6) Demuestre que:

a) $\operatorname{Log}(-ie)=1-\frac{\pi}{2}i \longrightarrow \operatorname{Log}(-ie)=\ln|e|+i\left(-\frac{\pi}{2}+2\pi n\right), n\in\mathbb{Z}$

$=1+i\left(-\frac{\pi}{2}+2\pi n\right)$

$z=0-ie \quad |z|=e$

$\tan\theta=\frac{-1}{0}\to\theta=-\frac{\pi}{2}$

$n=0,\ \operatorname{Log}(-ie)=1-\frac{\pi}{2}i$

b) $\operatorname{Log}(1-i)=\frac{1}{2}\ln(2)-\frac{\pi}{4}i \longrightarrow \operatorname{Log}(1-i)=\ln(2^{1/2})+i\left(-\frac{\pi}{4}+2\pi n\right), n\in\mathbb{Z}$

$=\frac{1}{2}\ln(2)+i\left(-\frac{\pi}{4}+2\pi n\right)$

$z=1-i \quad |z|=\sqrt{1^2+1^2}=\sqrt{2}$

$\tan\theta=\frac{-1}{1}\to\theta=-\frac{\pi}{4}$

$n=0,\ \operatorname{Log}(1-i)=\frac{1}{2}\ln(2)-\frac{\pi}{4}i$

c) $\operatorname{Log}(e)=1+2n\pi i \longrightarrow \operatorname{Log}(e)=\ln|e|+i(0+2\pi n), n\in\mathbb{Z}$

$=1+i(2\pi n)$

$z=e \quad |z|=e$

$\tan\theta=\frac{0}{1}\to\theta=0$

d) $\operatorname{Log}(i)=\left(2n+\frac{1}{2}\right)\pi i \longrightarrow \operatorname{Log}(i)=\ln|1|+i\left(\frac{\pi}{2}+2\pi n\right), n\in\mathbb{Z}$

$=i\left(\frac{\pi}{2}+2\pi n\right)$

$=\left(\frac{1}{2}+2n\right)\pi i$

$z=0+i \quad |z|=\sqrt{0^2+1^2}=1$

$\tan\theta=\frac{1}{0}\to\theta=\frac{\pi}{2}$

10) Las funciones hiperbólicas se definen como: $\cosh x=\frac{e^x+e^{-x}}{2}$, $\operatorname{senh} x=\frac{e^x-e^{-x}}{2}$ y de manera análoga a las funciones trigonométricas tendremos el resto de funciones. a) Muestre las siguientes equivalencias:

Se tiene que $e^{i\theta}+e^{-i\theta}=2\cos\theta$ y $e^{i\theta}-e^{-i\theta}=2i\operatorname{sen}\theta \to \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2},\ \operatorname{sen}\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$

$\cosh(x)=\cos(ix)$ $\quad \cos(ix)=\frac{e^{i(ix)}+e^{-i(ix)}}{2}=\frac{e^{-x}+e^{x}}{2}=\cosh x$

$i\operatorname{senh}(x)=\operatorname{sen}(ix)$ $\quad \operatorname{sen}(ix)=\frac{e^{i(ix)}-e^{-i(ix)}}{2i}=\frac{e^{-x}-e^{x}}{2i}=-\frac{e^{x}-e^{-x}}{2i}=-\frac{\operatorname{senh}x}{i}=i\operatorname{senh}x$

$\cos(x)=\cosh(ix)$ $\quad \cosh(ix)=\frac{e^{ix}+e^{-ix}}{2}=\cos x$

$i\operatorname{sen}(x)=\operatorname{senh}(ix)$ $\quad \operatorname{senh}(ix)=\frac{e^{ix}-e^{-ix}}{2}=(i)\frac{e^{ix}-e^{-ix}}{2(i)}=i\operatorname{sen}x$

b) Muestre las siguientes identidades:

$\cosh^2x-\operatorname{senh}^2x=1 \to \cosh^2x-\operatorname{senh}^2x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{4}=\frac{4}{4}=1$

$\operatorname{sech}^2x=1-\tanh^2x \to 1-\tanh^2x=1-\frac{\operatorname{senh}^2x}{\cosh^2x}=\frac{\cosh^2x-\operatorname{senh}^2x}{\cosh^2x}=\frac{1}{\cosh^2x}=\operatorname{sech}^2x$

$\cosh(2x)=\cosh^2x+\operatorname{senh}^2x \to \cosh^2x+\operatorname{senh}^2x=\left(\frac{e^x+e^{-x}}{2}\right)^2+\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{(e^{2x}+2+e^{-2x})+(e^{2x}-2+e^{-2x})}{4}=\frac{e^{2x}+e^{-2x}}{2}=\cosh(2x)$

c) Resuelva las siguientes ecuaciones hiperbólicas:

$\cosh x-5\operatorname{senh}x-5=0 \to y=e^x$

$\frac{y+1/y}{2}-5\frac{y-1/y}{2}-5=0$

$\frac{y^2+1}{2y}-5\frac{y^2-1}{2y}-5=0$

$y^2+1-5y^2+5-10y=0$

$2y^2+5y-3=0$

$y=\frac{-5\pm\sqrt{25-4(2)(-3)}}{4}=\frac{-5\pm7}{4}$

$y_1=1/2 \quad y_2=-3$ X $\to y=e^x>0$

$e^x=1/2\to x=\ln(1/2)=-\ln 2$

$2\cosh(4x)-8\cosh(2x)+5=0$

$2(2\cosh^2(2x)-1)-8\cosh(2x)+5=0 \quad t=\cosh(2x)$

$4t^2-2-8t+5=0$

$4t^2-8t+3=0$

$t=\frac{8\pm\sqrt{64-4(4)(3)}}{8}=\frac{8\pm4}{8}$

$t_1=\frac{3}{2} \quad t_2=\frac{1}{2}$ X $\cosh(x)\geq1$

$\cosh(2x)=\frac{3}{2}\to 2x=\pm\operatorname{arccosh}(3/2)$

$x=\pm\frac{1}{2}\operatorname{arccosh}(3/2)$

$\cosh(x)=\operatorname{senh}(x)+2\operatorname{sech}(x)\to y=e^x$

$\frac{y+1/y}{2}=\frac{y-1/y}{2}+2\frac{2}{y-1/y}$

$\frac{y^2+1}{2y}=\frac{y^2-1}{2y}+\frac{4y}{y^2+1}$

$y^2+1=y^2-1+\frac{8y^2}{y^2+1}$

$2(y^2+1)=8y^2$

$2=6y^2$

$\frac{1}{\sqrt{3}}=y \longrightarrow e^x=1/\sqrt{3}\to x=\ln\left(\frac{1}{\sqrt{3}}\right)=-\frac{1}{2}\ln3$

4) Encuentre las derivadas

a) $\frac{z-i}{z+i}$ en i

$f'(z)=\frac{1(z+i)-(z-i)(1)}{(z+i)^2}=\frac{z+i-z+i}{(z+i)^2}=\frac{2i}{(z+i)^2}$

$f'(i)=\frac{2i}{(2i)^2}=-\frac{i}{2}$

b) $(z-4i)^8$ en $3+4i$

$f'(z)=8(z-4i)^7$

$z_0=3+4i\to z_0-4i=3\to f'(3+4i)=8(z_0-4i)^7=8\cdot3^7=17496$

c) $\frac{1.5z+2i}{3iz-4}$ para todo z

$f'(z)=\frac{1.5(3iz-4)-(1.5z+2i)(3i)}{(3iz-4)^2}=\frac{4.5iz-6-4.5iz+6}{(3iz-4)^2}=0$

$f'(z)=0\ \forall z$ con $3iz-4\neq0$

d) $i(1-z)^n$ en 0

$f'(z)=i(n)(1-z)^{n-1}(-1)=-in(1-z)^{n-1}$

$f'(0)=-in(1-0)^{n-1}=-in$

e) $(iz^3+3z^2)^3$ en $2i$

$f'(z)=3(iz^3+3z^2)^2(3iz^2+6z)$

$f'(2i)=3(i(2i)^3+3(2i)^2)^2(3i(2i)^2+6(2i))$

$=3(8-12)^2(-12i+12i)=3(-4)^2(0)=0$

f) $\frac{z^3}{(z+i)^3}$ en i

$f'(z)=\frac{3z^2(z+i)^3-3z^3(z+i)^2}{(z+i)^6}$

$f'(i)=\frac{3(i^2)(2i)^3-3(i^3)(2i)^2}{(2i)^6}=\frac{24i-12i}{-32}=-\frac{3i}{16}$

Cuáles de las siguientes funciones son analíticas?

a) $f(z)=iz\bar{z}$

$f(z)=i|z|^2=i(x^2+y^2)$

$\frac{\partial u}{\partial x}=0 \quad \frac{\partial u}{\partial y}=0 \quad \frac{\partial v}{\partial x}=2x \quad \frac{\partial v}{\partial y}=2y$

$0=2y$ y $0=-2x$ solo se cumple en $(0,0)$

↳ No es analítica!

b) $f(z)=e^{-2x}(\cos2y-i\operatorname{sen}2y)$

$\frac{\partial u}{\partial x}=-2e^{-2x}\cos2y \quad \frac{\partial v}{\partial x}=2e^{-2x}\operatorname{sen}2y$

$\frac{\partial u}{\partial y}=-2e^{-2x}\operatorname{sen}2y \quad \frac{\partial v}{\partial y}=-2e^{-2x}\cos2y$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad -\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$, las 4 derivadas son continuas

↳ Es analítica!

c) $f(z)=e^x(\cos y-i\operatorname{sen}y)$

$\frac{\partial u}{\partial x}=e^x\cos y \quad \frac{\partial v}{\partial x}=-e^x\operatorname{sen}y$

$\frac{\partial u}{\partial y}=-e^x\operatorname{sen}y \quad \frac{\partial v}{\partial y}=-e^x\cos y$

$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y} \quad -\frac{\partial u}{\partial y}\neq\frac{\partial v}{\partial x}$

↳ No es analítica!

d) $f(z) = \mathrm{Re}(z^2) - i\,\mathrm{Im}(z^2) \rightarrow f(z) = (x^2-y^2) - 2ixy$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = -2y \quad \frac{\partial v}{\partial y} = -2x$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad -\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$

↳ **No es analítica!**

e) $f(z) = 1/(z-z^5)$

$z^5 = (x+iy)^5 = (x^5 - 10x^3y^2 + 5xy^4) + i(y^5 - 10x^2y^3 + 5x^4y)$

$z - z^5 = (x - x^5 + 10x^3y^2 - 5xy^4) + i(y - y^5 + 10x^2y^3 - 5x^4y)$

$f(z) = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} \rightarrow u(x,y) = \frac{a}{a^2+b^2} \quad v(x,y) = -\frac{b}{a^2+b^2}$

Ya que Cauchy-Riemann se cumpliría (pues a y b son polinomios que vienen de una sola variable $z = x+iy$) pero no en $z(1-z^4) = 0 \rightarrow z = 0, \pm 1, \pm i$

↳ **Es analítica excepto en $z = 0, \pm 1, \pm i$!**

f) $f(z) = \ln|z| + i\,\mathrm{Arg}(z) \rightarrow f(z) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x)$

$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, pero no están definidas en el origen

↳ **Es analítica excepto en $(0,0)$!**

g) $f(z) = \cos x \cosh y - i \,\mathrm{sen}\, x \,\mathrm{senh}\, y$

$\frac{\partial u}{\partial x} = -\mathrm{sen}\,x\cosh y \qquad \frac{\partial v}{\partial x} = -\cos x\,\mathrm{senh}\,y$

$\frac{\partial u}{\partial y} = \cos x\,\mathrm{senh}\,y \qquad \frac{\partial v}{\partial y} = -\mathrm{sen}\,x\cosh y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, las 4 derivadas son continuas

↳ **Es analítica!**