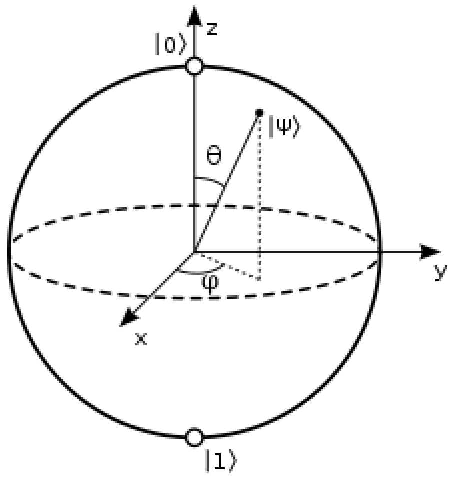
Intro to Quantum Computing

Entanglement II



Recall two Qbit Systems

- The general state of a 2-Qbit system is

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

- Such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

- In general, there is no distance limitation between the Qbits, they could be anywhere, but still we can define the system.
- What would happen if we want to measure the 1st Qbit?

Recall two Qbit Systems

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

- Probability that the 1st Qbit measures $|0\rangle$ is?
- Probability that the 1st Qbit measures $|1\rangle$ is?
- Suppose we measure it $|0\rangle$.
- What can we say about the 2nd Qbit?
- We have narrowed down the possibilities:

$$a|00\rangle + b|01\rangle$$

- I.e., the state of the system is now in the superposition

$$\boldsymbol{a}|\mathbf{0}\rangle|\mathbf{0}\rangle + \boldsymbol{b}|\mathbf{0}\rangle|\mathbf{1}\rangle = |\mathbf{0}\rangle(\boldsymbol{a}|\mathbf{0}\rangle + \boldsymbol{b}|\mathbf{1}\rangle) = |\mathbf{0}\rangle \otimes + \frac{\boldsymbol{a}|\mathbf{0}\rangle + \boldsymbol{b}|\mathbf{1}\rangle}{\sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2}}$$

- ***Partial Measurement rule!***

Operations on 2-Qbits

- CNOT operator (Controlled not)

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

- What does it do?

Operations on 2-Qbits

- What if we wanted a flip (NOT) operator on the 2nd Qbit?

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

- Can be written as $\boldsymbol{I} \otimes \boldsymbol{NOT}$

$$\boldsymbol{I} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \boldsymbol{NOT} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix},$$

NOT simply flips the value of the Qbit.

Operations on 2-Qbits

- What if we wanted a flip (NOT) operator on the 1st Qbit?

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

- Can be written as $\boldsymbol{NOT} \otimes \boldsymbol{I}$
- ***The amplitude of $|\mathbf{00}\rangle$ of the input becomes the amplitude of the $|\mathbf{10}\rangle$ of the output.***

Operations on 2-Qbits

- Most of the time, in a multi-Qbit system we would like to perform operations to ***some*** of the Qbits and leave the other Qbits intact.
- For example: “apply Hadamard on the third Qbit”

$$\boldsymbol{H} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

- **H** does the following: it maps the $|\mathbf{0}\rangle, |\mathbf{1}\rangle$ basis to the $|{\rightarrow}\rangle, |{\leftarrow}\rangle$**.**
- Remember

$$|{\rightarrow}\rangle = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \end{bmatrix}, |{\leftarrow}\rangle = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \\ \frac{\mathbf{-1}}{\sqrt{\mathbf{2}}} \end{bmatrix}$$

Operations on 2-Qbits

- I.e. $\boldsymbol{H}|\mathbf{0}\rangle = |\rightarrow\rangle$ **and** $\boldsymbol{H}|\rightarrow\rangle = |\mathbf{0}\rangle$**.**
- Similarly for $\boldsymbol{H}|\mathbf{1}\rangle = |\leftarrow\rangle$ and $\boldsymbol{H}|\leftarrow\rangle = |\mathbf{1}\rangle$.
- Note that some books use the **+,-** sign for the same basis:
$$|+\rangle = |\rightarrow\rangle, |-\rangle = |\leftarrow\rangle$$
- Note that by this operator, we have two orthogonal and complementary bases:
- Being certain in one basis means that we have to be uncertain in the other basis.
- That was ***Heisenberg's uncertainty principle.***

Operations on 2-Qbits

- “apply Hadamard on the third Qbit”
- What we really mean is apply the unitary matrix

$$I \otimes I \otimes H \otimes I \cdots \otimes I$$

Theorem: $\boldsymbol{U_1, U_2}$ are unitary $\Rightarrow$ $\boldsymbol{U_1 \otimes U_2}$ is a unitary.

- So we can use tensor products to built larger unitary matrices.
- Except CNOT operator (where the 1^{st} Qbit affects the 2^{nd}).
- We cannot built the CNOT tensoring elementary matrices.

Operations on 2-Qbits

- Let’s see what we can do with these operations.
- Let’s start with 2-Qbits in $|00\rangle$.
- Apply Hadamard on the 1st Qbit =

$$(\boldsymbol{H} \otimes \boldsymbol{I})|\mathbf{00}\rangle = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \\ \mathbf{0} \\ \mathbf{1}/\sqrt{\mathbf{2}} \\ \mathbf{0} \end{bmatrix} = |\rightarrow\rangle \otimes |\mathbf{0}\rangle$$

Operations on 2-Qbits

- Now apply CNOT with 1st Qbit the control bit and 2nd Qbit the target:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \text{Bell state!}$$

- It is really no wonder we ended up with an entangled state (started from an unentangled) given that we have applied the CNOT operator.

Measure in different bases and EPR paradox

- Now, let's take the previous bell state.
- What would happen if we do not want to measure in the $|0\rangle, |1\rangle$ basis, but instead we want to measure in the $|{\rightarrow}\rangle, |{\leftarrow}\rangle = |+\rangle, |-\rangle$?

- **Theorem:**

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

Measure in different bases and EPR paradox

- What if Alice measure in her $|+\rangle, |-\rangle$ basis?
- What if Alice measure in her $|0\rangle, |1\rangle$ basis?
- In each case, Bob's state will collapse accordingly.
- Einstein, Podolsky and Rosen argued that this is very troubling.
- Bob's Qbit seems to have ***simultaneously exact values for both bases***.
- It seems to violate uncertainty principle ***and*** it looks like faster than light communication!

EPR appeared to have contrived a means to establish the exact values of either the momentum or the position of B due to measurements made on particle A, without the slightest possibility of particle B being physically disturbed. ***(wiki)***