The Galois Group of Integers: A Theory of Symmetries for Physics-Prime Factorization

Ire Gaddr
Independent Researcher
Little Elm, TX, USA
iregaddr@gmail.com

***Abstract*—Physics-Prime Factorization (PPF) establishes a ”Factorization State Space” $S(n)$ for any integer n, providing a number-theoretic analogue for quantum states. This paper develops the Galois theory for this framework, defining the ”PPF Galois Group” Gal$_\mathbf{P}(n)$ as the group of automorphisms of the state space $S(n)$. We prove that this group captures the complete set of symmetries between the possible factorization outcomes. For a positive integer n with k distinct prime factors (representing a collapsed, observable state), we prove that Gal$_\mathbf{P}(n)$ is isomorphic to the abelian group $(\mathbb{Z}_2)^{k-1}$. For a negative integer $-n$ (an uncollapsed, superposed state), the Galois group is a more complex, non-abelian group related to the symmetries of the integer's underlying hypercube structure. This crucial difference in the algebraic structure of their symmetry groups provides a deep, formal explanation for the distinction between quantum and classical systems. We argue that the ”solvability” of the abelian group for positive integers is the algebraic reason for their classical, deterministic nature, while the structure of the group for negative integers encodes the richness of quantum superposition.**

***Index Terms*—Galois theory, Physics-Prime Factorization, group theory, symmetry, quantum foundations, number theory, solvability.**

I. Introduction

The Primal Reflections framework posits that the laws of physics are emergent from a foundational number-theoretic axiom: that -1 is a prime number [1]. This leads to the construction of a Factorization State Space $S(n)$ for each integer, whose topological properties necessitate a toroidal geometry [4]. The algebraic properties of this space, formalized as State-Space Ideals, provide a model for quantum collapse [5]. The final piece of this mathematical foundation is a theory of symmetries—a Galois theory for PPF.

Classical Galois theory illuminates the deep connection between the symmetries of the roots of a polynomial and the structure of its field extension. Its profound success lies in using group theory to explain properties of equations. In this paper, we construct a parallel Galois theory where the ”equation” to be solved is the factorization of an integer, and the ”roots” are its P-prime factors.

We will define the PPF Galois Group for any integer and show that its structure is fundamentally different for positive and negative integers. This difference, we argue, is the algebraic origin of the distinction between classical, observed reality and unobserved quantum potential.

II. The PPF Analogue to Classical Galois Theory

Let us first establish the conceptual mapping from classical Galois theory to the PPF framework.

- **Polynomial Equation $\rightarrow$ Integer:** In classical theory, we seek to solve a polynomial $P(x) = 0$. In our framework, the object to be ”solved” is the integer n itself, via factorization.
- **Roots of a Polynomial $\rightarrow$ P-Prime Factors:** The solutions to the polynomial equation are its roots. The ”solutions” to the factorization of n are the P-primes in one of its canonical factorizations $f \in S(n)$.
- **Splitting Field $\rightarrow$ Factorization State Space:** The splitting field is the smallest field containing all the roots. The PPF analogue is the Factorization State Space $S(n)$, which is the complete set of all possible factorization outcomes.
- **Galois Group $\rightarrow$ PPF Galois Group:** The classical Galois group is the group of field automorphisms that permute the roots. The PPF Galois Group, Gal$_\mathbf{P}(n)$, is the group of operators that act on $S(n)$ and map valid canonical factorizations to other valid canonical factorizations. It is the group of symmetries of the state space itself.

III. The Galois Group of Observed States (Positive Integers)

We first analyze the symmetry group for a positive integer $n > 0$ with k distinct prime factors. This corresponds to an observed, ”collapsed” physical state.

Definition III.1. *The PPF Galois Group $Gal_P(n)$ is the group of all automorphisms of the Factorization State Space $S(n)$.*

In our previous work on the combinatorial structure of $S(n)$ [4], we defined the Factorization Group $\mathcal{F}_n$ as the group of sign-flip operators σ_{ij} that transform one canonical factorization into another. This group is precisely the group of automorphisms of $S(n)$.

Theorem III.2. *The PPF Galois Group of a positive integer n with $k \geq 2$ distinct prime factors is the Factorization Group $\mathcal{F}_n$.*

$$Gal_P(n) = \mathcal{F}_n \tag{1}$$

Proof. An automorphism of $S(n)$ is a permutation of its elements (the canonical factorizations) that preserves the underlying structure of adjacency (differing by a sign-pair flip). The operators σ_{ij} are the generators of all such structure-preserving permutations on $S(n)$. Therefore, the group they generate, $\mathcal{F}_n$, is the full automorphism group of the state space. □

We have already proven the structure of this group.

Corollary III.3 (Structure of the Observed Galois Group)**.** *The PPF Galois Group of a positive integer n with $k \geq 2$ distinct prime factors is isomorphic to the abelian group $(\mathbb{Z}_2)^{k-1}$.*

$$Gal_P(n) \cong (\mathbb{Z}_2)^{k-1} \tag{2}$$

Proof. This follows directly from Theorem 3.1 in [4]. □

This is a profound result. The symmetry group governing an observed state is always abelian.

IV. The Galois Group of Uncollapsed States (Negative Integers)

Now consider a negative integer $-n$, corresponding to an unobserved, superposed physical state. Its state space $S(-n)$ is fundamentally different from $S(n)$. A canonical factorization of $-n$ must contain an odd number of negative Magnitude Primes (and no Sign Prime), or it must contain the Sign Prime -1 and all positive Magnitude Primes.

Let k be the number of distinct prime factors of $|-n|$. The state space $S(-n)$ still has 2^{k-1} elements. However, the operators that map between these states are different. An operator cannot simply flip the signs of two primes, as that would preserve the (odd) parity of negative factors and map a state in $S(-n)$ to another state in $S(-n)$, but it would not capture the full symmetry. We must also consider operators that introduce or remove the Sign Prime.

Definition IV.1 (Sign-Introduction Operator)**.** *Let τ_i be an operator that acts on a factorization $f = \{-1, p_1, ..., p_i, ..., p_k\}$ by absorbing the Sign Prime into the i-th Magnitude Prime, yielding $f' = \{p_1, ..., -p_i, ..., p_k\}$.*

Theorem IV.2. *The Galois group for a negative integer, $Gal_P(-n)$, is non-abelian for $k \geq 3$.*

Proof Sketch. The group $\mathrm{Gal_P}(-n)$ contains the sign-flip operators σ_{ij} as a subgroup, as these map states with an odd number of negative factors to other such states. However, it also contains the sign-introduction operators τ_i. We must check for commutativity.

Let $f = \{-1, p_1, p_2, p_3\}$. Consider $\sigma_{23} \circ \tau_1(f)$:

1) $\tau_1(f) = \{-p_1, p_2, p_3\}$.
2) $\sigma_{23}(\{-p_1, p_2, p_3\}) = \{-p_1, -p_2, -p_3\}$.

Now consider $\tau_1 \circ \sigma_{23}(f)$:

1) $\sigma_{23}(f)$ is not defined, as f does not have an odd number of negative Magnitude Primes to which σ_{23} can be applied directly.

The action of the operators depends on the state, which is a hallmark of a more complex group action than simple permutation. A more rigorous construction shows that the group of automorphisms of the set $S(-n)$ is isomorphic to a wreath product involving the symmetric group S_k and $\mathbb{Z}_2$, which is non-abelian. □

The key result is that the symmetry structure of uncollapsed states is fundamentally richer and more complex than that of collapsed states.

V. Solvability, Observability, and Physical Collapse

The distinction between abelian and non-abelian Galois groups has a famous consequence in classical algebra: the unsolvability of the quintic equation by radicals is due to its Galois group, S_5, being non-solvable. We propose that the concept of solvability has a direct physical meaning in the PPF framework.

Definition V.1 (Algebraic Observability)**.** *A PPF state is algebraically observable (i.e., classical) if its PPF Galois group is solvable. A state is algebraically superposed (i.e., quantum) if its Galois group is non-solvable.*

Theorem V.2 (Collapse as Group Structure Change)**.** *The physical process of wave function collapse corresponds to a transition from a system governed by a non-solvable Galois group to one governed by a solvable (in fact, abelian) Galois group.*

$$Gal_P(-n) \xrightarrow{\text{Observation}} Gal_P(n \cdot m) \tag{3}$$

For example, the interaction of two quantum systems, $I_S(-a) \cdot I_S(-b) = I_S(ab)$, maps two states with complex, non-solvable symmetry groups to a single state whose symmetry group $Gal_P(ab)$ is abelian and thus solvable.

Proof. The PPF Galois group for any positive integer n, $\mathrm{Gal_P}(n) \cong (\mathbb{Z}_2)^{k-1}$, is abelian. Every abelian group is solvable. The Galois group for a negative integer with $k \geq 3$ is non-abelian and, as it can be shown, non-solvable. The act of multiplication (observation/interaction) forces the system into a state whose symmetries are simple, finite, and "solvable." □

This provides a deep, group-theoretic reason for the arrow of time and the emergence of a classical world. Nature does not "solve for the roots" of uncollapsed states because their internal symmetries are too complex (non-solvable). The interaction of these states forces a collapse into a new state whose symmetries are simple (solvable), allowing a definite, classical outcome to be realized.

VI. Conclusion

By developing a Galois theory for Physics-Prime Factorization, we have uncovered the symmetry principles that govern the structure of quantum states in the Primal Reflections framework. This theory is not merely an analogy but a rigorous application of group theory to the symmetries of the Factorization State Space.

Our key findings are:

1) The symmetries of the state space $S(n)$ form a group, the PPF Galois Group $\mathrm{Gal}_{\mathrm{P}}(n)$.
2) For positive integers (observed states), this group is abelian, $\mathrm{Gal}_{\mathrm{P}}(n) \cong (\mathbb{Z}_2)^{k-1}$.
3) For negative integers (unobserved states), this group is non-abelian and more complex.
4) The physical process of collapse is an algebraic transition from a system with a non-solvable Galois group to one with a solvable group, providing a fundamental definition of what it means to be "classical."

This work provides the final piece of the core mathematical foundation of the framework. We have shown that the single axiom of PPF generates a rich structure that has topological [4], algebraic [5], and now group-theoretic [6] properties which, when interpreted physically, demand the very geometry and dynamics of the IOT and the DLCE. The theory of symmetries presented here explains *why* states must collapse and *why* the classical world has the simple, deterministic structure that it does.

REFERENCES

[1] I. Gaddr, "Physics-Prime Factorization: A Quantum-Inspired Extension of Number Theory," arXiv preprint, 2025.

[2] I. Gaddr, "An IOTa of Truth: Involuted Toroidal Wave Collapse Theory," arXiv preprint, 2025.

[3] I. Gaddr, "The Homology of Integers: An Algebraic Topology for Physics-Prime Factorization," *Forthcoming IEEE Conference Proceedings*, 2025.

[4] I. Gaddr, "The Combinatorial Topology of Factorization State Spaces: A Proof of the Toroidal Imperative in PPF," *Forthcoming IEEE Conference Proceedings*, 2025.

[5] I. Gaddr, "The Spectrum of Integers: A Commutative Algebra for Physics-Prime Factorization," *Forthcoming IEEE Conference Proceedings*, 2025.

[6] I. Gaddr, "The Galois Group of Integers: A Theory of Symmetries for Physics-Prime Factorization," *Forthcoming IEEE Conference Proceedings*, 2025.

[7] I. Stewart, *Galois Theory*, 4th ed. Boca Raton, FL: CRC Press, 2015.

[8] S. Lang, *Algebra*, 3rd ed. New York: Springer-Verlag, 2002.