ExerciseSet1_solutions

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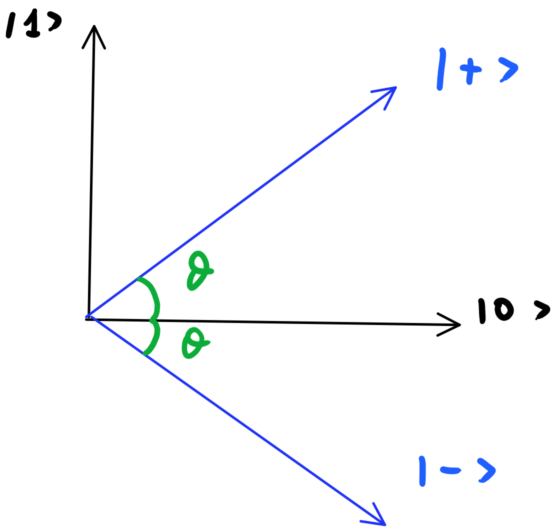
Ex. 1

$$|\rightarrow\rangle \equiv |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

"$\equiv$" : EQUAL BY DEFINITION

$$|\leftarrow\rangle \equiv |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



$\theta = \frac{\pi}{4}$

1. $\langle 1 || \rightarrow \rangle \equiv \langle 1 | \rightarrow \rangle = \langle 1 | \left[\frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \right] =$

$$= \langle 1 | \frac{1}{\sqrt{2}} |0\rangle + \langle 1 | \frac{1}{\sqrt{2}} |1\rangle =$$

$$= \frac{1}{\sqrt{2}} \underbrace{\langle 1 | 0 \rangle}_{0} + \frac{1}{\sqrt{2}} \underbrace{\langle 1 | 1 \rangle}_{1} =$$

$\langle 0|1\rangle = \langle 1|0\rangle = 0$
$\langle 0|0\rangle = \langle 1|1\rangle = 1$

$$= \frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

2. $|0\rangle\langle\rightarrow| \equiv |0\rangle \otimes \langle\rightarrow| \overset{(*)}{=}$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \frac{1}{\sqrt{2}} \\ 0 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

(*)

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(\overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{|0\rangle} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{|1\rangle}\right) =$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\langle\rightarrow| = |\rightarrow\rangle^{+} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}^{T} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

3. $\langle\rightarrow|\leftarrow\rangle \equiv \langle +|-\rangle =$

$$= \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right) \cdot \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) =$$

$$= \frac{1}{2}\left(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle\right) =$$

$$= \frac{1}{2}\left(1 - 0 + 0 - 1\right) = 0$$

4. $|\rightarrow\rangle\langle\leftarrow| \equiv |+\rangle\langle -| =$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(\langle 0| - \langle 1|\right) =$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right) \overset{(*)}{=}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(*)

$$|0\rangle\langle 0| \equiv |0\rangle \otimes \langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|0\rangle\langle 1| \equiv |0\rangle \otimes \langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

5. $|\rightarrow\rangle \otimes |\leftarrow\rangle \equiv |\rightarrow\rangle|\leftarrow\rangle \equiv |\rightarrow\leftarrow\rangle =$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) =$$

$$= \frac{1}{2}\left(\overbrace{|0\rangle \otimes |0\rangle}^{|00\rangle} - \overbrace{|0\rangle \otimes |1\rangle}^{|01\rangle} + \overbrace{|1\rangle \otimes |0\rangle}^{|10\rangle} - \overbrace{|1\rangle \otimes |1\rangle}^{|11\rangle}\right)$$

$$= \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

6. $(X \otimes I_2)(|\rightarrow\rangle \otimes |\leftarrow\rangle) =$

$$= X|\rightarrow\rangle \otimes I_2|\leftarrow\rangle =$$

$$= \frac{1}{\sqrt{2}} X\left(|0\rangle + |1\rangle\right) \otimes |\leftarrow\rangle =$$

$$= \overbrace{\frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right)}^{|\rightarrow\rangle} \otimes |\leftarrow\rangle =$$

$$= |\rightarrow\rangle \otimes |\leftarrow\rangle$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

7. $(X \otimes Y)(|\rightarrow\rangle \otimes |\leftarrow\rangle) =$

$$= X|\rightarrow\rangle \otimes Y|\leftarrow\rangle =$$

$$= X\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes Y\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] =$$

$$= \frac{1}{\sqrt{2}}(X|0\rangle + X|1\rangle) \otimes \frac{1}{\sqrt{2}}(Y|0\rangle - Y|1\rangle) =$$

$$= \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \otimes \frac{1}{\sqrt{2}}(i|1\rangle - (-i|0\rangle)) =$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} i(|0\rangle + |1\rangle) =$$

$$= \frac{i}{2}|0\rangle \otimes |0\rangle + \frac{i}{2}|0\rangle \otimes |1\rangle + \frac{i}{2}|1\rangle \otimes |0\rangle + \frac{i}{2}|1\rangle \otimes |1\rangle$$

$$= \frac{i}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$\boxed{\text{Ex. 2}}$

$$|T_1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$|T_2\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

1. LET'S TAKE $|T_2\rangle$.

ASSUME THAT THERE EXIST SINGLE QBIT STATES $|q_1\rangle$ AND $|q_2\rangle$, SUCH THAT

$$|T_2\rangle = |q_1\rangle \otimes |q_2\rangle$$

LET $|q_1\rangle = a|0\rangle + b|1\rangle, \quad a^2+b^2=1$

$|q_2\rangle = c|0\rangle + d|1\rangle, \quad c^2+d^2=1$ $\quad (a,b,c,d \in \mathbb{R})$

$$|q_1\rangle \otimes |q_2\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle \otimes d|1\rangle) =$$

$$= ac|0\rangle|0\rangle + ad|0\rangle|1\rangle + bc|1\rangle|0\rangle + bd|1\rangle|1\rangle =$$

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

SINCE WE ASSUME $|T_2\rangle = |q_1\rangle \otimes |q_2\rangle$

WE MUST HAVE:

$$0\cdot|00\rangle + \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle + 0\cdot|11\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$$\Rightarrow \begin{cases} ac = 0 \\ bd = 0 \\ ad = \frac{1}{\sqrt{2}} \\ bc = -\frac{1}{\sqrt{2}} \end{cases} \Rightarrow$$ THIS MEANS THAT NEITHER a, b, c, d CAN BE ZERO

WHICH MEANS THAT $a \cdot c$, $b \cdot d$ CAN NEVER BE ZERO ∇

THEN THE ASSUMPTION $|T_2\rangle = |q_1\rangle \otimes |q_2\rangle$ is FALSE FOR ANY SINGLE QBIT STATE $|q_1\rangle, |q_2\rangle$.

THE SOLUTION IS IDENTICAL FOR $|T_1\rangle$.

2. WE NEED TO SHOW THAT $(U \otimes U)|T_2\rangle = |T_2\rangle$

U is A 1-QBIT UNITARY OPERATOR. WE HAVE THAT

$$U|0\rangle = a|0\rangle + b|1\rangle$$
$$U|1\rangle = c|0\rangle + d|1\rangle$$

SUCH THAT

$$a^2 + b^2 = 1$$
$$c^2 + d^2 = 1$$

$$(U \otimes U)|T_2\rangle = (U \otimes U)\left(\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle\right) =$$

$$= (U \otimes U)\left(\frac{1}{\sqrt{2}}|0\rangle \otimes |1\rangle - \frac{1}{\sqrt{2}}|1\rangle \otimes |0\rangle\right) =$$

$$= \frac{1}{\sqrt{2}} U|0\rangle \otimes U|1\rangle - \frac{1}{\sqrt{2}} U|1\rangle \otimes U|0\rangle =$$

$$= \frac{1}{\sqrt{2}}\Big[(a|0\rangle + b|1\rangle)\otimes(c|0\rangle + d|1\rangle) - (c|0\rangle + d|1\rangle)\otimes(a|0\rangle + b|1\rangle)\Big] =$$

$$= \frac{1}{\sqrt{2}}\Big[\cancel{ac|00\rangle} + ad|01\rangle + bc|10\rangle + \cancel{bd|11\rangle} - \cancel{ac|00\rangle} - bc|01\rangle - ad|10\rangle - \cancel{bd|11\rangle}\Big] =$$

$$= \frac{1}{\sqrt{2}}\Big[ad|01\rangle + bc|10\rangle - bc|01\rangle - ad|10\rangle\Big] =$$

$$= \frac{1}{\sqrt{2}}\Big[(ad-bc)|01\rangle + (bc-ad)|10\rangle\Big] =$$

$$= \frac{1}{\sqrt{2}}\Big[(ad-bc)|01\rangle - (ad-bc)|10\rangle\Big] =$$

$$= \frac{1}{\sqrt{2}}(ad-bc)\big(|01\rangle - |10\rangle\big) =$$

$$= (ad-bc)\Big[\overbrace{\frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)}^{|T_2\rangle}\Big] = (ad-bc)|T_2\rangle$$

BUT SINCE U IS UNITARY $\Rightarrow$ $U\otimes U$ IS UNITARY

$$\Rightarrow \quad (ad-bc)^2 = 1 \quad \Rightarrow \quad (ad-bc) = \pm 1$$

$\Rightarrow \quad (U\otimes U)|T_2\rangle = |T_2\rangle$ (UP TO A PHASE)

$\boxed{\text{Ex. 3}}$

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}\,|1\rangle$$

What are the probabilities that we will measure |+> or |-> as they are defined in Exercise 1, if we perform the measurement in this new basis?

We want to perform a measurement in the new basis |+>, |->.

One way to do that is to write state |psi> into the new basis. In order to do that, we need to write |0>, |1> in terms of |+>, |->.

Remember that: $|+\rangle \equiv |\rightarrow\rangle$, $|-\rangle \equiv |\leftarrow\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad (A) \qquad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \quad (B)$$

From the sum (A) + (B) we get:

$$|+\rangle + |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) =$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle =$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|0\rangle = \frac{2}{\sqrt{2}}|0 = \sqrt{2}|0\rangle$$

$$\Rightarrow \quad |+\rangle + |-\rangle = \sqrt{2}\,|0\rangle \quad \Rightarrow \quad \boxed{|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)}$$

We wrote $|0\rangle$ in terms of $\{|+\rangle, |-\rangle\}$.

Now, we take (A) − (B) in order to get the state $|1\rangle$ in terms of $\{|+\rangle, |-\rangle\}$.

$$|+\rangle - |-\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle - \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) =$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle =$$

$$= \frac{2}{\sqrt{2}}|1\rangle = \sqrt{2}\,|1\rangle$$

$$\Rightarrow \quad |+\rangle - |-\rangle = \sqrt{2}\,|1\rangle$$

$$\Rightarrow \quad \boxed{|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)}$$

Now, we only need to substitute $|0\rangle$ and $|1\rangle$ in the state $|\psi\rangle$ with the relations found.

$$|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle =$$

$$= \sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) =$$

$$= \frac{1}{\sqrt{3}}\left(|+\rangle + |-\rangle\right) + \frac{1}{\sqrt{6}}\left(|+\rangle - |-\rangle\right) =$$

$$= \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|+\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|-\rangle$$

Then,

$$Pr\left[\,|\psi\rangle \text{ collepse to } |+\rangle\right] = \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)^2 \approx 0.97$$

$$Pr\left[\,|\psi\rangle \text{ collepse to } |-\rangle\right] = \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)^2 \approx 0.03$$

$\Pr[\,|\psi\rangle \text{ collapse to } |-\rangle\,] = \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right| \sim 0.03$

Check by yourself that those to probabilities sum to 1.

Another way to solve it is by constructing the matrix for the basis change $\{|0\rangle, |1\rangle\} \xrightarrow{B} \{|+\rangle, |-\rangle\}$

$$B = [\,|+\rangle \quad |-\rangle\,] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = H$$

"Hadamard Gate"

$$B^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = B$$

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}\,|1\rangle = |\psi\rangle_{01}$$

$|\psi\rangle$ in the base $\{|0\rangle, |1\rangle\}$

$$|\psi\rangle_{+-} = B^T|\psi\rangle = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2}{3}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} \end{bmatrix} = \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|+\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|-\rangle$$

$$\Rightarrow \quad |\psi\rangle = \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|+\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|-\rangle$$

This is exactly what we got with the previous method.

You can also use one of the postulates, to find those probabilities.

In general, a 1-qubit state written in the
ORTHONORMAL
basis $\{|x\rangle, |y\rangle\}$ is

$$|\psi\rangle = a|x\rangle + b|y\rangle$$

And,

$$\langle x|\psi\rangle = a\overset{1}{\langle x|x\rangle} + b\overset{0}{\langle x|y\rangle} = a$$

$$\text{Prob}[\text{observe } x] = \underbrace{(\langle x|\psi\rangle)^2 = a^2}_{}$$

BORN'S RULE

Then, in our case:

$$\text{Pr}[|\psi\rangle \text{ collapse to } |+\rangle] = (\langle +|\psi\rangle)^2$$

$$\text{Pr}[|\psi\rangle \text{ collapse to } |-\rangle] = (\langle -|\psi\rangle)^2$$