QUANTUM MEASUREMENT

THE MEASUREMENT POSTULATE OF QUANTUM MECHANICS OR BORN'S RULE.

- QUANTUM MEASUREMENTS HAVE ONLY PROBABILISTIC OUTCOMES AND THEY ARE INVASIVE: THE "CORRUPT" (OR COLLAPSE) THE QUANTUM STATE TO A DETERMINISTIC OUTCOME.

- UNLIKE ALL OTHER OPERATORS THAT DESCRIBE THE EVOLUTION OF A QUANTUM SYSTEM (AXIOM 4 OF QUANTUM MECHANICS) WHICH ARE UNITARY, THE MEASUREMENT IS NOT A UNITARY PROCESS. IN OTHER WORDS, IT IS NOT INVERTIBLE! ONCE WE PERFORM A MEASUREMENT, WE CANNOT RECONSTRUCT THE ORIGINAL STATE OF THE QUANTUM SYSTEM ON WHICH WE HAVE PERFORMED A MEASUREMENT.

THE BASIC BORN'S RULE (OR COMPLETE PROJECTIVE MEASUREMENTS):

- GIVEN IS A QUANTUM STATE $|\psi\rangle$ ON SOME HILBERT SPACE OF DIMENSION $= n$.
- LET $B = \{|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle\}$ BE AN ARBITRARY ORTHONORMAL BASIS OF THAT SPACE:

$\langle e_i|e_i\rangle = 1 \quad \forall\ i \in [n]$

$\langle e_i|e_j\rangle = 0 \quad \forall\ i \neq j \text{ in } [n]$

- THEN $|\psi\rangle = \alpha_1|e_1\rangle + \dots + \alpha_n|e_n\rangle = \sum_{i=1}^{n} \alpha_i|e_i\rangle$

- QUANTUM MEASUREMENT OF $|\psi\rangle$ RELATIVE TO THE BASIS $B = \{|e_1\rangle, \ldots, |e_n\rangle\}$:
- WE HAVE n POSSIBLE OUTCOMES $j = 1, \ldots, n$, EACH SUCH OUTCOME CORRESPONDING TO THE BASIC VECTOR $|e_j\rangle$.
- PROBABILITY [WE OBSERVE j] $= |\langle e_j|\psi\rangle|^2 = |\alpha_j|^2$

- THIS IS A VALID PROBABILITY DISTRIBUTION: $|\psi\rangle$ IS A VALID QUANTUM STATE $\Rightarrow$ IT IS IN A COHERENT SUPERPOSITION OF ITS BASIC STATES:

$$|\psi\rangle = \sum_{i=1}^{n} \alpha_i |e_i\rangle \quad \text{SUCH THAT} \quad \sum_{i=1}^{n} |\alpha_i|^2 = 1$$

- IF WE PERFORM A MEASUREMENT ON $|\psi\rangle$ RELATIVE TO B, AND WE OBSERVE j, THEN THE STATE OF THE SYSTEM IS NO LONGER $|\psi\rangle$: BUT THE STATE HAS "COLLAPSED" TO $|\psi'\rangle = |e_j\rangle$

- IF WE KEEP MEASURING $|\psi'\rangle$ WE WILL KEEP OBSERVING $|e_j\rangle$ (OUTCOME j). $|\psi\rangle$ IS <u>NOT</u> A PURELY QUANTUM STATE. IT IS A DETERMINISTIC ONE.

PROJECTION OPERATORS.

- WE CAN DEFINE MEASUREMENTS THROUGH PROJECTION OPERATORS
- THESE ARE OPERATORS (MATRICES) P FOR WHICH $P \cdot P = P^2 = P$.

- FOR ANY $|e_j\rangle$ WE CAN CONSTRUCT A PROJECTIVE OPERATOR THAT PROJECTS A GIVEN VECTOR $|\psi\rangle$ ONTO $|e_j\rangle$
- THIS OPERATOR IS SIMPLY THE <u>OUTER PRODUCT</u> OF $|e_j\rangle$ WITH ITSELF:

$$P_j = |e_j\rangle\langle e_j|$$

(OR IN VECTOR NOTATION $P_j = e_j \cdot e_j^T$)

- P_j IS A $n \times n$ MATRIX ($|e_j\rangle$ IS A VECTOR IN A n-DIMENSIONAL HILBERT SPACE).

- FOR $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, ITS CORRESPONDING PROJECTION OPERATOR IS $|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

- IF WE HAVE AN ORTHONORMAL BASIS $B = \{|e_1\rangle, \ldots, |e_n\rangle\}$ THEN INDEED WE CAN SHOW EASILY THAT <u>ALL</u> $|e_j\rangle\langle e_j|$ ARE PROJECTOR OPERATORS:

$$P_j = |e_j\rangle \cdot \langle e_j|$$

$$P_j^2 = P_j \cdot P_j = (|e_j\rangle\langle e_j|)(|e_j\rangle\langle e_j|) =$$

$$= |e_j\rangle \langle e_j|e_j\rangle \langle e_j| \quad (\langle e_j|e_j\rangle = 1)$$

$$= |e_j\rangle\langle e_j| = P_j \quad \checkmark$$

- PROBABILITY [OBSERVE j] $= \langle\psi|P_j P_j|\psi\rangle = \langle\psi|P_j|\psi\rangle$

- POST MEASUREMENT STATE: $|\psi_j\rangle = \dfrac{P_j|\psi\rangle}{\sqrt{\langle\psi|P_j|\psi\rangle}}$ NORMALIZE!

MEASURING IN DIFFERENT BASIS.

- ASSUME WE HAVE A SINGLE QBIT STATE
  $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$
- THIS QBIT IS EXPRESSED IN TERM OF THE BASIS
  $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ AND $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- IT IS ALSO KNOWN COMMONLY AS THE STANDARD OR COMPUTATIONAL BASIS.
- WHEN WE PERFORM A MEASUREMENT ON $|\psi\rangle$, THEN THE SUPERPOSITION IS DESTROYED AND THE STATE OF THE QBIT COLLAPSES TO ONE OF THE BASIC STATES $|0\rangle$ OR $|1\rangle$ WITH PROBABILITY α_0^2 AND α_1^2 RESPECTIVELY ($\alpha_0, \alpha_1 \in \mathbb{R}$).
- WE CAN REWRITE THE AMPLITUDES AS FOLLOWS:

$$\alpha_0 = \langle 0|\psi\rangle \quad , \quad \alpha_1 = \langle 1|\psi\rangle.$$

- REMEMBER THAT $\langle u|v\rangle$ IS SIMPLY THE INNER PRODUCT BETWEEN u AND v.

- GIVEN THIS, THE CORRESPONDING PROBABILITIES CAN BE WRITTEN AS:

$$a_0^2 = a_0\alpha_0 = \langle\psi|0\rangle\langle 0|\psi\rangle.$$
$$a_1^2 = a_1a_1 = \langle\psi|1\rangle\langle 1|\psi\rangle.$$

BUT WHAT DOES IT MEAN GEOMETRICALLY:

$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ IS A VECTOR IN THE UNIT CYCLE.

IF WE PROJECT $|\psi\rangle$ ONTO $|0\rangle$ (X-AXIS) WE GET A VECTOR WITH LENGNT a_0^2.

ALSO: THE DISTANCE FROM $|\psi\rangle$ TO THE PROJECTION IS a_1^2.
a_0^2 AND α_1^2 FORM A RIGHT TRIANGLE $\Rightarrow$
PYTHAGOREAN THEOREM: $a_0^2 + a_1^2 = 1$

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THIS REASONING CAN BE EXTENDED TO ANY ORTHONORMAL BASIS $\{|v\rangle, |v'\rangle\}$ I.E. $|v\rangle, |v'\rangle$ ARE UNIT ORTHOGONAL VECTORS :

$\langle v|v'\rangle = 0$.

▷ WE CAN REWRITE $|\psi\rangle$ IN $\{|v\rangle, |v'\rangle\}$:

$|\psi\rangle = b_0|v\rangle + b_1|v'\rangle$.

▷ THERE IS AN ASSUMPTION OF CORRESPONDENCE: $|v\rangle, |0\rangle$ CORRESPOND TO THE SAME TRUTH VALUE. SIMILARLY $|v'\rangle$ AND $|1\rangle$.

▷ WE CAN REWRITE THE AMPLITUDES b_0, b_1 AS:

$b_0 = \langle v|\psi\rangle$, $b_1 = \langle v'|\psi\rangle$,

WHERE $|\psi\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$.

⇒ PROJECTING $|\psi\rangle$ ONTO $|v\rangle$ AND $|v'\rangle$ GIVES AS, AS BEFORE, A RIGHT TRIANGLE WITH LENGHT OF ITS LEGS b_0^2 AND b_1^2 SUCH THAT $b_0^2 + b_1^2 = 1$ (AGAIN BY PYTHAGORAS).

So... $|\psi\rangle = a_0|0\rangle + a_1|1\rangle = b_0|v\rangle + b_1|v'\rangle$.

⇒ IF WE MEASURE OUR QBIT IN BASIS $\{|0\rangle, |1\rangle\}$ THEN WE HAVE PROBABILITY a_0^2 TO SEE $|0\rangle$ AND a_1^2 TO SEE $|1\rangle$.

IF, ON THE OTHER HAND, WE MEASURE IN $\{|v\rangle, |v'\rangle\}$ THEN WE SEE $|v\rangle$ WITH PROBABILITY b_0^2 AND $|v'\rangle$ WITH PROB. b_1^2.

SINCE $|0\rangle, |v\rangle$ ARE ASSUMED EQUIVALENT, THE PROBABILITY WE WILL GET THE TRUTH-VALUE 0 DEPENDS ON WHETHER WE ARE MEASURING IN $\{|0\rangle, |1\rangle\}$ OR $\{|v\rangle, |v'\rangle\}$.

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REMEMBER: UNITARY TRANSFORMATIONS ARE ~~[illegible]~~ EQUIVALENT TO CHANGE OF BASIS.

- THIS MEANS THAT IF WE HAVE ANY ORTHONORMAL BASIS IN $\mathbb{R}^2$ (FOR EXAMPLE). $\{|v\rangle, |v'\rangle\}$ AND ANY UNITARY OPERATION $U \in \mathbb{R}^{2\times 2}$, THEN:

$$\left.\begin{array}{l} U|v\rangle = |w\rangle \\ U|v'\rangle = |w'\rangle \end{array}\right\} \ \{|w\rangle, |w'\rangle\} \text{ IS ALSO ORTHONORMAL BASIS IN } \mathbb{R}^2:$$

$$\langle w|w\rangle = \langle w'|w'\rangle = 1 \quad \text{AND}$$
$$\langle w|w'\rangle = \langle w'|w\rangle = 0.$$

For Example: CONSIDER, AGAIN, THE HADAMARD TRANSFORM. (verify it is unitary):

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

WE HAVE $H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = |+\rangle$

Similarly: $H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle.$

THE HADAMARD MATRIX CAN BE CONSTRUCTED BY OUTER PRODUCTS:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= (|+\rangle\langle 0|) + (|-\rangle\langle 1|).$$

$\Rightarrow$ IF WE HAVE $\{|v\rangle, |v'\rangle\}$ THERE MUST BE UNITARY TRANSFORMATION U SUCH THAT $U|v\rangle = |0\rangle$, $U|v'\rangle = 1$.

$$\Rightarrow \boxed{U = |0\rangle\langle v| + |1\rangle\langle v'|}$$

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WHICH FURTHER MEANS THAT IF $|\psi\rangle = b_0|v\rangle + b_1|v'\rangle$, THEN IF WE MEASURE THE STATE $U|\psi\rangle = b_0|0\rangle + b_1|1\rangle$
GIVES $|0\rangle$ WITH PROBABILITY ~~[illegible]~~ b_0^2 AND $|1\rangle$ WITH PROBABILITY b_1^2.

So, for Example if $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
then if we measure in $\{|0\rangle, |1\rangle\}$ will gives us outcome $|0\rangle, |1\rangle$ with Equal probability $\frac{1}{2}$.

- If we rotate our basis $45°$ clockwise
  $\Rightarrow$ $Prob[|0\rangle] = 0$
  $Prob[|1\rangle] = 1$.

- Rotate the original basis $45°$ counter clockwise:
  $\Rightarrow$ $Prob[|0\rangle] = 1$
  $Prob[|1\rangle] = 0$.

CHANGING MEASUREMENT BASIS $\Rightarrow$ CHANGES THE PROBABILITY WE WILL SEE $|0\rangle, |1\rangle$.