The Combinatorial Topology of Factorization State Spaces: A Proof of the Toroidal Imperative in PPF

Ire Gaddr
Independent Researcher
Little Elm, TX, USA
iregaddr@gmail.com

***Abstract*—This paper provides the complete combinatorial proof for a central claim of the Primal Reflections framework: that the Factorization State Space $S(n)$ for any positive integer n is topologically equivalent to a torus, and therefore has an Euler characteristic of zero. We begin by formalizing the algebraic structure of $S(n)$ for $n > 0$, demonstrating that the set of operators that transition between factorization states forms a finite abelian group. We then prove that this "Factorization Group" is isomorphic to the group $(\mathbb{Z}_2)^{k-1}$, where k is the number of distinct prime factors of n. By constructing the Cayley graph of this group, we reveal its structure to be the skeleton of the $(k-1)$-dimensional torus, T^{k-1}. We then build the full cell complex for this space and calculate its Euler characteristic, proving it is zero for all $k \geq 2$. This result is not a postulate but a direct mathematical consequence of the Physics-Prime Factorization (PPF) axiom. It provides the rigorous, deductive link that necessitates the toroidal geometry of the Involuted Oblate Toroid (IOT), thus solidifying the framework's logical consistency.**

***Index Terms*—Physics-Prime Factorization, combinatorics, algebraic topology, group theory, Cayley graph, Euler characteristic, toroidal geometry.**

I. Introduction

In a previous work [1], we introduced the concept of the Factorization Simplex $K(n)$ and asserted, with a deferred proof, that its Euler characteristic $\chi(K(n))$ is zero for any positive integer n. This assertion is the mathematical linchpin connecting the abstract, number-theoretic axiom of Physics-Prime Factorization (PPF) [2] to the specific, toroidal geometry of the IOT [3]. A theory that derives geometry from number theory cannot leave such a link as an unproven assertion. The purpose of this paper is to provide that missing proof.

We will demonstrate that the set of allowed state transitions within the Factorization State Space $S(n)$ forms a well-defined abelian group. By proving this group is isomorphic to $(\mathbb{Z}_2)^{k-1}$, we can use the powerful tools of geometric group theory, specifically the construction of a Cayley graph, to reveal the inherent topology of the space. This approach replaces the initial, flawed combinatorial sketches with a rigorous and elegant argument, proving that the toroidal structure is not a choice, but a mathematical imperative.

II. The Factorization Group

We begin by formalizing the relationships between the canonical P-factorizations that constitute the Factorization State Space $S(n)$ for a positive integer n. Let n be a positive integer with k distinct prime factors in its standard prime factorization, $|n| = p_1 p_2 \cdots p_k$.

Definition II.1. *A canonical P-factorization $f \in S(n)$ can be uniquely represented by the subset of its Magnitude Primes $\{p_1, ..., p_k\}$ which are taken to be negative. By the rules of PPF, the cardinality of this subset must be even.*

Definition II.2 (Sign-Flip Operator)**.** *Let σ_{ij} be an operator that acts on a factorization $f \in S(n)$ by flipping the signs of its i-th and j-th Magnitude Primes.*

If a factorization f has signs $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ where $\epsilon_i \in \{+1, -1\}$, then σ_{ij} maps this to $(\epsilon_1, \ldots, -\epsilon_i, \ldots, -\epsilon_j, \ldots, \epsilon_k)$. Since we flip two signs, the parity of the number of negative factors is unchanged. If f had an even number of negative factors, the resulting factorization also does, and is therefore also a valid state in $S(n)$. The operator σ_{ij} is a map from $S(n)$ to itself.

Theorem II.3 (The Factorization Group)**.** *The set of all possible compositions of sign-flip operators, endowed with the operation of composition, forms a finite abelian group, which we call the Factorization Group, $\mathcal{F}_n$.*

Proof. We verify the group axioms:

1) **Closure:** The composition of any two sequences of sign-flips is itself a sequence of sign-flips, which maps $S(n)$ to $S(n)$.
2) **Identity:** The identity element is the operator σ_{ii}, or simply doing nothing, which leaves every factorization unchanged.
3) **Inverses:** For any operator σ_{ij}, we have $\sigma_{ij} \circ \sigma_{ij} = \text{id}$. Every element is its own inverse.
4) **Associativity:** Operator composition is associative.
5) **Commutativity:** The operators σ_{ij} and σ_{lm} commute. Flipping the pair (i, j) and then the pair (l, m) yields the same result as flipping them in the reverse order. Thus, the group is abelian.

□

III. Group Isomorphism to $(\mathbb{Z}_2)^{k-1}$

We now prove the central theorem that reveals the algebraic identity of the Factorization Group.

Theorem III.1. *For an integer n with $k \geq 2$ distinct prime factors, the Factorization Group $\mathcal{F}_n$ is isomorphic to the additive group $(\mathbb{Z}_2)^{k-1}$.*

Proof. Let us choose a basis for the group $\mathcal{F}_n$. We select a set of $k-1$ generators by fixing one prime, say p_1, as an anchor. The generators are:

$$g_1 = \sigma_{12}, \quad g_2 = \sigma_{13}, \quad \ldots, \quad g_{k-1} = \sigma_{1k} \tag{1}$$

Each generator g_i is its own inverse, so it corresponds to an element of order 2. They commute with each other. Thus, they generate a group isomorphic to $(\mathbb{Z}_2)^{k-1}$.

We must show that any operator σ_{ij} can be expressed as a composition of these generators. Consider an arbitrary sign-flip σ_{ij} for $i, j \neq 1$. We can write it as:

$$\sigma_{ij} = \sigma_{i1} \circ \sigma_{1j} = \sigma_{1i} \circ \sigma_{1j} = g_{i-1} \circ g_{j-1} \tag{2}$$

The first equality holds because applying σ_{i1} flips the signs of (p_i, p_1) and applying σ_{1j} flips the signs of (p_1, p_j). The sign of p_1 is flipped twice, returning it to its original state, while the signs of p_i and p_j are each flipped once. The second equality holds because the operators commute.

Since any element in $\mathcal{F}_n$ can be written as a composition of the basis elements $\{\sigma_{ij}\}$, and each σ_{ij} can be written as a composition of the generators $\{g_1, ..., g_{k-1}\}$, this set of generators spans the group. The isomorphism $\phi : (\mathbb{Z}_2)^{k-1} \to \mathcal{F}_n$ is thus established. □

This isomorphism reveals that the seemingly complex structure of state transitions is in fact the elementary structure of a $(k-1)$-dimensional vector space over the field of two elements.

IV. The Cayley Graph and the Toroidal Complex

Having identified the algebraic structure of the factorization state space, we can now construct its corresponding topological space.

Definition IV.1 (Factorization Cayley Graph)**.** *The Factorization Cayley Graph for $S(n)$ is the graph whose vertices are the elements of $S(n)$ and whose edges connect any two vertices f_1, f_2 such that $f_2 = g_i(f_1)$ for some generator g_i from the basis $\{\sigma_{12}, \ldots, \sigma_{1k}\}$.*

Theorem IV.2. *The Factorization Cayley Graph for $S(n)$ is the 1-skeleton of the $(k-1)$-dimensional torus, T^{k-1}.*

Proof. The Cayley graph of the group $(\mathbb{Z}_2)^m$ with the standard generating set is the skeleton of the m-dimensional hypercube, Q_m. Our group is $(\mathbb{Z}_2)^{k-1}$. Its Cayley graph is the skeleton of the hypercube Q_{k-1}.

A torus T^m can be constructed as a CW complex by taking the cube Q_m and identifying opposite faces. This identification means that the 1-skeleton of the torus is precisely the graph of the hypercube with opposite vertices identified, which is the structure of the Cayley graph of $(\mathbb{Z}_2)^{k-1}$. Each of the $k-1$ generators corresponds to traversing the torus along one of its fundamental, non-contractible loops. Starting from a vertex and applying a generator g_i takes you along one loop; applying it again brings you back to the start, as expected for a torus. □

To be precise, we define the full cell complex, which we will call the **Factorization Torus** $T(n)$.

Definition IV.3 (Factorization Torus $T(n)$)**.** *The Factorization Torus $T(n)$ is the CW complex for the group $(\mathbb{Z}_2)^{k-1}$.*

- *The 0-cells (vertices) are the 2^{k-1} elements of $S(n)$.*
- *The 1-cells (edges) are the pairs $(f, g_i(f))$ for each $f \in S(n)$ and generator g_i.*
- *The 2-cells (faces) are the quadrilaterals formed by paths $(f, g_i(f), g_j(g_i(f)), g_j(f))$.*
- *In general, an m-cell corresponds to an m-dimensional sub-hypercube in the Cayley graph.*

This construction shows that the topological space associated with $S(n)$ is not merely "like" a torus; it *is* a $(k-1)$-dimensional torus.

V. The Euler Characteristic and the Toroidal Imperative

We can now provide the definitive proof for the Euler characteristic.

Theorem V.1. *For any positive integer n with $k \geq 2$ distinct prime factors, the Euler characteristic of its Factorization Torus $T(n)$ is zero.*

Proof. We have established a topological equivalence between the Factorization Torus $T(n)$ and the standard $(k-1)$-dimensional torus, T^{k-1}. The Euler characteristic is a topological invariant, so we only need to compute $\chi(T^{k-1})$.

The Euler characteristic of a product of spaces is the product of their Euler characteristics:

$$\chi(X \times Y) = \chi(X) \times \chi(Y) \tag{3}$$

The torus T^{k-1} is the product of $k-1$ circles, S^1:

$$T^{k-1} = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{k-1 \text{ times}} \tag{4}$$

The Euler characteristic of a single circle, S^1, is 0. (It can be built from one 1-cell and one 0-cell, so $\chi = 1 - 1 = 0$). Therefore, for any $k-1 \geq 1$ (which corresponds to $k \geq 2$):

$$\chi(T(n)) = \chi(T^{k-1}) = (\chi(S^1))^{k-1} = 0^{k-1} = 0 \tag{5}$$

This completes the proof. □

Corollary V.2 (The Toroidal Imperative)**.** *Any physical theory that uses the PPF framework as its mathematical foundation, and which manifests this mathematical structure geometrically, must use a geometry whose topology has an Euler characteristic of zero. For a compact, orientable 3-manifold, the primary candidate is the torus.*

VI. Conclusion

This paper has rigorously established the previously asserted claim that the topological space associated with Physics-Prime Factorization has an Euler characteristic of zero. By formalizing the algebraic structure of the Factorization State Space $S(n)$, we proved its governing group of transformations is isomorphic to $(\mathbb{Z}_2)^{k-1}$. The natural topological realization of this group is the $(k-1)$-torus, a space whose Euler characteristic is necessarily zero.

This result is fundamental to the Primal Reflections framework. It demonstrates that the choice of the Involuted Oblate Toroid as the geometric arena for our physical theory is not a convenient or arbitrary postulate. It is a mathematical necessity, a direct and unavoidable consequence of the single foundational axiom that -1 is a prime number. The logic is now demonstrably complete: the number theory dictates the topology, and the topology dictates the geometry.

References

[1] I. Gaddr, "The Homology of Integers: An Algebraic Topology for Physics-Prime Factorization," *Forthcoming IEEE Conference Proceedings*, 2025.

[2] I. Gaddr, "Physics-Prime Factorization: A Quantum-Inspired Extension of Number Theory," arXiv preprint, 2025.

[3] I. Gaddr, "An IOTa of Truth: Involuted Toroidal Wave Collapse Theory," arXiv preprint, 2025.

[4] M. A. Armstrong, *Basic Topology*. New York: Springer-Verlag, 1983.

[5] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed. Hoboken, NJ: Wiley, 2004.