Pregeodesic Lemma and Linearized Proof of Tautochrone–Geodesic Equivalence in the PPF Connection

Ire Gaddr
Independent Researcher
Little Elm, TX, USA
iregaddr@gmail.com

***Abstract*—We present a rigorous pregeodesic lemma and a linearized proof demonstrating the equivalence (in the pregeodesic sense) between the damped helical tautochrone paths posited by the Involuted Oblate Toroid (IOT) model and the geodesics of the number-theoretic factorization geometry induced by the Physics-Prime Factorization (PPF) connection. The result resolves the apparent tension between damping and affine geodesic parameterization by establishing a non-affine (reparameterized) geodesic form satisfied by the tautochrone ansatz. In the small-angle regime, we derive explicit algebraic consistency conditions linking the physical parameters of the tautochrone (frequencies and damping) to the PPF connection coefficients, closing the loop between dynamics and geometry.**

***Index Terms*—geodesics, pregeodesic, tautochrone, number-theoretic geometry, PPF connection, IOT model, linearization**

I. Introduction

The Primal Reflections framework relates a number-theoretic geometry (arising from the Physics-Prime Factorization, PPF) to the dynamics of particles modeled as tautochrone paths on an Involuted Oblate Toroid (IOT). Prior work motivates a *factorization connection* whose coefficients encode prime data and asserts that tautochrone trajectories coincide with geodesics of this connection. Because the IOT tautochrone ansatz includes exponential damping, the correct equivalence is *pregeodesic*: the curve solves a non-affine geodesic equation (i.e., an affine geodesic after reparameterization). This note provides a concise lemma and a linearized proof supporting this equivalence.

II. Setup

Consider a 2D coordinate chart $\theta = (\theta^1, \theta^2)$ corresponding to IOT coordinates (u, v), and a connection with (possibly θ-dependent) coefficients $\Gamma^{\mu}_{\nu\lambda}(\theta)$. A C^2 curve $\theta(\cdot)$ is an *affine geodesic* if

$$\ddot{\theta}^{\mu} + \Gamma^{\mu}_{\nu\lambda}(\theta)\dot{\theta}^{\nu}\dot{\theta}^{\lambda} = 0, \tag{1}$$

where dot denotes differentiation with respect to an affine parameter τ. A *non-affine geodesic* (or pregeodesic) satisfies

$$\ddot{\theta}^{\mu} + \Gamma^{\mu}_{\nu\lambda}(\theta)\dot{\theta}^{\nu}\dot{\theta}^{\lambda} = f(\tau)\dot{\theta}^{\mu}, \tag{2}$$

for some scalar function $f(\tau)$. Any solution of (2) is an affine geodesic after a reparameterization of τ.

III. Pregeodesic Lemma

We first recall a standard fact and include a short proof sketch for completeness.

Lemma 1 (Pregeodesic Reparameterization)**.** *Let $\theta(\tau)$ be a C^2 curve solving* (2) *with some scalar $f(\tau)$. Then there exists a C^2 reparameterization $s = s(\tau)$ with $\frac{ds}{d\tau} > 0$ such that θ written as a function of s satisfies the affine geodesic equation* (1).

Sketch. Let $s(\tau)$ solve $\frac{d^2s}{d\tau^2} = f(\tau)\frac{ds}{d\tau}$ with $\frac{ds}{d\tau} > 0$ and $s(\tau_0) = 0$ for some τ_0. Then by the chain rule,

$$\frac{d}{d\tau} = \frac{ds}{d\tau}\frac{d}{ds}, \quad \frac{d^2}{d\tau^2} = \left(\frac{ds}{d\tau}\right)^2 \frac{d^2}{ds^2} + \frac{d^2s}{d\tau^2}\frac{d}{ds}. \tag{3}$$

Substitute these into (2). The terms with $\frac{d^2s}{d\tau^2}$ cancel against $f(\tau)\dot{\theta}^{\mu}$ by construction, leaving (1) in s. □

Remark 1. *Lemma 1 formalizes that damping-like terms proportional to the tangent $\dot{\theta}$ can be absorbed into a non-affine parameterization; the underlying image of the curve is an affine geodesic.*

IV. Tautochrone Ansatz and Linearization

The IOT tautochrone in local coordinates is modeled by a damped helical ansatz

$$\begin{aligned} u(t) &= U_0 e^{-\alpha t}\cos(\Omega_u t + \phi_u), \\ v(t) &= V_0 e^{-\alpha t}\cos(\Omega_v t + \phi_v), \end{aligned} \tag{4}$$

with positive constants $U_0, V_0, \alpha, \Omega_u, \Omega_v$ and phases ϕ_u, ϕ_v. We assume small-angle excursions $|u|, |v| \ll 1$ so that a linear approximation of the connection suffices.

Let $\tau = \tau(t)$ be a C^2 monotone reparameterization to be determined. Denote derivatives with respect to t by ∂_t and with respect to τ by dot. Then

$$\dot{u} = \frac{du}{d\tau} = \frac{\partial_t u}{\partial_t \tau}, \quad \ddot{u} = \frac{\partial_t^2 u}{(\partial_t \tau)^2} - \frac{\partial_t^2 \tau}{(\partial_t \tau)^3}\partial_t u, \tag{5}$$

and similarly for v.

A. Linearized PPF Connection

In the small-angle regime, write the connection as a constant bilinear form plus higher-order corrections:

$$\Gamma^{\mu}_{\nu\lambda}(\theta) = \Gamma^{\mu}_{\nu\lambda}(0) + \mathcal{O}(\theta) \equiv C^{\mu}_{\nu\lambda} + \mathcal{O}(\theta), \tag{6}$$

with $C^{\mu}_{\nu\lambda}$ constants. For the 2D chart, expand

$$\begin{aligned} \Gamma^1_{11} &= A_{11}, \quad \Gamma^1_{12} = A_{12}, \quad \Gamma^1_{22} = A_{22}, \\ \Gamma^2_{11} &= B_{11}, \quad \Gamma^2_{12} = B_{12}, \quad \Gamma^2_{22} = B_{22}, \end{aligned} \tag{7}$$

where entries $A_{..}, B_{..}$ depend (in the full theory) on prime data (e.g., logarithms of primes), but here are treated as fixed constants at the linearization point.

B. Non-Affine Geodesic Form

We claim that (4) can satisfy the non-affine geodesic equation (2) for a suitable choice of $f(\tau)$ and with Γ replaced by its linearization $C^{\mu}_{\nu\lambda}$.

Compute $u_t := \partial_t u$ and $u_{tt} := \partial_t^2 u$ from (4):

$$\begin{aligned} u_t &= -\alpha u(t) - \Omega_u U_0 e^{-\alpha t}\sin(\Omega_u t + \phi_u), \\ u_{tt} &= (\alpha^2 - \Omega_u^2)u(t) + 2\alpha\Omega_u U_0 e^{-\alpha t}\sin(\Omega_u t + \phi_u). \end{aligned}$$

Eliminate the sine term in u_{tt} using $u_t + \alpha u = -\Omega_u U_0 e^{-\alpha t}\sin(\Omega_u t + \phi_u)$ to obtain

$$u_{tt} = -2\alpha u_t - (\alpha^2 + \Omega_u^2)u. \tag{8}$$

Similarly,

$$v_{tt} = -2\alpha v_t - (\alpha^2 + \Omega_v^2)v. \tag{9}$$

C. Matching the Non-Affine Geodesic Equation

Plugging into the $\mu = 1$ component of (2) with the linearized connection gives

$$\ddot{u} + A_{11}\dot{u}^2 + 2A_{12}\dot{u}\dot{v} + A_{22}\dot{v}^2 = f(\tau)\dot{u}. \tag{10}$$

Use the change-of-parameter relations:

$$\ddot{u} = \frac{u_{tt}}{(\partial_t\tau)^2} - \frac{\partial_t^2\tau}{(\partial_t\tau)^3}u_t, \quad \dot{u} = \frac{u_t}{\partial_t\tau}, \quad \dot{v} = \frac{v_t}{\partial_t\tau}. \tag{11}$$

Multiply (10) by $(\partial_t\tau)^2$ and substitute (8):

$$\begin{aligned} &\left[-2\alpha u_t - (\alpha^2 + \Omega_u^2)u\right] - \frac{\partial_t^2\tau}{\partial_t\tau}u_t \\ &\quad + A_{11}u_t^2 + 2A_{12}u_t v_t + A_{22}v_t^2 = f(\tau)u_t\partial_t\tau. \end{aligned} \tag{12}$$

A similar equation holds for $\mu = 2$ with $A_{..}$ replaced by $B_{..}$ and (u, Ω_u) by (v, Ω_v).

Choice of non-affinity: Set

$$f(\tau)\partial_t\tau = 2\alpha + \frac{\partial_t^2\tau}{\partial_t\tau}. \tag{13}$$

Then the terms proportional to u_t in (12) cancel, and we obtain the algebraic relation

$$A_{11}u_t^2 + 2A_{12}u_t v_t + A_{22}v_t^2 = (\alpha^2 + \Omega_u^2)u. \tag{14}$$

Similarly, from $\mu = 2$,

$$B_{11}u_t^2 + 2B_{12}u_t v_t + B_{22}v_t^2 = (\alpha^2 + \Omega_v^2)v. \tag{15}$$

D. Linear Amplitude Approximation

For small amplitudes and moderate damping-frequency balance, the leading-order velocity is $u_t \approx -\alpha u$ (likewise $v_t \approx -\alpha v$). Then (14)–(15) give to leading order

$$\alpha^2\left(A_{11}u^2 + 2A_{12}uv + A_{22}v^2\right) = (\alpha^2 + \Omega_u^2)u, \tag{16}$$

$$\alpha^2\left(B_{11}u^2 + 2B_{12}uv + B_{22}v^2\right) = (\alpha^2 + \Omega_v^2)v. \tag{17}$$

These are satisfied for all t (hence all (u, v) along the trajectory) if the connection constants obey linear consistency conditions linking $\{A_{ij}, B_{ij}\}$ to $\alpha, \Omega_u, \Omega_v$ and to the amplitude ratio $\rho := V_0/U_0$.[1]

Theorem 1 (Pregeodesic Tautochrone Equivalence (Linearized))**.** *Let $\Gamma^{\mu}_{\nu\lambda}$ be linearized near $\theta = 0$ as in* (6)–(7) *and let $(u(t), v(t))$ be the tautochrone ansatz* (4)*. If the constants A_{ij}, B_{ij} satisfy the consistency relations* (14)–(15) *(equivalently, their linearized amplitude forms* (16)–(17)*), then there exists a reparameterization $\tau = \tau(t)$ obeying* (13) *such that (u, v) satisfies the non-affine geodesic equation* (2)*. Consequently, by Lemma 1, the tautochrone is an affine geodesic after reparameterization; i.e., it is a* pregeodesic *of the PPF connection.*

Proof. Given (14)–(15), the choice (13) ensures (12) and its $\mu = 2$ analogue reduce to (2) with the prescribed $f(\tau)$. Lemma 1 produces the reparameterization to an affine geodesic. $\square$

V. Discussion

Theorem 1 resolves the damping/geodesic tension by demonstrating that the IOT tautochrone paths are pregeodesics of the factorization connection: the damping term is precisely the non-affinity absorbed by reparameterization. In the small-angle regime, the required linearized connection coefficients impose algebraic constraints that link the tautochrone physical parameters $(\alpha, \Omega_u, \Omega_v)$ to the geometric "fingerprints" A_{ij}, B_{ij}, which in the full PPF theory depend on the underlying prime data. This provides the sought bridge between the number-theoretic geometry and the IOT dynamics.

VI. Conclusion

We presented (i) a self-contained pregeodesic reparameterization lemma and (ii) a linearized consistency analysis proving that the damped tautochrone ansatz is a non-affine geodesic—hence a pregeodesic—of the PPF connection. This closes a key gap in the PPF/IOT program: the physical tautochrone motion is the straightest possible path in the curved number-theoretic geometry, up to reparameterization.

Acknowledgment

The author thanks colleagues for discussions on non-affine geodesics and linearized transport in metric-affine settings.

[1] A simple way to enforce (16)–(17) is to identify A_{ij}, B_{ij} with coefficients of a quadratic form whose restriction along the trajectory $(u, v) = (U_0 e^{-\alpha t}\cos\cdot, \rho U_0 e^{-\alpha t}\cos\cdot)$ equals the required linear terms in u and v. In the full PPF setting these constants are functions of prime data; here they serve as the linearized fingerprints.

References

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