Probabilistic Modeling and Bayesian NEtworks

Probability Distributions

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There are many distributions available; for each application we have to select the most appropriate

DOMAIN

- Categorical, Ordinal or Continuous
- Bounded or Unbounded

SHAPE

- Parameter-dependant
- Represented by the **moments**
- Best known: mean and variance

Moments Around 0

$$\mu'_k = E[X^k] = \int_{-\infty}^{\infty} x^k f(x)dx$$

Moments Around Mean

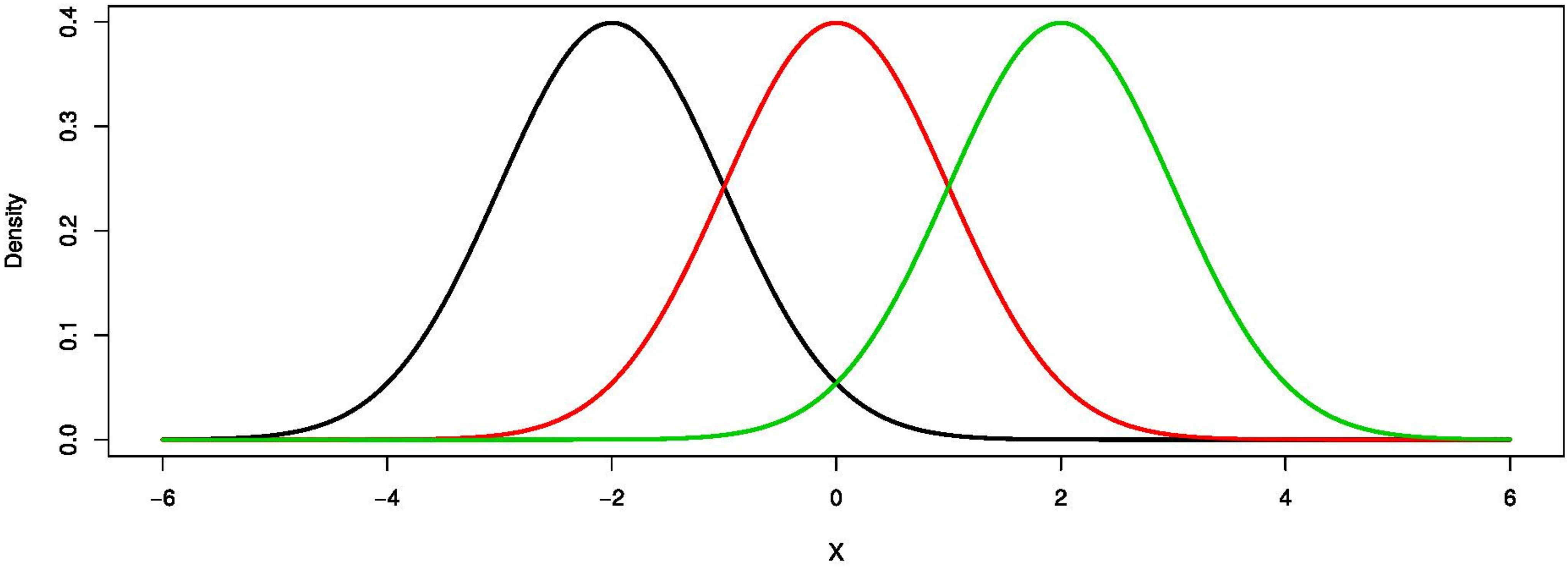
$$\mu_k = E[(X - \mu'_1)^k] = \int_{-\infty}^{\infty} (x - \mu'_1)^k f(x)dx$$

Sample Moments

$$m'_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k \qquad m_k = \frac{1}{n} \sum_{i=1}^{n} (x_i - m'_i)^k$$

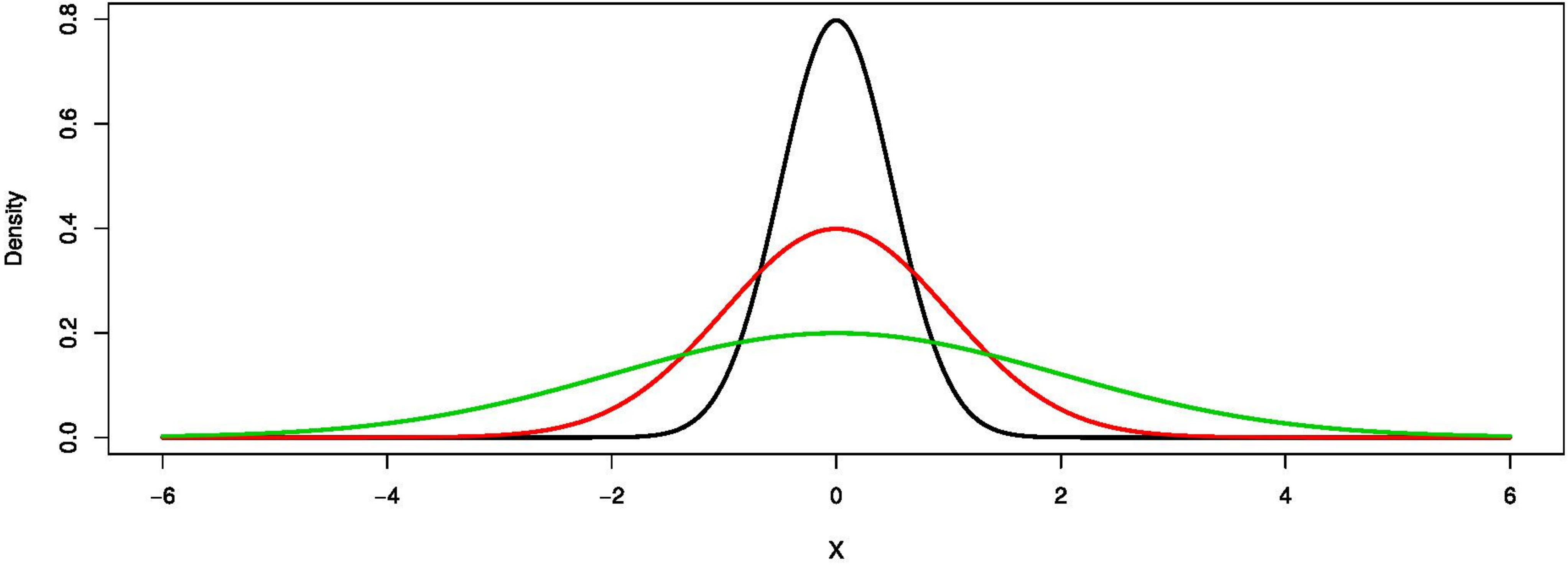
1st Moment: Mean

Information about the **LOCATION**



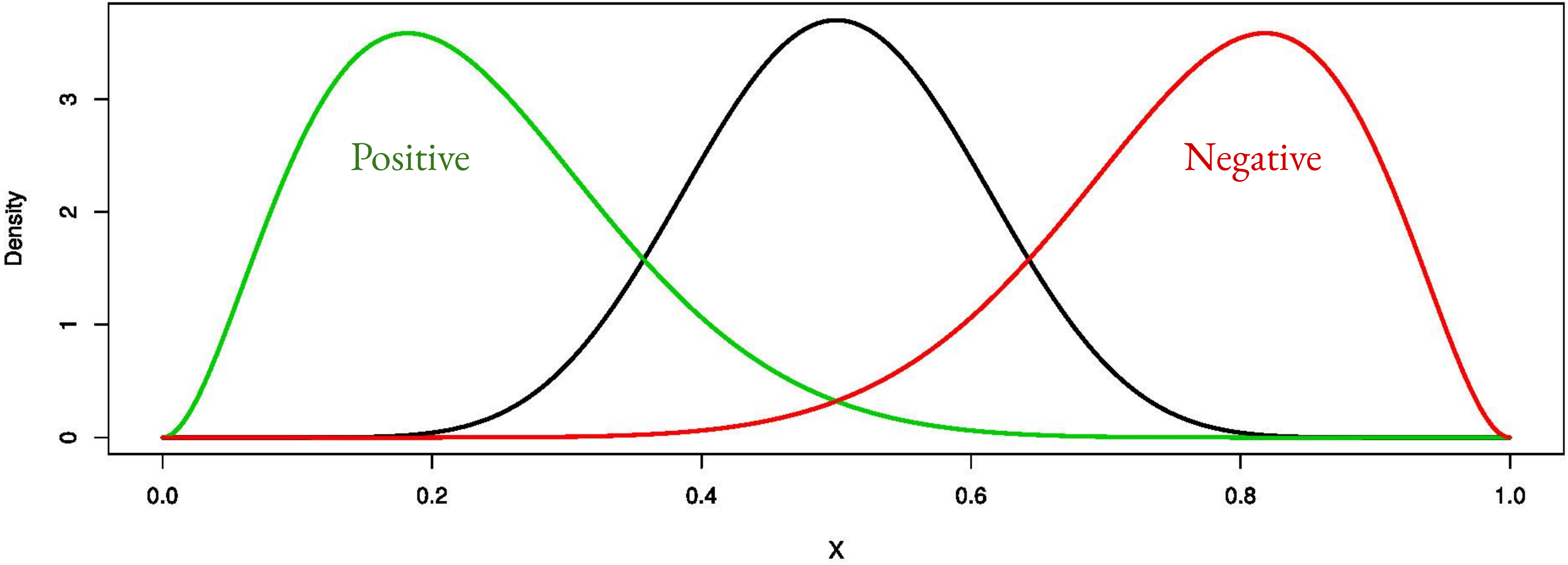
2^{ND} MOMENT: VARIANCE

Information about the **SPREADNESS**



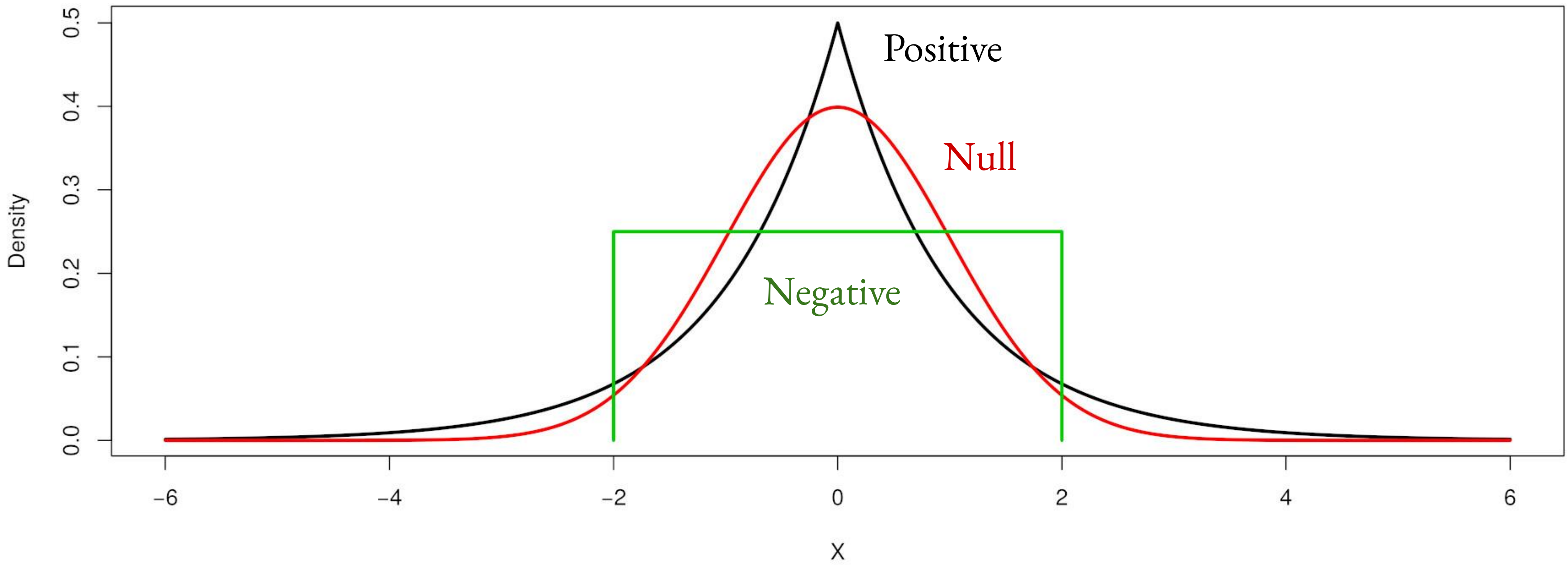
3^{RD} Moment: Skewness

Information about the **SYMMETRY**



4^{TH} MOMENT: KURTOSIS

Information about the tendency to produce **OUTLIERS**



Some Useful Properties

Expectation

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Variance

$$VAR(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

$$VAR(c) = 0$$

Probability Distributions

Discrete Variables

Multinomial Distribution

Probability distribution associated to categorical variables (discrete values with no particular order)

Domain $X \in \{1, \ldots, r\}$

Parameters $0 \leq p_1, \ldots, p_r \leq 1; \sum_{i=1}^{r} p_i = 1$

Function $P(X = x_i) = p_i$

Special Cases Bernouilli distribution (r=2)
Uniform distribution

Binomial Distribution

In a sample of size n drawn from a Bernouilli distribution, the binomial counts the number of outcomes of a certain type

Domain $X \in \{0, 1, \ldots, n\}$

Parameters $n \geq 0; 0 \leq p \leq 1$

Function $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

Moments $\mu_1' = np; \mu_2 = np(1 - p)$

Hypergeometric Distribution

In a set of N elements there are M of a special type. If we draw n elements without replacement, this variable counts the number of elements of the special type

Domain $$max\{0, M - N + n\} \leq X \leq min\{n, N\}$$

Parameters $$N \geq 0; 0 \leq M \leq N; 0 \leq n \leq N$$

Function $$P(X = k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

Moments $$\mu_1' = \frac{nM}{N}; \mu_2 = \frac{nM}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1}$$

Poisson Distribution

This distribution counts the number of events that happen in a certain period of time as long a the event rate is constant and events are independent

Domain $X \in \{0, 1, 2, \ldots\}$

Parameters $\lambda > 0$

Function $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

Moments $\mu_1' = \lambda; \mu_2 = \lambda$

Probability Distributions

Continuous Variables

Normal Distribution

This distributions represents many natural situations. In particular, it represents the distribution of the sum of random iid samples.

Domain $X \in \mathbb{R}$

Parameters $\mu \in \mathbb{R}; \sigma > 0$

Function $f(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Moments $\mu_1' = \mu; \mu_2 = \sigma^2$

Gamma Distribution

Given a Poisson distribution, the time until the a-th event happens is represented by this type of distribution

Domain	$X > 0$
Parameters	$a > 0; b > 0$
Function	$f(X = x) = \frac{1}{\Gamma(a)b^a} e^{-\frac{x}{b}} x^{(a-1)}$
Moments	$\mu_1' = ab; \mu_2 = ab^2$

Beta Distribution

This distribution represents the ratio between two Gamma variables. Its main interest is that it is bounded between 0 and 1

Domain

$$0 \leq X \leq 1$$

Parameters

$$a > 0; b > 0$$

Function

$$f(X = x) = \frac{1}{Beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Moments

$$\mu_1' = \frac{\alpha}{\alpha+\beta}; \mu_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Chi Square Distribution

This distribution represents the sum of n Gaussian distributed variables with mean 0 and variance 1. It is used to model variance. It is also used in some statistical tests

Domain	$X > 0$
Parameters	$n > 0$
Function	$f(X = x) = \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{n/2-1}$
Moments	$\mu_1' = n; \mu_2 = 2n$

Probability Distributions

Random vectors

Bivariate Normal Distribution

Extension of the Gaussian distribution to two dimensions

Domain $(X, Y) \in \mathbb{R}^2$

Parameters $\mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$

Function

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-a}$$

$$a = -\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

The marginals of the bivariate distribution are Gaussian distributions

Joint Moments, bivariate case

First Order

$$E[X] = \int_{\Omega_X} \int_{\Omega_Y} x f(x,y) dx dy \qquad E[Y] = \int_{\Omega_X} \int_{\Omega_Y} y f(x,y) dx dy$$

Second order

$$E[X^2] = \int_{\Omega_X} \int_{\Omega_Y} x^2 f(x,y) dx dy \qquad E[Y^2] = \int_{\Omega_X} \int_{\Omega_Y} y^2 f(x,y) dx dy$$

$$E[XY] = \int_{\Omega_X} \int_{\Omega_Y} xy f(x,y) dx dy$$

Second order around the mean: Covariance

$$E[(X - E[X])(Y - E[Y])] = \int_{\Omega_X} \int_{\Omega_Y} (x - E[X])(y - E[Y]) f(x,y) dx dy$$

Useful properties

Variance of Sum

$$VAR(aX + bY + c) = a^2VAR(X]) + b^2VAR(Y) + 2abCOV(X, Y)$$

Correlation

$$\rho_{XY} = \frac{COV(X,Y)}{\sqrt{VAR(X)VAR(Y)}}$$

Independent variables

$$COV(X, Y) = 0 \rightarrow \rho_{XY} = 0$$

Dirichlet Distribution

Extension of the Beta distribution to two or more dimensions

Domain $(X_1, \ldots, X_n) \in [0,1]^n; \sum_i x_i = 1$

Parameters $(\alpha_1, \ldots, \alpha_n); \alpha_i > 0$

Function $f(x_1, \ldots, x_n) = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \prod_i x_i^{\alpha_i - 1}$

The marginals of the Dirichlet distribution are Beta distributions