4) Utilizando una representación paramétrica para los diferentes caminos C evalúe la integral $\int_C f(z)dz$

a) $f(z)=(z+2)/z$ y C:

1) El semicírculo $z=2e^{i\theta}$ con $0\le\theta\le\pi$

$$\int_C \frac{z+2}{z}dz=\int_C dz+2\int_C\frac{dz}{z}=\int_0^{\pi}Rie^{i\theta}d\theta+2\int_0^{\pi}\frac{Rie^{i\theta}}{Re^{i\theta}}d\theta$$
$$=2i\left(\frac{e^{i\theta}}{i}\Big|_0^{\pi}\right)+2i\int_0^{\pi}d\theta$$
$$=2(e^{i\pi}-1)+2i\pi$$
$$=-4+2i\pi$$

2) El semicírculo $z=2e^{i\theta}$ con $\pi\le\theta\le2\pi$

$$\int_C \frac{z+2}{z}dz=\int_C dz+2\int_C\frac{dz}{z}=\int_{\pi}^{2\pi}Rie^{i\theta}d\theta+2\int_{\pi}^{2\pi}\frac{Rie^{i\theta}}{Re^{i\theta}}d\theta$$
$$=2i\left(\frac{e^{i\theta}}{i}\Big|_{\pi}^{2\pi}\right)+2i\int_{\pi}^{2\pi}d\theta$$
$$=2(e^{i\pi}e^{i\pi}-e^{i\pi})+2i\pi$$
$$=4+2i\pi$$

3) El círculo $z=2e^{i\theta}$ con $0\le\theta\le2\pi$

$$\int_C \frac{z+2}{z}dz=\int_C dz+2\int_C\frac{dz}{z}=\int_0^{2\pi}Rie^{i\theta}d\theta+2\int_0^{2\pi}\frac{Rie^{i\theta}}{Re^{i\theta}}d\theta$$
$$=2i\left(\frac{e^{i\theta}}{i}\Big|_0^{2\pi}\right)+2i\int_0^{2\pi}d\theta$$
$$=2(e^{i\pi}e^{i\pi}-1)+2i(2\pi)$$
$$=4i\pi$$

b) $f(z)=z-1$ y C el arco que va desde z=0 a z=2 conformado por:

1) El semicirculo $z=1+e^{i\theta}$ con $\pi\le\theta\le2\pi\to f(z)=z-1=(1+e^{i\theta})-1=e^{i\theta}$

$$\int_C(z-1)dz=\int_{\pi}^{2\pi}e^{i\theta}(ie^{i\theta})d\theta=i\int_{\pi}^{2\pi}e^{2i\theta}d\theta=\frac{1}{2}\left(e^{2i\theta}\Big|_{\pi}^{2\pi}\right)=\frac{1}{2}(e^{4\pi i}-e^{2\pi i})=0$$

2) El segmento z=2 $(0\le x\le2)$ y el eje real

$$\int_C(z-1)dz=\int_0^2(z-1)dz=\left(\frac{1}{2}z^2-z\Big|_0^2\right)=2-2=0$$

c) $f(z)=\pi e^{\pi z^*}$ y C el contorno conformado por el cuadro con vértices en: 0, 1, 1+i, i y en sentido antihorario.

$0\to1$ (eje real) $\to z(t)=t,\ t\in[0,1]$

$$I_1=\int_{C_1}\pi e^{\pi z^*}dz=\int_0^1\pi e^{\pi t}dt=e^{\pi t}\Big|_0^1=e^{\pi}-1$$

$1\to1+i$: $z(t)=1+it,\ t\in[0,1]$

$$I_2=\int_{C_2}\pi e^{\pi z^*}dz=\int_0^1\pi e^{\pi(1-it)}(idt)$$
$$=i\pi e^{\pi}\int_0^1e^{-i\pi t}dt=-e^{\pi}\left(e^{-i\pi t}\Big|_0^1\right)$$
$$=-e^{\pi}(e^{-i\pi}-1)$$
$$=2e^{\pi}$$

$1+i\to i$: $z(t)=(1-t)+i,\ t\in[0,1]$

$$I_3=\int_0^1\pi e^{\pi(1-t-i)}(-dt)=-\pi e^{\pi}e^{-i\pi}\int_0^1e^{-\pi t}dt$$
$$=\pi e^{\pi}\left(-\frac{1}{\pi}e^{-\pi t}\Big|_0^1\right)$$
$$=-e^{\pi}(e^{-\pi}-1)=e^{\pi}-1$$

$i\to0$: $z(t)=i(1-t),\ t\in[0,1]$

$$I_4=\int_0^1\pi e^{\pi i(1+t)}(-idt)=-i\pi e^{\pi i}\int_0^1e^{\pi it}dt$$
$$=-e^{\pi i}\left(e^{\pi it}\Big|_0^1\right)=e^{\pi i}-1=-2$$

$$I=I_1+I_2+I_3+I_4=4(e^{\pi}-1)$$

d) $f(z)=\begin{cases}1, & y<0\\ 4y, & y>0\end{cases}$ cuando C es el arco que va desde z = -1 - i a z = 1 + i, conformado por la curva $y=x^3$

Parametrizamos: $z(x)=x+ix^3,\ x\in[-1,1]\to x\in[-1,0)\to y=x^3<0,\ f=1$
$\to x\in[0,1]\to y=x^3>0,\ f=4y=4x^3$

$$\int_C f(z)dz=\int_{-1}^0(1)(1+3ix^2)dx+\int_0^1(4x^3)(1+3ix^2)dx=\left(x+ix^3\Big|_{-1}^0\right)+4\left(\frac{1}{4}x^4+\frac{1}{2}ix^6\Big|_0^1\right)$$
$$=(0-(-1-i))+4\left(\frac{1}{4}+\frac{1}{2}i\right)$$
$$=(1+i)+(1+2i)=2+3i$$

11) Integre en sentido antihorario:

a) $\oint_C\frac{dz}{z^2+4}$, $C:4x^2+(y-2)^2=4\to x^2+\frac{(y-2)^2}{4}=1\to$ Centro $=(0,2)=0+2i$
$x\in[-1,1]\ \ y\in[0,4]$

$$\oint_C\frac{dz}{z^2+4}=\oint_C\frac{dz}{(z+2i)(z-2i)}$$
$\to$ Polos: $2i\to$ Dentro de C (centro)
$-2i\to$ Fuera de $C\to4(0)^2+(-2-2)^2=16>4$

$$I=\oint\frac{\frac{1}{z+2i}}{z-2i}dz=2\pi if(2i)=2\pi i\left(\frac{1}{4i}\right)=\frac{\pi}{2}$$

b) $\oint_C\frac{zdz}{z^2+4z+3}$, C: círculo con centro en -1 y radio 3 $\leftarrow x\in[-4,2],\ y\in[-3,3]$

$$\oint_C\frac{zdz}{z^2+4z+3}=\oint\frac{zdz}{(z+1)(z+3)}$$
$\to$ Polos: $-1=(-1,0)\in C$ (centro)
$-3=(-3,0)\in C$

$$=\oint_{C(-1)}\frac{\frac{z}{z+3}}{z+1}dz+\oint_{C(-3)}\frac{\frac{z}{z+1}}{z+3}dz=2\pi if(-1)+2\pi if(-3)=2\pi i\left(-\frac{1}{2}\right)+2\pi i\left(\frac{3}{2}\right)=-\pi i+3\pi i=2\pi i$$

c) $\oint\frac{z+2}{z-2}dz$, $C:|z-1|=2\leftarrow$ centro $=(1,0)$, radio $=2$
$x\in[-1,3],\ y\in[-2,2]$

$\oint\frac{z+2}{z-2}dz\to$ Polo: $2=(2,0)\in C$

$$I=\oint\frac{z+2}{z-2}dz=2\pi if(2)=2\pi i(4)=8\pi i$$

d) $\oint\frac{e^zdz}{ze^z-2iz}$, $C:|z|=0{,}6\to$ Centro $=(0,0)$, $r=0.6$
$x\in[-0.6,0.6]\ \ y\in[-0.6,0.6]$

$\oint\frac{e^zdz}{z(e^z-2i)}\to$ Polos: $0=(0,0)\in C$ (centro)
$e^z=2i\to z=\ln(2i)=\ln(2)+i\left(\frac{\pi}{2}+2\pi k\right)\to\ln(2)\approx0.69>0.6\to\notin C$

$$I=\oint\frac{\frac{e^z}{e^z-2i}}{z}dz=2\pi if(0)=2\pi i\left(\frac{1}{1-2i}\right)=2\pi i\left(\frac{1+2i}{1+4}\right)=\frac{2\pi}{5}i-\frac{4\pi}{5}$$

12) Demuestre que $\oint_C\frac{dz}{(z-z_1)(z-z_2)}=0$ para un camino cerrado simple C que contiene los puntos z_1 y z_2 arbitrarios.

$$\oint_C\frac{dz}{(z-z_1)(z-z_2)}=\oint_{C(z_1)}\frac{\frac{1}{z-z_2}}{z-z_1}dz+\oint_{C(z_2)}\frac{\frac{1}{z-z_1}}{z-z_2}dz=2\pi i\left(\frac{1}{z_1-z_2}\right)+2\pi i\left(\frac{1}{z_2-z_1}\right)=2\pi i\left(\frac{1}{z_1-z_2}\right)-2\pi i\left(\frac{1}{z_1-z_2}\right)=0$$