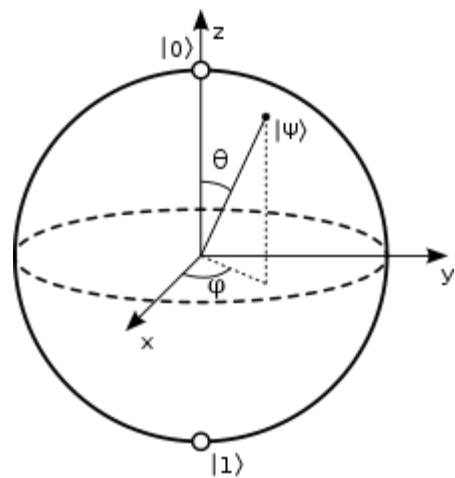


Intro to Quantum Computing

Maths of Qubits Part II



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- First, construct the matrix that correspond to the horizontal basis

$$H = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

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- Now, we compute $H^T |\uparrow\rangle$ to get the new probability amplitudes with respect to the horizontal basis:

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- So, when we measure $|\psi\rangle = |\uparrow\rangle$ in the horizontal direction we get that the electron is deflected either in left direction or in right direction, each with probability $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

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- ***Do you see why?***

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- **No experiment can distinguish them, no matter which basis we are using**

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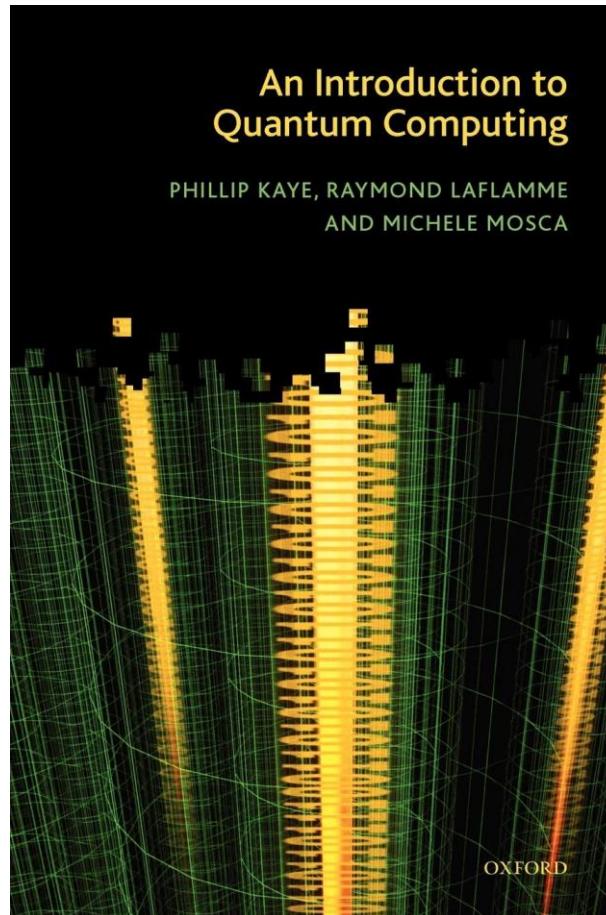
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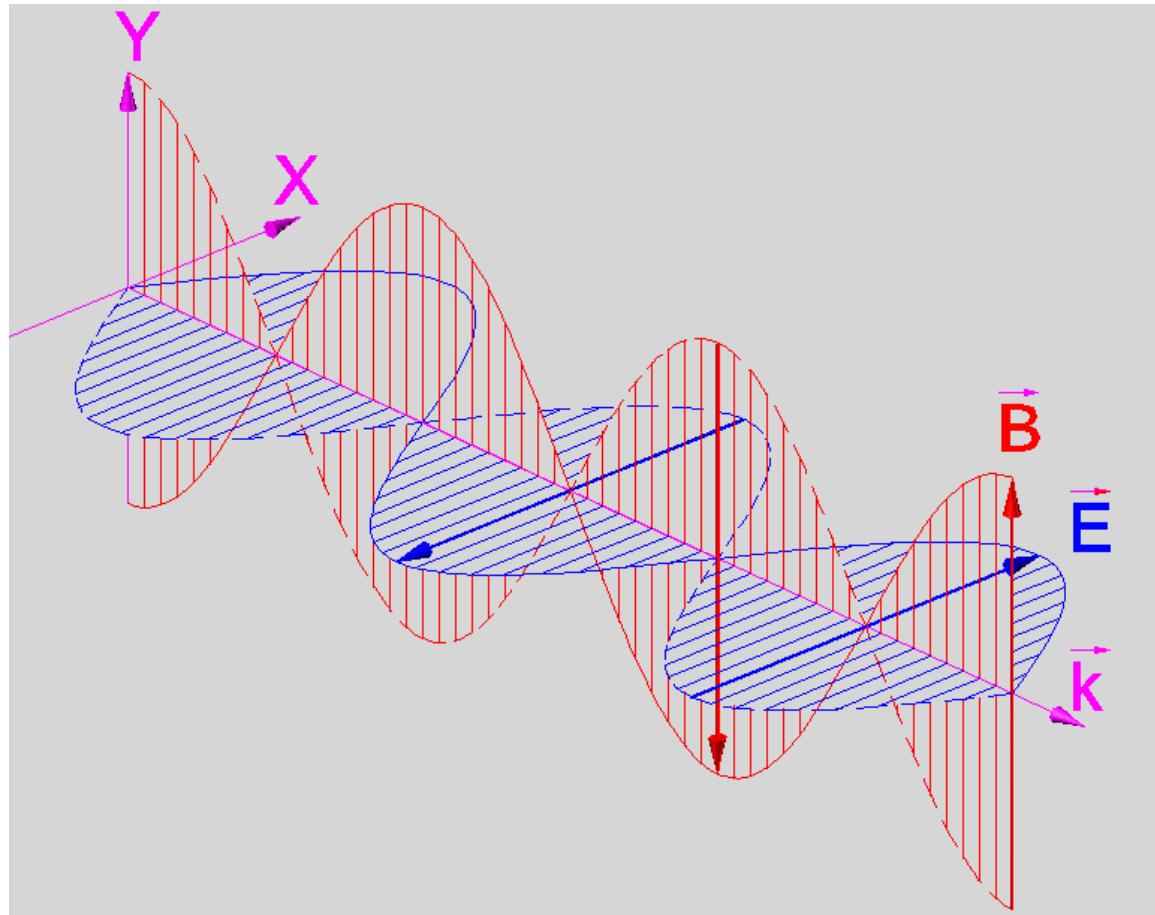
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- Is there any basis (rotation of the apparatus) that could help us distinguish them?

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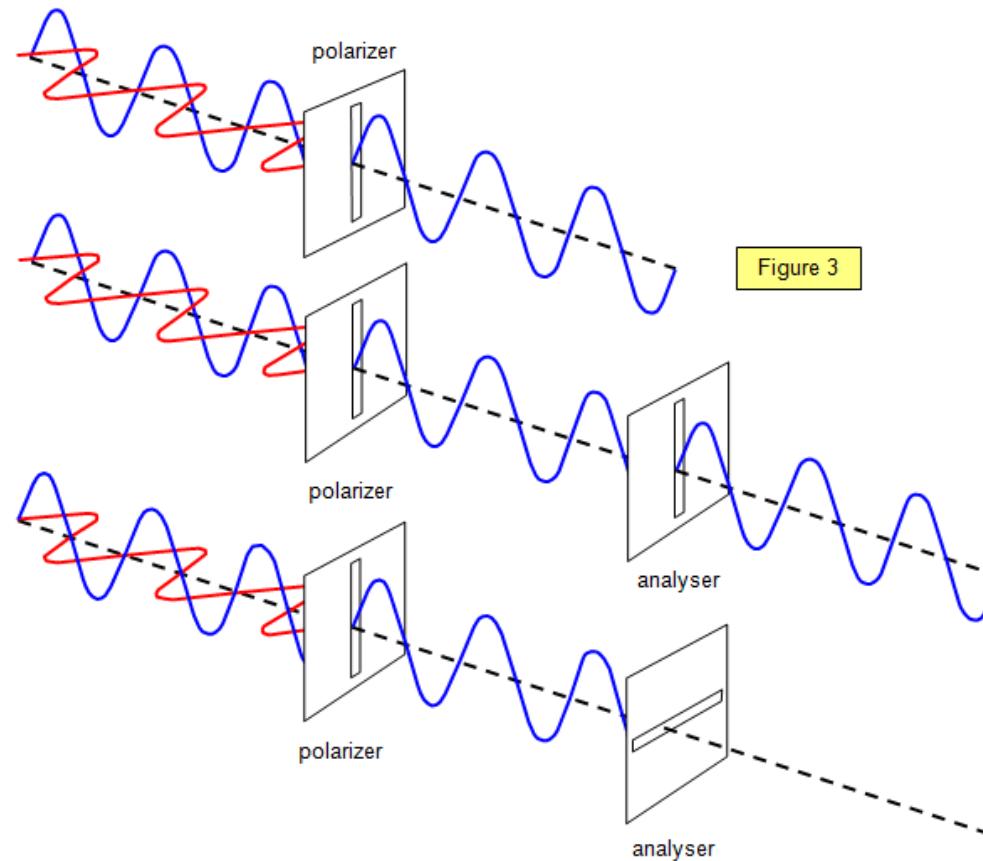
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- We can change the polarization of a photon by applying a **polaroid filter**.

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- Such a polaroid has an orientation (for example vertical)

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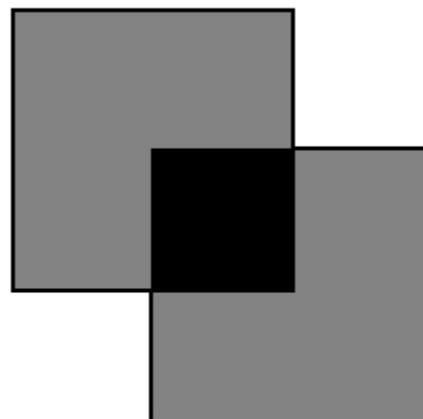
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- If a photon that has horizontal polarization tries to pass through that polaroid, ***it will be blocked with probability 1.***
- A photon with polarization 45° ***will pass through the polaroid with chance $\frac{1}{2}$ and will be blocked with probability $\frac{1}{2}.$***

Measuring polarization

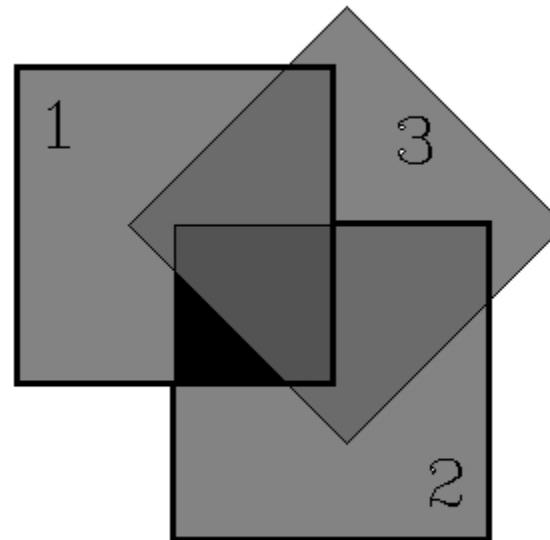
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Parallel axes.



Crossed axes.



Polarizer (3) between two crossed polarizers (1) and (2).

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- Are photons magically appearing?

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- Now it encounters the diagonally polarized filter.

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The photon passes the 1st filter with probability $\cos^2\theta$ and its new state is $|\uparrow\rangle$.

- Now it encounters the diagonally polarized filter.
- ***What is the chance it will pass through?***

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- The photon will pass the polaroid with probability $\frac{1}{2}$!
- So the total probability that the photon will survive the filters is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Equivalent states and Uncertainty principle

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$$\Delta x \Delta p \geq -\frac{\hbar}{2}$$

Equivalent states and Uncertainty principle

- We will motivate the uncertainty principle using qbits and bases.
- It will not be as precise as Heisenberg's principle.
- But at least it will give us an understanding on what it actually means
- At least in the context of qbits.

Equivalent states and Uncertainty principle

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- Let's assume that we have two different bases for our qbit.

Equivalent states and Uncertainty principle

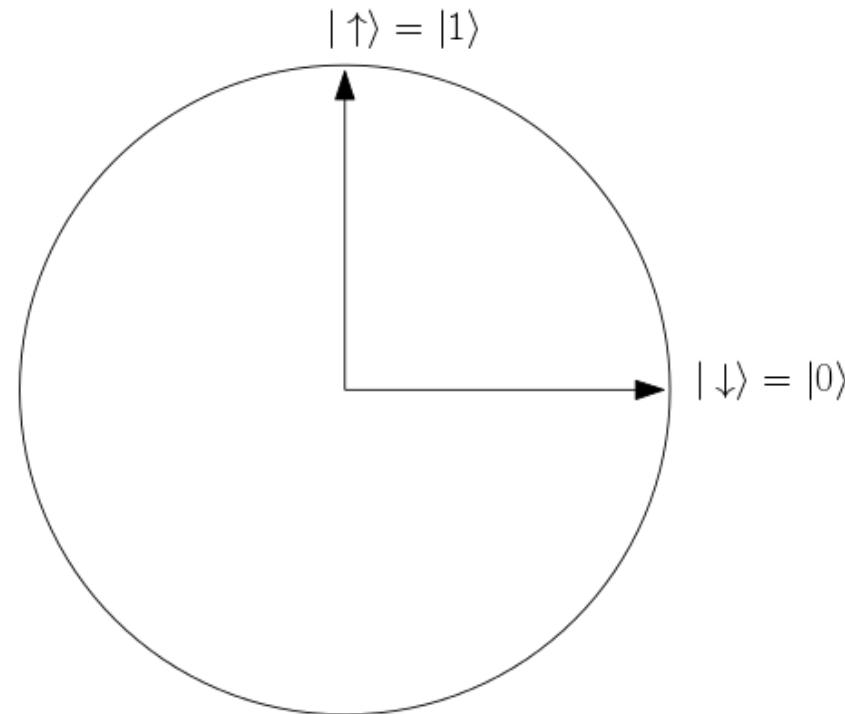
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 1. The “standard” vertical basis $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Equivalent states and Uncertainty principle

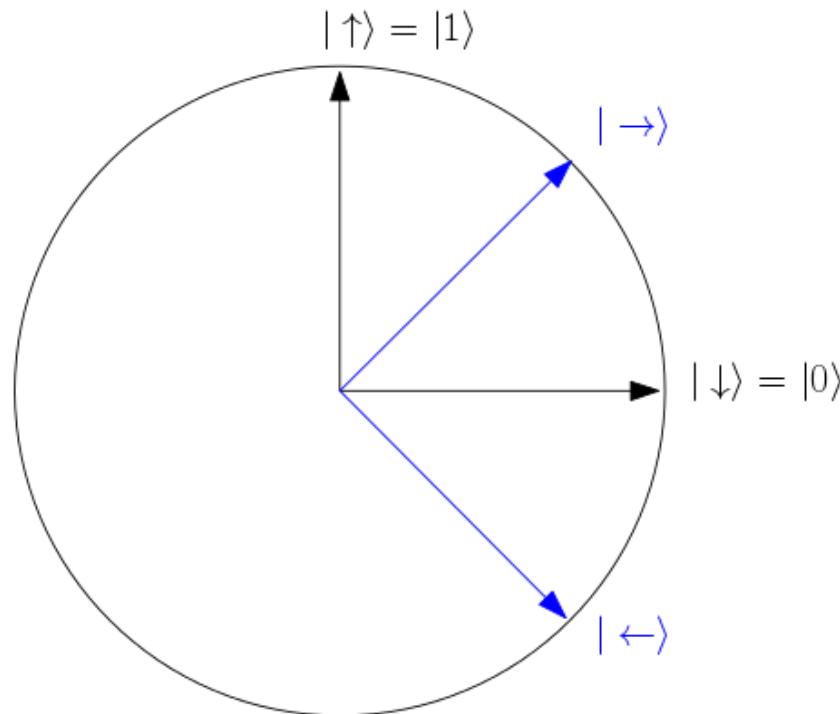
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 2. And the horizontal basis

$$|\rightarrow\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, |\leftarrow\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

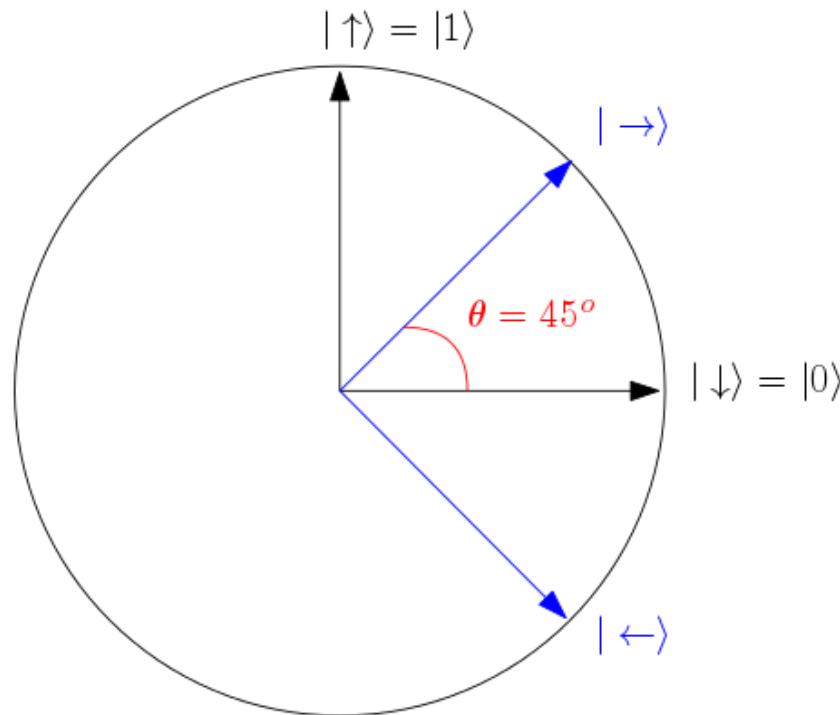
Equivalent states and Uncertainty principle



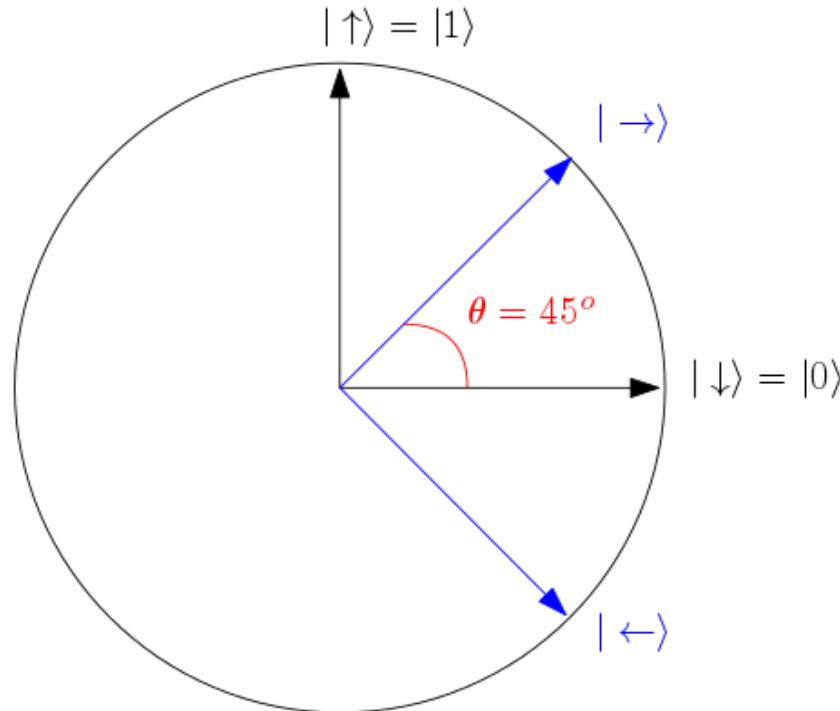
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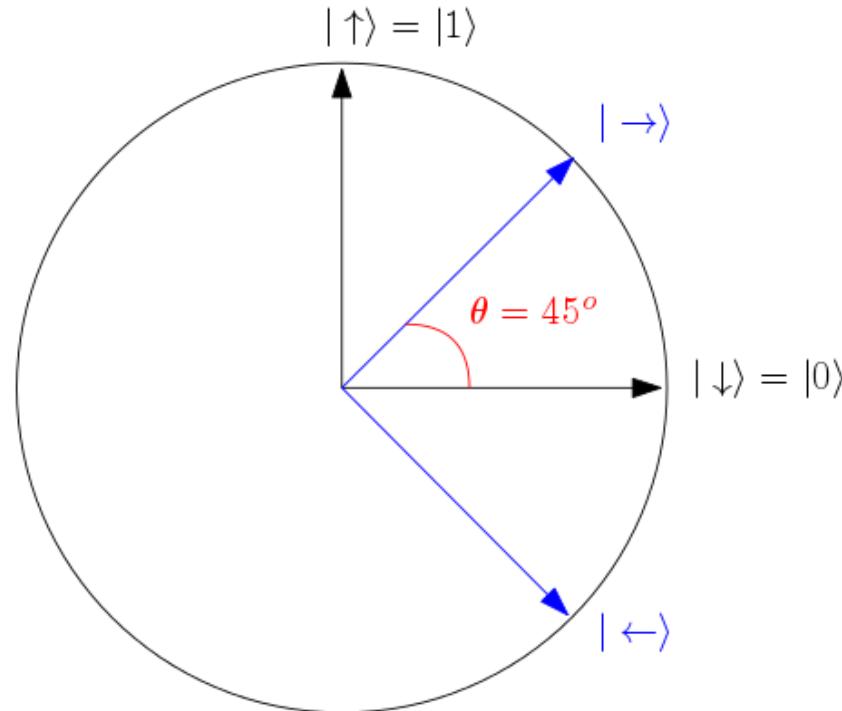


Equivalent states and Uncertainty principle



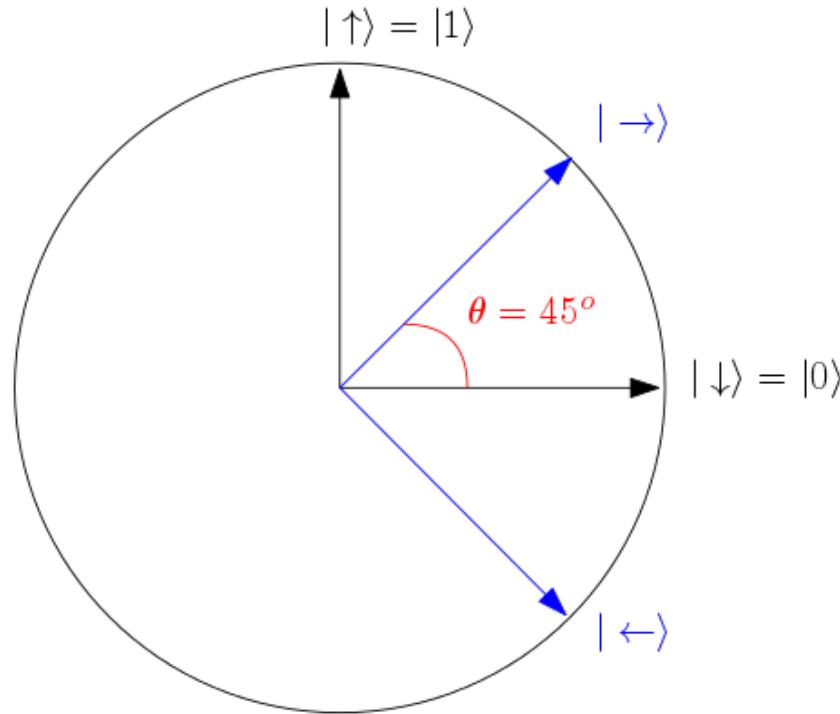
- The angle of the two bases is **not** the same as the angle we rotate the apparatus!!!

Equivalent states and Uncertainty principle



- 90° rotation of the apparatus \Rightarrow the basis rotates by 45° .

Equivalent states and Uncertainty principle



- In general: θ degrees rotation of the apparatus \Rightarrow the basis rotates by $\frac{\theta}{2}$.

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- The state $|\rightarrow\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle$ and $|\leftarrow\rangle = -\frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle$

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- Assume we have a state $|\psi\rangle$ and we measure it in the $|0\rangle$, $|1\rangle$ basis.
- $|\psi\rangle$ has some probability being in either of the two basic states.
- We can determine what is the value of $|\psi\rangle$ by simply measuring it and destroying its superposition.

Equivalent states and Uncertainty principle

- But besides the $|0\rangle$, $|1\rangle$ value of $|\psi\rangle$ we might be interested in some other physical quantity associated with $|\psi\rangle$.

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- Assume that the first 0-1 base corresponds to the position of the electron.
- And the second left-right base corresponds to its velocity.

Equivalent states and Uncertainty principle

- Question

Is it possible to know with perfect accuracy both the position and the velocity of the particle described by $|\psi\rangle$?

Equivalent states and Uncertainty principle

- Answer

Heisenberg

*says
NO!*

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- This is a consequence of the fact that the 2nd basis is a rotation of 45° of the 1st one.

Equivalent states and Uncertainty principle

- Quantitatively, let's assume that $|\psi\rangle = a_1|\uparrow\rangle + a_2|\downarrow\rangle$ in the first basis and $|\psi\rangle = b_1|\rightarrow\rangle + b_2|\leftarrow\rangle$ in the 2nd basis.
- Define $S_{\uparrow,\downarrow} = |a_1| + |a_2|$ the “spread” of the first basis.
- Similarly $S_{\rightarrow,\leftarrow} = |b_1| + |b_2|$.
- If we knew perfectly (i.e., measure) the “up-down” value of $|\psi\rangle$ then $S_{\uparrow,\downarrow} = 1$.
- Similarly, if we know (i.e., measure) the “left-right” value of $|\psi\rangle$ then $S_{\rightarrow,\leftarrow} = 1$.
- But if we are in the state $|\psi\rangle = |\rightarrow\rangle$ then $|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and in this case $S_{\uparrow,\downarrow} = \sqrt{2}$.

Equivalent states and Uncertainty principle

- Identically, if we are in the state $|\psi\rangle = |\downarrow\rangle$ then $|\psi\rangle = \frac{1}{\sqrt{2}}| \rightarrow \rangle + \frac{1}{\sqrt{2}}| \leftarrow \rangle$ and in this case $S_{\rightarrow,\leftarrow} = \sqrt{2}$.
- The more certain we are about one of the states, the more uncertain we are for the other.

$$S_{\rightarrow,\leftarrow}(|\psi\rangle) \cdot S_{\uparrow,\downarrow}(|\psi\rangle) \geq \sqrt{2}$$

Equivalent states and Uncertainty principle

- Identically, if we are in the state $|\psi\rangle = |\downarrow\rangle$ then $|\psi\rangle = \frac{1}{\sqrt{2}}|-\rangle + \frac{1}{\sqrt{2}}|+\rangle$ and in this case $S_{\rightarrow, \leftarrow} = \sqrt{2}$.
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