6) En el caso tridimensional tenemos que, si $\{e_i\}$ define un sistema de coordenadas (dextrógiro) y no necesariamente ortogonal, entonces demuestre que:

a) $e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$, i, j, k = 1, 2, 3 y sus permutaciones

Supongamos que deseamos construir la base recíproca $\{e^i\}$ asociada a la base directa dada $\{e_i\}$. Por definición tenemos que: $e^i \cdot e_j = \delta^i_j$, es decir, en este caso $e^i \cdot e_i = 1$, pero $e^i \cdot e_j = e^i \cdot e_k = 0$, pues ambos son perpendiculares a e^i. De esta manera,

(1) $e^i = \alpha (e_j \times e_k)$. Ya que $e_i \cdot e^i = 1 \rightarrow e_i \cdot (\alpha (e_j \times e_k)) = 1$

$$\alpha e_i \cdot (e_j \times e_k) = 1$$

$$\alpha = \frac{1}{e_i \cdot (e_j \times e_k)}$$

→ Al reemplazar en (1): $e^i = \alpha (e_j \times e_k) = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$ ✓

Análogamente, este resultado es valido para i,j,k=1,2,3 y sus permutaciones

b) Si los volúmenes $V = e_1 \cdot (e_2 \times e_3)$ y $\tilde{V} = e^1 \cdot (e^2 \times e^3)$, entonces $V\tilde{V} = 1$

Tomando $V = e_1 \cdot (e_2 \times e_3)$, nótese que $e^1 = \frac{e_2 \times e_3}{V}$, $e^2 = \frac{e_3 \times e_1}{V}$, $e^3 = \frac{e_1 \times e_2}{V}$. Ahora, intentemos calcular $\tilde{V} = e^1 \cdot (e^2 \times e^3)$ con esto:

$$\begin{aligned}
e^2 \times e^3 &= \frac{e_3 \times e_1}{V} \times \frac{e_1 \times e_2}{V} \\
&= \frac{1}{V^2} ((e_3 \times e_1) \times (e_1 \times e_2)) \rightarrow \begin{array}{l} a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \\ a = (e_3 \times e_1) \quad b = e_1 \quad c = e_2 \end{array} \\
&= \frac{1}{V^2} (e_1 ((e_3 \times e_1) \cdot e_2) - e_2 ((e_3 \times e_1) \cdot e_1)) \\
&= \frac{1}{V^2} (e_1 (e_1 \cdot (e_2 \times e_3)) - e_2 (0)) \\
&= \frac{1}{V^2} (e_1 (V)) \\
&= \frac{1}{V} e_1
\end{aligned}$$

$$\begin{aligned}
\rightarrow \tilde{V} &= e^1 \cdot (e^2 \times e^3) \\
\tilde{V} &= e^1 \cdot \left(\tfrac{1}{V} e_1\right) \\
\tilde{V} &= \tfrac{1}{V} (e^1 \cdot e_1) \\
\tilde{V} &= \tfrac{1}{V} \\
\tilde{V} V &= 1 \checkmark
\end{aligned}$$

c) ¿ Qué vector satisface $a \cdot e^i = 1$? Demuestre que a es único

La idea es buscar un vector a, para el cual $a\cdot e^i=1$ para $i=1,2,3$. Al expresar a en la base directa, tenemos que $a=a^j e_j$ con $j=1,2,3$, por lo cual:

$$\begin{aligned} a\cdot e^i &= (a^j e_j)\cdot(e^i)\\ 1 &= a^j(e_j\cdot e^i)\\ 1 &= a^j\delta_j^i\\ 1 &= a^i \text{ para } i=1,2,3 \end{aligned}$$

$\rightarrow$ Así, $a=a^ie_i=a^1e_1+a^2e_2+a^3e_3=e_1+e_2+e_3$ ✓

¿a es única? supongamos un b para la cual $b\cdot e^i=1$. Así, si $c=b-a$,

$$\begin{aligned} c\cdot e^i &= (b-a)\cdot e^i\\ &= b\cdot e^i-a\cdot e^i\\ &= 1-1\\ &= 0 \end{aligned}$$

$\rightarrow$ En consecuencia, $c=c^je_j$ y $c\cdot e^i=c^j(e_j\cdot e^i)$

$$\begin{aligned} 0 &= c^j\delta_j^i\\ 0 &= c^i \end{aligned}$$

Es decir, $c^i=b^i-a^i=0 \rightarrow b^i=a^i \leftarrow$ a es única ✓

d) Encuentre el producto vectorial de dos vectores a y b que están representados en un sistema de coordenadas oblicuo: Dada la base $w_1=4\hat{\imath}+2\hat{\jmath}+\hat{k}$, $w_2=3i+3\hat{\jmath}$, $w_3=2\hat{k}$. Entonces encuentre:

I) las bases recíprocas $\{e^i\}$

$$\begin{aligned} V &= w_1\cdot(w_2\times w_3)\\ &= (4,2,1)\cdot((3,3,0)\times(0,0,2))\\ &= (4,2,1)\cdot((6,-6,0))\\ &= 4(6)+2(-6)+1(0)\\ &= 12 \end{aligned}$$

$$e^1=\frac{w_2\times w_3}{V}=\frac{(6,-6,0)}{12}=\left(\tfrac{1}{2},-\tfrac{1}{2},0\right)$$

$$e^2=\frac{w_3\times w_1}{V}=\frac{(0,0,2)\times(4,2,1)}{12}=\frac{(-4,8,0)}{12}=\left(-\tfrac{1}{3},\tfrac{2}{3},0\right)$$

$$e^3=\frac{w_1\times w_2}{V}=\frac{(4,2,1)\times(3,3,0)}{12}=\frac{(-3,3,6)}{12}=\left(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2}\right)$$

Base recíproca $=\{e^1,e^2,e^3\}$
$=\{(\tfrac{1}{2},-\tfrac{1}{2},0),(-\tfrac{1}{3},\tfrac{2}{3},0),(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2})\}$

II) Las componentes covariantes y contravariantes del vector $a=\hat{\imath}+2\hat{\jmath}+3\hat{k}$

$a=(1,2,3)$ $\rightarrow$ Para los componentes covariantes: $a_i=a\cdot e_i$

$$\begin{aligned} a_1 &= (1,2,3)\cdot(4,2,1)=(1)(4)+2(2)+3(1)=11\\ a_2 &= (1,2,3)\cdot(3,3,0)=(1)(3)+2(3)+3(0)=9\\ a_3 &= (1,2,3)\cdot(0,0,2)=(1)(0)+2(0)+3(2)=6 \end{aligned}$$

$\rightarrow$ Componentes covariantes: $(a_1,a_2,a_3)=(11,9,6)$

$\rightarrow$ Para los componentes contravariantes: $a^i=a\cdot e^i$

$$\begin{aligned} a_1 &= (1,2,3)\cdot(\tfrac{1}{2},-\tfrac{1}{2},0)=1(\tfrac{1}{2})+2(-\tfrac{1}{2})+3(0)=-\tfrac{1}{2}\\ a_2 &= (1,2,3)\cdot(-\tfrac{1}{3},\tfrac{2}{3},0)=1(-\tfrac{1}{3})+2(\tfrac{2}{3})+3(0)=1\\ a_3 &= (1,2,3)\cdot(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2})=1(-\tfrac{1}{4})+2(\tfrac{1}{4})+3(\tfrac{1}{2})=\tfrac{7}{4} \end{aligned}$$

$\rightarrow$ Componentes contravariantes: $(a^1,a^2,a^3)=(-\tfrac{1}{2},1,\tfrac{7}{4})$

7) Considere una vez más el espacio vectorial de matrices hermíticas 2 x 2 y la definición de producto interno $\langle a|b\rangle \doteq Tr(A^{+}B)$ que introdujimos en los ejercicios de la sección 2.2.4. Hemos comprobado que la matriz unitaria y las matrices de Pauli $\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}$ forman base para ese espacio. Encuentre entonces la base dual asociada a las base de Pauli y, adicionalmente, dado un vector genérico en este espacio vectorial encuentre su 1-forma asociada.

$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\sigma_0\equiv I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\sigma^0 \rightarrow \langle\sigma^0,\sigma_0\rangle=1\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{0*}&\sigma_3^{0*}\\\sigma_2^{0*}&\sigma_4^{0*}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{0*}&\sigma_3^{0*}\\\sigma_2^{0*}&\sigma_4^{0*}\end{pmatrix}=\sigma_1^{0*}+\sigma_4^{0*}=1\rightarrow\sigma_1^{0*}=1-\sigma_4^{0*}$

$\langle\sigma^0,\sigma_1\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{0*}&\sigma_3^{0*}\\\sigma_2^{0*}&\sigma_4^{0*}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_3^{0*}&\sigma_1^{0*}\\\sigma_4^{0*}&\sigma_2^{0*}\end{pmatrix}=\sigma_2^{0*}+\sigma_3^{0*}=0\rightarrow\sigma_2^{0*}=-\sigma_3^{0*}$

$\langle\sigma^0,\sigma_2\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{0*}&\sigma_3^{0*}\\\sigma_2^{0*}&\sigma_4^{0*}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)=Tr\begin{pmatrix}i\sigma_3^{0*}&-i\sigma_1^{0*}\\i\sigma_4^{0*}&-i\sigma_2^{0*}\end{pmatrix}=i\sigma_3^{0*}-i\sigma_2^{0*}=0\rightarrow\sigma_2^{0*}=\sigma_3^{0*}=0$

$\langle\sigma^0,\sigma_3\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{0*}&\sigma_3^{0*}\\\sigma_2^{0*}&\sigma_4^{0*}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{0*}&-\sigma_3^{0*}\\\sigma_2^{0*}&-\sigma_4^{0*}\end{pmatrix}=\sigma_1^{0*}-\sigma_4^{0*}=0\rightarrow(1-\sigma_4^{0*})-\sigma_4^{0*}=0$

$\sigma_4^{0*}=1/2,\ \sigma_1^{0*}=1/2$

$\sigma^0=\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}$

$\sigma^1\rightarrow\langle\sigma^1,\sigma_0\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{1*}&\sigma_3^{1*}\\\sigma_2^{1*}&\sigma_4^{1*}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{1*}&\sigma_3^{1*}\\\sigma_2^{1*}&\sigma_4^{1*}\end{pmatrix}=\sigma_1^{1*}+\sigma_4^{1*}=0$

$\sigma_1^{1*}=-\sigma_4^{1*}$

$\langle\sigma^1,\sigma_1\rangle=1\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{1*}&\sigma_3^{1*}\\\sigma_2^{1*}&\sigma_4^{1*}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_3^{1*}&\sigma_1^{1*}\\\sigma_4^{1*}&\sigma_2^{1*}\end{pmatrix}=\sigma_2^{1*}+\sigma_3^{1*}=1$

$\sigma_2^{1*}=1-\sigma_3^{1*}$

$\langle\sigma^1,\sigma_2\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{1*}&\sigma_3^{1*}\\\sigma_2^{1*}&\sigma_4^{1*}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)=Tr\begin{pmatrix}i\sigma_3^{1*}&-i\sigma_1^{1*}\\i\sigma_4^{1*}&-i\sigma_2^{1*}\end{pmatrix}=i\sigma_3^{1*}-i\sigma_2^{1*}=0$

$\sigma_2^{1*}=\sigma_3^{1*}=\frac{1}{2}$

$\langle\sigma^1,\sigma_3\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{1*}&\sigma_3^{1*}\\\sigma_2^{1*}&\sigma_4^{1*}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{1*}&-\sigma_3^{1*}\\\sigma_2^{1*}&-\sigma_4^{1*}\end{pmatrix}=\sigma_1^{1*}-\sigma_4^{1*}=0$

$\sigma_1^{1*}=\sigma_4^{1*}=0$

$\sigma^1=\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}$

$\sigma^2\rightarrow\langle\sigma^2,\sigma_0\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{2*}&\sigma_3^{2*}\\\sigma_2^{2*}&\sigma_4^{2*}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{2*}&\sigma_3^{2*}\\\sigma_2^{2*}&\sigma_4^{2*}\end{pmatrix}=\sigma_1^{2*}+\sigma_4^{2*}=0$ $\quad\sigma^2=\begin{pmatrix}0&-i/2\\i/2&0\end{pmatrix}$

$\langle\sigma^2,\sigma_1\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{2*}&\sigma_3^{2*}\\\sigma_2^{2*}&\sigma_4^{2*}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_3^{2*}&\sigma_1^{2*}\\\sigma_4^{2*}&\sigma_2^{2*}\end{pmatrix}=\sigma_3^{2*}+\sigma_2^{2*}=0\rightarrow\sigma_2^{2*}=-\sigma_3^{2*}$

$\langle\sigma^2,\sigma_2\rangle=1\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{2*}&\sigma_3^{2*}\\\sigma_2^{2*}&\sigma_4^{2*}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)=Tr\begin{pmatrix}i\sigma_3^{2*}&-i\sigma_1^{2*}\\i\sigma_4^{2*}&-i\sigma_2^{2*}\end{pmatrix}=i\sigma_3^{2*}-i\sigma_2^{2*}=1\rightarrow 2i\sigma_3^{2*}=1$

$\sigma_3^{2*}=-i/2\quad\sigma_2^{2*}=i/2$

$\langle\sigma^2,\sigma_3\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{2*}&\sigma_3^{2*}\\\sigma_2^{2*}&\sigma_4^{2*}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{2*}&-\sigma_3^{2*}\\\sigma_2^{2*}&-\sigma_4^{2*}\end{pmatrix}=\sigma_1^{2*}-\sigma_4^{2*}=0\rightarrow\sigma_1^{2*}=\sigma_4^{2}=0$

$\sigma^3\rightarrow\langle\sigma^3,\sigma_0\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{3*}&\sigma_3^{3*}\\\sigma_2^{3*}&\sigma_4^{3*}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{3*}&\sigma_3^{3*}\\\sigma_2^{3*}&\sigma_4^{3*}\end{pmatrix}=\sigma_1^{3*}+\sigma_4^{3*}=0\rightarrow\sigma_1^{3*}=-\sigma_4^{3*}$

$\langle\sigma^3,\sigma_1\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{3*}&\sigma_3^{3*}\\\sigma_2^{3*}&\sigma_4^{3*}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_3^{3*}&\sigma_1^{3*}\\\sigma_4^{3*}&\sigma_2^{3*}\end{pmatrix}=\sigma_3^{3*}+\sigma_2^{3*}=0\rightarrow\sigma_2^{3*}=-\sigma_3^{3*}$

$\langle\sigma^3,\sigma_2\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{3*}&\sigma_3^{3*}\\\sigma_2^{3*}&\sigma_4^{3*}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)=Tr\begin{pmatrix}i\sigma_3^{3*}&-i\sigma_1^{3*}\\i\sigma_4^{3*}&-i\sigma_2^{3*}\end{pmatrix}=i\sigma_3^{3*}-i\sigma_2^{3*}=0\rightarrow\sigma_2^{3*}=\sigma_3^{3*}=0$

$\langle\sigma^3,\sigma_3\rangle=0\rightarrow Tr\left(\begin{pmatrix}\sigma_1^{3*}&\sigma_3^{3*}\\\sigma_2^{3*}&\sigma_4^{3*}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)=Tr\begin{pmatrix}\sigma_1^{3*}&-\sigma_3^{3*}\\\sigma_2^{3*}&-\sigma_4^{3*}\end{pmatrix}=\sigma_1^{3*}-\sigma_4^{3*}=1$

$-2\sigma_4^{3*}=1\rightarrow\sigma_4^{3*}=-1/2,\ \sigma_1^{3*}=1/2$

$\sigma^3=\begin{pmatrix}1/2&0\\0&-1/2\end{pmatrix}$

Base dual: $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\} = \left\{ \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}$

Para el segundo ejercicio, tengamos en cuenta que $\mathcal{F}_A[|\sigma_i\rangle] = \langle A|\sigma_i\rangle = \langle a_j \sigma^j | \sigma_i \rangle = a_j \langle \sigma^j | \sigma_i \rangle = a_j \delta^j_i = a_i \leftarrow$ Componentes contravariantes

$\mathcal{F}_A[|\sigma_i\rangle] = \langle A|\sigma_i\rangle = Tr(A^\dagger \sigma_i)$

Así, si $A = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \in$ matrices 2x2 hermíticas: $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$

$a_0 = Tr(A^\dagger \sigma_0) = Tr\left(\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = Tr \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} = \alpha + \gamma$

$a_1 = Tr(A^\dagger \sigma_1) = Tr\left(\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = Tr \begin{pmatrix} \beta & \alpha \\ \gamma & \beta^* \end{pmatrix} = \beta + \beta^* = 2Re(\beta)$

$a_2 = Tr(A^\dagger \sigma_2) = Tr\left(\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) = Tr \begin{pmatrix} \beta i & -\alpha i \\ \gamma i & -\beta^* i \end{pmatrix} = \beta i - \beta^* i = -2Im(\beta)$

$a_3 = Tr(A^\dagger \sigma_3) = Tr\left(\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = Tr \begin{pmatrix} \alpha & -\beta \\ \beta^* & -\gamma \end{pmatrix} = \alpha - \gamma$

$\rightarrow \mathcal{F}_A = a_0 \langle \sigma_0 | + a_1 \langle \sigma_1 | + a_2 \langle \sigma_2 | + a_3 \langle \sigma_3 |$

$= (\alpha + \gamma) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2Re(\beta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - 2Im(\beta) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + (\alpha - \gamma) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2\alpha & 2\beta \\ 2\beta^* & 2\gamma \end{pmatrix} = \underline{\underline{2A}}$