Analytical Investigations in Rate-Induced Tipping
IPC Handout

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1 Mathematical Derivations

Consider a dynamical system with a time varying parameter $p(t)$.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}f_1(x_1(t), x_2(t), x_3(t), p(t))\\ f_2(x_1(t), x_2(t), x_3(t), p(t))\\ f_3(x_1(t), x_2(t), x_3(t), p(t))\end{bmatrix} \tag{1}$$

Note: The equations we derive will be valid for any N-dimensional dynamical system but for simplicity we will go through the process for a 3-dimensional system.

Note: There may be other parameters in the system but for now we only vary one of them and keep the others constant. The final equations can be appropriately altered to consider multiple parameters varying simultaneously.

Let the current time instant be t. Say the system has a fixed point. We denote it by $X^*(t) = [x_1^*(t), x_2^*(t), x_3^*(t)]^T$.

Note: The fixed point may be stable or unstable.

At the fixed point we have,

$$f_i(x_1^*(t), x_2^*(t), x_3^*(t), p(t)) = 0 \quad \text{for i = 1 to 3} \tag{2}$$

Let us perturb the system from this fixed point. Let the perturbations be $[x_1'(t), x_2'(t), x_3'(t)]^T$.

Assumption: Assuming small perturbations and linearizing the dynamics.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3}\\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}\end{bmatrix}^* \begin{bmatrix}x_1'(t)\\ x_2'(t)\\ x_3'(t)\end{bmatrix} \tag{3}$$

Assumption: For studying the effect of rate in the scenario where it doesn't affect the stability through the Jacobian matrix we want the Jacobian matrix to be independent of the parameter p. That is we need $\frac{\partial f_i}{\partial x_j}$ to not be a function of p. This will happen if and only if,

$$f_i(x_1(t), x_2(t), x_3(t), p(t)) = g_i(x_1(t), x_2(t), x_3(t)) + h_i(p(t)) \quad \text{for i = 1 to 3} \tag{4}$$

Note: Even if the parameter is involved in the Jacobian or if the functions f_i's cannot be written as in Eq.4 the final results will still hold but for now we make it independent of the parameter for simplicity.

Therefore, we can use Eq.4 to reduce Eq.3 to the following,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{bmatrix} \tag{5}$$

Denote:

$$G^*(t) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{bmatrix}, \quad X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}, \quad X'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{bmatrix}$$

We have,

$$\dot{X}(t) = G^*(t)X'(t) \tag{6}$$

Consider a time step Δt.

$$X(t + \Delta t) = X(t) + \dot{X}(t)\Delta t$$

Using Eq.6 we have,

$$X(t + \Delta t) = X(t) + G^*(t)X'(t)\Delta t \tag{7}$$

Note: This relationship is approximate. But if $\Delta t \to dt$ then it will be exact. We will take this limit finally, but for now we work with the Δt.

Using Eq.7 we know how $X(t)$ evolves. We will now find out how $X^*(t)$ evolves which is required as we are interested in $X'(t)$, which is by definition:

$$X'(t) = X(t) - X^*(t) \tag{8}$$

Making use of Eq.4 in Eq.2 we will get,

$$g_i(x_1^*(t), x_2^*(t), x_3^*(t)) + h_i(p(t)) = 0 \quad \text{for i = 1 to 3} \tag{9}$$

We will use these equations to find out the evolution of $X^*(t)$.

Assumption: $p(t) = p_0 + \mu t$. This assumption can be relaxed to have $\frac{dp}{dt} = \mu(t)$ but for simplicity we use this for demonstration.

Considering g_1 and h_1 at the fixed point state at time instants t and $t + \Delta t$ we have,

$$g_1(x_1^*(t), x_2^*(t), x_3^*(t)) + h_1(p(t)) = 0 \tag{10a}$$
$$g_1(x_1^*(t + \Delta t), x_2^*(t + \Delta t), x_3^*(t + \Delta t)) + h_1(p(t + \Delta t)) = 0 \tag{10b}$$

Using first-order approximation to the taylor series expansion repeatedly for h_1,

$$h_1(p(t + \Delta t)) = h_1\left(p(t) + \frac{\partial p}{\partial t}\Delta t\right) = h_1(p(t) + \mu\Delta t)$$

$$h_1(p(t) + \mu\Delta t) = h_1(p(t)) + \left.\frac{\partial h_1}{\partial p}\right|_{p(t)} \mu\Delta t \tag{11}$$

Following a similar procedure for g_1,

$$\begin{aligned}
&g_1(x_1^*(t+\Delta t), x_2^*(t+\Delta t), x_3^*(t+\Delta t)) \\
&= g_1(x_1^*(t) + \frac{\partial x_1^*}{\partial t}\Delta t, x_2^*(t) + \frac{\partial x_2^*}{\partial t}\Delta t, x_3^*(t) + \frac{\partial x_3^*}{\partial t}\Delta t) \\
&= g_1(x_1^*(t), x_2^*(t), x_3^*(t)) + \frac{\partial g_1}{\partial x_1^*}\frac{\partial x_1^*}{\partial t}\Delta t + \frac{\partial g_1}{\partial x_2^*}\frac{\partial x_2^*}{\partial t}\Delta t + \frac{\partial g_1}{\partial x_3^*}\frac{\partial x_3^*}{\partial t}\Delta t \\
&= g_1(x_1^*(t), x_2^*(t), x_3^*(t)) + \begin{bmatrix} \frac{\partial g_1}{\partial x_1^*} & \frac{\partial g_1}{\partial x_2^*} & \frac{\partial g_1}{\partial x_3^*} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1^*}{\partial t} \\ \frac{\partial x_2^*}{\partial t} \\ \frac{\partial x_3^*}{\partial t} \end{bmatrix} \Delta t
\end{aligned} \tag{12}$$

Using Eq.11 and Eq.12 in Eq.10b and using Eq.10a we get,

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1^*} & \frac{\partial g_1}{\partial x_2^*} & \frac{\partial g_1}{\partial x_3^*} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1^*}{\partial t} \\ \frac{\partial x_2^*}{\partial t} \\ \frac{\partial x_3^*}{\partial t} \end{bmatrix} + \frac{\partial h_1}{\partial p}\mu = 0$$

Combining all three equations we will get,

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1^*} & \frac{\partial g_1}{\partial x_2^*} & \frac{\partial g_1}{\partial x_3^*} \\ \frac{\partial g_2}{\partial x_1^*} & \frac{\partial g_2}{\partial x_2^*} & \frac{\partial g_2}{\partial x_3^*} \\ \frac{\partial g_3}{\partial x_1^*} & \frac{\partial g_3}{\partial x_2^*} & \frac{\partial g_3}{\partial x_3^*} \end{bmatrix}^* \begin{bmatrix} \frac{\partial x_1^*}{\partial t} \\ \frac{\partial x_2^*}{\partial t} \\ \frac{\partial x_3^*}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial p} \\ \frac{\partial h_2}{\partial p} \\ \frac{\partial h_3}{\partial p} \end{bmatrix}\Bigg|_{p(t)} \mu = 0 \tag{13}$$

Denoting,

$$H^*(t) = \begin{bmatrix} \frac{\partial h_1}{\partial p} \\ \frac{\partial h_2}{\partial p} \\ \frac{\partial h_3}{\partial p} \end{bmatrix}\Bigg|_{p(t)}$$

Eq.13 can be written as,

$$G^*(t)\frac{\partial X^*}{\partial t} + H^*(t)\mu = 0$$

From which we can get,

$$\frac{\partial X^*}{\partial t} = -[G^*(t)]^{-1}H^*(t)\mu \tag{14}$$

Defining,

$$A^*(t) = -[G^*(t)]^{-1}H^*(t)$$

That is the rate of motion of the fixed point is given by,

$$\frac{\partial X^*}{\partial t} = A^*(t)\mu \tag{15}$$

For a time step Δt we can write,

$$X^*(t+\Delta t) = X^*(t) + \frac{\partial X^*}{\partial t}\Delta t$$

Using Eq.15 we have,

$$X^*(t+\Delta t) = X^*(t) + \mu A^*(t)\Delta t \tag{16}$$

From the definition of $X'(t)$

$$\begin{aligned} X'(t+\Delta t) &= X(t+\Delta t) - X^*(t+\Delta t) \\ &= X(t) + G^*(t)X'(t)\Delta t - X^*(t) - \mu A^*(t)\Delta t \\ &= X'(t) + [G^*(t)X'(t) - \mu A^*(t)]\Delta t \end{aligned} \tag{17}$$

2 Methods of Attack

2.1 Analyzing the norm

Taking the transpose of Eq.17 and multiplying with itself to get the norm of $X'(t)$ and neglecting the higher order terms in powers of Δt,

$$\begin{aligned} \|X'(t+\Delta t)\| &= \{X'(t)^T + [X'(t)^T G^*(t)^T - \mu A^*(t)^T]\Delta t\} \times \{X'(t) + [G^*(t)X'(t) - \mu A^*(t)]\Delta t\} \\ &= \|X'(t)\| + [X'(t)^T G^*(t)^T - \mu A^*(t)^T]X'(t)\Delta t + X'(t)^T[G^*(t)X'(t) - \mu A^*(t)]\Delta t \end{aligned}$$

$$\begin{aligned} \frac{\|X'(t+\Delta t)\| - \|X'(t)\|}{\Delta t} &= X'(t)^T G^*(t)^T X'(t) - \mu A^*(t)^T X'(t) \\ &\quad + X'(t)^T G^*(t) X'(t) - \mu X'(t)^T A^*(t) \\ &= X'(t)^T [G^*(t)^T + G^*(t)] X'(t) - 2\mu A^*(t)^T X'(t) \end{aligned} \tag{18}$$

If $S^*(t)$ is the symmetric part of $G^*(t)$ then,

$$G^*(t)^T + G^*(t) = 2S^*(t)$$

Therefore we get,

$$\frac{\|X'(t+\Delta t)\| - \|X'(t)\|}{\Delta t} = 2X'(t)^T S^*(t) X'(t) - 2\mu A^*(t)^T X'(t) \tag{19}$$

If the eigenvalues of $S^*(t)$ are all negative then it is a **negative-definite** matrix and $x^T S^*(t) x < 0 \ \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. This would imply that the first term in the RHS of Eq.19 is always negative and contributes to a decrement in the norm. The second term will therefore decide whether the norm increases or decreases.

<u>Note:</u> If $G^*(t)$ has all eigenvalues negative it does not imply that $S^*(t)$ will have all eigenvalues negative as well.

2.2 Differential equation for the error

From Eq.17 we have,

$$X'(t+\Delta t) = X'(t) + [G^*(t)X'(t) - \mu A^*(t)]\Delta t$$
$$\frac{X'(t+\Delta t) - X'(t)}{\Delta t} = G^*(t)X'(t) - \mu A^*(t)$$

$$\dot{X}'(t) = G^*(t)X'(t) - \mu A^*(t) \tag{20}$$