Exercise Sheet -2-

Sample Solutions

Ex. 1

We have a two-qbit system $0.8|00\rangle + 0.6|11\rangle = |\psi\rangle$

- Are they entangled or not?

Assume they are not entangled $\Rightarrow |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.
I.E $|\psi\rangle$ can be written as the tensor product of two individual 1-qbit states.

Let $\left.\begin{aligned} |\psi_1\rangle &= a|0\rangle + b|1\rangle \\ |\psi_2\rangle &= c|0\rangle + d|1\rangle \end{aligned}\right\} \Rightarrow$

$$|\psi_1\rangle \otimes |\psi_2\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) =$$

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$$= |\psi\rangle = 0.8|00\rangle + 0.6|11\rangle$$

$\Rightarrow$ $ac = 0.8$, $bd = 0.6$ $\Rightarrow a, b, c, d \neq 0$
$ad = bc = 0.$ $\Rightarrow (a \text{ or } d = 0)$ and $(b \text{ or } c = 0)$.

$\Rightarrow$ Contradiction ! No way to find a, b, c, d satisfying the four above equations

$\Rightarrow$ The qbits are entangled.

- We apply the Pauli-X gate (transform) on the second qbit.

$$X = \text{Bit flip} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ is a single qbit transformation.

I.e we leave the 1st qbit unchanged (which is equivalent to apply the identity transformation to it) and X on 2nd qbit

⇒ The two-qbit unitary transform describing the above, is the tensor product of the two individual operations:

$$I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = B$$

- Now we apply B to $|\psi\rangle$: $B|\psi\rangle =$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0 \\ 0 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8 \\ 0.6 \\ 0 \end{bmatrix} = \boxed{0.8|01\rangle + 0.6|10\rangle}$$

Ex. 2 Show that

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A|\psi\rangle) \otimes (B|\phi\rangle).$$

A, B 2x2 matrices, $|\psi\rangle, |\phi\rangle$ single Qbit states.

• Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$,

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\phi\rangle = c|0\rangle + d|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}.$$

We have: $A \otimes B = \begin{bmatrix} a_1 \cdot B & a_2 \cdot B \\ a_3 \cdot B & a_4 \cdot B \end{bmatrix} =$

$$= \begin{bmatrix} a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2 \\ a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_4b_1 & a_4b_2 \\ a_3b_3 & a_3b_4 & a_4b_3 & a_4b_4 \end{bmatrix} \rightsquigarrow 4 \times 4 \text{ matrix.}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

So, $(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) =$

$$= \begin{bmatrix} a_1b_1 \cdot ac + a_1b_2 \cdot ad + a_2b_1 \cdot bc + a_2b_2 \cdot bd \\ a_1b_3 \cdot ac + a_1b_4 \cdot ad + a_2b_3 \cdot bc + a_2b_4 \cdot bd \\ a_3b_1 \cdot ac + a_3b_2 \cdot ad + a_4b_1 \cdot bc + a_4b_2 \cdot bd \\ a_3b_3 \cdot ac + a_3b_4 \cdot ad + a_4b_3 \cdot bc + a_4b_4 \cdot bd \end{bmatrix}$$

↳ 4x1 VECTOR.

- WE DO THE SAME FOR $(A|\psi\rangle) \otimes (B|\phi\rangle)$

$$= \begin{bmatrix} a_1 \cdot a + a_2 \cdot b \\ a_3 \cdot a + a_4 b \end{bmatrix} \otimes \begin{bmatrix} b_1 \cdot c + b_2 \cdot d \\ b_3 \cdot c + b_4 \cdot d \end{bmatrix}.$$

$$= \begin{bmatrix} (a_1 \cdot a + a_2 \cdot b) \cdot \begin{bmatrix} b_1c + b_2d \\ b_3c + b_4d \end{bmatrix} \\ (a_3 \cdot a + a_4 \cdot b) \cdot \begin{bmatrix} b_1c + b_2d \\ b_3 \cdot c + b_4 \cdot d \end{bmatrix} \end{bmatrix} =$$

$$= \begin{bmatrix} a_1b_1 \cdot ac + a_1b_2 \cdot ad + a_2b_1 \cdot bc + a_2b_2 \cdot bd \\ a_1b_3 \cdot ac + a_1b_4 \cdot ad + a_2b_3 \cdot bc + a_2b_4 \cdot bd \\ a_3b_1 \cdot ac + a_3b_2 \cdot ad + a_4b_1 \cdot bc + a_4b_2 \cdot bd \\ a_3b_3 \cdot ac + a_3b_4 \cdot ad + a_4b_3 \cdot bc + a_4b_4 \cdot bd \end{bmatrix}$$ ✓

THE SAME!

Ex. 3

- If 1st Qbit is $|0\rangle$, then 2nd Qbit is unchanged
- If ~~2nd~~ 1st Qbit is $|1\rangle$, then 2nd Qbit passes through Hadamard.

This is part of a more general process:
- If 1st Qbit is $|0\rangle \to$ Do nothing on 2nd
  " " " $|1\rangle \to$ apply some unitary U on the second.

This leads to the following mapping of the states:

$|00\rangle$ is mapped to $|00\rangle$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |1\rangle \otimes U|0\rangle$

$|11\rangle \longrightarrow |1\rangle \otimes U|1\rangle$

where $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ is a 1-Qbit unitary transform.

So, the matrix representing the above "Controlled-U" operator is.

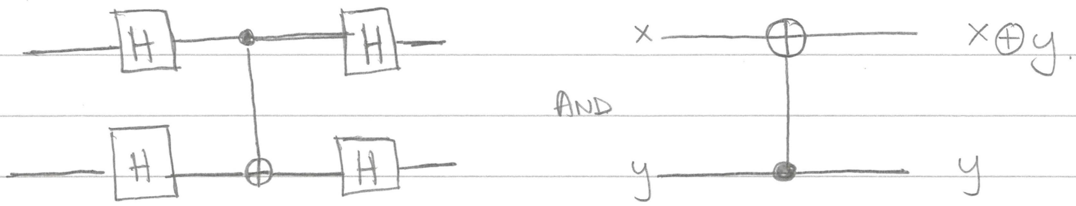
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_1 & u_2 \\ 0 & 0 & u_3 & u_4 \end{bmatrix}$$

, if $U = H$ we replace

$$\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \text{ with } \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = H$$

Ex. 4

We are asked to show that the two circuits in the figure are equivalent:



Observe that what we do on the 1st circuit is to first perform a change of basis step (from $\{|0\rangle, |1\rangle\}$ to $\{|+\rangle, |-\rangle\}$ basis) apply CNOT, perform again a reverse change of basis step.

In order to see what is happening, we work with each of the 4 basic Hadamard states:

$\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

So, if both qbits are in $|+\rangle$ state after Hadamard, then, the corresponding state in $\{|0\rangle, |1\rangle\}$ basis is

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right).$$

- We perform CNOT on the state above with control bit the 1st, and target bit the 2nd:

$$\text{CNOT}\left(\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right) = \frac{1}{2}\left(|00\rangle + |01\rangle + |11\rangle + |10\rangle\right)$$

$$= |++\rangle.$$

We continue the same way $\longrightarrow$

- $|+-\rangle$ corresponds to $\frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$

apply CNOT and we get

$$\frac{1}{2}\left(|00\rangle - |01\rangle + |11\rangle - |10\rangle\right) = |--\rangle$$

- $|-+\rangle \rightarrow \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$

apply CNOT and we get

$$\frac{1}{2}\left(|00\rangle + |01\rangle - |11\rangle - |10\rangle\right) = |-+\rangle$$

- $|--\rangle \rightarrow \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$

which, after we apply CNOT, becomes

$$\frac{1}{2}\left(|00\rangle - |01\rangle - |11\rangle + |10\rangle\right) = |+-\rangle$$

So... the application of CNOT to $\{|+\rangle, |-\rangle\}$ basis has the following effect: it maps

$$\left.\begin{aligned} |++\rangle &\rightarrow |++\rangle \\ |+-\rangle &\rightarrow |--\rangle \\ |-+\rangle &\rightarrow |-+\rangle \\ |--\rangle &\rightarrow |+-\rangle \end{aligned}\right\}$$

we see that the 2ND QBIT is always UNCHANGED, the 1ST QBIT flips from $|-\rangle$ to $|+\rangle$ or from $|+\rangle$ to $|-\rangle$ if the 2ND QBIT is $|-\rangle$!

$\Rightarrow$ It has the same effect as performing a CNOT with 2ND QBIT as control and 1st as target !

You could also prove it algebraically:
- We can describe the circuit in terms of matrix multiplications:

$$(H \otimes H) \cdot CNOT \cdot (H \otimes H) = H^{\otimes 2} \cdot CNOT \cdot H^{\otimes 2}$$

↳ this is the right most Hadamard !

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

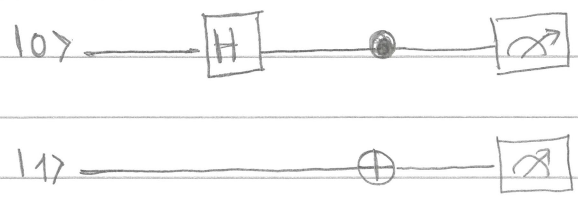
⤳ And this matrix corresponds to the 2nd circuit !

$\boxed{\text{Ex. 5}}$

This is easy one !
If Alice measures her qubit, then its state will collapse to either $|0\rangle$ or $|1\rangle$. If she keeps measuring after that, then she will keep observing the initial result of her measurement over and over again...

EX. 6

- The circuit is depicted below:



- The initial state of the system is $|\psi_0\rangle = |0\rangle \otimes |1\rangle$
$= |01\rangle$

- We apply H on 1st qbit and leave the 2nd qbit unchanged:

$$(H \otimes I)|0\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\psi_1\rangle$$

- Apply CNOT to $|\psi_1\rangle$:

$$|\psi_2\rangle = \text{CNOT}(|\psi_1\rangle) = \frac{1}{\sqrt{2}}\text{CNOT}(|01\rangle) + \frac{1}{\sqrt{2}}\text{CNOT}(|11\rangle)$$
$$= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle = |\psi_3\rangle.$$

- We perform measurement:

- with probability $(1/\sqrt{2})^2 = 1/2$ we observe $|01\rangle$

" " " $1/2$ we observe $|10\rangle$