QUANTUM ENTANGLEMENT.

- ASSUME THAT ALICE IS IN POSSESION OF A QBIT $|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$.
- Similarly BOB HAS A QBIT DESCRIBED BY $|\beta\rangle = b_0|0\rangle + b_1|1\rangle$
- THE STATE THAT DESCRIBES BOTH QBITS IS THE TENSOR PRODUCT OF THE TWO STATES:

$$
\begin{aligned}
|\psi\rangle &= |\alpha\rangle \otimes |\beta\rangle \\
&= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle). \\
&= \alpha_0 b_0|0\rangle|0\rangle + \alpha_0 b_1|0\rangle|1\rangle + \alpha_1 b_0|1\rangle|0\rangle + \alpha_1 b_1|1\rangle|1\rangle.
\end{aligned}
$$

- For notational convenience, WE DENOTE $|s\rangle \otimes |t\rangle = |s\rangle|t\rangle = |st\rangle$.
- So, WE CAN write $|\psi\rangle = \alpha_0 b_0|00\rangle + \alpha_0 b_1|01\rangle + \alpha_1 b_0|10\rangle + \alpha_1 b_1|11\rangle$.
- Note that if $\alpha_0^2 + \alpha_1^2 = b_0^2 + b_1^2 = 1$ THEN WE HAVE that $(\alpha_0 b_0)^2 + (\alpha_0 b_1)^2 + (\alpha_1 b_0)^2 + (\alpha_1 b_1)^2 = 1$
- Do you see why?
- Let's REPLACE $r = a_0 b_0$

$s = a_0 b_1$

$t = a_1 b_0$

$u = a_1 b_1$.

I.E $|\psi\rangle = r|00\rangle + s|01\rangle + t|10\rangle + u|11\rangle$.

- WE KNOW $r^2 + s^2 + t^2 + u^2 = 1$.
- WE ALSO KNOW that $r \cdot u = s \cdot t$. (CHECK!).
- So FAR NOTHING NEW!

- We will describe states of two qbits such that:
- $|\psi\rangle = r|00\rangle + s|01\rangle + t|10\rangle + u|11\rangle$
- $r^2 + s^2 + t^2 + u^2 = 1$
- ru is not necessarily equal to st !

1) If $ru = st \Rightarrow$ the two qbits that are described by $|\psi\rangle$ are UNENTANGLED.

2) If $ru \neq st \Rightarrow$ the two qbits are ENTANGLED.

- But suppose we have

$$|\psi\rangle = \frac{1}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle.$$

- We can easily see that the two qbits described by $|\psi\rangle$ are UNENTANGED.
- $|\psi\rangle$ says that with probability $\frac{1}{8}$ Alice and Bob will both see 0; with prob $1/8$ they will see 1 and 0 respectively, with prob $3/8$ 0 and 1 and with probability $3/8$ 1 and 1.

- But what happens when Alice makes her measurement?

- In order to answer that, we need to reconstruct her state of her qbit

- Similarly for Bob.

THIS IS HOW IT IS DONE.

- TAKE $|\psi\rangle = a_0 b_0 |00\rangle + \dots$
- REWRITE THE TENSOR PRODUCT FROM ALICE'S viewpoint:

$$|0\rangle\left(\frac{1}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|1\rangle\right) + |1\rangle\left(\frac{1}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|1\rangle\right)$$

- REMEMBER: THER QBIT IS THE left most ONE.

- WE WOULD like the Expressions IN THE parentesis to BE valid quantumstates I.E. UNIT VECTOR.

- To DO that, Simply DIVIDE By their lengthes INSIDE Parentesis...
- ... AND multiply By the Same NUMBER OUTSIDE:

$$\Rightarrow \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right).$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\cdot\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)$$

$$= |\alpha\rangle \cdot |\beta\rangle.$$

$\Rightarrow$ WE Immediately GET THE PROBABILITIES OF THE MEASUREMENT OF THEIR INDIVIDUAL QBITS.

Now, suppose we have the following situation:

$$|\psi\rangle = \frac{1}{2}|0\rangle|0\rangle + \frac{1}{2}|0\rangle|1\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle \quad (+\,0|1\rangle|1\rangle)$$

NOTE: In general, Alice and Bob can use different measurement bases.
For example, Alice could be measuring in $\{|a_0\rangle, |a_1\rangle\}$ and Bob in $\{|b_0\rangle, |b_1\rangle\}$ etc.
The situation is actually identical.

Back to $|\psi\rangle$. We easily see that the two qbits of Alice and Bob must be entangled:

$$r \cdot u = \frac{1}{2} \cdot 0 \neq s \cdot t = \frac{1}{2} \cdot \frac{1}{\sqrt{2}}.$$

When Alice and Bob measure their qbits, $|\psi\rangle$ tells us that they see:
- Both 0 with probability 1/4
- Alice 0 and Bob 1 w. pr. 1/4
- Alice 1 and Bob 0 w. pr. 1/2.

- Assume that Alice now makes a measurement by herself.
- Bob does not.

We rewrite $|\psi\rangle$ as follows (from Alice's perspective).

$$|\psi\rangle = |0\rangle\underbrace{\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right)}_{\text{Bob's}} + |1\rangle\left(\frac{1}{\sqrt{2}}|0\rangle + 0\cdot|1\rangle\right)$$

- We normalize (divide by lenght and multiply outside parenthesis):

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(|0\rangle\right).$$

- If $|\psi\rangle$ was not entangled, the terms in the parentesis would be the same and we could take common factors.

- But in entangled states they are not same.

- The amplitudes $\left(\frac{1}{\sqrt{2}}\right)$ in front of alice's bits tell us that Alice, when she makes a measurement, she sees $|0\rangle$ or $|1\rangle$ with equal probability $= 1/2$.

- If she sees $|0\rangle$ then Bob's qbit jumps to state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

- If she sees $|1\rangle$, Bob's qbit jumps to $|1\rangle$

- In either case, after alice measures her qbit, Bob's qbit is no longer entangled with Alice's.

- Does this mean that Alice and Bob can communicate faster than light?