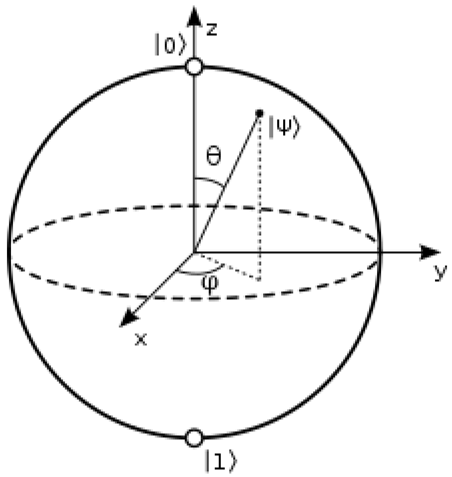
Intro to Quantum Computing

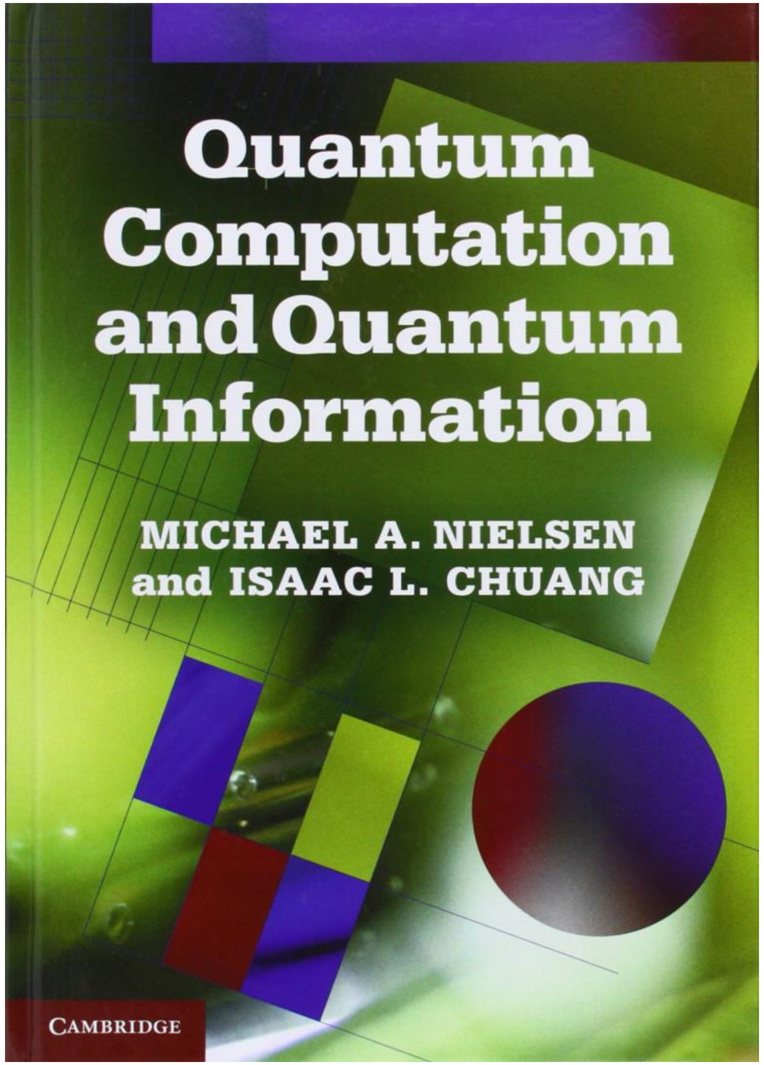
Maths of Qbits Part III



Today

- A *very gentle* intro to Postulates (axioms) of Quantum Mechanics
- At least through the lenses of Qbits
- Intro to Operators
- Projection operators, Pauli matrices. The Hadamard matrix

Today


Quantum Computation and Quantum Information
MICHAEL A. NIELSEN
and ISAAC L. CHUANG
CAMBRIDGE

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QM is the <u>operating system</u> that all physical theories run as application software

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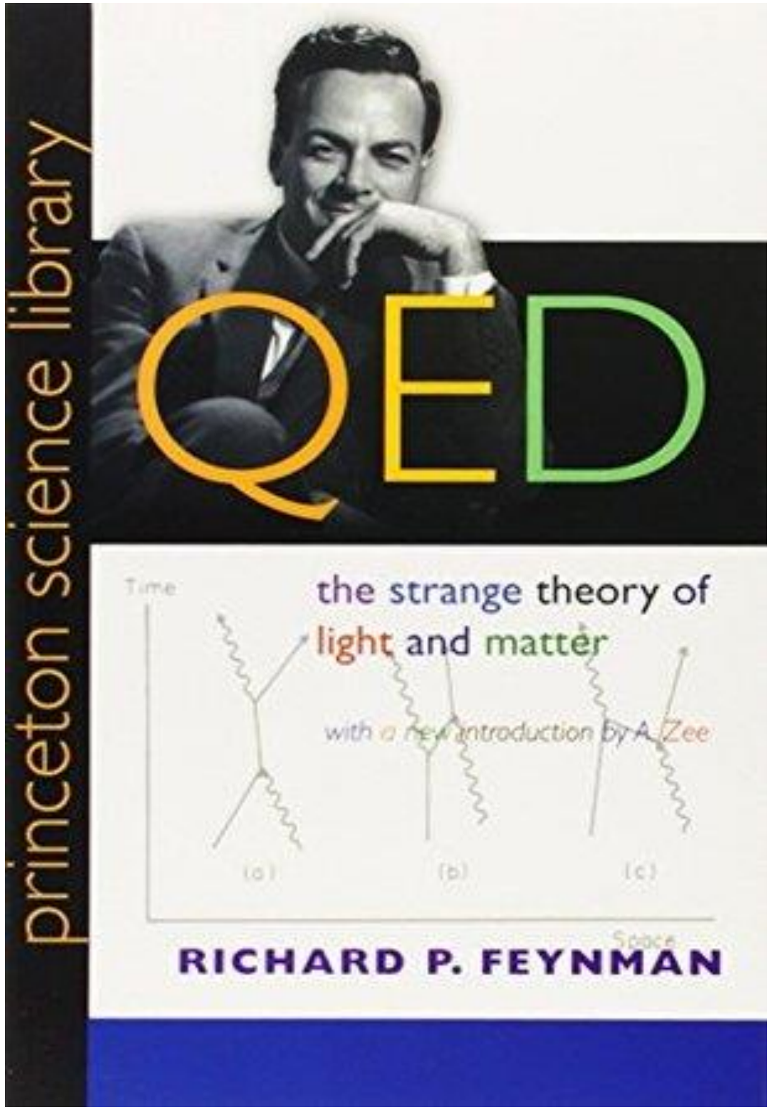
- One example of successful portability is ***Quantum Electrodynamics***
- Richard Feynman: [this theory is] "*The jewel of physics*"

Quantum Mechanics



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2. ***Evolution***: how a quantum state is allowed to change over time?
3. ***Measurement***: the effect on a quantum state of interaction with a classical system (apparatus) that yields *classical* information.
4. ***Composition***: How quantum systems can be composed?

State space of Quantum System

- ***Postulate 1***

Associated to any isolated physical system is a (complex) vector space equipped with an inner product (i.e., a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.

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- We have discussed physical realizations.

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The evolution over time of a quantum system is described by Schrödinger equation

$$ih\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = \boldsymbol{H}|\psi\rangle$$

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- Some authors in Quantum Computing call Postulate 2’ as the ***Fundamental Postulate of Quantum state evolution***

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- We want to transform $|\boldsymbol{\psi_1}\rangle = \boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle$ **to** $|\boldsymbol{\psi_2}\rangle = \boldsymbol{\gamma}|\mathbf{0}\rangle + \boldsymbol{\delta}|\mathbf{1}\rangle$**.**

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- The amplitudes of $|\psi_2\rangle$ must have the property that $\gamma^2 + \delta^2 = 1$.

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$\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \boldsymbol{U}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, where $\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = |\boldsymbol{\psi}_{\mathbf{2}}\rangle$ and $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = |\boldsymbol{\psi}_{\mathbf{1}}\rangle$ such that

$\left|\left||\boldsymbol{\psi}_2\rangle\right|\right|_2 = \left|\left||\boldsymbol{\psi}_{\mathbf{1}}\rangle\right|\right|_2 = 1$ (unit length vectors in ℓ_2)

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- All eigenvalues of unitary matrices have absolute value one.

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- The two “paths” leading to $|\boldsymbol{0}\rangle$ interfere destructively (cancel out).

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- I.e., if we apply it twice, it is the same as if we apply it once.

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- This is called the ***collapse of the wave function***.

The Pauli Matrices

- They are denoted by $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and they are important operators acting on one qbit.
- $\boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- It has the following effect on the standard basis:
- $\boldsymbol{X}|\mathbf{0}\rangle = |\mathbf{1}\rangle$
- $\boldsymbol{X}|\mathbf{1}\rangle = |\mathbf{0}\rangle$
- **Flip operator.**

The Pauli Matrices

- $\boldsymbol{Y} = i \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$
- It has the following effect on the standard basis:
- $\boldsymbol{Y}|\mathbf{0}\rangle = \boldsymbol{i}|\mathbf{1}\rangle$
- $\boldsymbol{Y}|\mathbf{1}\rangle = -\boldsymbol{i}|\mathbf{0}\rangle$
- Keep the bit unchanged but induces a phase change (in the complex plane)

The Pauli Matrices

- $\boldsymbol{Z} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}$
- It has the following effect on the standard basis:
- $\boldsymbol{Z}|\mathbf{0}\rangle = |\mathbf{0}\rangle$
- $\boldsymbol{Z}|\mathbf{1}\rangle = -|\mathbf{1}\rangle$
- The effect of $\boldsymbol{Z}$ (and $\boldsymbol{Y}$)is fundamentally quantum: it leaves the truth value (bit) unchanged but induces a phase change of -1 of the qbit is true.
- We can write $\boldsymbol{Y} = \boldsymbol{iXZ}$ which switches the two bit values and also adds a phase of $+i, -i$.

The Pauli Matrices

The Hadamard matrix

- It is defined as

$$\boldsymbol{H} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix}$$

- It has the following effect on the two basic states:
- $\boldsymbol{H}|\mathbf{0}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\,|\mathbf{0}\rangle + |\mathbf{1}\rangle)$
- $\boldsymbol{H}|\mathbf{1}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\,|\mathbf{0}\rangle - |\mathbf{1}\rangle)$.
- It puts the basis into a superposition.
- Useful when we want to prepare a totally random state.
- H also interferes the superposition so we can recover the original states.

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- Suppose we have now two (q)bits $\boldsymbol{x_1}, \boldsymbol{x_2}$.

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- We can even write $|\boldsymbol{x_1 x_2}\rangle = |\mathbf{3}\rangle$ meaning $|\boldsymbol{x_1 x_2}\rangle = |\mathbf{11}\rangle$ etc.

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- For example

$$|\mathbf{00}\rangle = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, |\mathbf{01}\rangle = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, |\mathbf{10}\rangle = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, |\mathbf{11}\rangle = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

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$$U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Composition of Quantum Systems

- ***Postulate 4***

$\boxed{\text{The state space of a composite physical system } |\psi\rangle \text{ is the tensor product of the state spaces of the constituent components physical systems, i.e., } |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle}$

I.e., if $|\psi\rangle$ is a 2-qbit system consisting of $|q_1\rangle, |q_2\rangle$ then $|\psi\rangle = |q_1\rangle \otimes |q_2\rangle$