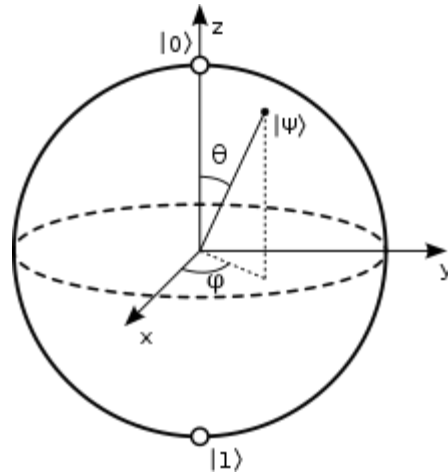


Intro to Quantum Computing

Quantum Algorithms I: Deutsch's Algorithm



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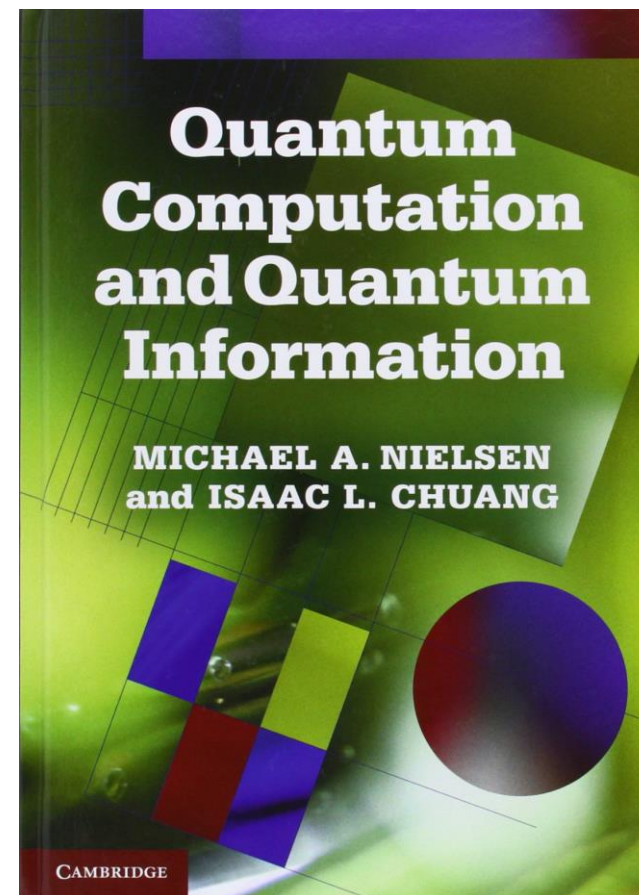
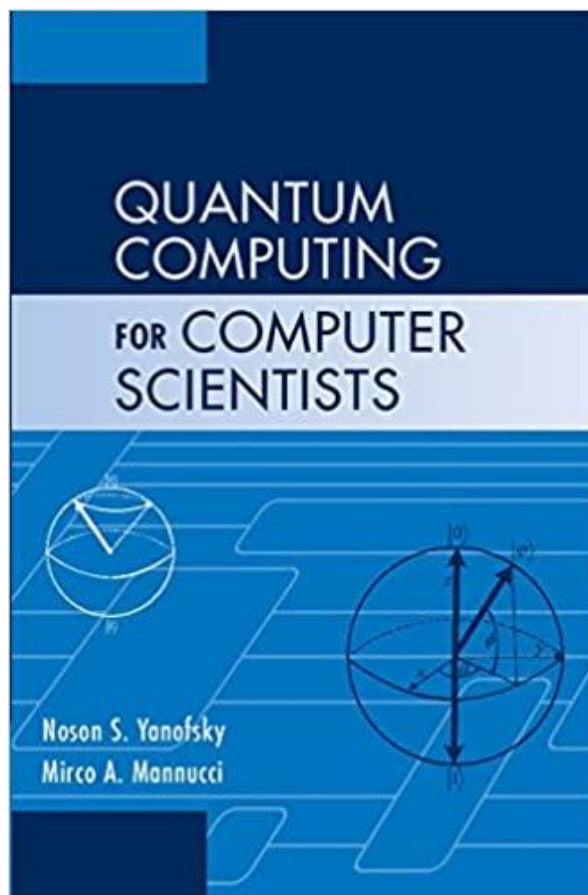
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- We will present an approach based on the latter version.
- Next lecture, we will generalize it to Deutsch-Josza Algorithm (a bit more difficult)

Today's Topic

- Sources



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- Yes! Deutsch algorithm is one of these (not many) tasks!

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- The **query complexity** is **2**.
- ***What if we had a quantum computer? Can we do better?***

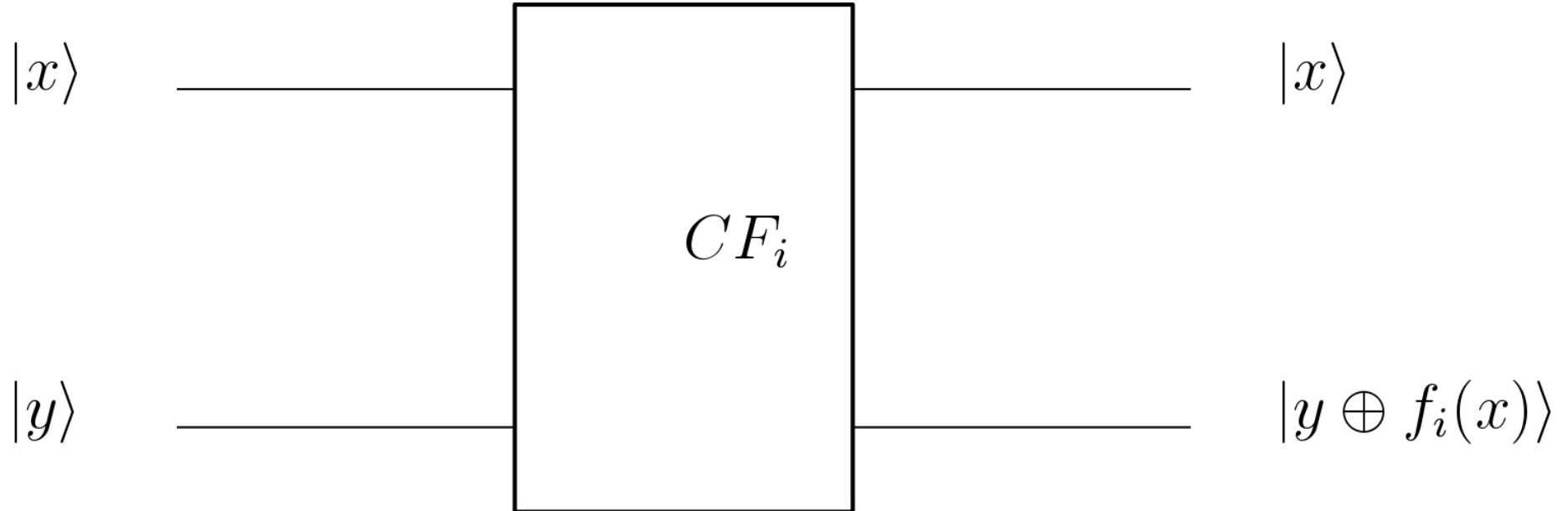
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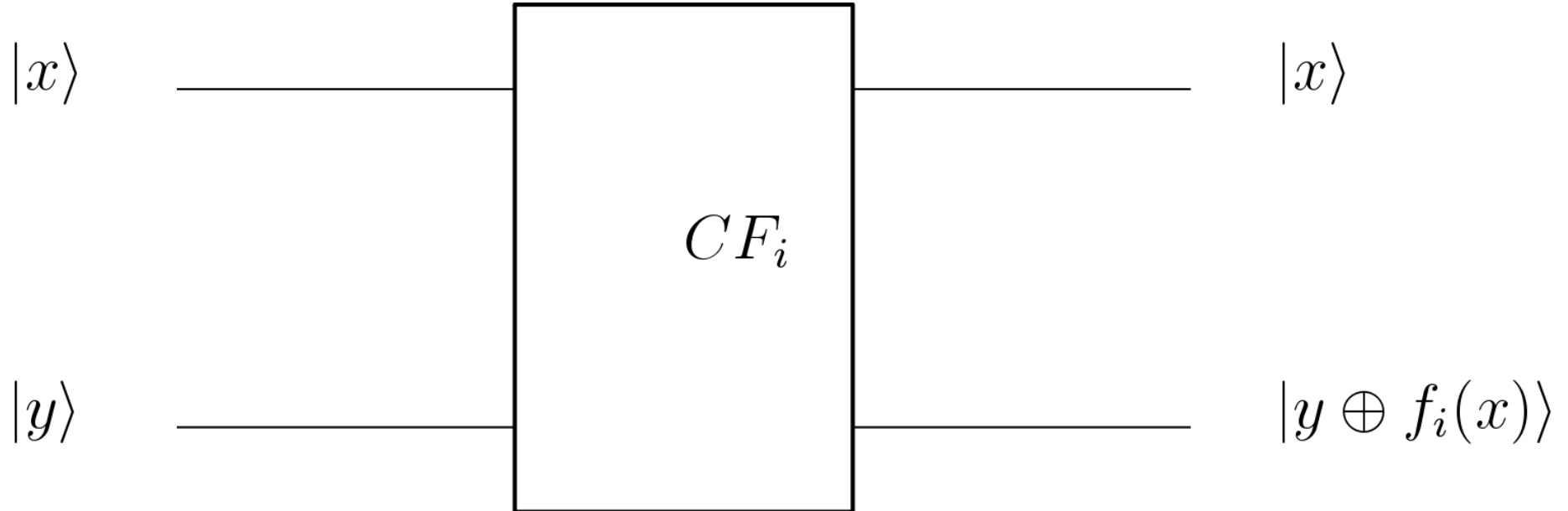
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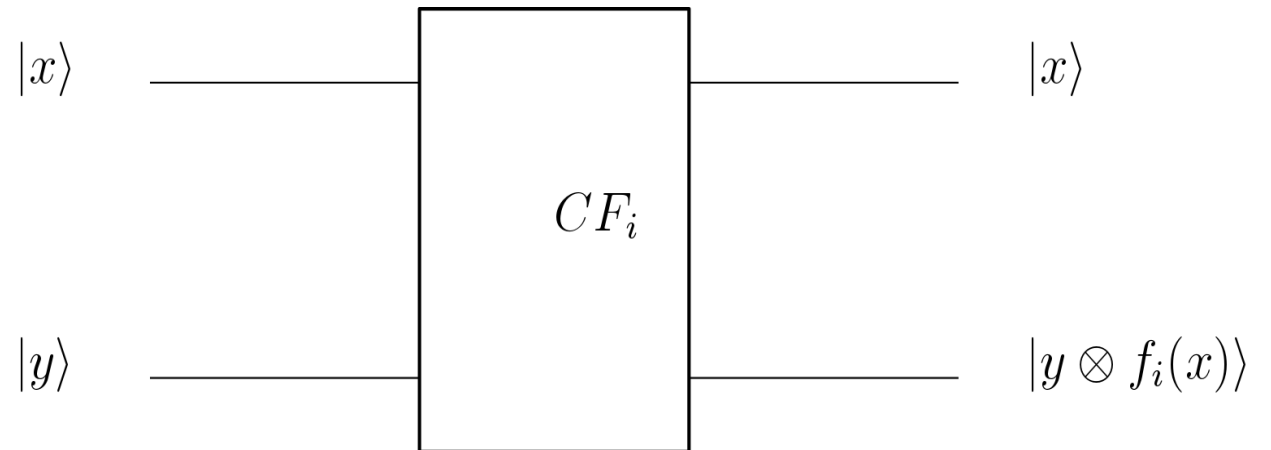
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- $|x\rangle$ is the input bit and $|y\rangle$ is a control bit: it controls the output $|y \oplus f_i(x)\rangle$.

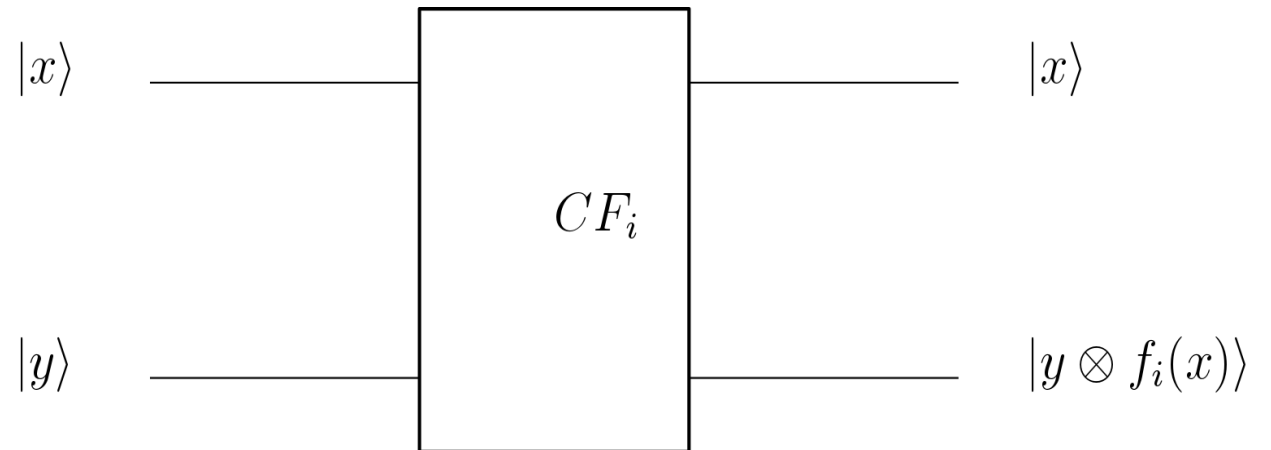
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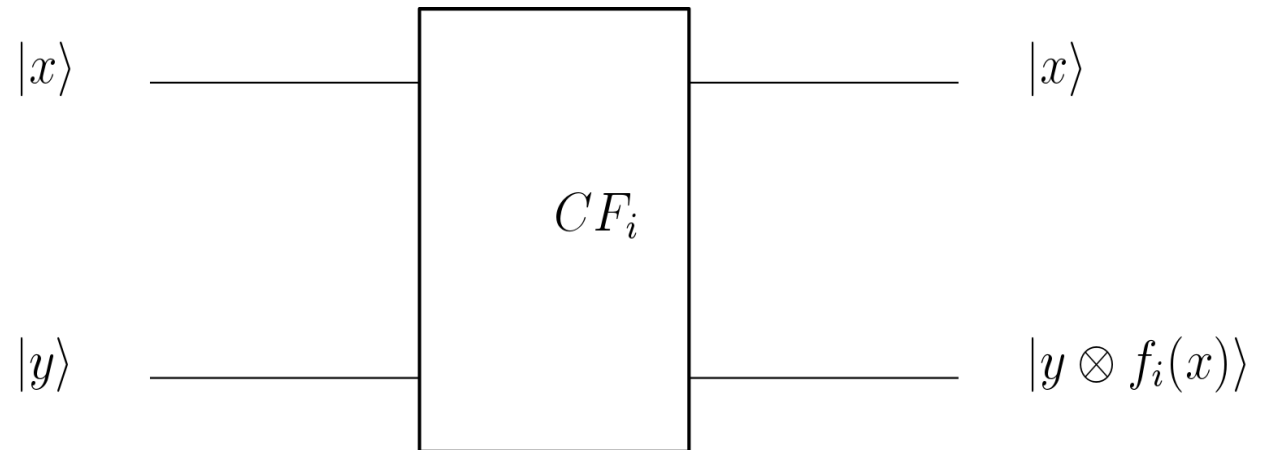
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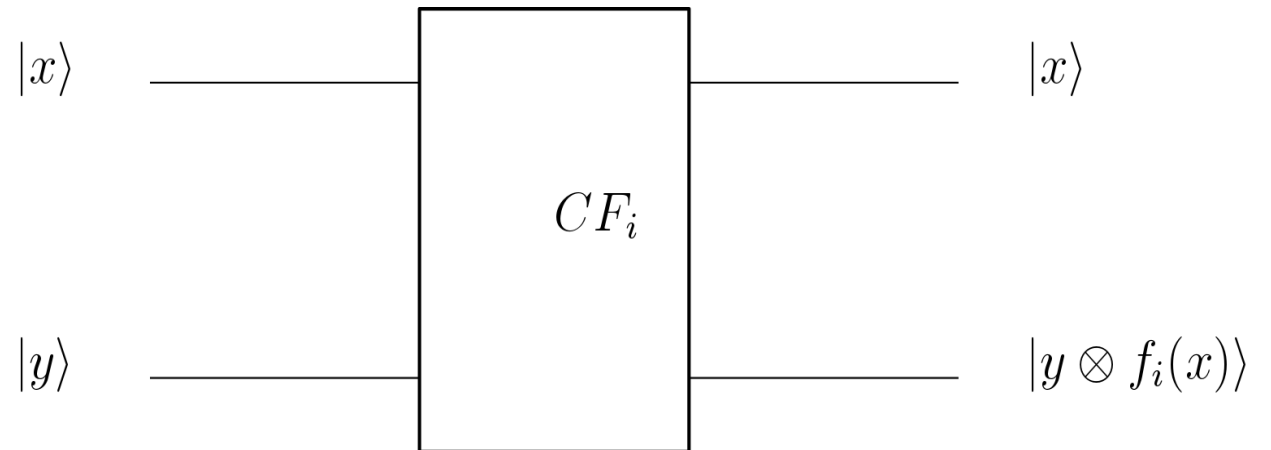
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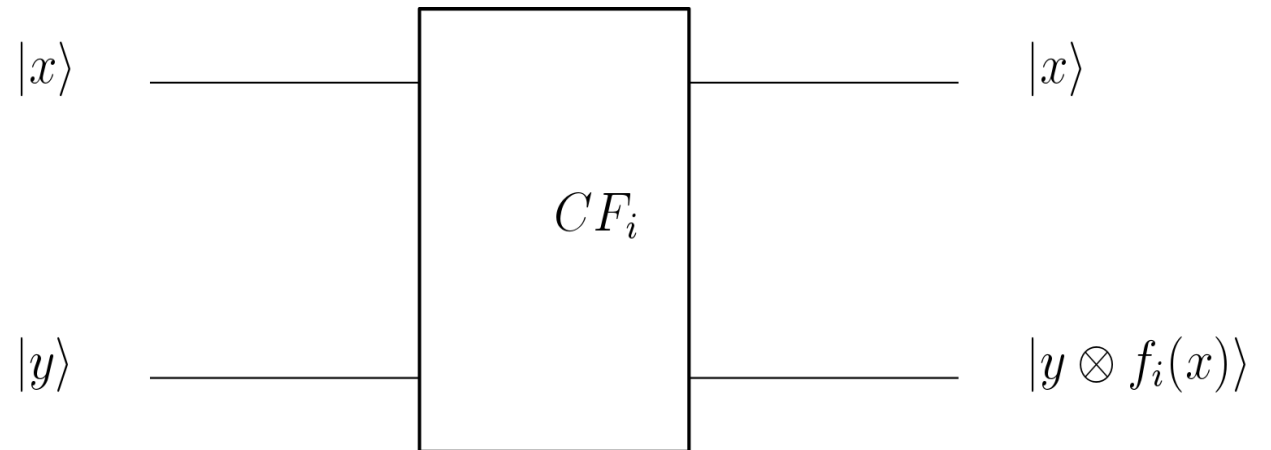
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- **So what?**
- For every i , exactly one of $|f_i(0)\rangle$ and $|f_i(0) \oplus 1\rangle$ is equal to 0. The other is equal to 1.
- Similarly exactly one of $|f_i(1)\rangle$ and $|f_i(1) \oplus 1\rangle$ is equal to 0. The other is equal to 1.

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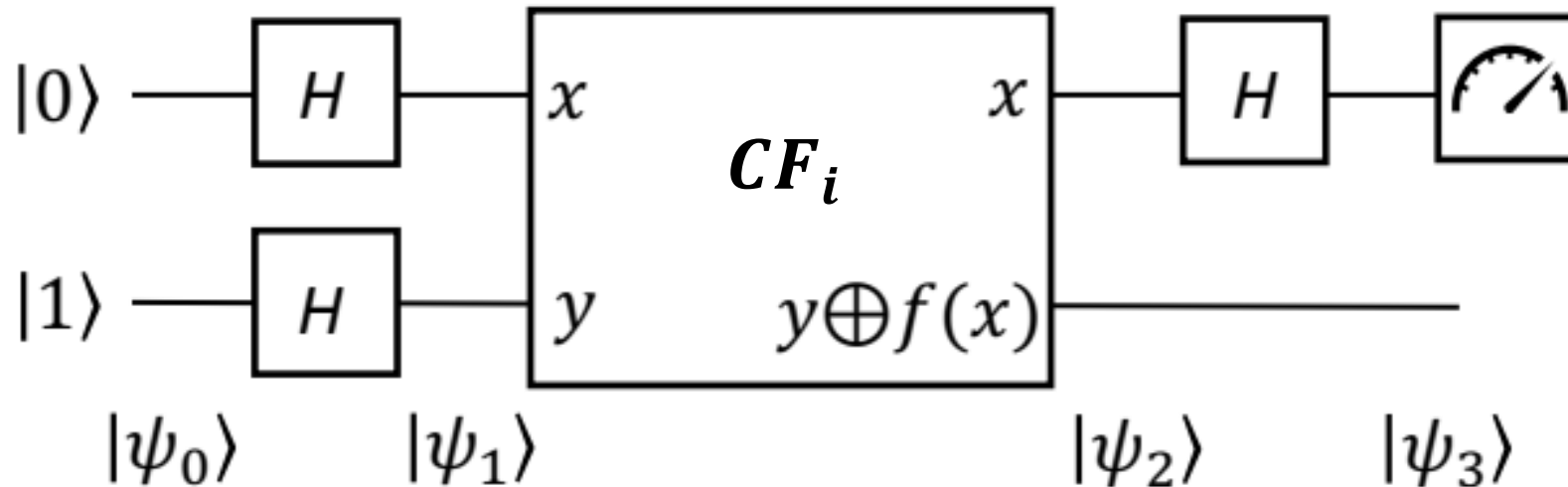
Given one of these four possible circuits, at random, how many times do we need to use the gate in order to determine if the underlying f_i is constant or not?

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- This circuit leaves the 1st Qbit alone and the 2nd Qbit becomes $|y \oplus f_i(x)\rangle$.

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- Similarly $|f_i(1)\rangle - |f_i(1) \oplus 1\rangle = (-1)^{f_i(1)} (|0\rangle - |1\rangle)$

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i.e...the two Qbits are not entangled

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 2. For f_1 we have $f_1(\mathbf{0}) = \mathbf{0}, f_1(\mathbf{1}) = \mathbf{1}$ and so $|\mathbf{x}\rangle = \frac{1}{\sqrt{2}} (|\mathbf{0}\rangle - |\mathbf{1}\rangle)$

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- Next, we send this Qbit $|x\rangle$ through another Hadamard gate and the outcome is $|\mathbf{0}\rangle$ if $|x\rangle = \frac{1}{\sqrt{2}} (|\mathbf{0}\rangle + |\mathbf{1}\rangle)$, or $|\mathbf{1}\rangle$ if $|x\rangle = \frac{1}{\sqrt{2}} (|\mathbf{0}\rangle - |\mathbf{1}\rangle)$

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 4. If $i = 3$ the Qbit is $-|0\rangle$
- Now we measure (in the standard basis)
- We get 0 if $i = 0$ or 3 i.e., when the function is **constant**
- We get 1 if $i = 1$ or 2 i.e., when the function is **non-constant!**
- ***And we got the answer by asking the circuit only once!***