Part 2 - Mathematical Preliminaries and the General Approach to RO Valuation

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Reading

Dixit, A and R. Pindyck (1994). *Investment under Uncertainty*
Chapters 3-5

Overview

Use of Option Pricing Methods

- Can be complicated to handle options with NPV methods.
  - The value of the asset and/or the cost of investment may go up or down in the future as market conditions change.
  - Unlike call option on stock, the option to invest may be very long-lived, or even perpetual.
- Need rigorous option pricing methods.

Project Value and Investment Decision

- To solve the problem, we must model the value of the project and its evolution $\boxed{\text{over time}}$.
- Given the dynamics of the project's value, we can value the option to invest in the project.
- Valuing the option to invest requires that we find the *optimal investment rule*; i.e., the rule for **when** to invest (or abandon etc.).
  - "When" does not mean determining the point in time that investment should occur.
  - It means finding the critical value of the project that should trigger investment.

Overview

Notation

- Value of the project V will evolve over time.
- Let r be the expected return on V. This is consistent with the project's risk. For all our applications, we let r be the risk-free rate of interest; i.e., all risk diversifiable.
- Let μ be the expected rate of capital gain.
- $\delta := r - \mu$ is the payout rate on the project. We will discuss this further later. Dividends, interest payment ...

Dynamics on the Project Value V

$\sigma = 0$

- Say there is no uncertainty over the future evolution. Then the rate of capital gain is

expected rate of capital gain

$$\frac{\Delta V}{V} = \mu \underbrace{\Delta t}_{(V_{t+1} - V_t)}.$$

$\sigma > 0$

- Now suppose V is risky; i.e., future values are uncertain. Then

$\sigma \approx$ Qualitivity

$$\frac{\Delta V}{V} = \mu \Delta t + \boxed{\sigma e_t}$$

where e_t is random noise and follows a standard normal distribution $e_t \sim N(0, 1)$.

Dynamics on the Project Value V (cont).

change of time is tiny
$\Delta t = 0$

For ease of analysis, we work in continuous time and let Δt become infinitesimally small. We then write this process as

$$\boxed{\frac{dV}{V} = \mu dt + \sigma dB_t,}$$

where $dB_t = e_t\sqrt{dt}$ is the increment of a Brownian motion.

These dynamics imply V follows a geometric Brownian motion (GBM). All applications we will consider assume these dynamics.

never negative

Brownian Motion

The Brownian motion is a stochastic process in continuous time and is the workhorse of applications in of continuous stochastic processes of economics and finance.

Definition

An adapted process $(B_t)_{t \geq 0}$ taking values in $\mathbb{R}$ is a Brownian motion if:

1. $B_t - B_s$ is independent of the past;
2. $B_t - B_s \sim N(0, \underbrace{t - s}_{\text{lag of timeperiod}})$, for all $0 \leq s < t$.

Unless stated otherwise, assume that $\boxed{B_0 = 0}$

From the definition of BM,

$dB_t = B_t - B_0$ $=0$ because it's random

$$E(dB_t) = 0 \text{ and } \mathrm{Var}(dB_t) = dt.$$

Recall $\mathrm{Var}(x) = E[x^2] - (E[x])^2$. Thus

移拔公式

$$\mathbb{E}(dB_t^2) = \mathrm{Var}(dB_t) + [\mathbb{E}(dB_t)]^2 = dt.$$

$dt + 0^2$ very small $=0.00000\ldots1$

Note that 波动小

高次項Var小越接近E(x)

$$Var(dB_t^2) = 2dt^2.$$

smaller than Expected value

So variance of dB_t^2 decreases faster than its mean for dt negligible. Thus, dB_t^2 can be treated as deterministic and approximated by its mean; i.e.,

$$\boxed{(dB_t)^2 \approx dt \text{ or } dB_t \approx \sqrt{dt}.}$$

Geometric Brownian Motion

V=隨机变数 μ=資產漂移率(平均增長率)
σ=資產波动率(標準差) dB_t=隨机波动 dt=無窮小時間長

The geometric Brownian motion is most widely used to model the evolution of stock prices since it ensures non-negativity of the underlying random variable. The SDE for a GBM is

前面公式同×V

$$\boxed{\frac{dV}{V} = \mu dt + \sigma dB_t,} \Rightarrow dV = \mu V dt + \sigma V \underline{dB_t}, \tag{1}$$

$V_0 = v$ and $\sigma > 0$. Percentage changes in V, dV/V are normally distributed and absolute changes in V, dV are lognormally distributed.

GMB (cont).

The solution to (1) is given by

$\log \frac{V_t}{v} = (\mu - \frac{1}{2}\sigma^2) + \sigma B_t$

$$V_t = ve^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}. = V e^{(\mu - \frac{1}{2}\sigma^2)t} \times e^{\sigma B_t}$$

確定性部份 $E(e^{\sigma B_t}) = e^{\frac{1}{2}\sigma^2 t}$

隨机性

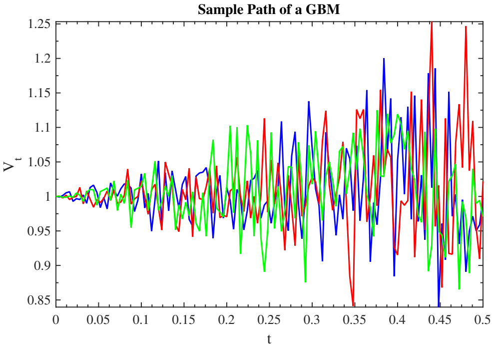
$$E_{t=0}[V_t] = ve^{\mu t} = V e^{t\mu - \cancel{\frac{1}{2}\sigma^2 t} + \cancel{\frac{1}{2}\sigma^2 t}} = V e^{\mu t}$$

and the expected present value of V_t (over an infinite period) is given by

$$E_{t=0}\left[\int_0^\infty V_t e^{-rt} dt\right] = \int_0^\infty ve^{\mu t} e^{-rt} dt = \int_0^\infty ve^{-(r-\mu)t} dt = \frac{v}{r-\mu}$$

If $\delta = r - \mu < 0$, then it is not optimal to ever exercise the option to invest so we assume that $r > \mu$. Inf. Stream Case

Sample Paths of a GBM



Ito's Lemma

Suppose $(V_t)_{t\geq 0}$ follows the GBM

$$dV = \mu V dt + \sigma V dB, \;\; V_0 = v.$$

Consider a function $F(V)$ that is at least twice differentiable in V. We want to find the total differential of this function i.e. $dF(V)$.

The usual rules of calculus define this differential in terms of first-order changes in V i.e.

$$dF(V) = \frac{\partial F(V)}{\partial V} dV$$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

$x^2+x+1 = a(x-10)^2+b(x-10)+c$

But suppose we also include higher order terms for changes in V:

Via a Taylor series expansion, we obtain

可以微無限次

$$dF = \frac{\partial F}{\partial V} dV + \underset{\frac{1}{2!}}{\frac{1}{2}} \frac{\partial^2 F}{\partial V^2}(dV)^2 + \underset{\frac{1}{3!}}{\frac{1}{6}} \frac{\partial^3 F}{\partial V^3}\underset{=0}{(dV)^3} + \dots$$

Now $dV = \mu V dt + \sigma V dB$ so

$$\begin{aligned}(dV)^2 =& \mu^2 V^2 (dt)^2 + 2\mu\sigma V^2 dt dB_t + \sigma^2 V^2 (dB_t)^2 \\ \approx& \underset{=0}{\cancel{\mu^2 V^2 (dt)^2}} + \underset{=0}{\cancel{2\mu\sigma V^2 (dt)^{3/2}}} + \sigma^2 V^2 dt\end{aligned}$$

as $dB_t \approx \sqrt{dt}$. ≈ 0

Now terms in $(dt)^2$ and $(dt)^{3/2}$ tend to zero faster than dt as dt becomes infinitesimally small (i.e. tend to zero). Thus we can ignore terms in dt raised to a power higher than 1.

Hence

$$(dV)^2 \approx \sigma^2 V^2 dt$$

All $(dV)^3$ terms will also be ignored. Hence, Ito's Lemma gives the differential

change of F(V

$$\begin{aligned} dF(V) &= \left[\mu V \frac{\partial F}{\partial V} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2}\right] dt + \sigma V \frac{\partial F}{\partial V} dB \\ &= \left[\mu V F_V + \frac{1}{2}\sigma^2 V^2 F_{VV}\right] dt + \underline{\sigma V F_V dB} \end{aligned} \tag{2}$$

$E[dF] = (\mu V F_V + \frac{1}{2}\sigma^2 V^2 F_{VV})dt$

$E[\sigma V F_V dB] = \sigma V F_V \underbrace{E[dB]}_{=0}$

Overview

Investment Problem -Basic Outline

Invest I and get one off payment V where

$$dV = \mu V dt + \sigma V dB.$$

value of waiting

We want to determine the value of the investment opportunity $F(V)$ and the investment rule (i.e., threshold) V^*.

Problem is to determine the optimal time to undertake the project; i.e. find a stopping time τ^* such that

investment

$$E_{t=0}[e^{-r\tau^*}(V^* - I)] = \sup_t E_{t=0}[e^{-rt}(V_t - I)]. \qquad (3)$$

discount factor

to maximize the NPV

τ^* = the time when $V = V^*$

Bellman Principle

If the decision maker has not invested before time t, the value of the investment opportunity is given by the maximum of the value of undertaking the project and the value of waiting for another period dt:

$A_1 V^{b_1}$

$$F(V_t) = \max\{V_t - I, e^{-rdt} E[F(V_{t+dt})]\} \tag{4}$$

where $r > 0$ is the discount rate.

This is called *Bellman's principle of optimality*.

change of $F(V_t)$

Note $F(V_{t+dt}) = F(V_t) + \underbrace{F(V_{t+dt}) - F(V_t)}_{dF(V_t)} = F(V_t) + dF(V_t)$.

If the second argument in equation (4) is larger, then optimal to wait.

Working out the value of waiting:

$$\begin{aligned}
F(V_t) =& e^{-rdt}\left(F(V_t) + E[dF(Y_t)]\right) \\
=& (1 - rdt + \cancelto{0}{o(dt)})\left(F(V_t) + E[dF(V_t)]\right) \\
=& F(V_t) + E[dF(V_t)] - rdtF(V_t) - rdtE[dF(V_t)] \\
\iff rdtF(V_t) =& E[dF(V_t)] - rdtE[dF(V_t)].
\end{aligned}$$

Ito's Lemma From Ito's Lemma

$$
\begin{aligned}
dF(V) =& \frac{dF}{dV} dV + \frac{1}{2} \frac{d^2 F}{dV^2} (dV)^2 \\
=& F_V dV + \frac{1}{2} F_{VV} (dV)^2
\end{aligned}
$$

Replacing for $dV = \mu V dt + \sigma V dB$, we get

$$
dF(V) = \mu V F_V dt + \sigma V F_V dB + \frac{1}{2} \sigma^2 V^2 F_{VV} dt
$$

so

$$
E[dF(V)] = \mu V F_V dt + \frac{1}{2} \sigma^2 V^2 F_{VV} dt
$$

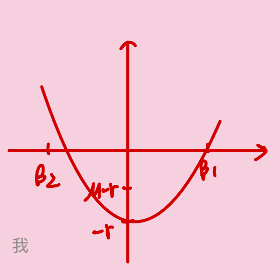
because $E[dB] = 0$

We replace $E[dF(V)]$ in the Bellman equation to get Bellman

$$rF(V_t) = \mu V F_V(V_t) + \frac{1}{2}\sigma^2 V^2 F_{VV}(V_t)$$

or, equivalently

$$\boxed{\frac{1}{2}\sigma^2 V^2 F_{VV} + \mu V F_V - rF = 0.}$$



The solution to this equation is $F(V)$ and represe
the investment option.

解微分方程(二階)

general solution

特徵解

The general solution to Eq. (5) is given by

$$F(V) = A_1 V^{\beta_1} + A_2 V^{\beta_2} \tag{6}$$

常數　常數

where $\beta_1 > 1$ and $\beta_2 < 0$ are the (real) roots to the fundamental quadratic equation given by

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \cancel{\alpha}\mu\beta - r = 0 \tag{7}$$

and A_1 and A_2 are constants to be determined.

Don't worry - We will examine this further in the exercises!!

Boundary Condition

Once V hits zero, the value of the option to invest is worthless. Therefore, the following condition must hold:

$$F(0) = 0. \tag{8}$$

Replacing for $V = 0$ in Eq. (6)

$$F(0) = A_1(0)^{\beta_1} + A_2(0)^{\beta_2} = 0 + \infty = \infty \neq 0$$

$\Downarrow$ $0^{-c} = \infty$

since $\beta_1 > 1$ and $\beta_2 < 0$.

$+$ $-$

When will the condition to be satisfied? If, and only if, $A_2 = 0$!

Therefore

$$F(V) = A_1 V^{\beta_1}.$$

largest solution

Additional Conditions

The investment and continuation regions are separated by a threshold value V^* and the following conditions must also hold

1. *Value Matching*

$$\underline{A_1(V^*)^{\beta_1} = V^* - I} \tag{9}$$

2. *Smooth Pasting*

first-order condition
微分

边界條件

$$F_V(V^*) = \frac{d}{dV}(V - I)\boxed{\big|_{V=V^*}} \tag{10}$$

in the condition $V=V^*$

$$\beta_1 A_1(V^*)^{\beta_1 - 1} = 1$$

= constant

From the VM condition

$$A_1 = (V^*)^{-\beta_1}(V^* - I)$$

Substituting for A_1 into the SP equation and rearranging to solve for V^* yields:

$$V^* = \frac{\beta_1}{\beta_1 - 1} I.$$

and

$$F(V_t) = \left(\frac{V_t}{V^*}\right)^{\beta_1} (V^* - I).$$

$V = A_1 (V^*)^{\beta_1}$

$$\beta_1 A_1 (V^*)^{\beta_1 - 1} = 1$$

$$\beta_1 \cdot \cancel{(V^*)^{-\beta_1}} (V^* - I) \cancel{(V^*)^{\beta_1}} \times (V^*)^{-1} = 1$$

$$\beta_1 \frac{V^* - I}{V^*} = 1$$

$$1 - \frac{I}{V^*} = \frac{1}{\beta_1}$$

$$1 - \frac{1}{\beta_1} = \frac{I}{V^*}$$

$$\frac{\beta_1 - 1}{\beta_1} = \frac{I}{V^*}$$

$$\frac{\beta_1}{\beta_1 - 1} = \frac{V^*}{I}$$

$$V^* = \frac{\beta_1}{\beta_1 - 1} \cdot I \quad \#$$

$\rightarrow \beta_1 > 1$

$\because \beta_1 / \beta_1 - 1 > 1$

$\therefore (\beta_1 / \beta_1 - 1) I > I \Rightarrow V^* > V$

NPV method

we invest in V^* → wait longer to invest

If NPV method, threshold $V - I \geq 0$ invest $\Rightarrow V \geq I$

value of waiting

$$F(V) = A_1 V^{\beta_1}$$

$$= (V^*)^{-\beta_1} (V^* - I) \cdot V^{\beta_1}$$

$$= \left(\frac{V_t}{V^*}\right)^{\beta_1} (V^* - I) \quad \#$$

$V < V^* \Rightarrow$ waiting value

"

$E[e^{-rT}]$ discount factor

$$F(V) = E[e^{-rT}] \cdot (V^* - I)$$

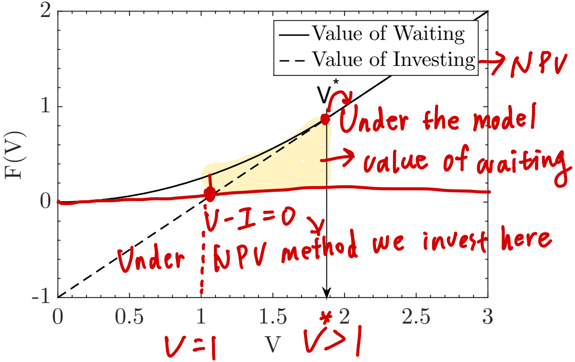
So we have solved for the critical point V^* at which it is optimal to invest. We will examine the characteristics of this solution later, but for now it is important to realise that since $\beta_1 > 1$, $\beta_1/(\beta_1 - 1) > 1$, implying $V^* > I$.

Thus, the simple NPV rule proposes much earlier investment $\longrightarrow$ uncertainty and irreversibility drive a wedge, of size $\beta_1/(\beta_1 - 1)$, between V^* and I. It is important to examine the magnitude of the wedge for realistic values of the underlying parameters, and its response to changes in the parameters.

$V^* > V_{(NPV)}$

$\rightarrow$ influence V^* through β_1

Value Function

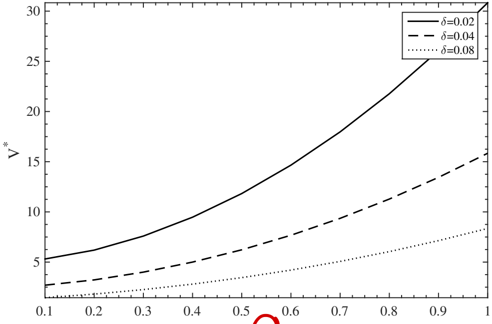


Overview

Effect of σ and δ on V^* Lump Sum case

$\sigma \uparrow V^* \uparrow$
$\delta \uparrow V^* \downarrow$

$\delta = r - \mu \rightarrow$ cash payout rate
opportunity cost of investing (vale waiting)



higher the uncertainty
higher the value of waiting

Infinite Stream Set-Up

→ value of waiting is same as lump sum payment. the only difference is the NPV

In many applications it is not realistic to assume that the payoff from investment is received as a one-off lump sum. For example, if the investment project is the introduction of a new product, then the payoff will be received as a flow over time. Assuming the project is infinitely lived and the sunk costs equal $I > 0$, then the expected PV of the project is given by

discount factor

$$E_{t=0}\left[\int_0^\infty e^{-rt} V_t dt\right] \underbrace{-I}_{= v} = \frac{v}{r-\mu} - I = \boxed{\frac{v}{\delta} - I} \tag{11}$$

paying today

where $v = V_0$, $\delta = r - \mu > 0$. Exp. PV formula

NPV ⇒ $V = I$

Threshold $V = \delta I$

Solution

We will show in the exercises that the solution to this problem is

$$V^* = \frac{\beta_1}{\beta_1 - 1}\delta I$$

and the value of the investment opportunity is given by

$$F(V_t) = \left(\frac{V_t}{V^*}\right)^{\beta_1}\left(\frac{V^*}{\delta} - I\right)$$

Convenience Yield

The parameter $\delta = r - \mu$, referred to previously as the payout rate of the project, is more commonly known as the *convenience yield* and has a clear economic interpretation, linked to the analogy between real and financial options.

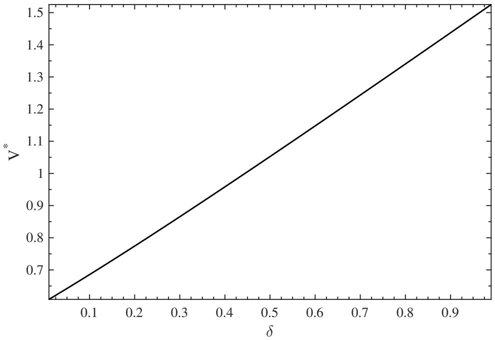
We can view the convenience yield from two perspectives:

1. as a dividend paid on the underlying project; or
2. as an opportunity cost of keeping the option alive

Note Effect of δ in this case!

$\delta \uparrow$ $V^* \uparrow$ different from lump-sum case

Lump sum plot



Effect of δ:

	LS	Flow
NPV	$V-I$	$\frac{V}{\delta}-I$
Waiting	$A_1V^{\beta_1}$	$A_1V^{\beta_1}$
V^* effect	<0	>0

dominate $A_1V^{\beta_1}$

δ change $A_1V^{\beta_1}$ → $\delta\uparrow V^*\uparrow$

will not effect $V-I$