- EPR (EINSTEIN-PODOLSKI-ROSEN) PARADOX
- A THOUGHT EXPERIMENT.
- IT SUGGESTED THAT QUANTUM MECHANICS ALLOW SYSTEMS VERY FAR AWAY FROM EACH OTHER TO INSTANTANEOUSLY COORDINATE THEIR ACTIONS.
- SEEMINGLY VIOLATING EINSTEIN'S GENERAL RELATIVITY THEORY THAT NOTHING CAN TRAVEL FASTER THAN LIGHT.
- AND ALSO HEISENBERG'S UNCERTAINTY PRINCIPLE.
- THE EPR PARADOX WAS BASED ON THE OBSERVATION THAT THE ENTANGLED STATE

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

IS EQUAL TO

$$\frac{1}{\sqrt{2}}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle \quad \left(= \frac{1}{\sqrt{2}}|\rightarrow\rightarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\leftarrow\rangle \right).$$

- IN 1964 JOHN BELL SHOWED HOW WE CAN ACTUALLY TEST THIS THOUGHT EXPERIMENT
- TWO SYSTEMS FAR AWAY FROM EACH OTHER HAVE A SHARED QUANTUM STATE I.E., A SYSTEM OF TWO QBITS.
- THIS SHARED STATE ALLOWS THEM TO COORDINATE THEIR ACTIONS IN A WAY THAT IS PROVABLY IMPOSSIBLE IN A NON-QUANTUM SYSTEM.

- BELL'S THEOREM HAS BEEN SIMPLIFIED BY CLAUSER, HORNE, SIMONY, HOLT (1969) PHYSICAL REVIEW LETTERS

- THE SYSTEMS INVOLVED DO NOT TRANSMIT INFORMATION, THEY ACT ON INFORMATION ALREADY SHARED, IN THE FORM OF QUANTUM SUPERPOSITION.

THE SET-UP : ALICE AND BOB PLAY A GAME.

1) ALICE AND BOB ARE SEPARATED BY A LONG DISTANCE
2) WE CHOOSE TWO RANDOM BITS $x, y \in \{0,1\}$.
3) WE GIVE x TO ALICE AND
   y TO BOB.
4) ALICE RESPONDS WITH A BIT a.
   BOB RESPONDS WITH A BIT b.
5) ALICE AND BOB WIN THE GAME iff
   $a \oplus b = x \wedge y$
   $\oplus$ is THE XOR (EXCLUSIVE-OR) OPERATION
   $\wedge$ is THE logical AND OPERATION.

OBSERVATION: THERE IS A STRATEGY WHERE ALICE AND BOB CAN WIN WITH PROBABILITY $\geq 3/4$.

SURPRIZINGLY (OR NOT) THIS IS THE BEST THEY CAN DO.!

- ALICE AND BOB DEFINE THEIR OWN STRATEGIES $f, g : \{0,1\} \to \{0,1\}$ SUCH THAT THEIR ANSWERS DEPEND ONLY ON THE INFORMATION THEY HAVE SEEN (THEY ARE NOT ALLOWED TO EXCHANGE INFO).
- Namely: $a = f(x)$
  $b = g(y)$.
- ASSUME THAT THERE IS A PAIR OF STRATEGIES f, g, THAT allows THEM TO WIN WITH PROBABILITY $> 3/4$.
- Without loss of GENERALITY, THESE STRATEGIES ARE DETERMINISTIC (NO RANDOMNESS).
- THIS IS BECAUSE THERE SHOULD BE A FIXED CHOICE OF THE player's Randomness that SUCCEEDS with at least THE SAME PROBABILITY.

$\longrightarrow$

- Now, f (Alice's function) can be one of four possible functions:
  (1) $f(x)=0$ (allways output 0 no matter what x is)
  (2) $f(x)=1$
  (3) $f(x)=x$
  (4) $f(x)=1-x$

- Let's say $f(x)=x$ (the other three cases are handled identically).
- So, $f(x)=x=a$.
- The players Alice and Bob win iff $(x \wedge y) \oplus a = b$.

- Bob only knows y, and nothing else.
- If $y=1$, then $x \wedge y = x$
- They will win if $b = (x \wedge y) \oplus x = x \oplus x = 0$.
- With probability 1!
- If $y=0$, then $(x \wedge y) \oplus a = a$
- But Bob does not know a!.
- So the probability $b=a$ is $1/2$.

$\Rightarrow$ Total probability of win $\leq \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{3}{4}}$

- The same game sharing quantum information.

- Alice and Bob can share a two-qubit system
- They prepare their qubits to the Bell state

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

- Each one takes her/his qubit and move away.

- THEY KEEP THEIR QBITS SEALED BUT THEY DO NOT MEASURE THEM YET!
- NOW, ALICE RECEIVES x.
  - IF $x=1$ SHE ROTATES HER QBIT BY $\pi/8 = 22.5°$ I.E SHE applies THE Unitary Transformation U THAT ROTATES HER MEASURING BASE BY $\pi/8$. SHE DOESN'T CARE WHAT BOB DOES.
  - IF $x=0$, SHE DOES NOTHING.
- BOB RECEIVES y.
  - IF $y=1$, HE ROTATES BY $-\pi/8 = -22.5°$ $\Rightarrow$ ROTATES HIS MEASURING BASE BY $-\pi/8$.
- ALICE AND BOB MEASURE THEIR QBITS IN THEIR NEW BASES, AND OUTPUT a, b.

LET'S ANALYSE THE PROBABILITY THEY WILL WIN THE GAME:

- THEY WIN IF $a \neq b$ WHENEVER $x=y=1$
  OR $a=b$ WHENEVER $x \neq y$. OR $x=y=0$.

INTUITION: UNLESS $x=y=1$, THE STATES OF THE TWO QBITS WILL BE CLOSE TO EACH OTHER: THEIR ANGLE WILL BE AT MOST $\pi/8$.

- IF $x=y=1$, THE STATES WILL BE FAR APART BY $\pi/4$.

WE WILL NOW CALCULATE THE FOLLOWING CASES:

- $x=y=0$.
  $\Rightarrow$ ALICE AND BOB PERFORM NO OPERATION ON THEIR QBITS.
  $\Rightarrow$ WHEN THEY MAKE MEASUREMENT THEIR STATES WILL COLLAPSE TO EITHER $|00\rangle$ OR $|11\rangle$.

IN EITHER CASE THEY WIN THE GAME WITH PROBABILITY 1.

▷ 2ND CASE $x \neq y$.
IN THIS CASE AGAIN ALICE AND BOB WIN THE GAME if they output the same answer $a = b$.

- Let's analyze the case $x=0$ and $y=1$.
- IN THIS CASE, ALICE DOES NOTHING TO HER QBIT BUT BOB ROTATES THIS MEASURING BASE BY $-\pi/8$.
- WITH PROBABILITY $1/2$ ALICE WILL MEASURE $|0\rangle$ and BOB'S QBIT will COLLAPSE TO $|0\rangle$ ROTATED BY $-\pi/8$.

⇒ BOB will OBTAIN $|0\rangle$ with PROBABILITY $\cos^2 \pi/8$.

- Similarly for the $|1\rangle$ CASE.

▷ 3RD CASE $x = y = 1$.
IN THIS CASE BOTH ALICE AND BOB ROTATE THEIR QBITS. NOW, THE 2-QBIT SYSTEM IS IN THE STATE:

$$\frac{1}{\sqrt{2}}\left(\cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle\right)\left(\cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle\right) +$$

$$\frac{1}{\sqrt{2}}\left(-\sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle\right)\left(\sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle\right).$$

$$= \left(\cos^2 \pi/8 - \sin^2 \pi/8\right)|00\rangle + \left(-2\sin\pi/8 \cos\pi/8\right)|01\rangle +$$

$$+ 2\sin\pi/8 \cos\pi/8 \,|10\rangle + \left(\cos^2\pi/8 - \sin^2\pi/8\right)|11\rangle.$$

all coefficients have same value ! use:

$$\cos^2 \pi/8 - \sin^2 \pi/8 = \cos \pi/4 = \frac{1}{\sqrt{2}} = \sin \pi/4 = 2\sin\pi/8 \cos\pi/8$$

⇒ WHEN MEASURED, THE SYSTEM WILL BE IN $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ WITH EQUAL PROBABILITY $= 1/4$.

⇒ IN THIS CASE, ALICE AND BOB WIN with PROB $1/2$.

Total probability of success: $\frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \cos^2 \pi/8 + \frac{1}{4} \cdot \frac{1}{2} = 0.8$!

Bell State and Rotations.

- We have seen that

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|{++}\rangle + \frac{1}{\sqrt{2}}|{--}\rangle$$

where $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, and

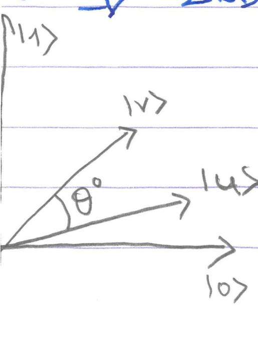
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

- This property of the Bell state holds for any other orthonormal basis u, u', where $\langle u|u'\rangle = 0$.
- Namely:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|uu\rangle + \frac{1}{\sqrt{2}}|u'u'\rangle.$$

- Now, if we measure the 1st Qbit in the $\{u, u'\}$ basis, the probability we will measure $u = 1/2$.
- If now we measure the 2nd Qbit in $\{u, u'\}$ the outcome of the measurement must be u.
- Always the same result, no matter $\{u, u'\}$.

- But what if we measure 2nd Qbit in $\{v, v'\}$. ?
- What is the probability that if we measure 1st Qbit in $\{u, u'\}$ and 2nd in $\{v, v'\}$ we will see "matching" outcomes? (uv or $u'v'$)?
- Say we measure 1st and we see u $\Rightarrow$ the new state of the system is $|uu\rangle$.
- $\Rightarrow$ 2nd Qbit is in $|u\rangle$. And assume $\text{angle}(|u\rangle, |v\rangle) = \theta$.
  - Measure 2nd Qbit $|u\rangle$ in $\{v, v'\}$ $\Rightarrow$ $\Pr[\text{2nd Qbit measured } |v\rangle] = \cos^2\theta$
  - Same: if state collapses to $|u'u'\rangle$ $\Rightarrow$ $\Pr[\text{2nd Qbit measures } |v'\rangle] = \cos^2\theta$.



- SO, IF WE HAVE A BELL STATE BUT WE MEASURE THE TWO QBITS IN TWO DIFFERENT BASES $\{u, u'\}$ AND $\{v, v'\}$, SUCH THAT THE TWO BASIS MAKE AN ANGLE θ° DEGREES, THEN

  PROBABILITY [OBSERVING MATCHING OUTCOMES] $= \cos^2\theta$