4) Suponga $\mathbb{A}\mathbb{B}=\mathbb{B}\mathbb{A}$, demuestre que:

a) $(\mathbb{A}+\mathbb{B})^2=\mathbb{A}^2+2\mathbb{A}\mathbb{B}+\mathbb{B}^2$

$$
\begin{aligned}
\rightarrow (\mathbb{A}+\mathbb{B})^2 &= (\mathbb{A}+\mathbb{B})(\mathbb{A}+\mathbb{B}) \\
&= \mathbb{A}(\mathbb{A}+\mathbb{B})+\mathbb{B}(\mathbb{A}+\mathbb{B}) \\
&= \mathbb{A}\mathbb{A}+\mathbb{A}\mathbb{B}+\mathbb{B}\mathbb{A}+\mathbb{B}\mathbb{B} \\
&= \mathbb{A}^2+\mathbb{A}\mathbb{B}+\mathbb{A}\mathbb{B}+\mathbb{B}^2 \\
&= \mathbb{A}^2+2\mathbb{A}\mathbb{B}+\mathbb{B}^2
\end{aligned}
$$

b) $(\mathbb{A}+\mathbb{B})^3=\mathbb{A}^3+3\mathbb{A}^2\mathbb{B}+3\mathbb{A}\mathbb{B}^2+\mathbb{B}^3$

$$
\begin{aligned}
\rightarrow (\mathbb{A}+\mathbb{B})^3 &= (\mathbb{A}+\mathbb{B})^2(\mathbb{A}+\mathbb{B}) \\
&= (\mathbb{A}^2+2\mathbb{A}\mathbb{B}+\mathbb{B}^2)(\mathbb{A}+\mathbb{B}) \\
&= \mathbb{A}^2(\mathbb{A}+\mathbb{B})+2\mathbb{A}\mathbb{B}(\mathbb{A}+\mathbb{B})+\mathbb{B}^2(\mathbb{A}+\mathbb{B}) \\
&= \mathbb{A}^2\mathbb{A}+\mathbb{A}^2\mathbb{B}+2\mathbb{A}\mathbb{B}\mathbb{A}+2\mathbb{A}\mathbb{B}\mathbb{B}+\mathbb{B}^2\mathbb{A}+\mathbb{B}^2\mathbb{B} \\
&= \mathbb{A}^3+\mathbb{A}^2\mathbb{B}+2\mathbb{A}\mathbb{A}\mathbb{B}+2\mathbb{B}\mathbb{A}\mathbb{B}+\mathbb{B}^2\mathbb{A}+\mathbb{B}^3 \\
&= \mathbb{A}^3+\mathbb{A}^2\mathbb{B}+2\mathbb{A}^2\mathbb{B}+2\mathbb{B}\mathbb{B}\mathbb{A}+\mathbb{B}^2\mathbb{A}+\mathbb{B}^3 \\
&= \mathbb{A}^3+\mathbb{A}^2\mathbb{B}+2\mathbb{A}^2\mathbb{B}+2\mathbb{B}^2\mathbb{A}+\mathbb{B}^2\mathbb{A}+\mathbb{B}^3 \\
&= \mathbb{A}^3+3\mathbb{A}^2\mathbb{B}+3\mathbb{B}^2\mathbb{A}+\mathbb{B}^3
\end{aligned}
$$

5) Suponga que un operador $\mathbb{L}$ puede ser escrito como la composición de otros dos operadores $\mathbb{L}=\mathbb{L}_-\mathbb{L}_+$ con $[\mathbb{L}_-,\mathbb{L}_+]=\mathbb{I}$. Demostrar que:

Si $\mathbb{L}|x\rangle=\lambda|x\rangle$ y $|y\rangle=\mathbb{L}_+|x\rangle$ entonces $\mathbb{L}|y\rangle=(\lambda+1)|y\rangle$

Queremos relacionar $\mathbb{L}$ y $\mathbb{L}_+$ $\rightarrow$

$$
\begin{aligned}
[\mathbb{L},\mathbb{L}_+] &= [\mathbb{L}_-\mathbb{L}_+,\mathbb{L}_+] = \mathbb{L}_-\mathbb{L}_+\mathbb{L}_+ - \mathbb{L}_+\mathbb{L}_-\mathbb{L}_+ \\
&= \mathbb{L}_-\mathbb{L}_+\mathbb{L}_+ - \underbrace{\mathbb{L}_-\mathbb{L}_+\mathbb{L}_+ + \mathbb{L}_-\mathbb{L}_+\mathbb{L}_+}_{\text{nuevo término}} - \mathbb{L}_+\mathbb{L}_-\mathbb{L}_+ \\
&= \mathbb{L}_-[\mathbb{L}_+\mathbb{L}_+ - \mathbb{L}_+\mathbb{L}_+] + [\mathbb{L}_-\mathbb{L}_+ - \mathbb{L}_+\mathbb{L}_-]\mathbb{L}_+ \\
&= \mathbb{L}_-[\mathbb{L}_+,\mathbb{L}_+] + [\mathbb{L}_-,\mathbb{L}_+]\mathbb{L}_+ \\
&= \mathbb{L}_+
\end{aligned}
$$

$$
\longrightarrow [\mathbb{L},\mathbb{L}_+] = \mathbb{L}\mathbb{L}_+ - \mathbb{L}_+\mathbb{L} = \mathbb{L}_+
$$

$$
\mathbb{L}\mathbb{L}_+ = \mathbb{L}_+ + \mathbb{L}_+\mathbb{L}
$$

Ahora, aplicamos

$$
\begin{aligned}
\mathbb{L}|y\rangle &= \mathbb{L}(\mathbb{L}_+|x\rangle) = (\mathbb{L}_+ + \mathbb{L}_+\mathbb{L})|x\rangle \\
&= \mathbb{L}_+|x\rangle + \mathbb{L}_+(\mathbb{L}|x\rangle) \\
&= \mathbb{L}_+|x\rangle + \mathbb{L}_+(\lambda|x\rangle) \\
&= |y\rangle + \lambda|y\rangle \\
&= (\lambda+1)|y\rangle \checkmark
\end{aligned}
$$

Del mismo modo, demuestre que $\mathbb{L}|x\rangle=\lambda|x\rangle$ y $|z\rangle=\mathbb{L}_-|x\rangle$ entonces $\mathbb{L}|z\rangle=(\lambda-1)|z\rangle$

Queremos relacionar $\mathbb{L}$ y $\mathbb{L}_-$ $\rightarrow$

$$
\begin{aligned}
[\mathbb{L},\mathbb{L}_-] &= [\mathbb{L}_-\mathbb{L}_+,\mathbb{L}_-] = \mathbb{L}_-\mathbb{L}_+\mathbb{L}_- - \mathbb{L}_-\mathbb{L}_-\mathbb{L}_+ \\
&= \mathbb{L}_-\mathbb{L}_+\mathbb{L}_- - \underbrace{\mathbb{L}_-\mathbb{L}_-\mathbb{L}_+ + \mathbb{L}_-\mathbb{L}_-\mathbb{L}_+}_{\text{nuevo término}} - \mathbb{L}_-\mathbb{L}_-\mathbb{L}_+ \\
&= \mathbb{L}_-[\mathbb{L}_+\mathbb{L}_- - \mathbb{L}_-\mathbb{L}_+] + [\mathbb{L}_-\mathbb{L}_- - \mathbb{L}_-\mathbb{L}_-]\mathbb{L}_+ \\
&= \mathbb{L}_-[\mathbb{L}_+,\mathbb{L}_-] + [\mathbb{L}_-,\mathbb{L}_-]\mathbb{L}_+ \\
&= -\mathbb{L}_-[\mathbb{L}_-,\mathbb{L}_+] \\
&= -\mathbb{L}_-
\end{aligned}
$$

$$
\longrightarrow [\mathbb{L},\mathbb{L}_-] = \mathbb{L}\mathbb{L}_- - \mathbb{L}_-\mathbb{L} = -\mathbb{L}_-
$$

$$
\mathbb{L}\mathbb{L}_- = \mathbb{L}_-\mathbb{L} - \mathbb{L}_-
$$

Ahora, aplicamos

$$
\begin{aligned}
\mathbb{L}|z\rangle &= \mathbb{L}(\mathbb{L}_-|x\rangle) = (\mathbb{L}_-\mathbb{L} - \mathbb{L}_-)|x\rangle \\
&= \mathbb{L}_-(\mathbb{L}|x\rangle) - \mathbb{L}_-|x\rangle \\
&= \mathbb{L}_-(\lambda|x\rangle) - |z\rangle \\
&= \lambda|z\rangle - |z\rangle \\
&= (\lambda-1)|z\rangle \checkmark
\end{aligned}
$$