4) Calcule los factores de escala, la métrica y los vectores base para los siguientes sistemas de coordenadas: a) Cilíndricas parabolicas definidas por $x=uv\cos(\varphi)$, $y=uv\,\text{sen}(\varphi)$, $z=\frac{1}{2}(u^2-v^2)$

$|r\rangle = x(u,v,\varphi)|\hat{i}\rangle + y(u,v,\varphi)|\hat{j}\rangle + z(u,v,\varphi)|\hat{k}\rangle$

$|e_u\rangle = \frac{\partial|r\rangle}{\partial u} = \frac{\partial x}{\partial u}|\hat{i}\rangle + \frac{\partial y}{\partial u}|\hat{j}\rangle + \frac{\partial z}{\partial u}|\hat{k}\rangle = v\cos(\varphi)|\hat{i}\rangle + v\,\text{sen}(\varphi)|\hat{j}\rangle + u|\hat{k}\rangle$

$h_u = \sqrt{v^2\cos^2(\varphi)+v^2\text{sen}^2(\varphi)+u^2} = \sqrt{v^2+u^2}$

$|\hat{e}_u\rangle = \frac{1}{\sqrt{u^2+v^2}}\left(v\cos(\varphi)|\hat{i}\rangle + v\,\text{sen}(\varphi)|\hat{j}\rangle + u|\hat{k}\rangle\right)$

$|e_v\rangle = \frac{\partial|r\rangle}{\partial v} = \frac{\partial x}{\partial v}|\hat{i}\rangle + \frac{\partial y}{\partial v}|\hat{j}\rangle + \frac{\partial z}{\partial v}|\hat{k}\rangle = u\cos(\varphi)|\hat{i}\rangle + u\,\text{sen}(\varphi)|\hat{j}\rangle - v|\hat{k}\rangle$

$h_v = \sqrt{u^2\cos^2(\varphi)+u^2\text{sen}^2(\varphi)+(-v)^2} = \sqrt{u^2+v^2}$

$|\hat{e}_v\rangle = \frac{1}{\sqrt{u^2+v^2}}\left(u\cos(\varphi)|\hat{i}\rangle + u\,\text{sen}(\varphi)|\hat{j}\rangle - v|\hat{k}\rangle\right)$

$|e_\varphi\rangle = \frac{\partial|r\rangle}{\partial \varphi} = \frac{\partial x}{\partial \varphi}|\hat{i}\rangle + \frac{\partial y}{\partial \varphi}|\hat{j}\rangle + \frac{\partial z}{\partial \varphi}|\hat{k}\rangle = -uv\,\text{sen}(\varphi)|\hat{i}\rangle + uv\cos(\varphi)|\hat{j}\rangle + 0|\hat{k}\rangle$

$h_\varphi = \sqrt{u^2v^2\text{sen}^2(\varphi)+u^2v^2\cos^2(\varphi)} = uv$

$|\hat{e}_\varphi\rangle = \frac{1}{uv}\left(-uv\,\text{sen}(\varphi)|\hat{i}\rangle + uv\cos(\varphi)|\hat{j}\rangle + 0|\hat{k}\rangle\right) = -\text{sen}(\varphi)|\hat{i}\rangle + \cos(\varphi)|\hat{j}\rangle$

$ds^2 = (h_1 dq^1)^2 + (h_2 dq^2)^2 + (h_3 dq^3)^2 \leftrightarrow ds^2 = (u^2+v^2)(du^2+dv^2) + u^2v^2 d\varphi^2$ $\quad g_{11} = g_{uu} = u^2+v^2 \quad g_{22} = g_{vv} = u^2+v^2 \quad g_{33} = g_{\varphi\varphi} = u^2v^2$

b) Hiperbólicas definidas por $x=\cosh(u)\cos(v)\cos(\varphi)$, $y=\cosh(u)\cos(v)\text{sen}(\varphi)$, $z=\text{senh}(u)\text{sen}(v)$

$|r\rangle = x(u,v,\varphi)|\hat{i}\rangle + y(u,v,\varphi)|\hat{j}\rangle + z(u,v,\varphi)|\hat{k}\rangle$

$|e_u\rangle = \frac{\partial|r\rangle}{\partial u} = \frac{\partial x}{\partial u}|\hat{i}\rangle + \frac{\partial y}{\partial u}|\hat{j}\rangle + \frac{\partial z}{\partial u}|\hat{k}\rangle = \text{senh}(u)\cos(v)\cos(\varphi)|\hat{i}\rangle + \text{senh}(u)\cos(v)\text{sen}(\varphi)|\hat{j}\rangle + \cosh(u)\text{sen}(v)|\hat{k}\rangle$

$h_u = \sqrt{\text{senh}^2(u)\cos^2(v)\cos^2(\varphi)+\text{senh}^2(u)\cos^2(v)\text{sen}^2(\varphi)+\cosh^2(u)\text{sen}^2(v)} = \sqrt{\text{senh}^2(u)\cos^2(v)+\cosh^2(u)\text{sen}^2(v)} = \sqrt{\text{senh}^2(u)\cos^2(v)+(\text{senh}^2(u)+1)\text{sen}^2(v)}$
$= \sqrt{\text{senh}^2(u)+\text{sen}^2(v)}$

$|\hat{e}_u\rangle = \frac{1}{\sqrt{\text{senh}^2(u)+\text{sen}^2(v)}}\left(\text{senh}(u)\cos(v)\cos(\varphi)|\hat{i}\rangle + \text{senh}(u)\cos(v)\text{sen}(\varphi)|\hat{j}\rangle + \cosh(u)\text{sen}(v)|\hat{k}\rangle\right)$

$|e_v\rangle = \frac{\partial|r\rangle}{\partial u} = \frac{\partial x}{\partial u}|\hat{i}\rangle + \frac{\partial y}{\partial u}|\hat{j}\rangle + \frac{\partial z}{\partial u}|\hat{k}\rangle = -\cosh(u)\text{sen}(v)\cos(\varphi)|\hat{i}\rangle - \cosh(u)\text{sen}(v)\text{sen}(\varphi)|\hat{j}\rangle + \text{senh}(u)\cos(v)|\hat{k}\rangle$

$h_v = \sqrt{\cosh^2(u)\text{sen}^2(v)\cos^2(\varphi)+\cosh^2(u)\text{sen}^2(v)\text{sen}^2(\varphi)+\text{senh}^2(u)\cos^2(v)} = \sqrt{\cosh^2(u)\text{sen}^2(v)+\text{senh}^2(u)\cos^2(v)} = \sqrt{(\text{senh}^2(u)+1)\text{sen}^2(v)+\text{senh}^2(u)\cos^2(v)}$
$= \sqrt{\text{senh}^2(u)+\text{sen}^2(v)}$

$|\hat{e}_v\rangle = \frac{1}{\sqrt{\text{senh}^2(u)+\text{sen}^2(v)}}\left(-\cosh(u)\text{sen}(v)\cos(\varphi)|\hat{i}\rangle - \cosh(u)\text{sen}(v)\text{sen}(\varphi)|\hat{j}\rangle + \text{senh}(u)\cos(v)|\hat{k}\rangle\right)$

$|e_\varphi\rangle = \frac{\partial|r\rangle}{\partial u} = \frac{\partial x}{\partial u}|\hat{i}\rangle + \frac{\partial y}{\partial u}|\hat{j}\rangle + \frac{\partial z}{\partial u}|\hat{k}\rangle = -\cosh(u)\cos(v)\text{sen}(\varphi)|\hat{i}\rangle + \cosh(u)\cos(v)\cos(\varphi)|\hat{j}\rangle + 0|\hat{k}\rangle$

$h_\varphi = \sqrt{\cosh^2(u)\cos^2(v)\text{sen}^2(\varphi)+\cosh^2(u)\cos^2(v)\cos^2(\varphi)} = \sqrt{\cosh^2(u)\cos^2(v)} = \cosh(u)\cos(v)$

$|\hat{e}_\varphi\rangle = \frac{1}{\cosh(u)\cos(v)}\left(-\cosh(u)\cos(v)\text{sen}(\varphi)|\hat{i}\rangle + \cosh(u)\cos(v)\cos(\varphi)|\hat{j}\rangle + 0|\hat{k}\rangle\right)$

$g_{11} = g_{uu} = \text{senh}^2 u + \text{sen}^2(v) \quad g_{22} = g_{vv} = \text{senh}^2 u + \text{sen}^2(v)$

$ds^2 = (h_1 dq^1)^2 + (h_2 dq^2)^2 + (h_3 dq^3)^2 \leftrightarrow ds^2 = (\text{senh}^2 u + \text{sen}^2(v))(du^2+dv^2) + \cosh^2(u)\cos^2(v)d\varphi^2$ $\quad g_{33} = g_{\varphi\varphi} = \cosh^2(u)\cos^2(v)$

10) Encuentre las curvas integrales y las trayectorias ortogonales de los siguientes campos vectoriales:

a) $f(x,y) = -\frac{x}{2}\hat{i} - \frac{y}{2}\hat{j} \rightarrow \frac{dy}{dx} = \frac{-y/2}{-x/2} = \frac{y}{x} \rightarrow \int\frac{dy}{y} = \int\frac{dx}{x}$

$\ln|y| = \ln|x| + C$

$|y| = e^C|x|$

Curvas integrales $\rightarrow y = Dx \rightarrow D = \pm e^C$

$f^{\perp}(x,y) = \frac{y}{2}\hat{i} - \frac{x}{2}\hat{j} \rightarrow \frac{dy}{dx} = \frac{-x/2}{y/2} = -\frac{x}{y} \rightarrow \int y\,dy = -\int x\,dx$

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

$y^2 + x^2 = D \rightarrow D = 2C$ ← Trayectorias ortogonales

b) $f(x,y) = xy\hat{i} + 2x\hat{j} \rightarrow \frac{dy}{dx} = \frac{2x}{xy} = \frac{2}{y} \rightarrow \int y\,dy = 2\int dx$

$\frac{1}{2}y^2 = 2x + C$

Curvas integrales $\rightarrow y^2 = 4x + D \rightarrow D = 2C$

$f^{\perp}(x,y) = -2x\hat{i} + xy\hat{j} \rightarrow \frac{dy}{dx} = \frac{xy}{-2x} = -\frac{y}{2} \rightarrow \int\frac{dy}{y} = -\frac{1}{2}\int dx$

$\ln|y| = -\frac{1}{2}x + C$

$y = De^{-x/2} \rightarrow D = e^C$ ← Trayectorias ortogonales

c) $f(x,y) = x^2\hat{i} - y\hat{j} \rightarrow \frac{dy}{dx} = \frac{-y}{x^2} \rightarrow \int\frac{dy}{y} = -\int\frac{dx}{x^2}$

$\ln|y| = \frac{1}{x} + C$

Curvas integrales $\rightarrow y = De^{1/x} \rightarrow D = e^C$

$f^{\perp}(x,y) = y\hat{i} + x^2\hat{j} \rightarrow \frac{dy}{dx} = \frac{x^2}{y} \rightarrow \int y\,dy = \int x^2 dx$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$

$y^2 = \frac{2}{3}x^3 + D \rightarrow D = 2C$ ← Trayectorias ortogonales

11) El vector área para una superficie s se define como $s = \iint_s ds$, encuentre el vector área de la superficie: $x^2+y^2+z^2=a^2$ con $z \leq 0$. Utilice el hecho de que en coordenadas esféricas $ds = a^2\text{sen}(\theta)d\theta d\varphi\hat{e}_1$

$|\hat{e}_1\rangle = \operatorname{sen}\theta\cos\varphi\,|\hat{\imath}\rangle + \operatorname{sen}\theta\operatorname{sen}\varphi\,|\hat{\jmath}\rangle + \cos\theta\,|\hat{k}\rangle$

$$S = \int_0^{2\pi}\int_{\pi/2}^{\pi} a^2 \operatorname{sen}\theta\,\hat{e}_r\,d\theta\,d\varphi = a^2\int_0^{2\pi}\int_{\pi/2}^{\pi}\left(\operatorname{sen}^2\theta\cos\varphi\,|\hat{\imath}\rangle + \operatorname{sen}^2\theta\operatorname{sen}\varphi\,|\hat{\jmath}\rangle + \operatorname{sen}\theta\cos\theta\,|\hat{k}\rangle\right)d\theta\,d\varphi$$

$$= a^2\int_0^{2\pi}\int_{\pi/2}^{\pi}\left[\left(\frac{1-\cos 2\theta}{2}\right)\cos\varphi\,|\hat{\imath}\rangle + \left(\frac{1-\cos 2\theta}{2}\right)\operatorname{sen}\varphi\,|\hat{\jmath}\rangle + \operatorname{sen}\theta\cos\theta\,|\hat{k}\rangle\right]d\theta\,d\varphi$$

$$= a^2\int_0^{2\pi}\left[\left(\frac{1}{2}x - \frac{\operatorname{sen}2\theta}{4}\Big|_{\frac{\pi}{2}}^{\pi}\right)\cos\varphi\,|\hat{\imath}\rangle + \left(\frac{1}{2}x - \frac{\operatorname{sen}2\theta}{4}\Big|_{\frac{\pi}{2}}^{\pi}\right)\operatorname{sen}\varphi\,|\hat{\jmath}\rangle + \left(\frac{1}{2}\operatorname{sen}^2\theta\Big|_{\pi/2}^{\pi}\right)|\hat{k}\rangle\right]d\varphi$$

$$= a^2\int_0^{2\pi}\left[\frac{\pi}{4}\cos\varphi\,|\hat{\imath}\rangle + \frac{\pi}{4}\operatorname{sen}\varphi\,|\hat{\jmath}\rangle - \frac{1}{2}|\hat{k}\rangle\right]d\varphi$$

$$= a^2\left[\frac{\pi}{4}\operatorname{sen}\varphi\,|\hat{\imath}\rangle - \frac{\pi}{4}\cos\varphi\,|\hat{\jmath}\rangle - \frac{1}{2}\varphi\,|\hat{k}\rangle\Big|_0^{2\pi}\right] = -\pi a^2|\hat{k}\rangle \leftarrow$$

Magnitud $= \pi a^2$

Dirección $= -\hat{k}$