stoke's theorem to eq. ① gives

$$\oint_L \vec{E}\cdot d\vec{l} = \int_S (\nabla\times\vec{E})\cdot d\vec{S} = 0$$

$$\nabla\times\vec{E} = 0 \quad \text{—② [differential form]}$$

Any vector field that satisfies ① and ② is said to be conservative or irrotational.

Eq. ① and ② is referred to as Maxwell's equation for static electric fields. [The second maxwell's equation].

From the way we define potential, $V = -\int \vec{E}\cdot d\vec{l}$, it follows that,

$$dV = -\vec{E}\cdot d\vec{l} = -E_x dx - E_y dy - E_z dz$$

But from calculus of multivariables, a total change in $V(x,y,z)$ is the sum of partial changes with respect to x, y, z variables:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Comparing the two expressions for dV, we obtain

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

Thus,

$$E = -\nabla V \quad \text{--- (3)}$$

i.e. electric field intensity is the gradient of V. The negative sign shows that the direction of $\vec{E}$ is opposite to the direction in which V increases; $\vec{E}$ is directed from higher to lower levels of V.

$\because$ curl of gradient of a scalar function is always zero ($\nabla \times \nabla V = 0$), eq. (2) obviously implies that $\vec{E}$ must be a gradient of some scalar function. Thus eq. (3) could have been derived from eq. (2).

- Dielectric constant and strength

The dielectric constant (or relative permittivity) ε_r is the ratio of the permittivity of the dielectric to that of free space.

Dielectric breakdown is said to have occurred when a dielectric becomes conducting.

The ~~diell diellet~~ dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without electrical breakdown.

A dielectric material (in which $\vec{D} = \varepsilon \vec{E}$ applies) is linear if ε does not change with the applied $\vec{E}$ field, homogeneous if ε does not change from point to point, and isotropic if ε does not change with direction.

- Continuity equation and Relaxation time

From the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume. Thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ_{in}}{dt} \quad \text{--- (1)}$$

where Q_{in} is the total charge enclosed by the closed surface. Invoking the divergence theorem, we write.

$$\oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J}\, dv \quad \text{--- (2)}$$

But,
$$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_V \rho_v\, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv \quad \text{--- (3)}$$

substituting (2) and (3) into (1) gives,

$$\int_V \nabla \cdot \vec{J}\, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$$

or
$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{--- (4)}$$

which is called the continuity of current equation or just continuity equation.

Having discussed the continuity equation and the properties σ (σ conductivity) and ε (permittivity) of materials, it is appropriate to consider the ~~effect~~ effect of introducing charge at some interior point of a given material (conductor, or dielectric).

We make use of eq. (4) in conjuction with Ohm's law

$$\vec{J} = \sigma \vec{E} \quad \text{--- (5)}$$

and Gauss's law
$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} \quad \text{--- (6)}$$

Substituting eq. (5) and (6) into (4)

$$\nabla \cdot \sigma \vec{E} = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

Or
$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0 \quad \text{--- (7)}$$

This is a homogeneous linear ordinary differential equation. By separating variables in eq. (7), we get.

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}\partial t$$

and integrating both sides gives.

$$\ln \rho_v = -\frac{\sigma}{\varepsilon}t + \ln \rho_{vo}$$

where $\ln \rho_{vo}$ is a constant of integration. Thus

$$\rho_v = \rho_{vo}\, e^{-t/T_r}$$

where

$$T_r = \frac{\varepsilon}{\sigma}$$ } T_r is the time constant in seconds. known as relaxation or rearrangement time.

ρ_{vo} is the initial charge density (i.e. ρ_v at $t=0$)

Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} (= 36.8%) of its initial value.

• Boundary Conditions

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions.
These conditions are helpful in finding the field on one side of the boundary if the field on the other side is known.
To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint_L \vec{E} \cdot d\vec{l} = 0 \quad \text{—— } (*)$$

and

$$\oint_S \vec{D} \cdot d\vec{S} = Q_{enc} \quad \text{—— } (**)$$

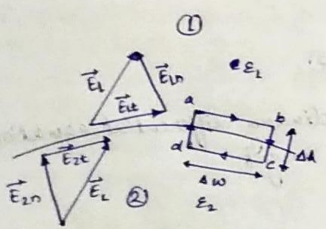
where Q_{enc} is the free charge enclosed by the surface S.
Also we need to decompose electric field intensity $\vec{E}$ into two orthogonal components,

$$\vec{E} = \vec{E}_t + \vec{E}_n \quad \Big\} \begin{array}{l} t \rightarrow \text{tangential component} \\ n \rightarrow \text{normal component.} \end{array}$$

Similar decomposition can be done for the electric flux density $\vec{D}$.

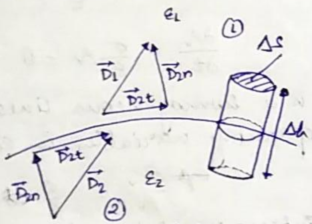
$$\vec{D} = \vec{D}_t + \vec{D}_n$$

→ Dielectric (ε_{r1}) – Dielectric (ε_{r2}) boundary conditions



(a)

determining $E_{1t} = E_{2t}$



(b)

determining $D_{1n} = D_{2n}$

Consider the $\vec{E}$ field existing in a region that consists of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$ as shown in fig (a). The fields $\vec{E}_1$ and $\vec{E}_2$ in media ① and ②, respectively, can be decomposed as,

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$
$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

We apply eq. (*) to the closed path abcda of fig(a), assuming that the path is very small with respect to the spatial variation of $\vec{E}$.
We obtain

$$\oint_L \vec{E} \cdot d\vec{l} = 0$$

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

where $E_t = |\vec{E}_t|$ and $E_n = |\vec{E}_n|$. The $\frac{\Delta h}{2}$ terms cancel and the above eq. becomes,

$$0 = (E_{1t} - E_{2t})\,\Delta w$$

or

$$E_{1t} = E_{2t} \quad \text{—— } ①$$

Thus the tangential components of $\vec{E}$ are the same on the two sides of the boundary.
In other words, $\vec{E}_t$ undergoes no change on the boundary and it is said to be continuous across the boundary.
Since, $\vec{D} = \varepsilon\vec{E} = \vec{D}_t + \vec{D}_n$ eq. ① can be written as

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

or

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

that is, D_t undergoes some change across the interface. Hence, D_t is said to be discontinuous across the interface.
Similarly, we apply eq. (**) to the pillbox (cylindrical gaussian surface) of fig. (b). The contribution due to sides vanishes.
Allowing $\Delta h \to 0$ gives

$$\Delta Q = \rho_s\,\Delta S = D_{1n}\,\Delta S - D_{2n}\,\Delta S$$

or

$$D_{1n} - D_{2n} = \rho_s \quad \text{—— } ②$$

where ρ_s is the free charge density placed deliberately at the boundary.
Eq. ② is based on assumption that $\vec{D}$ is directed from region ② to region ①.
If no free charges exist at the interface (i.e., charges are not deliberately placed there), $\rho_s = 0$, eq. ② becomes,

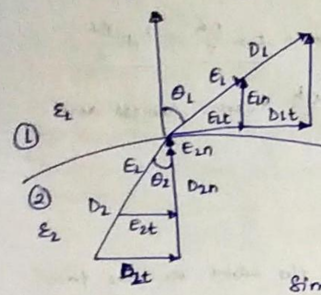
$$D_{1n} = D_{2n} \quad \text{—— } ③$$

Thus the normal component of $\vec{D}$ is continuous across the interface, i.e., D_n undergoes no change at the boundary.
Since, $\vec{D} = \varepsilon\vec{E}$, eq. ③ can be written as,

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

Showing that normal component of $\vec{E}$ is discontinuous at the boundary.
Eq. ①, ② and ③ are collectively referred to as ~~bo~~ boundary conditions.

Refraction of $\vec{D}$ or $\vec{E}$ at a dielectric - dielectric boundary



consider $\vec{D_1}$ or $\vec{E_1}$ and $\vec{D_2}$ or $\vec{E_2}$ making angles θ_1 and θ_2 with the normal to the interface as shown in fig.

Using eq. ① we have

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

or $$E_1 \sin\theta_1 = E_2 \sin\theta_2 \quad \text{— ⓐ}$$

Similarly, by applying $D_{1n} = D_{2n}$ and $\vec{D} = \varepsilon\vec{E}$, we get

$$\varepsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos\theta_2$$

or $$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \quad \text{— ⓑ}$$

Dividing eq. ⓐ by eq. ⓑ gives

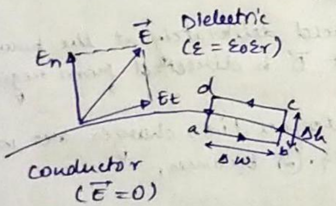
$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

Since ~~or~~ $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$

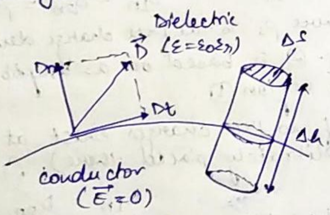
$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \quad \text{— ④}$$

This is the law of refraction of the electric field at a boundary free of charge ($\because \rho_s = 0$).

→ Conductor - Dielectric boundary condition



(a)



(b)

The conductor is assumed to be perfect (i.e., $\sigma \to \infty$ or $\rho_c \to 0$).

we may regard, copper and silver as though they were perfect conductors.

We have, $\vec{E} = 0$ inside the conductor.

Applying eq. ✱ to the closed path abcda of fig. (a)

$$\oint_L \vec{E} \cdot \vec{dl} = 0$$

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$, $\quad E_t = 0$ — ①

Similarly, by applying ✱✱ to the cylindrical pillbox of fig. (b) and letting $\Delta h \to 0$, we get,

$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S$$

because, $\vec{D} = \varepsilon\vec{E} = 0$, inside the conductor.

$$D_n = \frac{\Delta Q}{\Delta S} = \rho_s$$

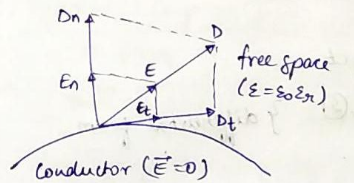
or $$D_n = \rho_s \quad \text{— ②}$$

Conclusions, about a perfect conductor, under static conditions;

1) No electric field may exist within a conductor
$$\rho_v = 0, \ \vec{E} = 0 \quad \text{— ③}$$
2) Since $\vec{E} = -\nabla V = 0$, there can be no potential difference between any two points in the conductor, i.e., a conductor is an equipotential body.
3) An electric field $\vec{E}$ must be external to the conductor and must be normal to its surface, i.e.,
$$D_t = \varepsilon_0\varepsilon_r E_t = 0, \quad D_n = \varepsilon_0\varepsilon_r E_n = \rho_s \quad \text{— ④}$$

An important application of the fact that $\vec{E} = 0$ inside a conductor is in electrostatic screening or shielding.

→ Conductor - Free space boundary conditions



It is a special case of conductor-dielectric conditions.

The boundary conditions at the interface between a conductor and free space can be obtained from eq. ④ by replacing $\varepsilon_r = 1$ (because free space may be regarded as a special dielectric for which $\varepsilon_r = 1$).

The electric field ~~$\vec{E}$~~ $\vec{E}$ must be external to the conductor and normal to its surface. Thus the boundary conditions are

$$D_t = \varepsilon_0 E_t = 0, \quad D_n = \varepsilon_0 E_n = \rho_s \quad \text{— ①}$$

Q! Two extensive homogeneous isotropic dielectric meet on plane $z=0$. For $z>0$, $\varepsilon_{r1} = 4$ and for $z<0$, $\varepsilon_{r2} = 3$. A uniform electric field $\vec{E_1} = 5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z$ kV/m exists for $z \geq 0$. Find.

(a) $\vec{E_2}$ for $z \leq 0$
(b) The angles $\vec{E_1}$ and $\vec{E_2}$ make with the interface.

Ans! Example 5.4 [Sadiku] page 204.

Q: A homogeneous dielectric ($\varepsilon_r = 2.5$) fills region 1 ($x<0$) while region 2 ($x>0$) is free space.

(a) If $\vec{D}_1 = 12\hat{a}_x - 10\hat{a}_y + 4\hat{a}_z$ nC/m², find $\vec{D}_2$ and θ_2.

(b) If $E_2 = 12$ V/m and $\theta_2 = 60°$, find E_1 and θ_1. Take θ_1 and θ_2 as defined in previous question. [Example 5.9]

Ans: Practice Exercise 5.9 [Sadiku] page 206.

Magnetostatic Fields

- Ampere's circuit law - Maxwell's equation

It states that the line integral of $\vec{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of $\vec{H}$ equals I_{enc}; that is,

$$\oint_L \vec{H}\cdot d\vec{l} = I_{enc}. \quad \text{--- (1)} \quad \} \text{ integral form}$$

Ampere's law is similar to Gauss's law, since ampere's law is easily applied to determine $\vec{H}$ when the current distribution is symmetrical.

Eq. (1) always holds regardless of whether the current distribution is symmetrical or not.

Ampere's law is a special case of Biot-Savart's law; the former may be derived from the latter.

By applying Stoke's theorem to the left-hand side of eq. (1), we obtain

$$I_{enc} = \oint_L \vec{H}\cdot d\vec{l} = \int_S (\nabla\times\vec{H})\cdot d\vec{S}$$

But
$$I_{enc} = \int_S \vec{J}\cdot d\vec{S}$$

Comparing the surface integrals, we get

$$\nabla\times\vec{H} = \vec{J} \quad \text{--- (2)} \quad \} \text{ differential form}$$

This is the third Maxwell's equation.

From eq. (2), we should observe that $\nabla\times\vec{H} = \vec{J} \neq 0$; i.e., a magnetostatic field is not conservative.

- Magnetic flux density - Maxwell's equation

The magnetic flux density $\vec{B}$ is similar to the electric flux density $\vec{D}$. As $\vec{D} = \varepsilon\vec{E}$ in free space, the magnetic flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ according to,

$$\vec{B} = \mu_0\vec{H}$$

where μ_0 is a constant known as the permeability of free space.

$$\mu_0 = 4\pi\times10^{-7} \text{ H/m}$$

The magnetic flux through a surface S is given by

$$\Psi = \int_S \vec{B}\cdot d\vec{S}$$

where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density $\vec{B}$ is in webers per square meter (Wb/m²) or Teslas (T).

An isolated magnetic charge does not exist. (division of bar magnet results in pieces with north and south poles).

Thus the total flux through a closed surface in a magnetic field must be zero; i.e.,

$$\oint_S \vec{B}\cdot d\vec{S} = 0 \quad \text{--- (1)}$$

This equation is referred to as the law of conservation of magnetic flux or Gauss's Law for magnetostatic fields.

magnetostatic field is not conservative, magnetic flux is conserved.

Applying divergence theorem to above equation, we obtain

$$\oint_S \vec{B}\cdot d\vec{S} = \int_v \nabla\cdot\vec{B}\, dv = 0$$

or
$$\nabla\cdot B = 0 \quad \text{--- (2)}$$

This is the fourth Maxwell's equation.

Eq. (1) and (2) shows that magnetostatic fields have no sources or sinks. Eq. (2) suggests that magnetic field lines are always continuous.

- Maxwell's equations for static fields (electric and magnetic)

Differential (or point) form	Integral form	Remarks
$\nabla\cdot\vec{D} = \rho_v$	$\oint_S \vec{D}\cdot d\vec{S} = \int_v \rho_v\, dv$	Gauss's law
$\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{S} = 0$	Nonexistence of magnetic monopoles
$\nabla\times\vec{E} = 0$	$\oint_L \vec{E}\cdot d\vec{l} = 0$	Conservative nature of electrostatic field.
$\nabla\times\vec{H} = \vec{J}$	$\oint_L \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S}$	Ampere's law

- Magnetic scalar and vector potentials

The magnetic potential could be scalar V_m or vector $\vec{A}$.

Just as $\vec{E} = -\nabla V$, we define the magnetic scalar potential V_m (in amperes) as related to $\vec{H}$ according to

$$\vec{H} = -\nabla V_m \quad \text{--- (1) if } \vec{J} = 0$$

combining eq. ① and $\nabla \times \vec{H} = \vec{J}$, gives

$$\vec{J} = \nabla \times \vec{H} = \nabla \times (-\nabla V_m) = 0$$

since V_m must satisfy the condition $\nabla \times \nabla V = 0$. Thus the magnetic scalar potential V_m is only defined in a region where $\vec{J} = 0$ as in eq. ①.

V_m also satisfies Laplace's equation just as V does for electrostatic fields, hence,

$$\nabla^2 V_m = 0, \quad (\vec{J} = 0)$$

We know that for a magnetostatic field, $\nabla \cdot \vec{B} = 0$. To satisfy equation $\nabla \cdot \vec{B} = 0$ and $\nabla \cdot (\nabla \times \vec{A}) = 0$, simultaneously, we can define the magnetic vector potential $\vec{A}$ (in Wb/m) such that

$$\vec{B} = \nabla \times \vec{A} \quad \text{—— ②}$$

Just as we defined

$$V = \int \frac{dQ}{4\pi \varepsilon_0 R}$$

we can define

$$\vec{A} = \int_L \frac{\mu_0 I \, d\vec{l}}{4\pi R} \quad \text{for line current}$$

$$\vec{A} = \int_S \frac{\mu_0 \vec{K} \, d\vec{S}}{4\pi R} \quad \text{for surface current}$$

$$\vec{A} = \int_v \frac{\mu_0 \vec{J} \, dv}{4\pi R} \quad \text{for volume current}$$

By substituting ② into $\Psi = \int_S \vec{B} \cdot d\vec{S}$ and applying Stoke's theorem, we obtain,

$$\Psi = \int_S \vec{B} \cdot d\vec{S} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_L \vec{A} \cdot d\vec{l}$$

or

$$\Psi = \oint_L \vec{A} \cdot d\vec{l}$$

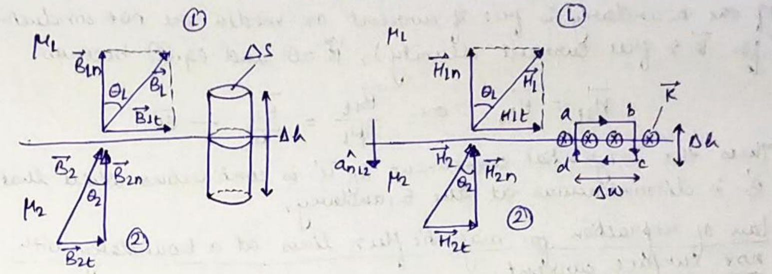
- <u>Magnetic Boundary Conditions</u>

Magnetic boundary conditions are the conditions that $\vec{H}$ (or $\vec{B}$) field must satisfy at the boundary between two different media. We make use of Gauss's law for magnetic fields,

$$\oint_S \vec{B} \cdot d\vec{S} = 0 \quad \text{—— (*)}$$

and Ampere's circuit law

$$\oint_L \vec{H} \cdot d\vec{l} = \vec{J} \quad \text{—— (**)}$$



(a) (b)

Applying eq. (*) to the pillbox (Gaussian surface) of fig (a) and allowing $\Delta h \to 0$, we obtain,

$$B_{1n} \Delta S - B_{2n} \Delta S = 0$$

Thus

$$\vec{B}_{1n} = \vec{B}_{2n} \quad \text{or} \quad \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n} \quad \text{—— ①}$$

since $\vec{B} = \mu \vec{H}$.

Eq. ① shows that the normal component of $\vec{B}$ is continuous at the boundary. It also shows that the normal component of $\vec{H}$ is discontinuous at the boundary; $\vec{H}$ undergoes some change at the interface.

Similarly, we apply eq. (**) to the closed path abcda of fig (b), where surface current $\vec{K}$ on the boundary is assumed normal to the path. we obtain,

$$K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} - H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$, this eq. leads to

$$H_{1t} - H_{2t} = K \quad \text{—— ②}$$

This shows that tangential component of $\vec{H}$ is also discontinuous at the boundary. Eq. ② can be written in terms of $\vec{B}$ as,

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K \quad \text{—— ③}$$

In the general case eq. ③ becomes,

$$(\vec{H}_1 - \vec{H}_2) \times \hat{a}_{n12} = \vec{K}$$

where $\hat{a}_{n12}$ is a unit vector normal to the interface and is directed from medium ① to medium ②.

If the boundary is free of current or media are not conductors (for $\vec{K}$ is free current density), $\vec{K} = 0$ and eq. ② becomes

$$\vec{H}_{1t} = \vec{H}_{2t} \quad \text{or} \quad \frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2} \quad \text{— ④}$$

Thus the tangential component of $\vec{H}$ is continuous while that of $\vec{B}$ is discontinuous at the boundary.

- Law of refraction for magnetic flux lines at a boundary with ~~no~~ surface current.

If the fields make an angle θ with the normal to the interface eq. ① results in

$$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2 \quad \text{— (a)}$$

while eq. ④ produces

$$\frac{B_1}{\mu_1} \sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2} \sin\theta_2 \quad \text{— (b)}$$

Dividing eq. (b) ~~an~~ by eq. (a) gives,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

- Maxwell's equations in final forms

Differential form	Integral form	Remarks
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho_v \, dv$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	Nonexistence of isolated magnetic charge or Gauss's law for magnetic fields.
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S}$	Faraday's law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S}$	Ampere's circuit law