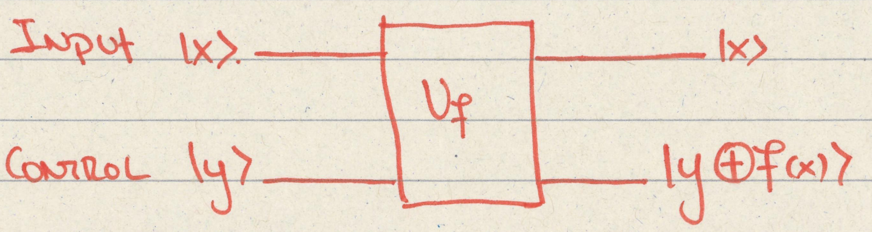
More Notes on Deutch-Jozsa Algorithm

- $f : \{0,1\}^n \to \{0,1\}$. Is f constant or balanced?

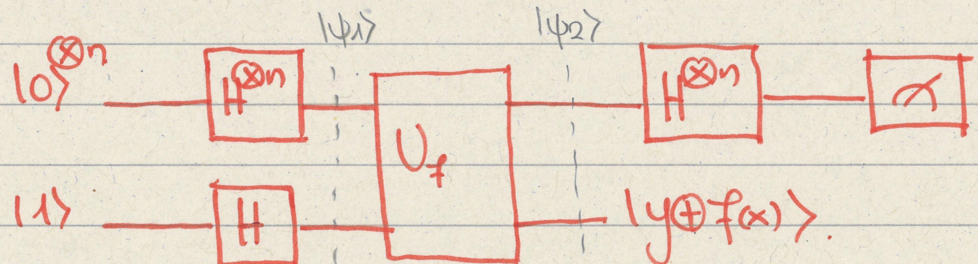
- Oracle access to f:



Unitary!

- If $|y\rangle = |0\rangle$ and $|x\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle)$ the output is
$$\frac{1}{\sqrt{2}}\Big[\, |0, f(0)\rangle + |1, f(1)\rangle \,\Big]$$
: Quantum Parallelism!

Deutch-Jozsa Circuit:



- $$H^{\otimes n}|0\rangle = (H|0\rangle) \otimes (H|0\rangle) \otimes \cdots \otimes (H|0\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =$$
$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

- $$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= \frac{1}{\sqrt{2^n}} \cdot \frac{1}{\sqrt{2}} \cdot \left[\sum_{x \in \{0,1\}^n} |x\rangle|0\rangle - \sum_{x \in \{0,1\}^n} |x\rangle|1\rangle \right]$$

$0 \oplus b = b \ \forall\, b \in \{0,1\}$

- $$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \cdot \frac{1}{\sqrt{2}} \cdot \left[\sum_{x\in\{0,1\}^n} |x\rangle|0\oplus f(x)\rangle - \sum_{x\in\{0,1\}^n} |x\rangle|1\oplus f(x)\rangle \right]$$

$$= \frac{1}{\sqrt{2^n}} \cdot \frac{1}{\sqrt{2}} \cdot \left[\sum_{x\in\{0,1\}^n} |x\rangle|f(x)\rangle - \sum_{x\in\{0,1\}^n} |x\rangle|1\oplus f(x)\rangle \right]$$

$$= \frac{1}{\sqrt{2^n}} \cdot \frac{1}{\sqrt{2}} \cdot \left[\sum_{x\in\{0,1\}^n} |x\rangle \otimes \left(|f(x)\rangle - |1\oplus f(x)\rangle \right) \right]$$

- $$|f(x)\rangle - |1\oplus f(x)\rangle = \begin{cases} |0\rangle - |1\rangle & \text{if } f(x) = 0 \\ |1\rangle - |0\rangle & \text{if } f(x) = 1 \end{cases}$$

$$\Rightarrow |f(x)\rangle - |1\oplus f(x)\rangle = (-1)^{f(x)} (|0\rangle - |1\rangle).$$

- So, $|\psi_2\rangle$ Becomes:

$$|\psi_2\rangle = \underbrace{\frac{1}{\sqrt{2^n}} \left[\sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle \right]}_{|q_1'\rangle} \otimes \underbrace{\frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right)}_{|q_2\rangle}.$$

- If f is a constant function, then $f(x) = 0 \ \forall x$ or $f(x) = 1 \ \forall x$.

$$\Rightarrow |\psi_2\rangle = \sum_{x\in\{0,1\}^n} |x\rangle \quad \text{or} \quad |\psi_2\rangle = -\sum_{x\in\{0,1\}^n} |x\rangle.$$

- In this case $H^{\otimes n}|q_1\rangle = |0\rangle^{\otimes n}$ or $H^{\otimes n}|q_1\rangle = -|0\rangle^{\otimes n}$

- In both cases, measuring all n input qbits gives $000\cdots0$ with probability 1!

- If f is not constant, then in our problem we know it must be balanced.

$\Rightarrow$ $|q_1\rangle$ contains an equal number of $+$ and $-$ terms.

- The amplitude of measuring $00\cdots0$ in this case (after we pass $|q_1\rangle$ through $H^{\otimes n}$) is the inner product of the vector corresponding to $|q_1\rangle$ (same number of $+,-$ terms) with the <u>1st row</u> of $H^{\otimes n}$ which is the all-1 row and the result is always zero !

(each $+$ term gets canceled be a $-$ term)

$\Rightarrow$ The result of the measurement is $00\cdots0$ if, and only if, f is <u>constant</u> !

$\hookrightarrow$ We must observe at least one 1 outcome somewhere.