EXERCISE SET III : SAMPLE SOLUTIONS

EX 1 The (4x4) matrix corresponding to SWAP with respect to $|0\rangle, |1\rangle$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You may construct this by noticing to SWAP performs the following mapping:

$|00\rangle \to |00\rangle$
$|01\rangle \to |10\rangle$
$|10\rangle \to |01\rangle$
$|11\rangle \to |11\rangle$

So, we simply put the new vectors in the corresponding columns of our matrix. So, the 1st column of SWAP is the column vector corresponding to $|00\rangle$, the 2nd column of SWAP is the column vector that corresponds to $|10\rangle$, the 3rd is $|01\rangle$ and 4th $|11\rangle$.

• In order to see that it is unitary, multiply it with its transpose:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ SWAP is unitary.

(2). For any single Qubit states $|\phi\rangle, |\psi\rangle$: $SWAP(|\psi\rangle, |\phi\rangle) =$
$= |\phi\rangle|\psi\rangle$.

Let $|\psi\rangle = a|0\rangle + b|1\rangle$
$|\phi\rangle = c|0\rangle + d|1\rangle$.

$|\psi\rangle|\phi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle = A$

$SWAP(|\psi\rangle|\phi\rangle) = SWAP(A)$.
$= ac\,SWAP(|00\rangle) + ad\,SWAP(|01\rangle) + bc\,SWAP(|10\rangle)$
$+ bd\,SWAP(|11\rangle)$

The above follows because SWAP is a LINEAR operator (a matrix). Remember that an operator B is linear if $B(\alpha x + \beta y) = \alpha B(x) + \beta B(y)$.

So, $SWAP(|\psi\rangle|\phi\rangle) = ac|00\rangle + ad|10\rangle + bc|01\rangle + bd|11\rangle$
$= |\phi\rangle|\psi\rangle$ ✓

(3). For the 1st case, we have the following sequence of transformations performed by the circuit:

$|0\rangle|\psi\rangle|\phi\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle|\phi\rangle$

↱ SWAP ! CONTROL BIT IS $|1\rangle$!

$\to \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\phi\rangle|\psi\rangle\right)$

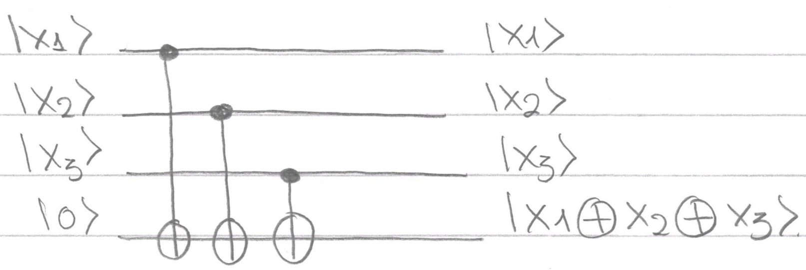
$\to \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\left(|0\rangle(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle) + |1\rangle(|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle)\right)$

Now $|\psi\rangle = 0$, $|\phi\rangle = 1$ so the above becomes
$\frac{1}{2}\left(|0\rangle(|01\rangle + |10\rangle) + |1\rangle(|01\rangle - |10\rangle)\right)$.

With probability $1/2$ measure the 1st Qbit to be $|0\rangle$ or $|1\rangle$.

For the 2nd part, perform similar calculations and observe that $|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle$ to get that the probability of measuring the 1st qubit at the state $|1\rangle$ is **zero**!

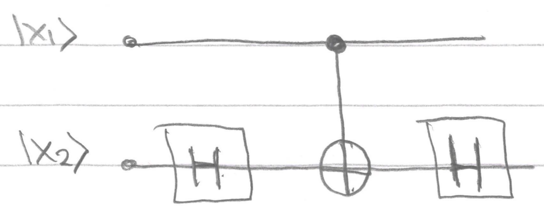
Ex. 2 This is a straightforward generalization of CNOT:
(remember that CNOT computes the transformation
$|x\rangle|\psi\rangle \longrightarrow |x\rangle|x \oplus \psi\rangle$
if $|x\rangle = 0$, then $|\psi\rangle$ is left unchanged
if $|x\rangle = 1$ $\quad |\psi\rangle$ flips
if $|x\rangle = |-\rangle$ $\quad$ then we have phase kickback!



Ex. 3 The Pauli Z gate leaves $|0\rangle$ unchanged but maps $|1\rangle$ to $-|1\rangle$.
The Hadamard transforms $|0\rangle \to |+\rangle$ and $|1\rangle \to |-\rangle$.
The effect of Z on $\{|+\rangle, |-\rangle\}$ is the same as the effect of flip X on $\{|0\rangle, |1\rangle\}$:

$$Z = HXH.$$

So, controlled Z is



- verify by performing the multiplication!

Ex 4. Grover with 4 items, one marked $x^* = 10$.

The matrix corresponding to the oracle operator $|x\rangle|b\rangle \longrightarrow |x\rangle|x \oplus b\rangle$ (which is the same as the phase inversion step) is simply

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix leaves the amplitudes of non-solutions 00, 01, 11 unchanged, and induces a phase flip of -1 to the marked element 10.

(2) The inversion about the mean is simply (see notes)

$H^{\otimes 2}(2|0\rangle\langle 0| - I)H^{\otimes 2}$. which is

$$\frac{1}{\sqrt{2}^2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}^2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Multiply everything, starting from $H^{\otimes 2}|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$$\underbrace{\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}}_{\text{Invert about mean}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Oracle/Phase shift}} \cdot \underbrace{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{H^{\otimes 2}|00\rangle} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

(4) PERFORMING ONE MORE STEP MEANS TO APPLY THE OPERATOR $H^{\otimes 2}[2|0\rangle\langle 0|-I]H^{\otimes 2}O_f$ TO THE PREVIOUSLY COMPUTED STATE WHICH GIVES:

$$H^{\otimes 2}[2|0\rangle\langle 0|-I]H^{\otimes 2}\cdot O_f\cdot|01\rangle = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\-1\end{bmatrix}$$

IF WE MEASURE THIS STATE, WE GET $x^* = |01\rangle$ WITH PROBABILITY EQUAL TO $\left[1/2\right]^2 = \frac{1}{4}$.

EX 5. THIS IS DISCUSSED IN THE NOTES ☺.