VARIABLE TRANSFORMATION

One variable

EXAMPLE

$$X \in \Omega_X = \{1, \ldots, 6\}$$

$$\forall x \in \Omega_X, P(X = x) = \frac{1}{6}$$

We have a dice, and we define:

$$Y = g(X) = |X - 3|$$

What's the distribution of *Y*?

$$P(Y = y) = \sum_{x \in \Omega_X / x = g^{-1}(y)} P(X = x)$$

CONTINUOUS VARIABLES

Suppose *Y* is an **increasing** function of *X*, then ...

$$F(Y = y) = P(Y \leq y) = P(g(X) \leq y) =$$
$$= P(X \leq g^{-1}(y)) = F(X = g^{-1}(y))$$

Taking the derivative

$$f(Y = y) = \frac{d}{dy} F(X = g^{-1}(y)) = f(X = g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

Similar for the decreasing case. In general:

$$f(Y = y) = f(X = g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

CONTINUOUS VARIABLES: EXAMPLE

$$X \sim Exp(\lambda) \rightarrow f_X(x) = \lambda e^{-\lambda x}$$

$$y = g(x) = kX,\ f_Y(y)??$$

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

$$g^{-1}(y) = \frac{y}{k},\ \frac{d}{dy}g^{-1}(y) = \frac{1}{k}$$

$$f_Y(y) = \lambda e^{-\lambda\frac{y}{x}}\frac{1}{k} = \frac{\lambda}{k}e^{-\frac{\lambda}{k}y}$$

Therefore:

$$X \sim Exp(\lambda) \rightarrow kX \sim Exp(\frac{\lambda}{k})$$

TWO VARIABLE CASE

$$f_{X,Y}(x,y)$$

$$Z = g_Z(X,Y);\ T = g_T(X,Y)$$

$$X = l_X(Z,T);\ Y = l_Y(Z,T)$$

$$|J| = \left|\frac{\partial(X,Y)}{\partial(Z,T)}\right| = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial T} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial T} \end{vmatrix}$$

$$f_{Z,T}(z,t) = f_{X,Y}(l_X(z,t), l_Y(z,t))||J||$$

Two variables: Example

$$X \sim Exp(\lambda_1) \rightarrow f_X(x) = \lambda_1 e^{-\lambda_1 x}$$

$$Y \sim Exp(\lambda_2) \rightarrow f_Y(y) = \lambda_2 e^{-\lambda_2 y}$$

$$X \perp Y$$

$$f_{XY}(x, y) = \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}$$

$$Z = X + Y \rightarrow f_Z(z) = ?$$

$$Z = X + Y$$
$$T = X$$

$$X = T$$
$$Y = Z - T$$

$$||J|| = 1$$

$$f_{ZT}(z, t) = f_{XY}(t, z - t)||J|| = \lambda_1 \lambda_2 e^{-\lambda_1 t - \lambda_2 (z - t)}$$

$$= \lambda_1 \lambda_2 e^{-(\lambda_1 - \lambda_2) t} e^{-\lambda_2 z}$$

TWO VARIABLES: EXAMPLE

$Z = X + Y \rightarrow f_Z(z) = ?$

$$\begin{aligned} Z &= X + Y \\ T &= X \end{aligned} \qquad \begin{aligned} X &= T \\ Y &= Z - T \end{aligned}$$

$f_{ZT}(z,t) = \lambda_1 \lambda_2 e^{-(\lambda_1 - \lambda_2)t} e^{-\lambda_2 z}$

$Y \geq 0 \rightarrow Z \geq T \rightarrow 0 \leq t \leq z$

$f_Z(z) = \int_{\Omega_T} f_{ZT}(z,t)dt$

$f_Z(z) = \int_0^z \lambda_1 \lambda_2 e^{-\lambda_2 z} e^{-(\lambda_1 - \lambda_2)t} dt \; = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}(e^{-\lambda_1 z} - e^{-\lambda_2 z})$

Hypoexponential distribution

$$E[X+Y] = E[Z] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

SAMPLING BASED APPROXIMATIONS

Monte Carlo Methods

APPROACHING THE EXPECTATION

$$X \sim Exp(\lambda_1) \rightarrow f_X(x) = \lambda_1 e^{-\lambda_1 x} \qquad X \perp Y$$
$$Y \sim Exp(\lambda_2) \rightarrow f_Y(y) = \lambda_2 e^{-\lambda_2 y} \qquad E[X+Y] =?$$

1.- Sample X and Y

2.- For each sample, get Z=X+Y

3.- Approach E[Z] with the sample mean

EXAMPLE: $\lambda_1 = 3, \lambda_2 = 2; E[X+Y] = 0.8\hat{3}$

10 samples → 1.168

100 samples → 0.850

1000 samples → 0.816

10000 samples → 0.835

100000 samples → 0.833

1000000 samples → 0.832

Sampling based approaches are great …

… if we can sample the distributions!

$$f(x) = e^{e^{-(x-\mu)/\beta}}$$

Sampling general Distributions

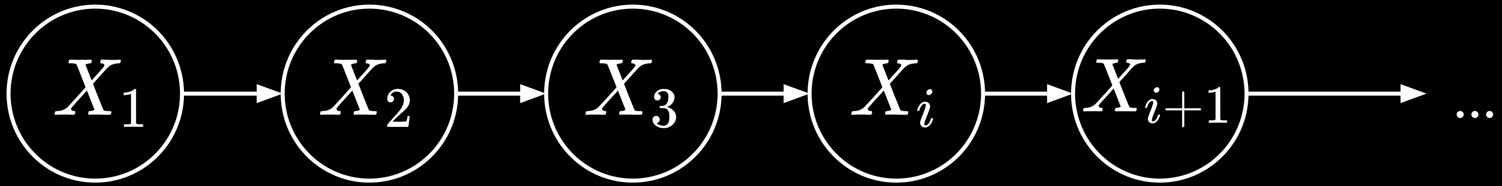
Markov Chain Monte Carlo

MARKOVIAN PROPERTY

The conditional probability distribution of **future states** of the process **depends only upon** the **present state**

MARKOV CHAIN

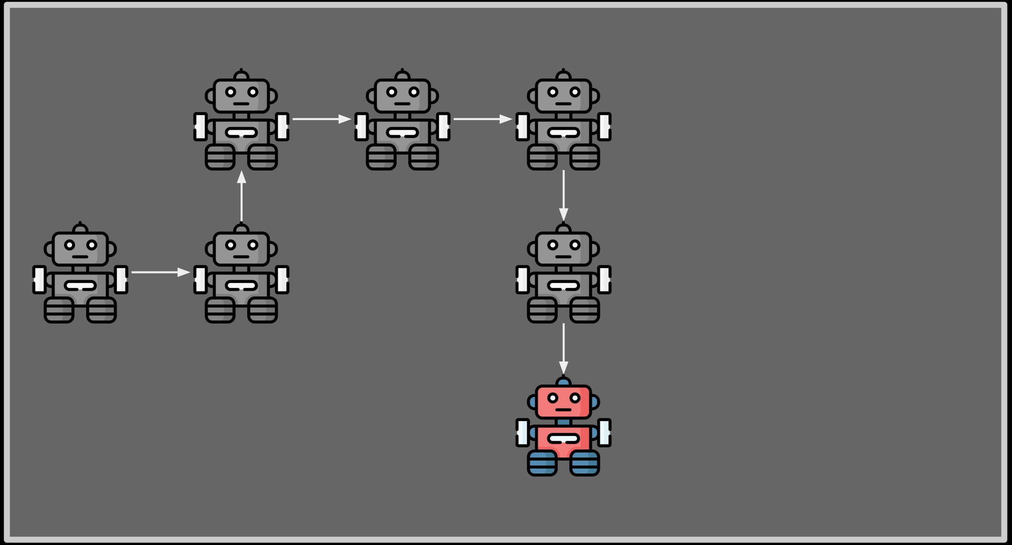
Stochastic process → Infinite sequence of random variables



Due to the Markovian property:

$$\forall i \;\; P(X_{i+1} | X_1, X_2 \ldots, X_i) = P(X_{i+1} | X_i)$$

MARKOV CHAIN: DISCRETE EXAMPLE



Random movement in discrete steps, according to:

$$X_i \in \Omega_X = \{\uparrow, \downarrow, \leftarrow, \rightarrow\}$$

$$\forall a, b \in \Omega_X \;\; P(X_{i+1} = a | X_i = b)$$

X_{i+1}

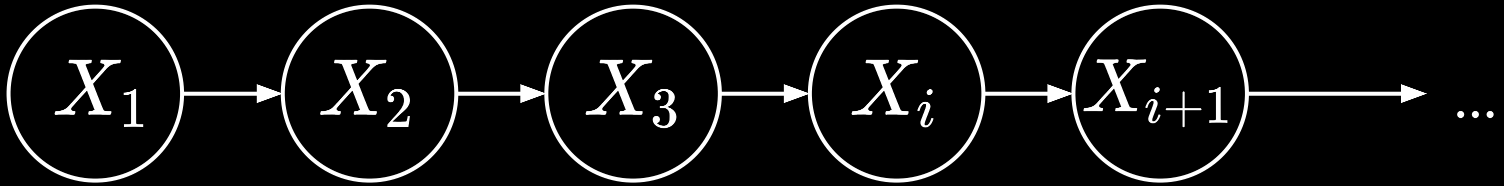
X_i

	↑	↓	←	→
↑	0.7	0.1	0.1	0.1
↓	0.1	0.7	0.1	0.1
←	0.1	0.1	0.7	0.1
→	0.1	0.1	0.1	0.7

If the transition probabilities are **time-independent**, then we say the chain is **homogeneous**

MARKOV CHAIN

Stochastic process → Infinite sequence of random variables



Due to the Markovian property:

$$\forall i \;\; P(X_{i+1} | X_1, X_2 \ldots, X_i) = P(X_{i+1} | X_i)$$

STATIONARY DISTRIBUTION

If, for *i* big enough, $P(X_{i+1}) = P(X_i) = \pi(X)$

We say that the Markov chain has a **stationary distribution**, $\pi(X)$

$$x_0\,, x_1\,, \ldots\ldots\,, \underset{\text{Sampling of } \pi(X)}{\underrightarrow{x_i\,, x_{i+1}\,, \ldots \qquad\qquad\qquad}}$$

If P is the transition probability matrix, then:

$$\pi^T = \pi^T P$$

DEFINITIONS AND PROPERTIES

Ergodic or irreducible: If it is possible to move between any two given states

Periodic: If it is possible to return to the same state at given intervals

If a Markov chain is **ergodic** and **aperiodic**, then it has a **unique stationary distribution**

SAMPLING ANY DISTRIBUTION

How do we sample this distribution?

$$f(x) = \begin{cases} 1 - |1 - x|, & 0 < x < 2 \\ 0, & \text{in other case} \end{cases}$$

Find a Markov chain with a unique stationary distribution such that:

$$\pi(x) = f(x)$$

Then, sample the Markov chain until convergence

MARKOV CHAIN MONTE CARLO

Metropolis-Hasting algorithm

METROPOLIS - HASTING ALGORITHM

We need two elements for our algorithm:

A mechanism to propose new values

$$g(x^*|x_{t-1})$$

A criterion to accept or reject them

$$\rho = \frac{f(x^*)g(x^*|x_{t-1})}{f(x_{t-1})g(x_{t-1}|x^*)} \quad p = \min\{1, \rho\}$$

At each step *t*

1.- Propose a new value x^* using *g* and the previous value x_{t-1}

2.- With probability p, $x_t = x^*$, $x_t = x_{t-1}$ otherwise

METROPOLIS - HASTING ALGORITHM

Two typical approaches for the proposal distribution:

$$g(x^*|x_{t-1}) = g(x^*)$$

$$g(x^*|x_{t-1}) = h(x^* - x_{t-1})$$ with *h* symmetric

In the latter case:

$$g(x^*|x_{t-1}) = g(x_{t-1}|x^*) \rightarrow \rho = \frac{f(x^*)}{f(x_{t-1})}$$

EXAMPLE

$$f(x) = \begin{cases} 1 - |1 - x|, & 0 < x < 2 \\ 0, & \text{in other case} \end{cases}$$

Proposal distribution

$$g(x^*|x_{t-1}) = g(x^*) = Unif(0, 2)$$

$$g(x^*) = g(x_{t-1}) \rightarrow \rho = \frac{f(x^*)}{f(x_{t-1})}$$

* We use *g* also to generate x_0

Probabilistic Modeling
and
Bayesian NEtworKs

Probability Distributions

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