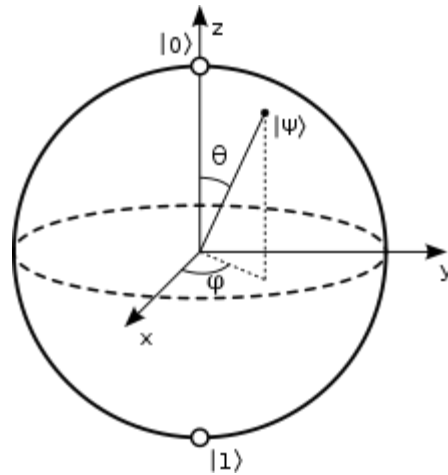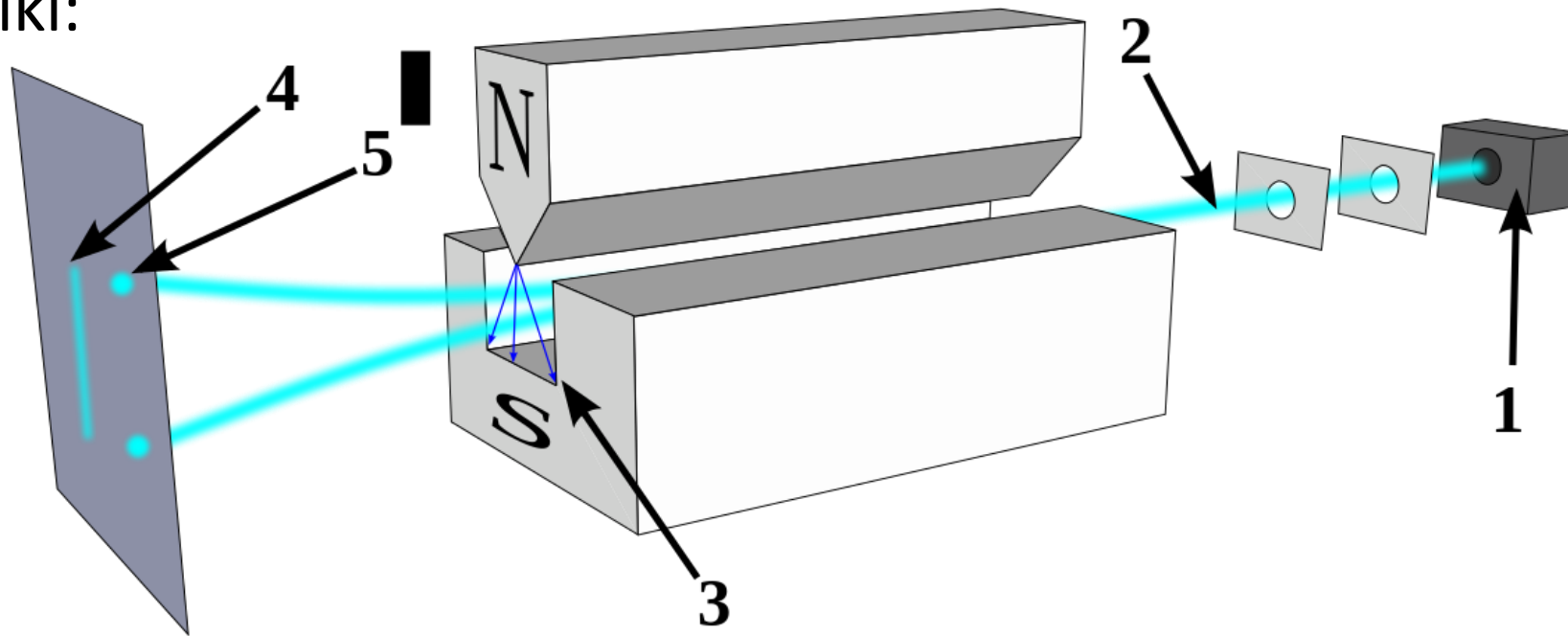# Intro to Quantum Computing

Maths of Qbits Part 1

Recall: The Stern-Gerlach Experiment

- From wiki:

S-G Experiment

- Instead of measuring a continuum of values of the spin of the magnetic field induced by silver atoms, we observe *only two values:*

S-G Experiment

- Instead of measuring a continuum of values of the spin of the magnetic field induced by silver atoms, we observe **only two values:**

Up value ↑ or down value ↓.

S-G Experiment

- Instead of measuring a continuum of values of the spin of the magnetic field induced by silver atoms, we observe *only two values:*

  Up value ↑ or down value ↓.

- The spin is a quantized quantity: it can be either up (its south pole is oriented upwards) or down (its north pole is oriented upwards)

S-G Experiment

- Instead of measuring a continuum of values of the spin of the magnetic field induced by silver atoms, we observe **only two values:**

    Up value ↑ or down value ↓.

- The spin is a quantized quantity: it can be either up (its south pole is oriented upwards) or down (its north pole is oriented upwards)

- Before the experiment, the spin has unknown orientation: it is **in a superposition** of both ↑ and ↓:

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

S-G Experiment

- Instead of measuring a continuum of values of the spin of the magnetic field induced by silver atoms, we observe **only two values:**

  Up value ↑ or down value ↓.

- The spin is a quantized quantity: it can be either up (its south pole is oriented upwards) or down (its north pole is oriented upwards)

- Before the experiment, the spin has unknown orientation: it is **in a superposition** of both ↑ and ↓:
$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

- Such that $a^2 + b^2 = 1.$

S-G Experiment

- a, b are called **amplitudes** of the quantum states $|\uparrow\rangle, |\downarrow\rangle$ respectively.

S-G Experiment

- a, b are called ***amplitudes*** of the quantum states $|\uparrow\rangle, |\downarrow\rangle$ respectively.
- An electron will be in quantum state $|\uparrow\rangle$ ($|\downarrow\rangle$) with probability $a^2 (b^2)$

S-G Experiment

- a, b are called **amplitudes** of the quantum states $|\uparrow\rangle, |\downarrow\rangle$ respectively.
- An electron will be in quantum state $|\uparrow\rangle$ $(|\downarrow\rangle)$ with probability a^2 (b^2)
- Observing its state constitutes a **measurement**:

S-G Experiment

- a, b are called ***amplitudes*** of the quantum states $|\uparrow\rangle, |\downarrow\rangle$ respectively.

- An electron will be in quantum state $|\uparrow\rangle$ ($|\downarrow\rangle$) with probability $a^2 (b^2)$

- Observing its state constitutes a ***measurement***:

- Measurement causes the superposition to ***collapse to one of its constituent states*** with the corresponding probability (Born's rule).

S-G Experiment

- a, b are called ***amplitudes*** of the quantum states $|\uparrow\rangle, |\downarrow\rangle$ respectively.
- An electron will be in quantum state $|\uparrow\rangle$ ($|\downarrow\rangle$) with probability $a^2 (b^2)$
- Observing its state constitutes a ***measurement***:
- Measurement causes the superposition to ***collapse to one of its constituent states*** with the corresponding probability (Born's rule).
- The act of measurement a quantum state is irreversible.

S-G Experiment

- a, b are called ***amplitudes*** of the quantum states $|\!\uparrow\rangle, |\!\downarrow\rangle$ respectively.

- An electron will be in quantum state $|\!\uparrow\rangle$ ($|\!\downarrow\rangle$) with probability a^2 (b^2)

- Observing its state constitutes a ***measurement***:

- Measurement causes the superposition to ***collapse to one of its constituent states*** with the corresponding probability (Born's rule).

- The act of measurement a quantum state is irreversible.

- Once we observe a state, for example $|\!\uparrow\rangle$, *the particle will stay in that state no matter how many times we measure it again*.

S-G Experiment

- a, b are called ***amplitudes*** of the quantum states $|\uparrow\rangle, |\downarrow\rangle$ respectively.

- An electron will be in quantum state $|\uparrow\rangle$ ($|\downarrow\rangle$) with probability $a^2 (b^2)$

- Observing its state constitutes a ***measurement***:

- Measurement causes the superposition to ***collapse to one of its constituent states*** with the corresponding probability (Born's rule).

- The act of measurement a quantum state is irreversible.

- Once we observe a state, for example $|\uparrow\rangle$, ***the particle will stay in that state no matter how many times we measure it again***.

- On the same orientation!

S-G Experiment – Rotating the magnet

- But if we ***rotate*** the apparatus, say by 90°, and we repeat the experiment, the electron that was measured $|\uparrow\rangle$ will be measured to have 50% chance to be measured in either of the two new directions.

S-G Experiment – Rotating the magnet

- But if we ***rotate*** the apparatus, say by 90°, and we repeat the experiment, the electron that was measured $|\uparrow\rangle$ will be measured to have 50% chance to be measured in either of the two new directions.
- **Which directions?**

S-G Experiment – Rotating the magnet

- But if we ***rotate*** the apparatus, say by 90°, and we repeat the experiment, the electron that was measured $|\uparrow\rangle$ will be measured to have 50% chance to be measured in either of the two new directions.

- **Which directions?**

- Since we have rotated the magnet by 90°, the two new orientations of N-S pole are $\rightarrow$ and $\leftarrow$.

S-G Experiment – Rotating the magnet

- But if we ***rotate*** the apparatus, say by 90°, and we repeat the experiment, the electron that was measured $|\uparrow\rangle$ will be measured to have 50% chance to be measured in either of the two new directions.

- **Which directions?**

- Since we have rotated the magnet by 90°, the two new orientations of N-S pole are $\rightarrow$ and $\leftarrow$.

- By rotating the apparatus we *induce a new base of measurement* and the spin of $|\psi\rangle$ (before measurement) is in superposition

$$c|\rightarrow\rangle + d|\leftarrow\rangle, \text{ with } c^2 + d^2 = 1.$$

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?
- Is it $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$?

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?

- Is it $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$?

- Or $|\psi\rangle = c|\rightarrow\rangle + d|\leftarrow\rangle$?

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?

- Is it $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$?

- Or $\quad |\psi\rangle = c|\rightarrow\rangle + d|\leftarrow\rangle$?

It depends in which direction we are making our measurement!

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?

- Is it $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$?

- Or $|\psi\rangle = c|\rightarrow\rangle + d|\leftarrow\rangle$?

It depends in which direction we are making our measurement!

- If we are interested in $\{\uparrow, \downarrow\}$ then it is in the **first**.

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?
- Is it $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$?
- Or $|\psi\rangle = c|\rightarrow\rangle + d|\leftarrow\rangle$?

It depends in which direction we are making our measurement!

- If we are interested in $\{\uparrow, \downarrow\}$ then it is in the **first**.
- If we are interested in $\{\rightarrow, \leftarrow\}$ then it is in the **second**.

Superposition in different bases

- So which one is the true superposition of $|\psi\rangle$?

- Is it $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$?

- Or $|\psi\rangle = c|\rightarrow\rangle + d|\leftarrow\rangle$?

It depends in which direction we are making our measurement!

- If we are interested in $\{\uparrow, \downarrow\}$ then it is in the **first**.

- If we are interested in $\{\rightarrow, \leftarrow\}$ then it is in the **second**.

- We can also measure on any other direction we like!

The state of a Qbit

- A superposition of *basic states* describes (much!) more than just probabilities.

The state of a Qbit

- A superposition of **basic states** describes (much!) more than just probabilities.
- For example:
  - $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$

The state of a Qbit

- A superposition of **basic states** describes (much!) more than just probabilities.

- For example:
  - $\frac{1}{\sqrt{2}}|{\uparrow}\rangle + \frac{1}{\sqrt{2}}|{\downarrow}\rangle$
  - $\frac{1}{\sqrt{2}}|{\uparrow}\rangle - \frac{1}{\sqrt{2}}|{\downarrow}\rangle$

The state of a Qbit

- A superposition of ***basic states*** describes (much!) more than just probabilities.

- For example:
  - $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$
  - $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$

  Both have ½ probability of being on either state when we make a measurement.

The state of a Qbit

- A superposition of *__basic states__* describes (much!) more than just probabilities.

- For example:
  - $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$
  - $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$

  Both have ½ probability of being on either state when we make a measurement.

- But they are very different superpositions!

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

- Let us say that ↑ spin encodes 0 and ↓ encodes 1.

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

- Let us say that ↑ spin encodes 0 and ↓ encodes 1.

- Before measurement it is in superposition of both:

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

- Let us say that ↑ spin encodes 0 and ↓ encodes 1.

- Before measurement it is in superposition of both:

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \text{ such that } a, b \in R \text{ and } a^2 + b^2 = 1 \left(|a|^2 + |b|^2 = \right.$$

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

- Let us say that ↑ spin encodes 0 and ↓ encodes 1.

- Before measurement it is in superposition of both:

$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ such that $a, b \in R$ and $a^2 + b^2 = 1 \left(|a|^2 + |b|^2 = \right.$

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

- Let us say that ↑ spin encodes 0 and ↓ encodes 1.

- Before measurement it is in superposition of both:

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \text{ such that } a, b \in R \text{ and } a^2 + b^2 = 1 \left(|a|^2 + |b|^2 = \right.$$

Using Spin to represent Qbit

- G-S experiment showed that electrons can have either ↑ or ↓ spin.

- Let us say that ↑ spin encodes 0 and ↓ encodes 1.

- Before measurement it is in superposition of both:

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \text{ such that } a, b \in R \text{ and } a^2 + b^2 = 1 \left(|a|^2 + |b|^2 = \right.$$
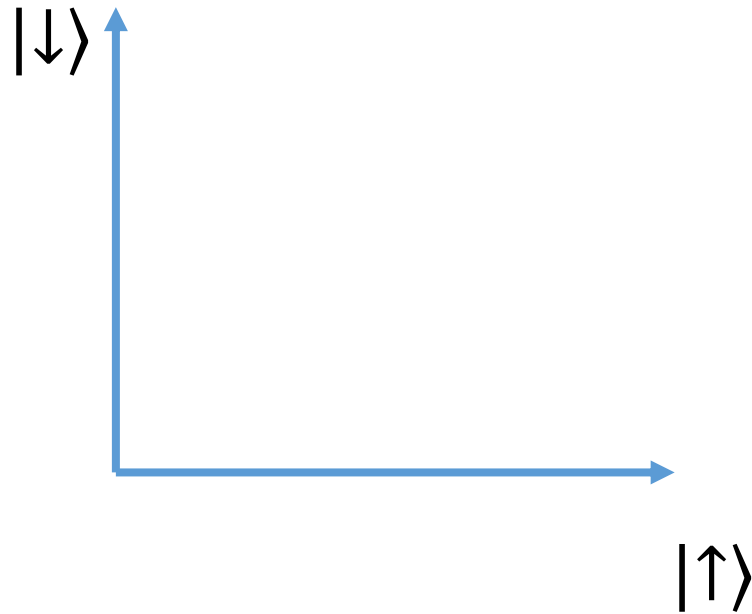
Geometrical Representation

- $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e., $a = 1, b = 0.$
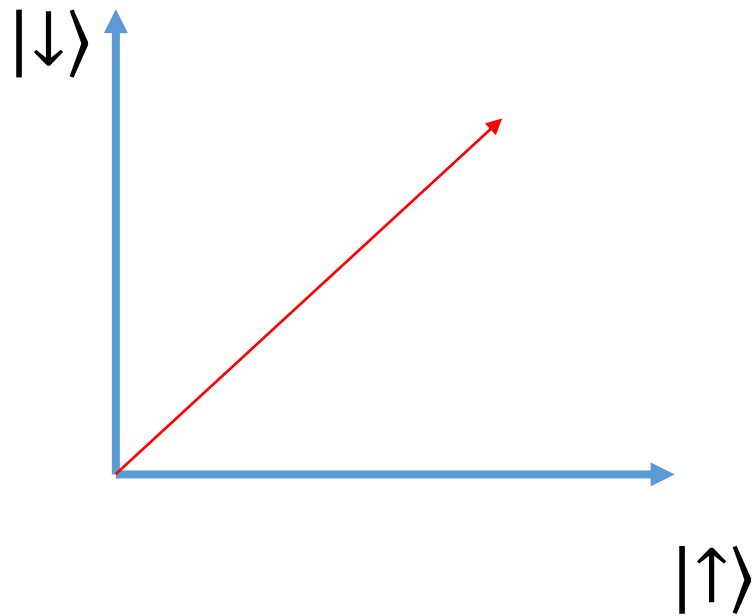
Geometrical Representation

- $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e., $a = 1, b = 0$.

- $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e., $a = 0, b = 1$.

Geometrical Representation

- $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e., $a = 1, b = 0$.

- $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e., $a = 0, b = 1$.

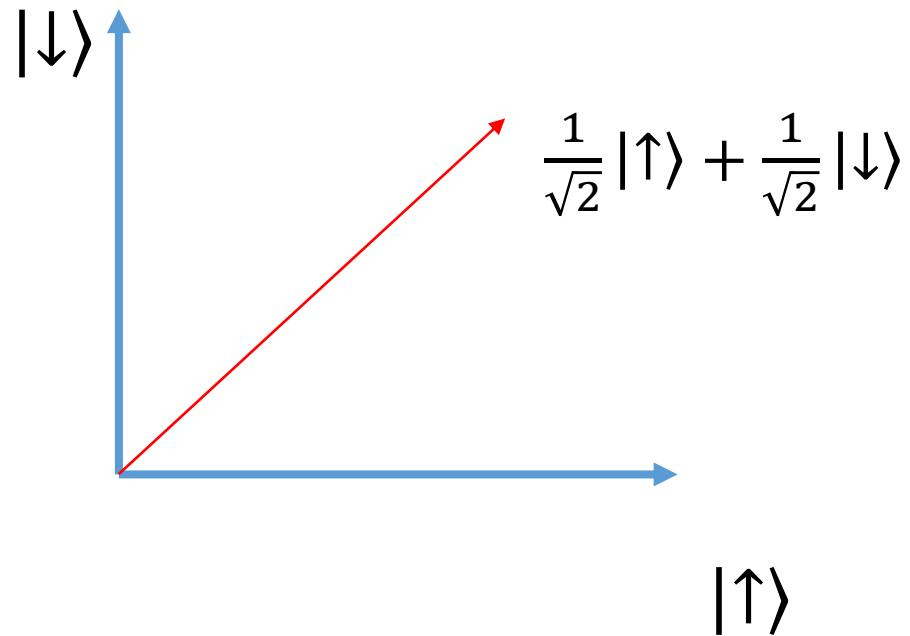$|\downarrow\rangle$

$|\uparrow\rangle$

Geometrical Representation

- $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e., $a = 1, b = 0.$

- $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e., $a = 0, b = 1.$

$|\downarrow\rangle$

$|\uparrow\rangle$

Geometrical Representation

- $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e., $a = 1, b = 0$.

- $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e., $a = 0, b = 1$.

$|\downarrow\rangle$

$\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$

$|\uparrow\rangle$

But what is $|\,\rangle$?

But what is $| \rangle$?

- This is called a ***ket.***

But what is $| \rangle$?

- This is called a **_ket._**
- From braket (bra-ket).

But what is | ⟩?

- This is called a **ket.**
- From braket (bra-ket).
- This is a bra ⟨·|.

But what is | ⟩?

- This is called a ***ket.***

- From braket (bra-ket).

- This is a bra ⟨·|.

- A ket is simply a column vector!

But what is | ⟩?

- This is called a ***ket.***

- From braket (bra-ket).

- This is a bra ⟨·|.

- A ket is simply a column vector!

- It is also known as Dirac notation.

But what is | ⟩?

- This is called a **_ket._**

- From braket (bra-ket).

- This is a bra ⟨·|.

- A ket is simply a column vector!

- It is also known as Dirac notation.

- So a Qbit is a unit vector in a 2-dimensional space.

But what is | ⟩?

- This is called a ***ket.***
- From braket (bra-ket).
- This is a bra ⟨·|.
- A ket is simply a column vector!
- It is also known as Dirac notation.
- So a Qbit is a unit vector in a 2-dimensional space.

Braket (Dirac's) notation

- Any quantum quantity or system ψ is described as (or by) $|\psi\rangle$.

- It is an element of a **complex Hilbert Space**.

- We will not deal (as discussed) with complex number but for now we will present the ideas using complex numbers for full generality.

Measurement (again)

- Assume a qbit $|\psi\rangle$ that is represented by a unit vector which makes angle θ with $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Measurement (again)

- Assume a qbit $|\psi\rangle$ that is represented by a unit vector which makes angle θ with $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- So then $|\psi\rangle = \cos\theta \, |0\rangle + \sin\theta \, |1\rangle$.

Measurement (again)

- Assume a qbit $|\psi\rangle$ that is represented by a unit vector which makes angle θ with $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- So then $|\psi\rangle = \cos\theta \, |0\rangle + \sin\theta \, |1\rangle$.

- If we measure:
  - We observe $|0\rangle$ with probability $\cos^2\theta$
 - Or $|1\rangle$ with probability $\sin^2\theta = \cos^2(\frac{\pi}{2} - \theta)$.

Measurement (again)

- Assume a qbit $|\psi\rangle$ that is represented by a unit vector which makes angle θ with $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- So then $|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$.

- If we measure:
  - We observe $|0\rangle$ with probability $\cos^2\theta$
  - Or $|1\rangle$ with probability $\sin^2\theta = \cos^2(\frac{\pi}{2} - \theta)$.

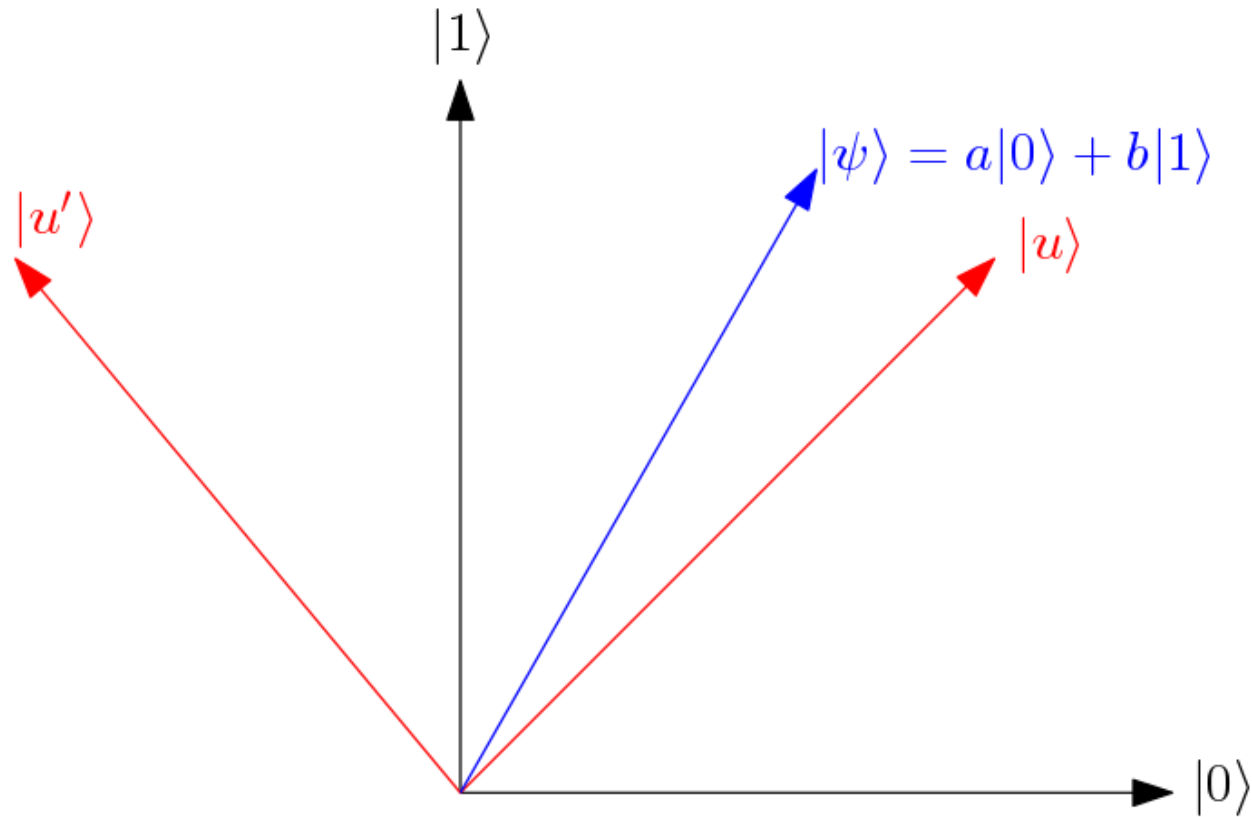- So a measurement is simply *a projection onto the standard bases with the corresponding probabilities.*
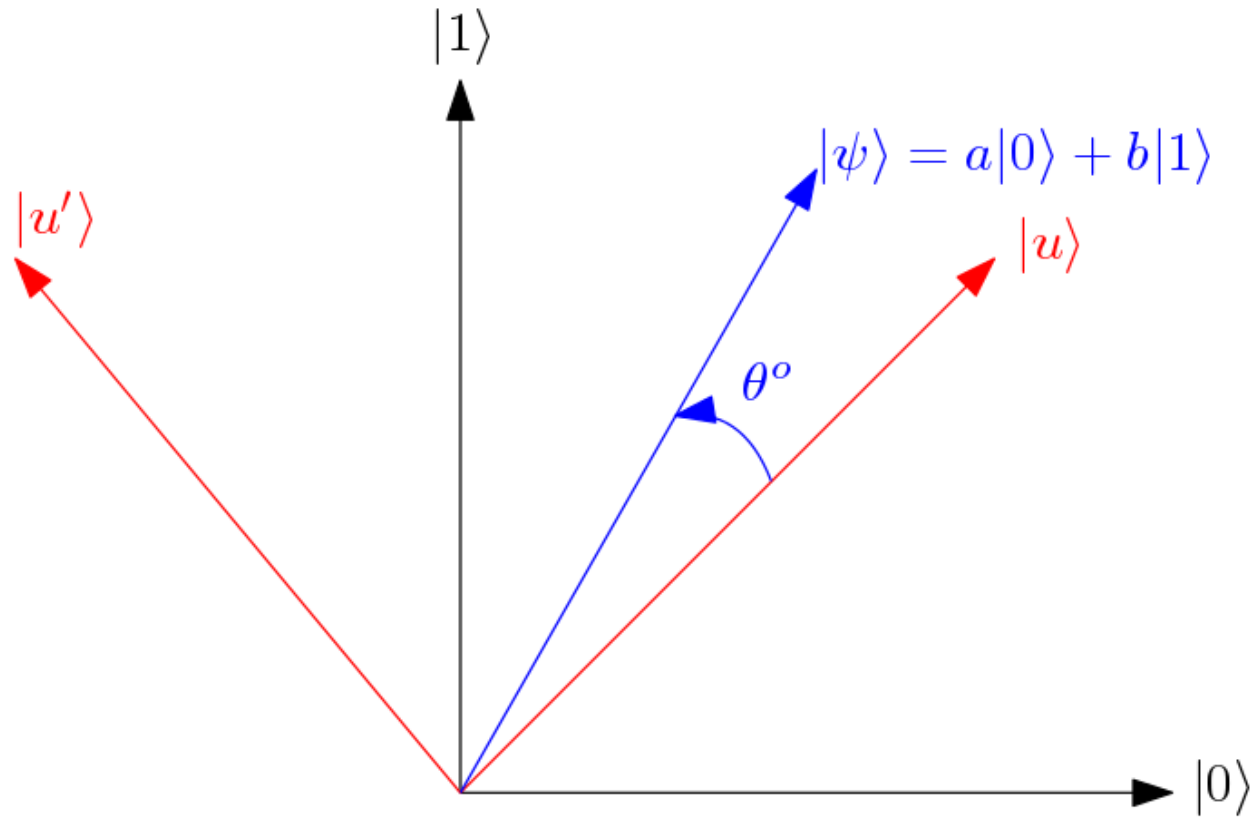
What if we want to measure in another basis?
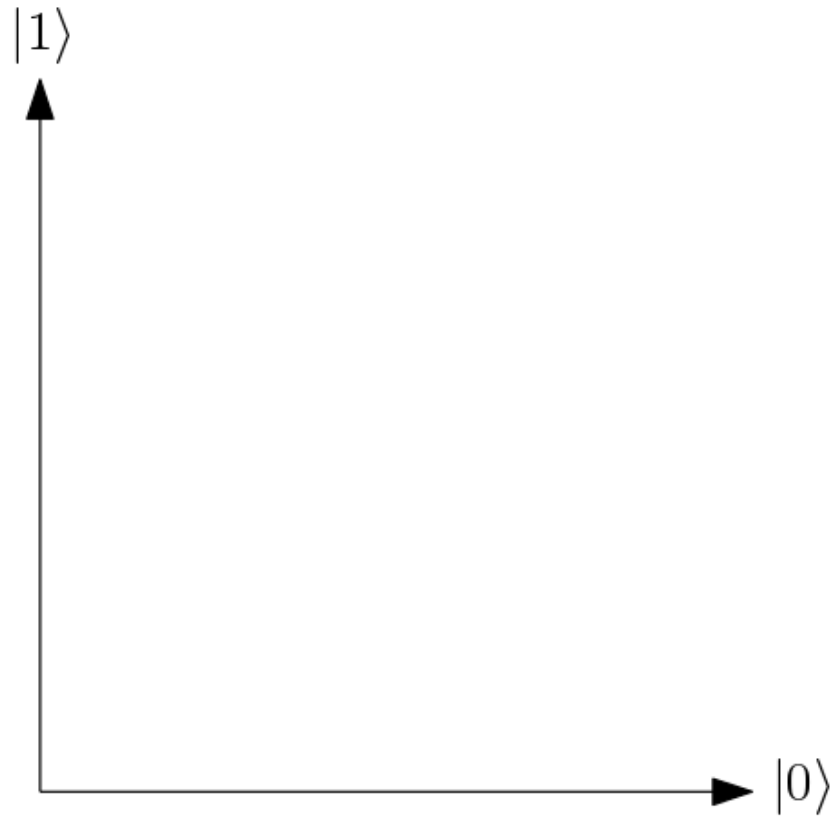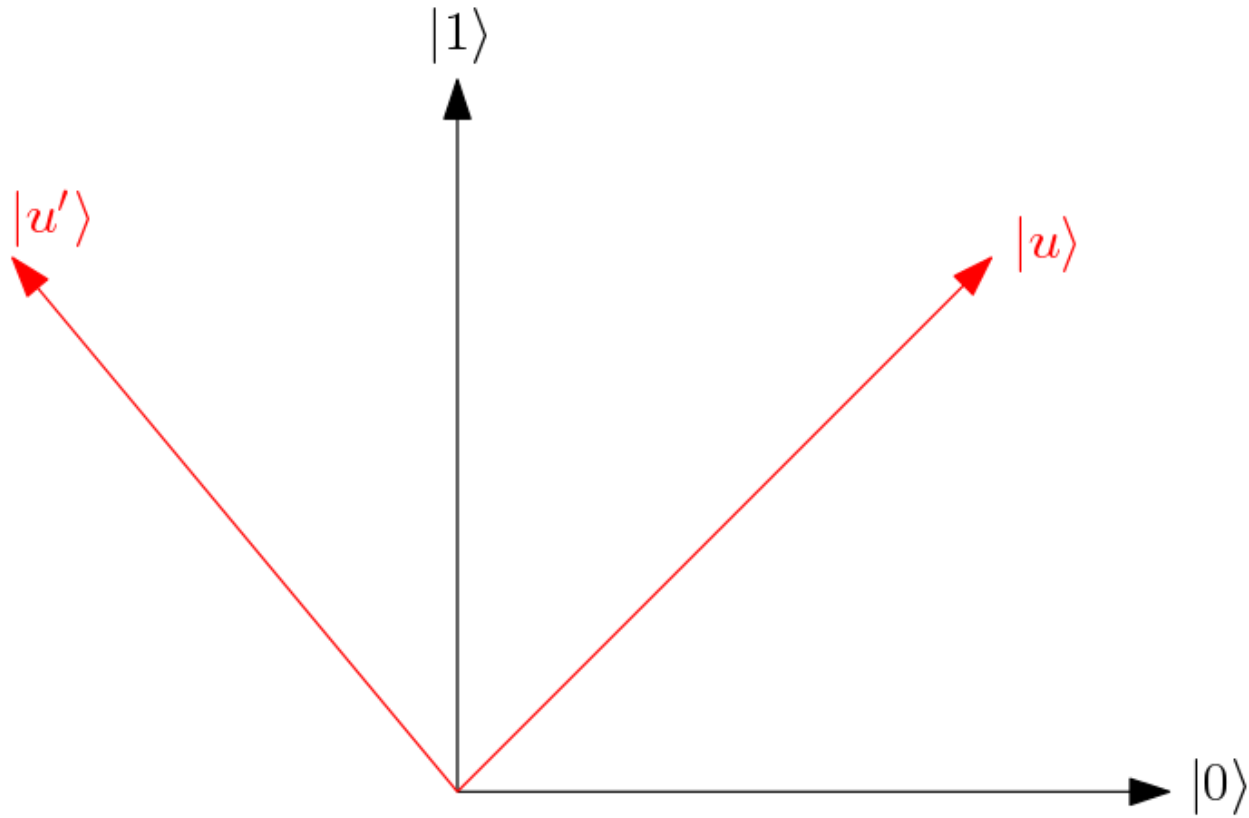
What if we want to measure in another basis?

What if we want to measure in another basis?

What if we want to measure in another basis?

What if we want to measure in another basis?

- We want to measure $|\psi\rangle$ in the new basis $\{\boldsymbol{u}, \boldsymbol{u}'\}$.

What if we want to measure in another basis?

- We want to measure $|\psi\rangle$ in the new basis $\{u, u'\}$.

- We expect to observe $|u\rangle$ with probability $\cos^2 \theta$ and $|u'\rangle$ with probability $\sin^2 \theta$.

What if we want to measure in another basis?

- We want to measure $|\psi\rangle$ in the new basis $\{u, u'\}$.

- We expect to observe $|u\rangle$ with probability $\cos^2\theta$ and $|u'\rangle$ with probability $\sin^2\theta$.

- What does that mean?

What if we want to measure in another basis?

- We want to measure $|\psi\rangle$ in the new basis $\{u, u'\}$.

- We expect to observe $|u\rangle$ with probability $\cos^2\theta$ and $|u'\rangle$ with probability $\sin^2\theta$.

- What does that mean?

- We measure to see if the Qbit will be in the $|u\rangle$ superposition of $|0\rangle, |1\rangle$ or in the $|u'\rangle$ superposition of $|0\rangle, |1\rangle$.

What if we want to measure in another basis?

- We want to measure $|\psi\rangle$ in the new basis $\{u, u'\}$.

- We expect to observe $|u\rangle$ with probability $\cos^2 \theta$ and $|u'\rangle$ with probability $\sin^2 \theta$.

- What does that mean?

- We measure to see if the Qbit will be in the $|u\rangle$ superposition of $|0\rangle, |1\rangle$ or in the $|u'\rangle$ superposition of $|0\rangle, |1\rangle$.

- I.e. $|u\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|u'\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

What if we want to measure in another basis?

- We want to measure $|\psi\rangle$ in the new basis $\{u, u'\}$.

- We expect to observe $|u\rangle$ with probability $\cos^2 \theta$ and $|u'\rangle$ with probability $\sin^2 \theta$.

- What does that mean?

- We measure to see if the Qbit will be in the $|u\rangle$ superposition of $|0\rangle, |1\rangle$ or in the $|u'\rangle$ superposition of $|0\rangle, |1\rangle$.

- I.e. $|u\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|u'\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

- So we do not measure anymore if the qbit has spin **up** or **down** but if it is in $|u\rangle$ state or $|u'\rangle$ state.

The study of Qbits uses a lot of linear algebra

- Quantum states are represented by *vectors*

The study of Qbits uses a lot of linear algebra

- Quantum states are represented by *vectors*
- These vectors form a (complex or real) *vector space*.

The study of Qbits uses a lot of linear algebra

- Quantum states are represented by *vectors*

- These vectors form a (complex or real) *vector space*.

- The transformations (evolution) of quantum systems is described by *linear operators*.

The study of Qbits uses a lot of linear algebra

- Quantum states are represented by *vectors*

- These vectors form a (complex or real) *vector space*.

- The transformations (evolution) of quantum systems is described by *linear operators*.

- We will give a short recap of the notions we will be using in the bra-ket vocabulary.

Bras and Kets

- A ***ket*** is a column vector: for example $|v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Bras and Kets

- A ***ket*** is a column vector: for example $|v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- A ***bra*** is a row vector: $\langle v| = [2 \ 3] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$.

Bras and Kets

- A *ket* is a column vector: for example $|v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- A *bra* is a row vector: $\langle v| = [2\ 3] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$.

- If $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $|\mathbf{b}\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then $|a+b\rangle = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix} = |b+a\rangle$.

Bras and Kets

- A **ket** is a column vector: for example $|v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- A **bra** is a row vector: $\langle v| = [2\ 3] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$.

- If $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $|\mathbf{b}\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then $|a+b\rangle = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} = |b+a\rangle$.

- The dot (inner) product between a bra and a ket is:

Bras and Kets

- A **ket** is a column vector: for example $|v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- A **bra** is a row vector: $\langle v| = [2\ 3] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$.

- If $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $|\mathbf{b}\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then $|a + b\rangle = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} = |b + a\rangle$.

- The dot (inner) product between a bra and a ket is:

- $\langle a||b\rangle = a_1 b_1 + a_2 b_2$.

Bras and Kets

- A **_ket_** is a column vector: for example $|v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- A **_bra_** is a row vector: $\langle v| = [2\ 3] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$.

- If $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $|b\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then $|a + b\rangle = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} = |b + a\rangle$.

- The dot (inner) product between a bra and a ket is:

- $\langle a||b\rangle = a_1 b_1 + a_2 b_2$.

- More compactly we denoted it as $\langle a|b\rangle$ (i.e., $a^T b$).

Bras and Kets

Length of a ket:

Bras and Kets

Length of a ket:

$$\left| |a\rangle \right| = \sqrt{\sum_i a_i^2} = \sqrt{\langle a|a \rangle}.$$

Bras and Kets

Length of a ket:

$$\left|\left|a\right\rangle\right| = \sqrt{\sum_i a_i^2} = \sqrt{\langle a|a\rangle}.$$

We will be dealing with unit vectors (kets with length =1)

Bras and Kets

Length of a ket:

$$\big|\,|a\rangle\,\big| = \sqrt{\sum_i a_i^2} = \sqrt{\langle a|a\rangle}.$$

We will be dealing with unit vectors (kets with length =1)

Orthogonality:

Bras and Kets

Length of a ket:

$$\big|\,|a\rangle\,\big| = \sqrt{\sum_i a_i^2} = \sqrt{\langle a|a\rangle}.$$

We will be dealing with unit vectors (kets with length =1)

Orthogonality:

Two kets $|a\rangle, |b\rangle$ are **orthogonal** if and only if $\langle a|b\rangle = 0$.

Bras and Kets

Length of a ket:

$$\big|\,|a\rangle\big| = \sqrt{\sum_i a_i^2} = \sqrt{\langle a|a\rangle}.$$

We will be dealing with unit vectors (kets with length =1)

Orthogonality:

Two kets $|a\rangle, |b\rangle$ are ***orthogonal*** if and only if $\langle a|b\rangle = 0$.

Geometrically: $\big|\,|a\rangle + |b\rangle\big|^2 = \big|\,|a\rangle\big|^2 + \big|\,|b\rangle\big|^2 + 2\langle a|b\rangle$.

Bras and Kets

Length of a ket:

$$\big|\,|a\rangle\big| = \sqrt{\sum_i a_i^2} = \sqrt{\langle a|a\rangle}.$$

We will be dealing with unit vectors (kets with length =1)

Orthogonality:

Two kets $|a\rangle, |b\rangle$ are **orthogonal** if and only if $\langle a|b\rangle = 0$.

Geometrically: $\big|\,|a\rangle + |b\rangle\big|^2 = \big|\,|a\rangle\big|^2 + \big|\,|b\rangle\big|^2 + 2\langle a|b\rangle$.

If $\langle a|b\rangle = 0$ then the above gives Pythagorean Theorem (and vice versa).

Orthonormal Bases

- Ortho (from orthogonal) and normal from normalized (i.e. unit).

Orthonormal Bases

- Ortho (from orthogonal) and normal from normalized (i.e. unit).
- For n-dimensional kets, such a basis will have n unit kets that are orthogonal to each other.

Orthonormal Bases

- Ortho (from orthogonal) and normal from normalized (i.e. unit).
- For n-dimensional kets, such a basis will have n unit kets that are orthogonal to each other.
- In $\boldsymbol{R^2}$ we need two kets $|\boldsymbol{b_1}\rangle, |\boldsymbol{b_2}\rangle$ such that
  - $\langle \boldsymbol{b_1}|\boldsymbol{b_1}\rangle = \boldsymbol{1}$
  - $\langle \boldsymbol{b_2}|\boldsymbol{b_2}\rangle = \boldsymbol{1}$
  - $\langle \boldsymbol{b_1}|\boldsymbol{b_2}\rangle = \boldsymbol{0}.$

Orthonormal Bases

- Ortho (from orthogonal) and normal from normalized (i.e. unit).

- For n-dimensional kets, such a basis will have n unit kets that are orthogonal to each other.

- In $\boldsymbol{R^2}$ we need two kets $|\boldsymbol{b_1}\rangle, |\boldsymbol{b_2}\rangle$ such that
    - $\langle \boldsymbol{b_1}|\boldsymbol{b_1}\rangle = \boldsymbol{1}$
    - $\langle \boldsymbol{b_2}|\boldsymbol{b_2}\rangle = \boldsymbol{1}$
    - $\langle \boldsymbol{b_1}|\boldsymbol{b_2}\rangle = \boldsymbol{0}.$

- $\begin{bmatrix} \boldsymbol{1} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{1} \end{bmatrix}$ **or** $\begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix}$ **or** $\begin{bmatrix} \dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \dfrac{-\sqrt{3}}{2} \\ \dfrac{1}{2} \end{bmatrix}$ are all orthonormal.

Orthonormal bases

- The first of the previous three orthonormal basis corresponds to a specific measuring direction i.e., $\{|\uparrow\rangle, |\downarrow\rangle\}$.

Orthonormal bases

- The first of the previous three orthonormal basis corresponds to a specific measuring direction i.e., $\{|\uparrow\rangle, |\downarrow\rangle\}$.

- The next two correspond to $\{|\rightarrow\rangle, |\leftarrow\rangle\}$ and $\{|\nearrow\rangle, |\swarrow\rangle\}$ respectively.

Linear combinations of basis vectors

- Given a ket $|a\rangle$ and some basis (all in the same space), can we express $|a\rangle$ as linear combination of the basis kets?

Linear combinations of basis vectors

- Given a ket $|a\rangle$ and some basis (all in the same space), can we express $|a\rangle$ as linear combination of the basis kets?

- For example $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

Linear combinations of basis vectors

- Given a ket $|a\rangle$ and some basis (all in the same space), can we express $|a\rangle$ as linear combination of the basis kets?

- For example $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- Trivially, $x = a_1$ and $y = a_2$.

Linear combinations of basis vectors

- Given a ket $|a\rangle$ and some basis (all in the same space), can we express $|a\rangle$ as linear combination of the basis kets?

- For example $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- Trivially, $x = a_1$ and $y = a_2$.

- What about the basis $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = |\rightarrow\rangle, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |\leftarrow\rangle$?

Linear combinations of basis vectors

- Given a ket $|a\rangle$ and some basis (all in the same space), can we express $|a\rangle$ as linear combination of the basis kets?

- For example $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

- Trivially, $x = a_1$ and $y = a_2$.

- What about the basis $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = |\rightarrow\rangle, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |\leftarrow\rangle$?

- We can (easily) show that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{a_1 - a_2}{\sqrt{2}} |\rightarrow\rangle + \frac{a_1 + a_2}{\sqrt{2}} |\leftarrow\rangle.$

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ is a basis for R^n.

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ is a basis for R^n.
- We want to write $|v\rangle = w_1|b_1\rangle + \cdots + w_n|b_n\rangle$.

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|\boldsymbol{b_1}\rangle, |\boldsymbol{b_2}\rangle, \ldots, |\boldsymbol{b_n}\rangle\}$ is a basis for $\boldsymbol{R^n}$.
- We want to write $|v\rangle = \boldsymbol{w_1}|\boldsymbol{b_1}\rangle + \cdots + \boldsymbol{w_n}|\boldsymbol{b_n}\rangle.$
- We can (easily) show that $\boldsymbol{w_i} = \langle\boldsymbol{b_i}|v\rangle$ and so

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ is a basis for R^n.

- We want to write $|v\rangle = w_1|b_1\rangle + \cdots + w_n|b_n\rangle$.

- We can (easily) show that $w_i = \langle b_i|v\rangle$ and so
$$|v\rangle = \langle b_1|v\rangle|b_1\rangle + \langle b_2|v\rangle|b_2\rangle + \cdots + \langle b_n|v\rangle|b_n\rangle.$$

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|\boldsymbol{b_1}\rangle, |\boldsymbol{b_2}\rangle, \ldots, |\boldsymbol{b_n}\rangle\}$ is a basis for $\boldsymbol{R^n}$.

- We want to write $|v\rangle = \boldsymbol{w_1}|\boldsymbol{b_1}\rangle + \cdots + \boldsymbol{w_n}|\boldsymbol{b_n}\rangle$.

- We can (easily) show that $\boldsymbol{w_i} = \langle\boldsymbol{b_i}|v\rangle$ and so
$$|v\rangle = \langle\boldsymbol{b_1}|v\rangle|\boldsymbol{b_1}\rangle + \langle\boldsymbol{b_2}|v\rangle|\boldsymbol{b_2}\rangle + \cdots + \langle\boldsymbol{b_n}|v\rangle|\boldsymbol{b_n}\rangle.$$

- In matrix notation:

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ is a basis for R^n.

- We want to write $|v\rangle = w_1|b_1\rangle + \cdots + w_n|b_n\rangle$.

- We can (easily) show that $w_i = \langle b_i|v\rangle$ and so
$$|v\rangle = \langle b_1|v\rangle|b_1\rangle + \langle b_2|v\rangle|b_2\rangle + \cdots + \langle b_n|v\rangle|b_n\rangle.$$

- In matrix notation:

  - Form $A = [|b_1\rangle \ldots |b_n\rangle]$ and take its transpose $A^T = \begin{bmatrix} \langle b_1| \\ \vdots \\ \langle b_n| \end{bmatrix}$.

Linear combinations of basis vectors

- Let $|v\rangle$ be a n-dimensional ket and $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ is a basis for R^n.

- We want to write $|v\rangle = w_1|b_1\rangle + \cdots + w_n|b_n\rangle$.

- We can (easily) show that $w_i = \langle b_i|v\rangle$ and so
$$|v\rangle = \langle b_1|v\rangle|b_1\rangle + \langle b_2|v\rangle|b_2\rangle + \cdots + \langle b_n|v\rangle|b_n\rangle.$$

- In matrix notation:

  - Form $A = [|b_1\rangle \ldots |b_n\rangle]$ and take its transpose $A^T = \begin{bmatrix} \langle b_1| \\ \vdots \\ \langle b_n| \end{bmatrix}$.

  - Then $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = A^T|v\rangle = \begin{bmatrix} \langle b_1|v\rangle \\ \vdots \\ \langle b_n|v\rangle \end{bmatrix}$.

Tensor Products

- To describe the evolution of a quantum system and, consequently, quantum computation, we consider a third form of multiplication on matrices, the ***tensor product***.

Tensor Products

- To describe the evolution of a quantum system and, consequently, quantum computation, we consider a third form of multiplication on matrices, the **tensor product**.

- Let A, B be two matrices of *any dimension*.

Tensor Products

- To describe the evolution of a quantum system and, consequently, quantum computation, we consider a third form of multiplication on matrices, the ***tensor product***.

- Let A, B be two matrices of *any dimension*.

- Then $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$.

Tensor Products

- To describe the evolution of a quantum system and, consequently, quantum computation, we consider a third form of multiplication on matrices, the **_tensor product_**.

- Let A, B be two matrices of **_any dimension_**.

- Then $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$.

- Example: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \otimes [1 \ \ 2 \ \ 3] = ?$

Tensor Products

- To describe the evolution of a quantum system and, consequently, quantum computation, we consider a third form of multiplication on matrices, the **tensor product**.

- Let A, B be two matrices of **any dimension**.

- Then $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$.

- Example: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \otimes [1 \ \ 2 \ \ 3] = ?$

- When tensor multiplying kets, the following are all equivalent: $|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle = |\psi\phi\rangle$. Tensor multiplication **is associative.**