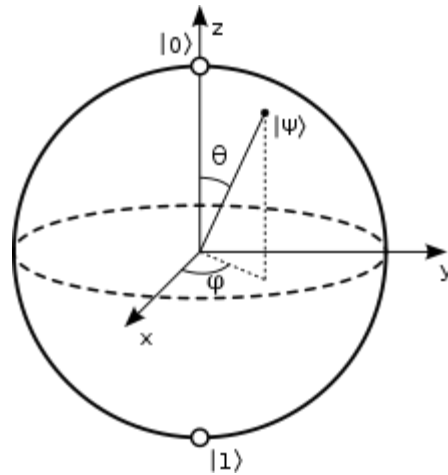


Intro to Quantum Computing

Maths of Qbits Part II



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- First, construct the matrix that correspond to the horizontal basis

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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- **Do you see why?**

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- **No experiment can distinguish them, no matter which basis we are using**

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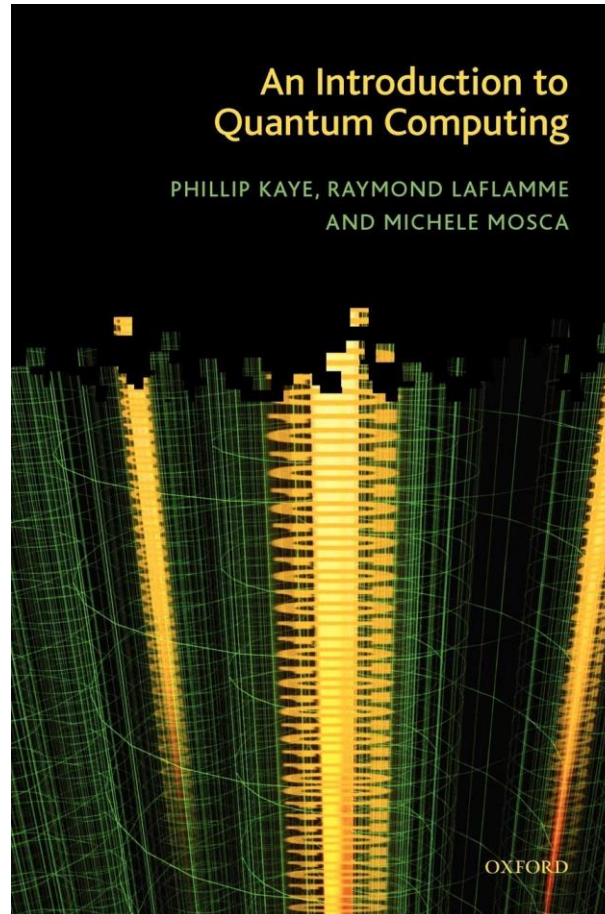
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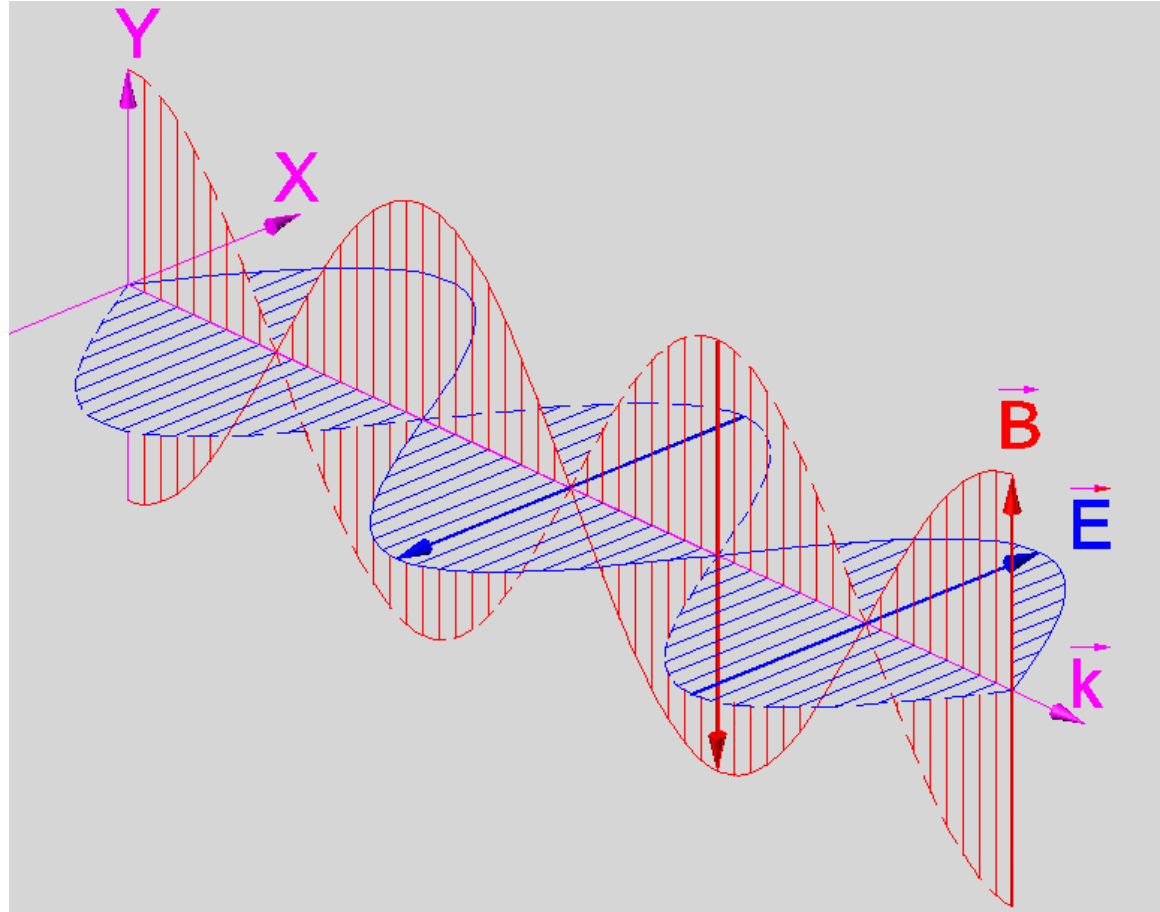
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- Is there any basis (rotation of the apparatus) that could help us distinguish them?

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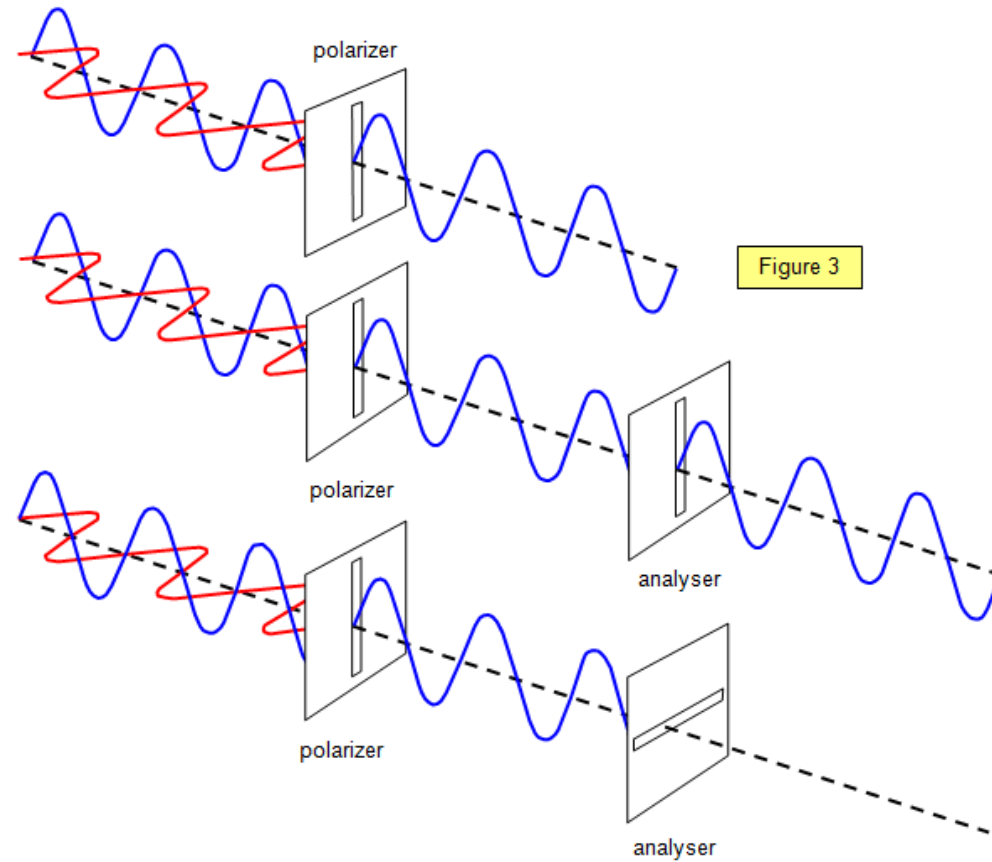
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- A photon can be ***polarized*** i.e., its electric field component can be vertically or horizontally oriented.
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- We can change the polarization of a photon by applying a **polaroid filter**.

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- Such a polaroid has an orientation (for example vertical)

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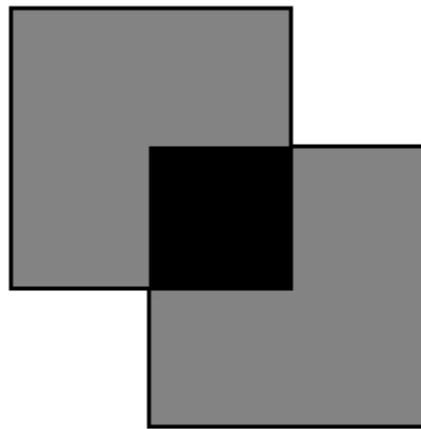
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- If a photon that has horizontal polarization tries to pass through that polaroid, *it will be blocked with probability 1.*
- A photon with polarization 45° *will pass through the polaroid with chance $\frac{1}{2}$ and will be blocked with probability $\frac{1}{2}$.*

Measuring polarization

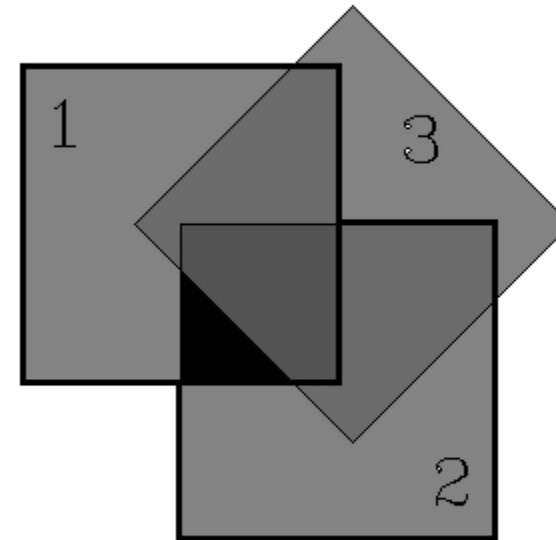
- Source: lock haven university



Parallel axes.



Crossed axes.



Polarizer (3) between two crossed polarizers (1) and (2).

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- Are photons magically appearing?

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- *What is the chance it will pass through?*

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- So the total probability that the photon will survive the filters is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Equivalent states and Uncertainty principle

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$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Equivalent states and Uncertainty principle

- We will motivate the uncertainty principle using qbits and bases.
- It will not be as precise as Heisenberg's principle.
- But at least it will give us an understanding on what it actually means
- At least in the context of qbits.

Equivalent states and Uncertainty principle

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 1. The “standard” vertical basis $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Equivalent states and Uncertainty principle

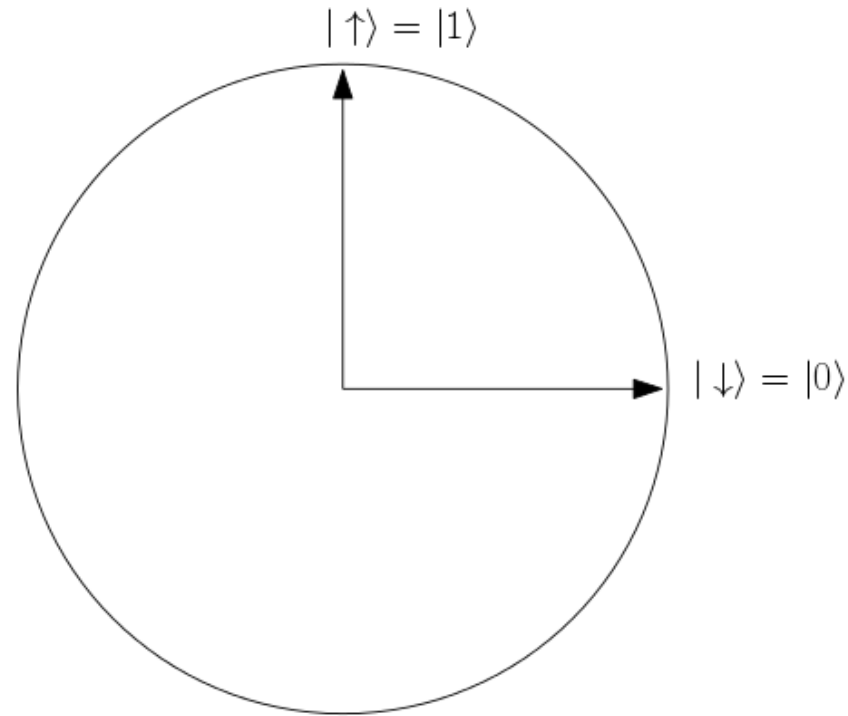
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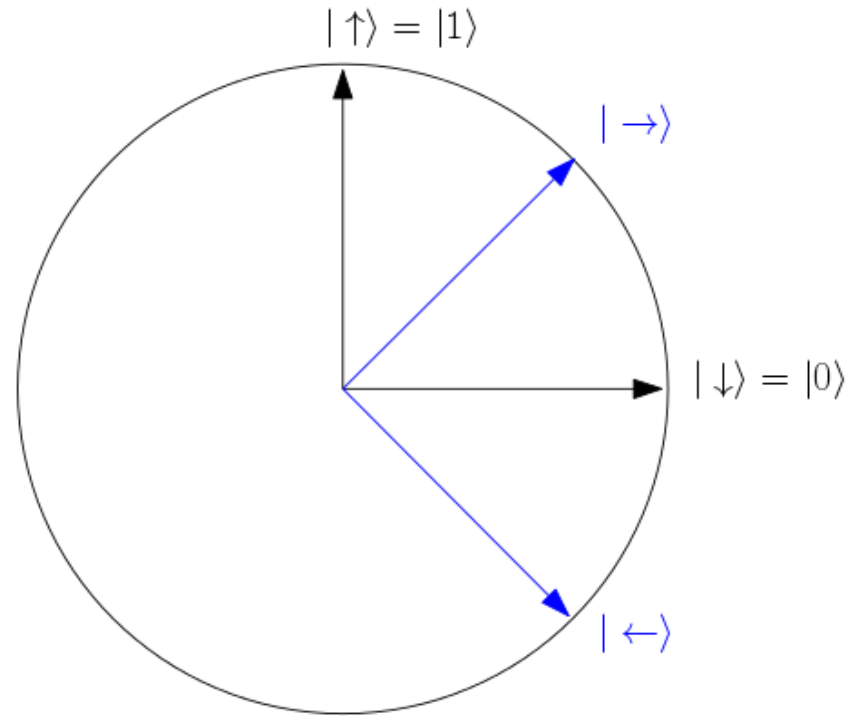
2. And the horizontal basis

$$|\rightarrow\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, |\leftarrow\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

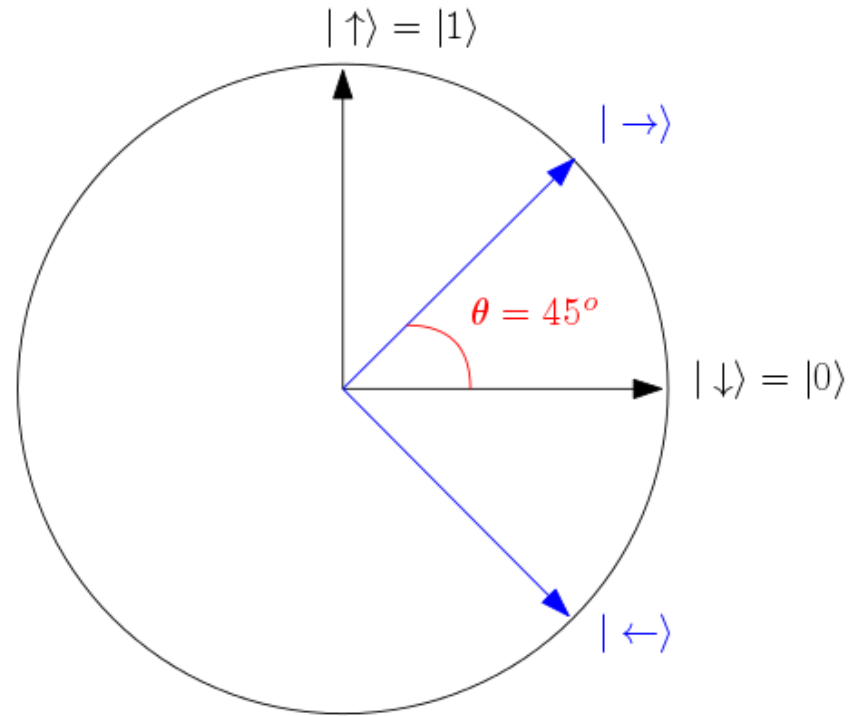
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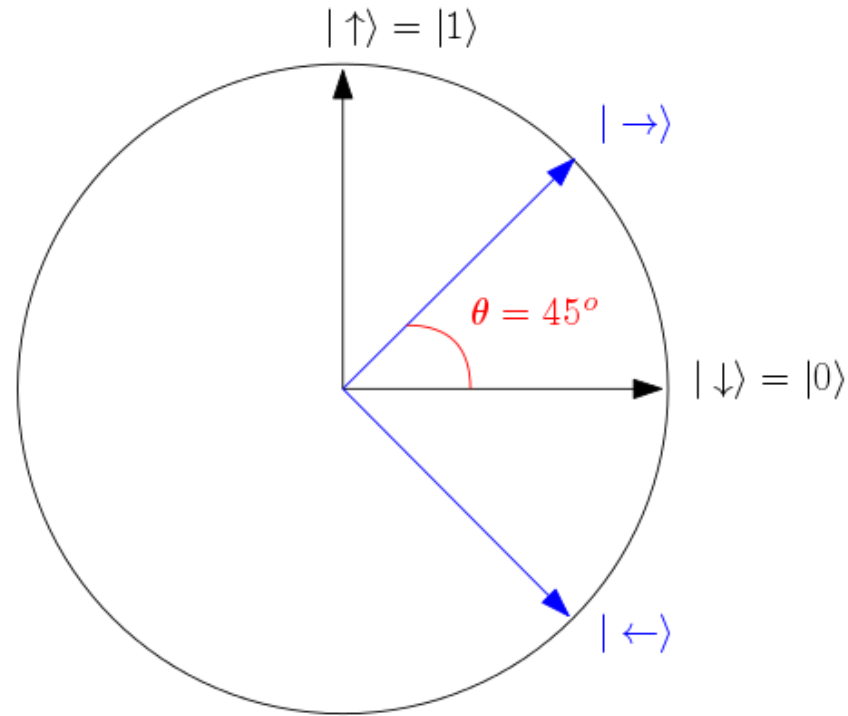
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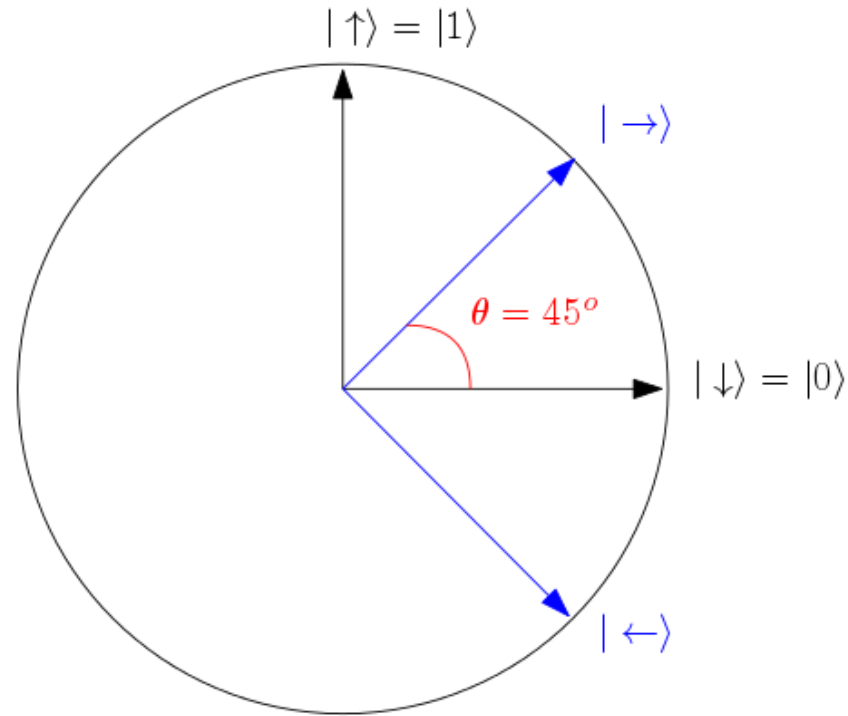


Equivalent states and Uncertainty principle



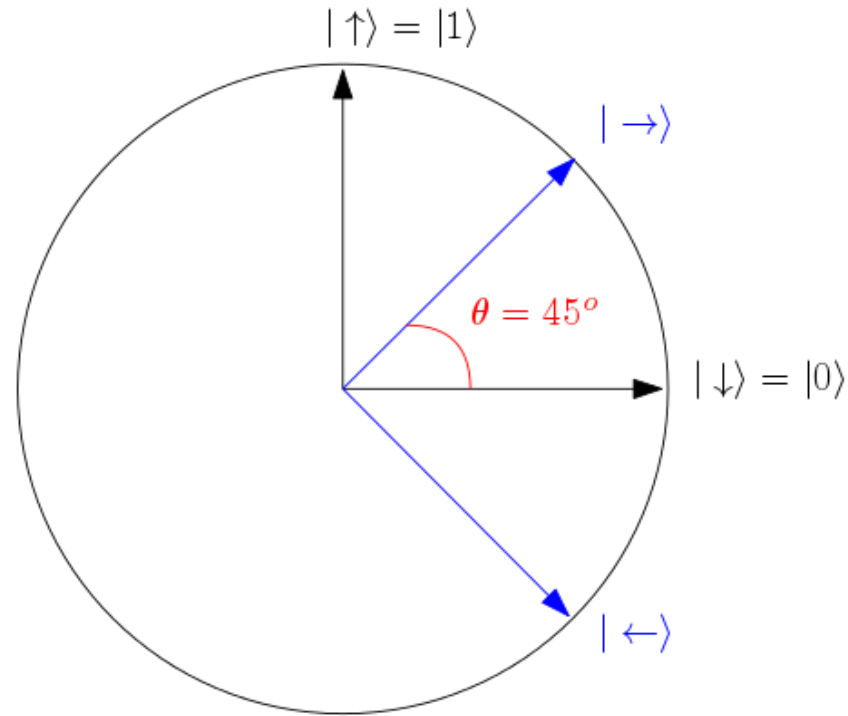
- The angle of the two bases is **not** the same as the angle we rotate the apparatus!!!

Equivalent states and Uncertainty principle



- 90° rotation of the apparatus \Rightarrow the basis rotates by 45° .

Equivalent states and Uncertainty principle



- In general: θ degrees rotation of the apparatus \Rightarrow the basis rotates by $\frac{\theta}{2}$.

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- Assume we have a state $|\psi\rangle$ and we measure it in the $|0\rangle, |1\rangle$ basis.
- $|\psi\rangle$ has some probability being in either of the two basic states.
- We can determine what is the value of $|\psi\rangle$ by simply measuring it and destroying its superposition.

Equivalent states and Uncertainty principle

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- Measuring $|\psi\rangle$ in the new basis does not give us $|0\rangle, |1\rangle$ but rather $|\leftarrow\rangle, |\rightarrow\rangle$ each with some probability.
- Assume that the first 0-1 base corresponds to the position of the electron.
- And the second left-right base corresponds to its velocity.

Equivalent states and Uncertainty principle

- Question

Is it possible to know with perfect accuracy both the position and the velocity of the particle described by $|\psi\rangle$?

Equivalent states and Uncertainty principle

- Answer

*Heisenberg
says
NO!*

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- Similar with $1 = |\uparrow\rangle$.
- This is a consequence of the fact that the 2nd basis is a rotation of 45° of the 1st one.

Equivalent states and Uncertainty principle

- Quantitatively, let's assume that $|\psi\rangle = \mathbf{a}_1|\uparrow\rangle + \mathbf{a}_2|\downarrow\rangle$ in the first basis and $|\psi\rangle = \mathbf{b}_1|\rightarrow\rangle + \mathbf{b}_2|\leftarrow\rangle$ in the 2nd basis.
- Define $\mathbf{S}_{\uparrow,\downarrow} = |\mathbf{a}_1| + |\mathbf{a}_2|$ the “spread” of the first basis.
- Similarly $\mathbf{S}_{\rightarrow,\leftarrow} = |\mathbf{b}_1| + |\mathbf{b}_2|$.
- If we knew perfectly (i.e., measure) the “up-down” value of $|\psi\rangle$ then $\mathbf{S}_{\uparrow,\downarrow} = 1$.
- Similarly, if we know (i.e., measure) the “left-right” value of $|\psi\rangle$ then $\mathbf{S}_{\rightarrow,\leftarrow} = 1$.
- But if we are in the state $|\psi\rangle = |\rightarrow\rangle$ then $|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and in this case $\mathbf{S}_{\uparrow,\downarrow} = \sqrt{2}$.

Equivalent states and Uncertainty principle

- Identically, if we are in the state $|\psi\rangle = |\downarrow\rangle$ then $|\psi\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\rangle$ and in this case $S_{\rightarrow,\leftarrow} = \sqrt{2}$.
- The more certain we are about one of the states, the more uncertain we are for the other.

$$S_{\rightarrow,\leftarrow}(|\psi\rangle) \cdot S_{\uparrow,\downarrow}(|\psi\rangle) \geq \sqrt{2}$$

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$$\Delta\chi\Delta\rho \geq \frac{\hbar}{2}$$