Field Theory Notes

June 26, 2016

1 Constants and Measures Used

Throughout it is convenient to use the constant

$$\mathcal{U} = \frac{r^{d/2-2}}{(4\pi D)^{d/2}}\Gamma(2 - d/2)$$

In I_8 to I_{10} for convenience we shorten the expression by using

$$\mathcal{T} = \tau_\omega(-i\omega + \epsilon') + \tau - \epsilon'\tau_\omega$$

The measure for integrals is consistently

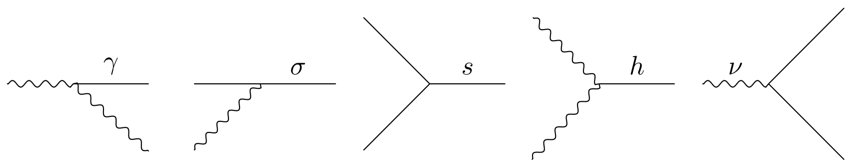
$$\mathbb{D} = \frac{d^d k}{2\pi}\frac{d\omega}{2\pi}$$

Also observe that one expects the theory to be consistent in the limit $\epsilon' \to 0$ so for example I_7 simplifies considerably.

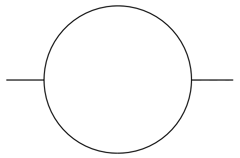
2 Propagation and Transmutation

$$\widehat{=} \frac{1}{-i\omega + \epsilon'}\Big(\tau - i\tau\omega\Big)\frac{1}{-i\omega + Dk^2 + r} \tag{1}$$

3 Disconnected Diagrams

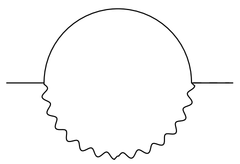


4 Integrals



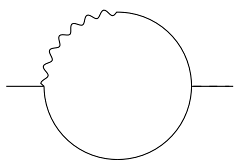
$$I_1 \widehat{=} \int \mathbb{D} \frac{1}{-i(\omega+\omega')+D(k+k')+r} \frac{1}{i\omega'+Dk'^2+r}$$

$$= \tfrac{1}{2}\mathcal{U}r\left(\frac{k^2 D}{4r}+\frac{2r-i\omega}{2r}\right)^{\frac{d}{2}-1} \frac{1}{1-\frac{d}{2}}$$

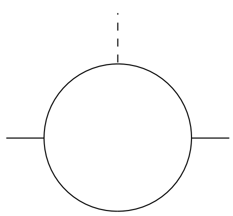


$$I_2 \widehat{=} \int \mathbb{D} \frac{1}{-i(\omega+\omega')+\epsilon'} \frac{1}{i\omega'+Dk'^2+r}$$

$$= \tfrac{1}{2}\mathcal{U}r\left(\frac{r+\epsilon'-i\omega}{r}\right)^{\frac{d}{2}-1} \frac{1}{1-\frac{d}{2}}$$

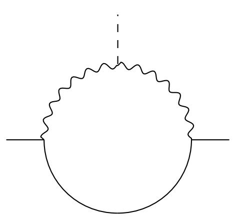


$I_3 \widehat{=} \quad ??$



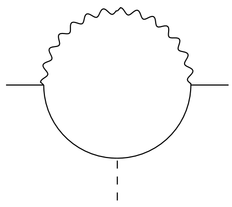
$$I_4 \widehat{=} \int \mathbb{D} \frac{1}{-i\omega+Dk^2+r} \frac{1}{(i\omega+Dk^2+r)^2}$$

$$= \tfrac{1}{4}\mathcal{U}$$



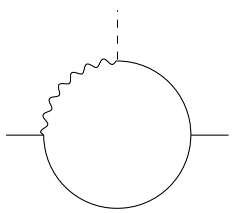
$$I_5 \widehat{=} \int \mathbb{D} \frac{1}{-i\omega+Dk^2+r} \frac{1}{(i\omega+\epsilon')^2}$$

$$= \left(\frac{r+\epsilon'}{r}\right)^{\frac{d}{2}-2} \mathcal{U}$$



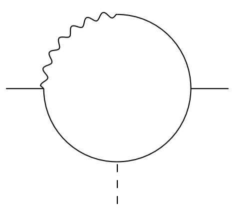
$$I_6 \widehat{=} \int \mathbb{D} \frac{1}{(i\omega + Dk^2 + r)^2} \frac{1}{-i\omega + \epsilon'}$$

$$= \left(\frac{r + \epsilon'}{r} \right)^{\frac{d}{2}-2} \mathcal{U} = I_5$$



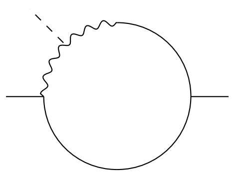
$$I_7 \widehat{=} \int \mathbb{D} \frac{1}{-i\omega + Dk^2 + r} \frac{1}{i\omega + \epsilon'} \frac{1}{i\omega + Dk^2 + r}$$

$$= \frac{\mathcal{U} r}{2\epsilon'} \left(\left(\frac{r + \epsilon'}{r} \right)^{\frac{d}{2}-1} - 1 \right) \frac{1}{\frac{d}{2} - 1}$$



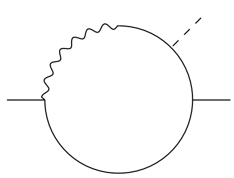
$$I_8 \widehat{=} \int \mathbb{D} \frac{1}{-i\omega + \epsilon'} \mathcal{T} \frac{1}{-i\omega + Dk^2 + r} \frac{1}{(i\omega + Dk^2 + r)^2}$$

$$= \frac{1}{4} \mathcal{U} \tau_\omega$$



$$I_9 \widehat{=} \int \mathbb{D} \frac{1}{-i\omega + \epsilon'} \mathcal{T} \frac{1}{-i\omega + Dk^2 + r} \frac{1}{-i\omega + \epsilon'} \frac{1}{i\omega + Dk^2 + r}$$

$$= \tau_\omega I_7$$



$$I_{10} \widehat{=} \int \mathbb{D} \frac{1}{-i\omega + \epsilon'} \mathcal{T} \frac{1}{(-i\omega + Dk^2 + r)^2} \frac{1}{i\omega + Dk^2 + r}$$

$$= \frac{1}{4} \mathcal{U} \tau_\omega = I_8$$

5 Some Rough Notes (WIP)

Some notes on working with the first order propagators follow. While the calculus of residues can be used very broadly from first principles, in the case where we are dealing with simple poles and in particular the Fourier space propagator products, we can normally make use of various short-cuts.
We can say that if we take a function (product of propagators) $\Pi(z) = q(z)/(z-\omega)p(z)$ where the function p and q are analytic at ω and in our propagator case q is likely to be 1 and $p(\omega) \neq 0$ then

$$Res(\Pi, \omega) = q(\omega)/p(\omega) \tag{4}$$

If h is holomorphic at ω at which point g has a simple pole, then the composite residue

$$Res(gh, \omega) = h(w)Res(g, \omega) \tag{5}$$

From this we can write a general rule for the simple propagators we encounter of the form $\prod_i 1/p_i(z)$ by ensuring the decomposition for the poles of interest

$$Res\Big(\frac{1}{f \cdot (z-\omega)\hat{q}(z)}, \omega\Big) = \frac{1}{f\hat{q}(z)} \tag{6}$$

where f is some normalising function that allows us to express in the correct form with $z - \omega$ and q is a remaining polynomial.[1]

Now, in Fourier space, we can usually simply read off the result as a sum of terms - taking care with signs.

Note although the residue is the sum of residues at the poles, in many cases it becomes possible to evaluate just one or a small number of poles in the region of interest and we can apply the residue theorem at this point.

We rewrite general propagators with possibly complicated terms for momenta and mass as

$$\int \mathbb{D}\frac{1}{\bullet + m} \tag{7}$$

where the momentum term $\bullet$ could be p^2 and the mass is some other mess - it turns out that we can convert to polar coordinates and integrate over one momentum dimension/term

$$\rightarrow S_d \int dx \frac{x^{d-1}}{x^2+m} \tag{8}$$

and in what follows the bullet is "irrelevant" when we use a Gamma formula i.e (noting a special case for simple propagators with $\beta = 1$)

$$\int dx \frac{x^\alpha}{(x+m)^\beta}\Big|_{\beta=1} = m^\alpha \frac{\Gamma(-\alpha)\Gamma(\alpha+1)}{\Gamma(1)} \tag{9}$$

[1]Combining these results suggests that f is the residue of a polynomial g that can be expressed as $z - \omega$ - in our case is its residue $f = i$?? - check signs but this sort of makes sense as it is the coefficient of the first term in the Laurent expansion.

Or for our case this looks like - with a change of the "irrelevant" variable -

$$\tfrac{1}{2}\int dy \frac{y^{\frac{1}{2}d-1}}{y+m} = \tfrac{1}{2}m^{d/2-1}\frac{\Gamma(-d/2+1)\Gamma(d/2)}{\Gamma(1)} \tag{10}$$

which does not depend on y (originally x) and $\frac{1}{2}dx = dy$. What this means that generally, in Fourier space

$$A\frac{S_d}{(2\pi)^d}\int d^dk \frac{x^{d-1}}{x^2+m} = A\frac{1}{(4\pi)^{d/2}}\Gamma(1-\tfrac{d}{2})m^{d/2-1} \tag{11}$$

after clean up.[2]

6 Useful Results

The Feynman identity is given as

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax+b(1-x)]^2} \tag{12}$$

and a generalisation[3]

$$\frac{1}{A^aB^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{[Ax+B(1-x)]^2}dx \tag{13}$$

We shall frequently reduce a d-dimensional integral to a single integral making use of K_d, the volume of a D-dimensional sphere.

$$\int d^dx \frac{1}{x^2+a} = K_d\int dx \frac{x^{d-1}}{x^2+a} \tag{14}$$

K_d can be expressed in terms of a Gamma function (see **??**) and this is often useful in calculations.

6.1 Complex analysis

Cauchy's coefficient formula

$$[z^n]f_z \sim \frac{1}{2\pi i}\int_\Gamma f_z \frac{dz}{z^{n+1}} \tag{15}$$

The integral of an anticlockwise closed contour, by the residue theorem, is

$$\oint dz f_z = 2\pi i\Sigma \tag{16}$$

where Σ is the sum of the residues from enclosed poles.

[2]Due to the value of $S_D = 2\pi^{D/2}/\Gamma(D/2)$ such that the clean up proceeds as $\frac{2\pi^{d/2}}{2^d\pi^d} \to \frac{2}{4^{d/2}\pi^{d-d/2}} \to \frac{1}{(4\pi)^{d/2}}$. The $\frac{1}{2}\Gamma(d/2)$ cancels with terms in S_D

[3]Feynman's parametric integral formula

6.2 Dirac Delta Function

The Dirac delta function can be defined as

$$\delta(t) = (2\pi)^{-1} \int e^{-ikz} dz \tag{17}$$

6.3 Gamma Functions

The Gamma function is in fact the Mellin transform of the negative exponential function i.e.

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1} e^{-t} \tag{18}$$

arises frequently in the evaluation of Feynman diagrams. For example, the Mellin-Barnes representation exploits formula for propagator forms. Take for example the formula

$$\frac{1}{(a+b)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda + z) \Gamma(-z) \frac{b^z}{a^{\lambda+z}} \tag{19}$$

where a, b take the place of terms such as $m^2, -k^2$ appearing in propagators. The Mellin-Barnes integral is related to various hyper-geometric series and plays a useful role in asymptomatic expansions. Some similar, and directly useful forms of Gamma-based integral formulae, include

$$\underbrace{\int \mathbb{D}_d \frac{1}{(a+b)^\lambda}}_{I_0} = \frac{\Gamma(\lambda - d/2)}{(4\pi)^{d/2}\Gamma(\lambda)} \frac{1}{(b)^{N-D/2}} \tag{20}$$

and

$$I_1 = aI_0 = \frac{\Gamma(\lambda - 1 - d/2)}{2(4\pi)^{d/2}\Gamma(\lambda)} \frac{d}{(b)^{\lambda-1-D/2}} \tag{21}$$

also

$$I_2 = a^\gamma I_0 = b^{\gamma-\lambda+1} \frac{\Gamma(-1(\gamma - \lambda + 1))\Gamma(\gamma + 1)}{\Gamma(\lambda)} \tag{22}$$

Other useful results for the Gamma function include $\Gamma(0) = \infty$, $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n) = (n-1)!$ and the recurrence

$$z\Gamma(z) - \Gamma(z+1) = 0 \tag{23}$$

The *Beta function* can be expressed in terms of Gamma functions

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{24}$$