Provable Surface Reconstruction from Noisy Samples *

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ABSTRACT

We present an algorithm for surface reconstruction in presence of noise. We show that, under a reasonable noise model, the algorithm has theoretical guarantees. Actual performance of the algorithm is illustrated by our experimental results.

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General Terms: Algorithms, Experimentation, Theory

Keywords: Voronoi diagram, Delaunay triangulation, surface reconstruction, noise, sampling

1. INTRODUCTION

The problem of surface reconstruction asks to approximate a surface from a set of point samples. This problem has been the focus of research across many fields because of its wide applications [5, 14, 16]. Many algorithms have been proposed recently for the problem. Some of these algorithms, following the work of Amenta and Bern [1], provide theoretical guarantees [2, 3, 6, 13]. These theoretical guarantees are based on the fact that the input sample is dense with respect to the *local feature size*. However, in practice this condition often does not hold. There are several reasons for undersampling as described in [10]. Of particular interest is the presence of noise that is typical for samples obtained by a scanning process. Algorithms for curve reconstruction [9] and surface normal estimations [20] from noisy samples have been designed. Also, some of the existing surface reconstruction algorithms work well in presence of noise [3, 11, 17]. However, there is no known algorithm that has theoretical guarantees for surface reconstruction in presence of noise.

In this work we present a surface reconstruction algorithm that has theoretical guarantees under some reasonable noise model. The model allows the points to be scattered around the sampled surface and the range of the scatter is restricted by the local feature size. The algorithm works with the Delaunay/Voronoi diagrams of the input point sample and draws upon some of the principles of the power crust algorithm [3]. In the power crust algorithm it is observed that the union of a set of Delaunay balls called polar balls approximates the solid bounded by the sampled surface. Obviously, this property does not hold in presence of noise. However, we observe that, under the assumed noise model, some of the Delaunay balls are relatively big and can play the role of the polar balls. These balls are identified and partitioned into inner and outer balls. We show that the boundary of the union of the outer (or inner) big Delaunay balls is homeomorphic to the sampled surface. This immediately gives a homeomorphic surface reconstruction though the reconstructed surface may not interpolate the sample points.

We extend the algorithm further to compute a homeomorphic surface interpolating through a subset of input sample points. These sample points reside on the outer (or inner) big Delaunay balls. The rest of the points are deleted. We show that the Delaunay triangulation of the chosen sample points restricted to the boundary of the chosen big Delaunay balls is homeomorphic to the sampled surface. Figure 1 illustrates this algorithm.

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2. PRELIMINARIES

2.1 Definitions

For a set $P \subseteq \mathbb{R}^3$ and a point $x \in \mathbb{R}^3$, let $d(x,P)$ denote the Euclidean distance of x from P; that is,

$$\begin{aligned} d(x,P) &= \min_{p \in P}\{\|p-x\|\} \text{ if } P \text{ is finite} \\ &= \inf_{p \in P}\{\|p-x\|\} \text{ otherwise.} \end{aligned}$$

The set $B_{r,x} = \{y \mid y \in \mathbb{R}^3, \|y-x\| \leq r\}$ is a *ball* with radius r and center x.

Voronoi and Delaunay diagram. For a finite point set $P \in \mathbb{R}^3$, the Voronoi diagram $\mathrm{Vor}\, P$ is the collection of *Voronoi*

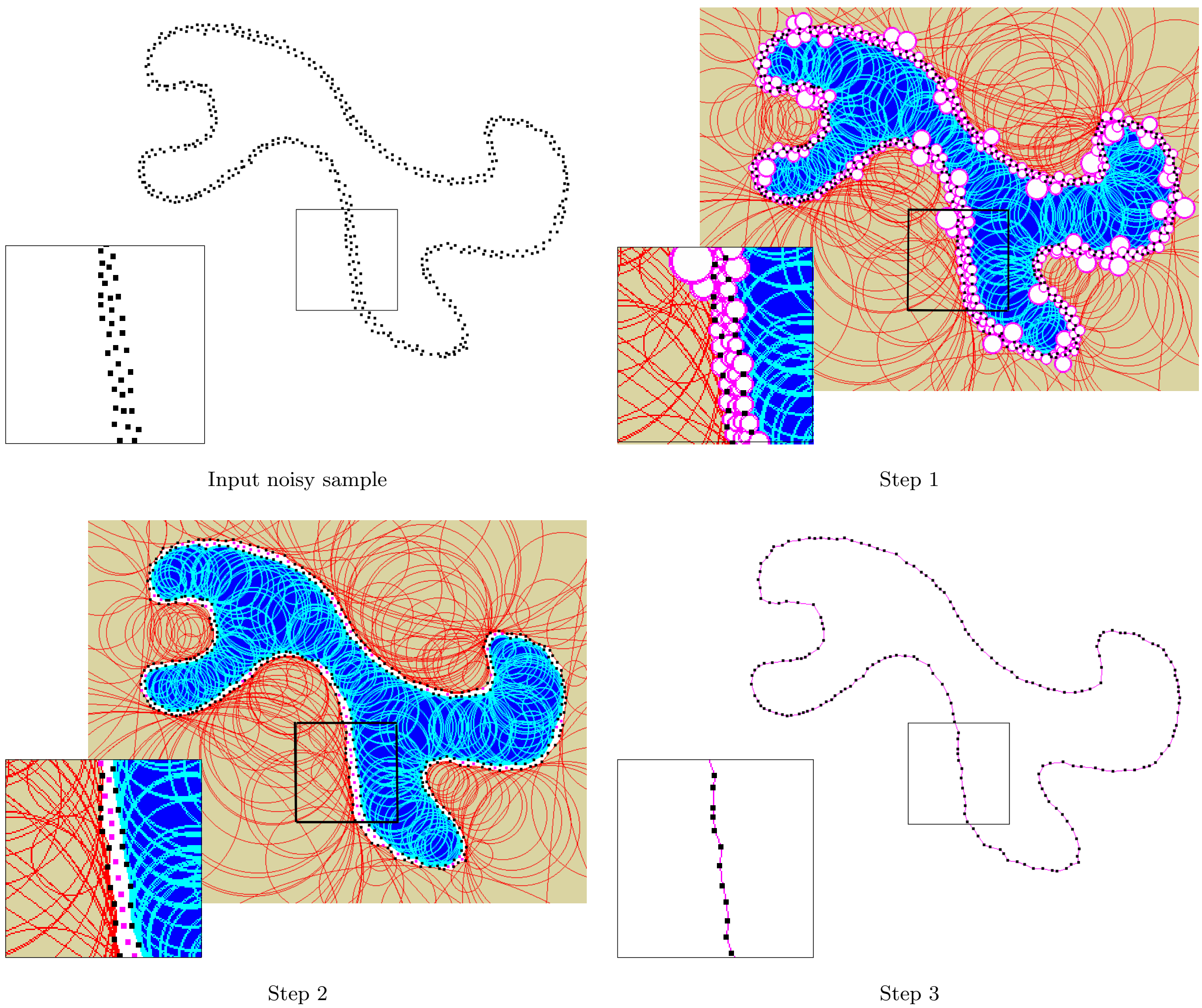

Figure 1: Step 1: Big Delaunay balls (shaded) are separated from small ones (unshaded), Step 2: Outer and inner big Delaunay balls are separated, Step 3: Only the points on the outer balls are retained and the surface is reconstructed from them.

faces defined as follows. The *Voronoi cell* for a point $p \in P$ is

$$V_p = \{x \in \mathbb{R}^3 \mid d(x, P) = \|x - p\|\}.$$

Closed faces shared by d Voronoi cells, $2 \leq d \leq 4$, are called $(4-d)$-dimensional Voronoi faces. The 0-, 1-, 2-, 3-dimensional Voronoi faces are called Voronoi *vertices, edges, facets* and *cells* respectively.

The *Delaunay diagram* of P is a dual to $\mathrm{Vor}\, P$. The convex hull of $d \leq 4$ points defines a $(d-1)$-dimensional *Delaunay face* if the intersection of their corresponding Voronoi cells is not empty. If the points are in general position 1-, 2-, 3-dimensional Delaunay faces are *Delaunay edges, triangles and tetrahedra* respectively. They define a decomposition of the convex hull of all points in P called the *Delaunay triangulation* $\mathrm{Del}\, P$. We always assume general position unless stated differently.

A set of points P on a surface Σ defines a *restricted Voronoi diagram* $\mathrm{Vor}\, P|_\Sigma$ as the collection of *restricted Voronoi cells* $\{V_p|_\Sigma = V_p \cap \Sigma\}$. Dual to the restricted Voronoi diagram is the *restricted Delaunay triangulation* $\mathrm{Del}\, P|_\Sigma$. It is a simplicial complex where $\sigma \in \mathrm{Del}\, P|_\Sigma$ if and only if it is the convex hull of a set of vertices $R \subseteq P$ and $\bigcap_{q \in R} V_q|_\Sigma \neq \emptyset$.

Sampled surface. Let $\Sigma \in \mathbb{R}^3$ be a compact smooth surface without boundary from which the input sample is derived possibly with noise.

The *medial axis* M of Σ is the closure of the set $X = \{x\}$ so that $d(x, \Sigma)$ is realized by two or more points of Σ. In other words, x is the center of a maximal empty ball that meets Σ only tangentially at two or more points. We call each such ball $B_{r,x}$ a *medial ball* where $r = d(x, \Sigma)$. For each point $x \in \Sigma$, define three functions $f, \rho_i, \rho_o : \Sigma \to \mathbb{R}$ where $f(x) = d(x, M)$ and $\rho_i(x), \rho_o(x)$ are the radii of the inner and outer medial balls respectively touching Σ at x. Certainly, $f(x) \leq \min\{\rho_i(x), \rho_o(x)\}$. Since Σ is smooth and compact, there exists a surface dependent constant Δ_1 so that $\inf_{x \in \Sigma} f(x) = \Delta_1$. Similarly, since Σ has no boundary, there is another surface constant Δ_2 so that $\sup_{x \in \Sigma} \rho_i(x) = \Delta_2$. Combining the two bounds we obtain:

FACT 1. *There exists a surface constant $\Delta = \Delta(\Sigma) \geq 1$ for Σ so that $\rho_i(x) \leq \Delta f(x)$ for all $x \in \Sigma$.*

In our proofs we only argue about inner medial balls and thus the above fact is sufficient for our proofs though a similar fact will also hold for outer medial balls if Σ is enclosed in a sufficiently large bounding sphere, a standard trick used in earlier works [3, 6, 10].

The function $f()$, also called the *local feature size*, satisfies the following Lipschitz property [1].

Lipschitz property. For any two points $x, y \in \Sigma$, $f(x) \leq f(y) + \|x - y\|$.

2.2 Sampling

A finite set of points $P \subset \Sigma$ is called an ε-sample of Σ if

$$d(x, P) \leq \varepsilon d(x, M) \quad \text{for each } x \in \Sigma.$$

We consider a noisy sample of Σ which may not be a subset of Σ, but we assume that it lies close to the surface. This proximity assumption alone cannot disambiguate the reconstruction process as sample points can collaborate to form arbitrary patterns. This is the reason why Cheng et. al [9] devised a probabilistic algorithm for curve reconstruction from noisy samples. Here our target is a deterministic algorithm for surface reconstruction from noisy samples. So, we need a locally uniform sampling condition such as in [13].

Let $\nu : \mathbb{R}^3 \to \Sigma$ be the map so that $\nu(x)$ is the closest point on Σ for a point $x \in \mathbb{R}^3$. Denote $\tilde{p} = \nu(p)$ and $\tilde{P} = \{\nu(p)\}_{p \in P}$.

Noisy (ε, κ)-sample. We say $P \in \mathbb{R}^3$ is a *noisy* (ε, κ)-sample of Σ for two positive constants k_1 and k_2 if the following sampling conditions hold.

(i) $\tilde{P}$ is an ε-sample of Σ,

(ii) $\|p - \tilde{p}\| \leq k_1 \varepsilon f(\tilde{p})$,

(iii) $\|p - q\| \geq k_2 \varepsilon f(\tilde{p})$ for any two points p, q in P where q is the κth nearest sample point to p.

The first condition says that the projection of the point set P on the surface makes a dense sample and the second says that it is close to the surface. The third condition makes the sampling locally uniform. Notice that, for $\kappa = 1$ this condition prohibits any two sample points to be arbitrarily close. But, that may be a severe restriction for point samples in practice. This is why we introduce κ to make the locally uniform condition less restrictive. This condition is very similar to the locally uniform conditions used in [13]. In practice we take κ in the range of three to five.

Sampling parameters. In the sequel we will formulate and use several $\varepsilon_i > 0$, $i = 1, ..., 9$ which have the property that $\lim_{\varepsilon \to 0} \varepsilon_i = 0$.

Some of the immediate consequences of the sampling requirements are the following lemmas.

LEMMA 1. *Any point $x \in \Sigma$ has a sample point within $\varepsilon_1 f(x)$ distance where $\varepsilon_1 = \varepsilon(1 + k_1 + k_1\varepsilon)$.*

PROOF. From the sampling condition (i), we must have a sample point p so that $\|x - \tilde{p}\| \leq \varepsilon f(x)$. Also, $\|p - \tilde{p}\| \leq k_1 \varepsilon f(\tilde{p}) \leq k_1 \varepsilon (1 + \varepsilon) f(x)$. Thus,

$$\begin{aligned} \|x - p\| &\leq \|x - \tilde{p}\| + \|\tilde{p} - p\| \\ &\leq \varepsilon f(x) + k_1 \varepsilon (1 + \varepsilon) f(x) \\ &\leq \varepsilon (1 + k_1 + k_1 \varepsilon) f(x). \end{aligned}$$

□

LEMMA 2. *Any sample point $p \in P$ has its κth closest sample point within $\varepsilon_2 f(\tilde{p})$ distance where*

$$\varepsilon_2 = \left(k_1 + \frac{4\kappa + k_1}{1 - 4\kappa\varepsilon}\right)\varepsilon.$$

PROOF. Consider the locally uniform sample $\tilde{P}$. It is an easy consequence of the sampling condition (i) and the Lipschitz property of $f()$ that, for each $x \in \Sigma$ there exists a sample point p so that $\|\tilde{p} - x\| \leq \frac{\varepsilon}{1-\varepsilon} f(\tilde{p})$. This means that, for sufficiently small ε, balls of radius $2\varepsilon f(\tilde{p}) > \frac{\varepsilon}{1-\varepsilon} f(\tilde{p})$ around each point $\tilde{p} \in \tilde{P}$ cover Σ. Consider the graph where a point

$\tilde{p} \in \tilde{P}$ is joined with $\tilde{q} \in \tilde{P}$ with an edge if the balls $B_{\tilde{p},r_1}$ and $B_{\tilde{q},r_2}$ intersect where $r_1 = 2\varepsilon f(\tilde{p})$ and $r_2 = 2\varepsilon f(\tilde{q})$. Consider a simple path Π of κ edges in this graph with one endpoint at $\tilde{p}$. An edge between two points $\tilde{p}$ and $\tilde{q}$ in the graph has a length at most $2\varepsilon(f(\tilde{p}) + f(\tilde{q}))$. The path Π thus has length at most

$$\ell = 2\varepsilon(f(\tilde{p}) + 2f(\tilde{q_1}) + ... + 2f(\tilde{q}_{\kappa-1}) + f(\tilde{q_\kappa}))$$

where $\tilde{q}_i, i = 1, ..., \kappa$ are the vertices ordered along the path. Denoting f_{max} as the maximum of the feature sizes of all vertices on the considered path we get

$$\begin{aligned} \ell &\leq 4\kappa\varepsilon f_{max} \\ &\leq \frac{4\kappa\varepsilon}{1-4\kappa\varepsilon} f(\tilde{p}) \end{aligned}$$

The distance from p to the farthest point, say q, among the κ closest points to p cannot be more than the length of Π and thus is within distance

$$\begin{aligned} d &\leq \|p - \tilde{p}\| + \|\tilde{p} - \tilde{q}\| + \|\tilde{q} - q\| \\ &\leq k_1\varepsilon f(\tilde{p}) + \frac{4\kappa\varepsilon}{1-4\kappa\varepsilon} f(\tilde{p}) + \frac{k_1\varepsilon}{1-4\kappa\varepsilon} f(\tilde{p}). \end{aligned}$$

We have $d \leq \varepsilon_2 f(\tilde{p})$. $\square$

2.3 Offset surfaces

Many standard arguments used in proving the guarantees of the surface reconstruction algorithms [1, 2, 6] cannot be applied here because P may not be a subset of Σ. We take the help of offset surfaces of Σ to overcome this difficulty. Let Σ be oriented so that the unit normal $\mathbf{n}_p$ at any $p \in \Sigma$ points locally towards the unbounded component of $\mathbb{R}^3 - \Sigma$. Consider the signed distance function

$$h : \mathbb{R}^3 \to \mathbb{R} \text{ where } h(x) = (x - \tilde{x}) \cdot \mathbf{n}_{\tilde{x}}.$$

This function is smooth everywhere except at the medial axis M. We consider two level sets avoiding the medial axis. Let

$$\Sigma_{-\varepsilon} = \{x \in \mathbb{R}^3 \mid |h(x)| = k_1\varepsilon\rho_i(\tilde{x}) \text{ and } h(x) < 0\}$$

and

$$\Sigma_{+\varepsilon} = \{x \in \mathbb{R}^3 \mid |h(x)| = k_1\varepsilon\rho_o(\tilde{x}) \text{ and } h(x) > 0\}.$$

These two level set surfaces are smooth and have M as the medial axis. Also, they lie close to the surface Σ, see Figure 2.

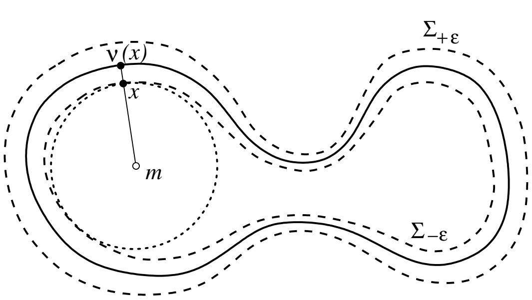

Figure 2: A point x on the offset surface $\Sigma_{-\varepsilon}$ has its closest point $\nu(x) = \tilde{x}$ on Σ. The points x, $\nu(x)$ and the medial axis point m are collinear.

Lemma 3. *The surfaces $\Sigma_{-\varepsilon}$ and $\Sigma_{+\varepsilon}$ are smooth and homeomorphic to Σ. Also, any point x on either $\Sigma_{-\varepsilon}$ or $\Sigma_{+\varepsilon}$ has $\tilde{x}$ within $\varepsilon_3 f(\tilde{x})$ distance and a sample point within $\varepsilon_4 f(\tilde{x})$ distance where $\varepsilon_3 = k_1\Delta\varepsilon$ and $\varepsilon_4 = (\varepsilon_1 + k_1\Delta\varepsilon)$.*

Proof. One can make the function h smooth by using arbitrarily small modifications near the medial axis M; for example using the partition of unity [7]. This does not affect $\Sigma_{-\varepsilon}$ or $\Sigma_{+\varepsilon}$ since they are far away from M. All the critical points of this modified height function occur near the medial axis. Thus, the level sets $\Sigma_{-\varepsilon}$, $\Sigma_{+\varepsilon}$ are smooth and homotopy equivalent to Σ [19]. Since any two homotopy equivalent surfaces without boundary are also homeomorphic, the claim of homeomorphism follows.

Let x be any point on $\Sigma_{-\varepsilon}$. The proof for the case when $x \in \Sigma_{+\varepsilon}$ is similar. It follows from the definition that $\|x - \tilde{x}\| \leq k_1\varepsilon\rho_i(\tilde{x}) \leq k_1\varepsilon\Delta f(\tilde{x})$. The point $\tilde{x}$ has a sample point within $\varepsilon_1 f(\tilde{x})$ distance (Lemma 1). Combining the two distance bounds we get the required bound. $\square$

The following observation is an easy consequence of our definitions.

Lemma 4. *The bounded (unbounded) component of $\mathbb{R}^3 - \Sigma_{-\varepsilon}$ ($\mathbb{R}^3 - \Sigma_{+\varepsilon}$ respectively) does not contain any point from P.*

Proof. If the bounded (unbounded) component of $\mathbb{R}^3 - \Sigma_{-\varepsilon}$ ($\mathbb{R}^3 - \Sigma_{+\varepsilon}$) could contain a point, say $p \in P$, we would have $d(p, \Sigma) > k_1\varepsilon\rho_i(\tilde{p}) > k_1\varepsilon f(\tilde{p})$, which is impossible due to the sampling condition (ii). $\square$

3. UNION OF BALLS

As we indicated before, our goal is to filter out a subset of points from P that lie on big Delaunay balls. We do this by choosing Delaunay balls that are big compared to the distances of the κth nearest neighbor from the sample points. Let λ_p denote the distance to the κth nearest neighbor of a sample point $p \in P$. For an appropriate constant $k_3 > 1$, we define

$\mathcal{B}$ = set of Delaunay balls $B_{r,x}$ where $r > k_3\lambda_p$ for all points $p \in P$ incident on the boundary of $B_{r,x}$.

Since we know that $\lambda_p \geq k_2\varepsilon f(\tilde{p})$ by the sampling condition (iii), we have:

Observation 1. *Let $B_{r,x} \in \mathcal{B}$ be a Delaunay ball with $p \in P$ on its boundary. Then, $r > K\varepsilon f(\tilde{p})$ where $K = k_3k_2$.*

Let $B_{r,x}$ be a ball in $\mathcal{B}$. We call $B_{r,x}$ *inner* if x lies in the closure of the bounded component of $\mathbb{R}^3 - \Sigma$. The balls in $\mathcal{B}$ that are not inner are called *outer*. Let $\mathcal{B} = \mathcal{B}_I \cup \mathcal{B}_O$ where $\mathcal{B}_I$ is the set of inner balls and $\mathcal{B}_O$ is the set of outer balls.

We will filter out those points from P that lie on the balls in $\mathcal{B}$. Partition of $\mathcal{B}$ induces a partition on these points, namely

$$\begin{aligned} P_I &= \{p \in P \cap B \mid B \in \mathcal{B}_I\} \text{ and} \\ P_O &= \{p \in P \cap B \mid B \in \mathcal{B}_O\}. \end{aligned}$$

Notice that P_I and P_O partition only the set of points incident to the balls in $\mathcal{B}$ and not necessarily the set P.

Next lemma is pivotal for later proofs.

LEMMA 5. *For each point $x \in \Sigma_{-\varepsilon}$ ($x \in \Sigma_{+\varepsilon}$) there is a Delaunay ball which contains the center of the medial ball touching $\Sigma_{-\varepsilon}$ ($\Sigma_{+\varepsilon}$ respectively) at x and whose boundary contains a sample point $p \in P_I$ ($p \in P_O$ respectively) within a distance $\varepsilon_5 f(\tilde{x})$ from x where $\varepsilon_5 = \sqrt{6\varepsilon_4\Delta}$ and ε is sufficiently small.*

PROOF. We prove the lemma for the case $x \in \Sigma_{-\varepsilon}$. The case where $x \in \Sigma_{+\varepsilon}$ can be proved similarly. Consider the medial ball $B = B_{r,m}$ for $\Sigma_{-\varepsilon}$ touching it at x and its shrunk copy $B^{3/4} = B_{3r/4,m}$. The ball B and hence $B^{3/4}$ are empty (Lemma 4). Translate $B^{3/4}$ rigidly by moving the center m along the direction $x - m$ until its boundary hits a sample point $p \in P$. Let this new ball be denoted $B' = B'_{3r/4,m'}$, refer to Figure 3.

Let x' be the closest point to x on the boundary of B'. We have $\|x' - x\| \leq \varepsilon_4 f(\tilde{x})$ since otherwise there is an empty ball centering x with radius $\varepsilon_4 f(\tilde{x})$ and thus x does not have a sample point within $\varepsilon_4 f(\tilde{x})$ distance violating Lemma 3.

Next, we show that both B and B' contain their centers in their intersection. Since B' has radius smaller than B, it is sufficient to show that B' contains m inside. During the rigid motion when the ball $B^{3/4}$ touches B at x, it still contains the point m inside since its radius is only three-fourth that of B. Since we move $B^{3/4}$ by the distance $\|x - x'\| \leq \varepsilon_4 f(\tilde{x})$, it is displaced slightly enough to retain m inside when ε is sufficiently small.

Now we show that x is close to p. The point p can only be on that part of the boundary of B' which is outside the empty ball B. This with the fact that the centers of B and B' are in their intersection imply that the largest distance from x to p is realized when p is on the circle of intersection of B and B'. So, consider this situation and let C and C' be the diametric circles of B and B' respectively on the plane passing through p, x and x' as in Figure 3. Among the two intersection points of C' with the line tangent to C at x, let a be closer to p. We have

$$\|p - x\| \leq \|p - x'\| \leq \|p - a\| + \|a - x'\|.$$

From the circle C' we get

$$\|a - x'\| = \frac{\|x - x'\|}{\cos\alpha} \quad \text{and} \quad \|m' - x'\|\cos\alpha = \frac{\|a - x'\|}{2}$$

which implies

$$\|a - x'\| = \sqrt{\frac{3r}{2}\|x - x'\|} \leq \sqrt{\left(\frac{3\Delta}{2}\varepsilon_4\right)} f(\tilde{x}).$$

It follows from circle geometry that the segment ap cannot intersect C other than at p since C' is smaller than C. Thus, $\|p - a\| \leq \|a - x\| \leq \|a - x'\|$ which implies

$$\|p - x\| \leq 2\|a - x'\| \leq \sqrt{6\Delta\varepsilon_4} f(\tilde{x}).$$

Next we argue that we can deform B' further to a Delaunay ball containing m inside. We deform B' as follows till it meets at least four sample points always keeping the point m inside. We refer to the new ball and its center during this deformation process also as B' and m' respectively. First, grow it by keeping p on its boundary and moving its center along the direction $m' - p$ until it hits a sample point, say, q. At this point certainly m is in the new ball as it contains the original B'. Let y be the midpoint of pq and Z be the disc with diameter pq and perpendicular to ym'. Deform B' by keeping p, q on its boundary and moving the center along the direction $m' - y$ if Z does not separate m and m' and along the direction $y - m'$ otherwise. Stop when the ball hits a third point, say s. At this point the ball B' has three points on its boundary and also has m in it. Let W be the circumdisk of the triangle pqs with the circumcenter w. Deform B' keeping p, q, s on its boundary and moving the center along the direction $m' - w$ if W does not separate m and m' and along the direction $w - m'$ otherwise. Again stop when B' hits the fourth point, say t. At this point B' is a Delaunay ball containing m inside. For any of $u \in \{p, q, s, t\}$, the radius r' of B' satisfies

$$\begin{aligned} r' \geq \frac{\|u - m\|}{2} &\geq \frac{\|\tilde{u} - m\| - \|u - \tilde{u}\|}{2} \\ &\geq \frac{(1 - k_1\varepsilon)}{2} f(\tilde{u}). \end{aligned}$$

Also we have $\lambda_u \leq \varepsilon_2 f(\tilde{u})$ from Lemma 2. Thus, B' is in $\mathcal{B}$ if

$$\begin{aligned} \frac{(1 - k_1\varepsilon)}{2} &> K\varepsilon_2, \quad \text{or} \\ 1 &> k_1\varepsilon + 2K\varepsilon_2, \end{aligned}$$

a condition which is satisfied for sufficiently small ε.

Next, we argue that the Delaunay ball $B' \in \mathcal{B}_I$ completing the claim that $p \in P_I$. We establish that $B' \in \mathcal{B}_I$ by showing that m' lies in the bounded component of $\mathbb{R}^3 - \Sigma$. Suppose not. Then, the segment joining m and m' must intersect Σ and let x be that intersection point. Certainly, $\|x - m\| \geq f(x)$. The ball centered at x with radius $\|x - m\|$ lies completely inside B' and thus cannot contain any sample point. This means x cannot have a sample point within $f(x)$ distance, a contradiction to Lemma 1 for sufficiently small ε. □

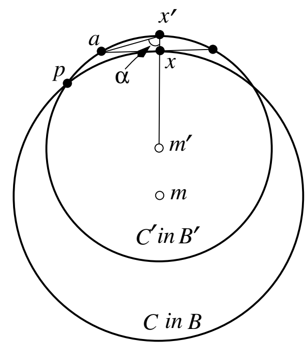

Figure 3: Each point x on $\Sigma_{-\varepsilon}$ has a nearby sample point p in $\mathcal{B}_I$.

4. PROXIMITY

We aim to prove that the boundary of $\bigcup \mathcal{B}_I$ is homeomorphic and close to Σ. The proof can be adapted in a straightforward manner for a similar result between the boundary of $\bigcup \mathcal{B}_O$ and Σ. We define

$$\begin{aligned} S_I &= \text{Boundary of } \textstyle\bigcup \mathcal{B}_I. \\ S_O &= \text{Boundary of } \textstyle\bigcup \mathcal{B}_O. \end{aligned}$$

We will use a result of Amenta, Choi and Kolluri [3] (Lemma 23) to claim the next lemma. Although the statement of this lemma is a little bit different from that of

Lemma 23 [3], the proof can be adapted with only minor modifications.

Lemma 6. *Let x be a point in S_I so that the following conditions hold:*

(i) $\|x - \tilde{x}\| = O(\varepsilon)f(\tilde{x})$,

(ii) each point $y \in \Sigma$ has a sample point within $O(\varepsilon)f(y)$ distance,

(iii) x is in a ball $B_{r,c} \in \mathcal{B}_I$ so that $r \geq K\varepsilon f(\tilde{x})$.

Then, the angle between the inner surface normal at $\tilde{x}$ and the vector $\vec{xc}$ is $O(\sqrt{1/K})$.

In the next two lemmas we establish the condition (i) of Lemma 6, that is, we show that each point in S_I has a nearby point on Σ.

Lemma 7. *Let x be a point lying in the unbounded component of $\mathbb{R}^3 - \Sigma$ where $x \in S_I$. Then, $\|x - \tilde{x}\| \leq \varepsilon_6 f(\tilde{x})$ where $\varepsilon_6 = \frac{\varepsilon_1}{1-2\varepsilon_1}$.*

Proof. Let $x \in B_{r,c}$ where $B_{r,c} \in \mathcal{B}_I$. The line segment joining x and c must intersect Σ since c lies inside or on the boundary of the bounded component of $\mathbb{R}^3 - \Sigma$ while x lies outside. Let this intersection point be z. We claim that $\|x - z\| \leq \varepsilon_1 f(z)$. Otherwise, there is a ball inside $B_{r,c}$ centering z and radius at least $\varepsilon_1 f(z)$. This ball is empty since $B_{r,c}$ is empty. This violates Lemma 1 for z. This means that the closest point $\tilde{x} \in \Sigma$ to x has a distance $\|x-\tilde{x}\| \leq \|x-z\| \leq \varepsilon_1 f(z)$. We also have $\|z-\tilde{x}\| \leq 2\|x-z\|$. Applying the Lipschitz property of $f()$, we get the desired bound for $\|x - \tilde{x}\|$. □

Lemma 8. *Let x be a point lying in the bounded component of $\mathbb{R}^3 - \Sigma$ where $x \in S_I$. Then $\|x - \tilde{x}\| \leq \varepsilon_7 f(\tilde{x})$ where*

$$\varepsilon_7 = \left(3\sqrt{\left(\frac{4\varepsilon_5^2}{(1-\varepsilon_3)^2} + \frac{4\varepsilon_5}{(1-\varepsilon_3)}\right)} + \varepsilon_3\right).$$

Proof. Let y be the point in $\Sigma_{-\varepsilon}$ closest to x. Observe that x, y and $\tilde{x}$ are collinear. Consider a Delaunay ball $B = B_{r,c} \in \mathcal{B}_I$ that contains the center m of the medial ball touching $\Sigma_{-\varepsilon}$ at y and has a sample point $p \in P$ on the boundary so that $\|y - p\| \leq \varepsilon_5 f(\tilde{x})$. Such a ball exists by Lemma 5. If point y is inside B, the entire segment ym is inside B. In that case x must be in between y and $\tilde{x}$ on the segment yx and thus $\|x - \tilde{x}\| \leq \|y - \tilde{x}\| \leq \varepsilon_3 f(\tilde{x})$ (Lemma 3).

Consider the other case when $y \notin B$. Since $y \notin B$ and $m \in B$, the segment ym intersect the boundary of B, say at z. If the point x lies in between $\tilde{x}$ and y, we have $\|x - \tilde{x}\| \leq \varepsilon_3 f(\tilde{x})$. So, assume x lies in between y and z and hence $\|y - x\| \leq \|y - z\|$. We bound the distance $\|y - z\|$.

The distance from y to z is no more than the length of the tangent from y to B. Using simple circle geometry the length of the tangent from y to B can be shown to be at most

$$\sqrt{(\varepsilon'^2 + 2\varepsilon')}r \tag{4.1}$$

where the closest point of y to the boundary of B is within $\varepsilon' r$ distance. Since B contains m, the segment pm is inside B. It intersects $\Sigma_{-\varepsilon}$ as p and m are in two different sides of $\Sigma_{-\varepsilon}$. Therefore, $\|p - m\| \geq \|y - m\|$ as y is the closest point to m on $\Sigma_{-\varepsilon}$. This means $\|p - m\| \geq (1 - \varepsilon_3)f(\tilde{x})$ (Lemma 3).

The radius r of B is at least $\|p - m\|/2$. Thus, $r \geq \frac{(1-\varepsilon_3)}{2} f(\tilde{x})$. The closest point of y to the boundary of B is no more than $\varepsilon_5 f(\tilde{x})$ since there is a sample point p on the boundary of B within that distance. This means we can apply $\varepsilon' = \frac{2\varepsilon_5}{(1-\varepsilon_3)}$ in the equantion 4.1 to obtain an upper bound on the distance $\|y - z\|$ in terms of r. We also need an upper bound on r to complete the proof. We claim that $r \leq 3f(\tilde{x})$ for sufficiently small ε.

The segment cp must intersect $\Sigma_{-\varepsilon}$ since c and p reside in different sides of $\Sigma_{-\varepsilon}$. Let the intersection point of cp and $\Sigma_{-\varepsilon}$ be u. The distance $\|u - p\|$ is at most $\varepsilon_3 f(\tilde{u})$ since otherwise there is an empty ball centering u with radius more than $\varepsilon_4 f(\tilde{u})$ violating the claim of Lemma 3 that u must have a sample point within $\varepsilon_4 f(\tilde{u})$ distance. Consider the empty ball centering c with radius $\|c - u\|$. This ball cannot be too large compared to the medial ball of $\Sigma_{-\varepsilon}$ at u. In fact, we can claim that its radius is at most twice the radius of this medial ball for a sufficiently small ε. This means $\|c-u\| \leq 2(1-\varepsilon_3)f(\tilde{u})$ and hence $\|c-p\| \leq (2-2\varepsilon_3+\varepsilon_4)f(\tilde{u})$. Since $\|u - y\| \leq \|u - p\| + \|p - y\| = O(\varepsilon)f(\tilde{u}) + O(\varepsilon)f(\tilde{x})$, it can be shown $f(\tilde{u}) \leq \frac{f(\tilde{x})}{1-O(\varepsilon)}$ by Lipschitz property of $f()$. Therefore,

$$r = \|c - p\| \leq \frac{(2 - 2\varepsilon_3 + \varepsilon_4)}{1 - O(\varepsilon)} f(\tilde{x}) \leq 3f(\tilde{x})$$

for sufficiently small ε.

Finally, we have

$$\begin{aligned} \|x - \tilde{x}\| &\leq \|y - z\| + \|y - \tilde{x}\| \\ &\leq \left(3\sqrt{\frac{4\varepsilon_5^2}{(1-\varepsilon_3)^2} + \frac{4\varepsilon_5}{(1-\varepsilon_3)}} + \varepsilon_3\right) f(\tilde{x}) \end{aligned}$$

as claimed.

□

From Lemma 7 and Lemma 8 we get the following lemma that satisfies the condition (i) of Lemma 6.

Lemma 9. *Each point x on S_I has a point in Σ within $\varepsilon_8 f(\tilde{x})$ distance where $\varepsilon_8 = \max\{\varepsilon_6, \varepsilon_7\}$.*

The condition (ii) of Lemma 6 is established in the next lemma.

Lemma 10. *Each point $y \in \Sigma$ has a sample point $p \in \mathcal{P}_I$ within $(\varepsilon_3 + \varepsilon_5)f(y)$ distance.*

Proof. Each $y = \tilde{x} \in \Sigma$ has a point x in $\Sigma_{-\varepsilon}$ within $\varepsilon_3 f(y)$ distance (Lemma 3) which in turn has a sample point $p \in \mathcal{P}_I$ within $\varepsilon_5 f(y)$ distance (Lemma 5). We obtain the desired bound from these two facts. □

The condition (iii) of Lemma 6 is established in the next lemma.

Lemma 11. *Let x be any point on the boundary of a ball $B_{r,c} \in \mathcal{B}_I$, we have $r \geq (K/2)\varepsilon f(\tilde{x})$ for sufficiently small ε.*

PROOF. Suppose the claim is not true. Then, consider a vertex $p \in P$ on the Delaunay ball $B_{r,c}$. Since this ball is in $\mathcal{B}_I$, we have $r \geq K\varepsilon f(\tilde{p})$. Since $\|x - p\| \leq 2r$, we have $\|x - p\| \leq K\varepsilon f(\tilde{x})$ by our assumption. This means $\|x - \tilde{p}\| \leq K\varepsilon f(\tilde{x}) + k_1\varepsilon f(\tilde{p})$. Since $\tilde{x}$ is closer to x than $\tilde{p}$, we have

$$\begin{aligned} \|\tilde{x} - \tilde{p}\| &\leq \|\tilde{x} - x\| + \|x - \tilde{p}\| \\ &\leq 2\left(K\varepsilon f(\tilde{x}) + k_1\varepsilon f(\tilde{p})\right). \end{aligned}$$

Using the Lipschitz property of $f()$ we get

$$f(\tilde{x}) \leq \left(\frac{1 + 2k_1\varepsilon}{1 + 2K\varepsilon}\right) f(\tilde{p})$$

Therefore by our assumption,

$$r \leq \left(\frac{K}{2}\right)\left(\frac{1 + 2k_1\varepsilon}{1 + 2K\varepsilon}\right) \varepsilon f(\tilde{p}).$$

We reach a contradiction if $\frac{K(1+2k_1\varepsilon)}{2(1+2K\varepsilon)} < K$, a condition which is satisfied for sufficiently small ε. □

5. HOMEOMORPHIC SURFACE

We have all ingredients to apply the proof of Lemma 25 in [3] which establishes the homeomorphism between Σ and S_I. For completeness and to accommodate the slight differences we include the proof.

THEOREM 1. *Let S be either S_I or S_O. The restriction of ν to S defines a homeomorphism between S and Σ for sufficiently small ε and sufficiently large K.*

PROOF. We prove it for S_I which also extends straightforwardly to S_O. Since S_I and Σ are both compact we only need to show that ν is continuous, one-to-one and onto. The discontinuity of ν occurs only at the medial axis of Σ. Since each point x in S_I is within $\varepsilon_8 f(\tilde{x})$ distance from $\tilde{x}$ (Lemma 9), all points of S_I are far away from the medial axis when ε is sufficiently small. Thus the restriction of ν to S_I is continuous.

To prove that ν is one-to-one, assume on the contrary that there are points x and x' in S_I so that $\tilde{x} = \nu(x) = \nu(x')$. Without loss of generality assume x' is further away from $\tilde{x}$ than x is. Let $x \in B_{r,c}$ where $B_{r,c} \in \mathcal{B}_I$. The line ℓ_x passing through x and x' is normal to Σ at $\tilde{x}$, and according to Lemma 6, ℓ_x makes an angle at most $\alpha = O(\sqrt{1/K})$ with the vector $\vec{xc}$. Thus, while walking on the line ℓ_x towards the inner medial axis starting from $\tilde{x}$, we encounter a segment of length at least $2r\cos\alpha$ inside $B_{r,c}$. By Lemma 9 both x and x' are within $\varepsilon_8 f(\tilde{x})$ distance from $\tilde{x}$. We reach a contradiction if $2r\cos\alpha$ is more than $\varepsilon_8 f(\tilde{x})$. Since $r > (K/2)\varepsilon f(\tilde{x})$ by Lemma 11 and $\alpha = O(\sqrt{1/K})$, this contradiction can be reached for sufficiently small ε and large K. Therefore, x and x' are same.

Now we argue that ν is also onto. Since S_I is a closed, compact surface without boundary and ν maps S_I continuously to Σ, $\nu(S_I)$ must consist of closed connected components of Σ. If Σ has a connected component, say σ, for which there is no sample point p where $\nu(p)$ is in σ, Lemma 1 is violated for sufficiently small ε. Thus $\nu(S_I)$ has a point in each connected component of Σ and therefore is onto. □

5.1 Ball separation

In order to apply the previous results, we need to separate the inner balls in $\mathcal{B}_I$ from the outer ones in $\mathcal{B}_O$. We achieve this by looking at how deeply the balls intersect. We call two balls in $\mathcal{B}_I$ ($\mathcal{B}_O$) *neighbors* if their boundaries intersect in S_I (S_O *respectively*). The neighbor balls in $\mathcal{B}_I$ or in $\mathcal{B}_O$ intersect deeply while two balls, one from $\mathcal{B}_I$ and the other from $\mathcal{B}_O$, can have only shallow intersection. We measure the depth of intersection by the angle at which two balls intersect. Let x be any point where the boundaries of two balls B_1 and B_2 intersect. We say B_1 intersects B_2 at an angle α if $\angle(\vec{xc_1}, \vec{xc_2}) = \alpha$ where c_1 and c_2 are the centers of B_1 and B_2 respectively.

LEMMA 12. *Any two neighbor balls B_1 and B_2 in $\mathcal{B}_I$ intersect at an angle $O(\sqrt{1/K})$.*

PROOF. Let $x \in B_1 \cap B_2$ be a point in S_I. The angle at which B_1 and B_2 intersect at x is equal to the angle between the vectors $\vec{xc_1}$ and $\vec{xc_2}$ where c_1 and c_2 are the centers of B_1 and B_2 respectively. By Lemma 6 both $\angle(\mathbf{n}_{\tilde{x}}, \vec{xc_1})$ and $\angle(\mathbf{n}_{\tilde{x}}, \vec{xc_2})$ are $O(\sqrt{1/K})$. This implies $\angle(\vec{xc_1}, \vec{xc_2}) = O(\sqrt{1/K})$. □

LEMMA 13. *Any two balls B_1 and B_2 intersect at an angle more than $\pi/2 - \arcsin((2/K)(1 + O(\varepsilon)))$ where $B_1 \in \mathcal{B}_I$ and $B_2 \in \mathcal{B}_O$.*

PROOF. The line segment joining the center c_1 of B_1 and the center c_2 of B_2 intersects Σ as c_1 lies in the bounded component of $\mathbb{R}^3 - \Sigma$ where c_2 lies in the unbounded one. Let this intersection point be x. Without loss of generality, assume that x lies inside B_1. Let C be circle of intersection of B_1 and B_2 and d be its radius. Clearly, d is smaller than the distance of x to the closest sample point as B_1 is empty. This fact and Lemma 1 imply

$$d \leq \varepsilon_1 f(x). \tag{5.2}$$

Next, we obtain a lower bound on the radius of B_1 in terms of $f(x)$. Let the segment c_1c_2 intersect the boundary of B_1 at y. Lemma 3 implies $\|x - y\| \leq \varepsilon_3 f(x)$. This also means $\|x - \tilde{y}\| \leq 2\varepsilon_3 f(x)$. By Lipschitz property of $f()$, we have

$$f(\tilde{y}) \geq (1 - 2\varepsilon_3) f(x).$$

The radius r of B_1 satisfies

$$\begin{aligned} r &> (K/2)\varepsilon f(\tilde{y}) && (5.3) \\ &> (K/2)\varepsilon(1 - 2\varepsilon_3) f(x). && (5.4) \end{aligned}$$

Combining 5.2 and 5.4 we obtain that, for a point z on the circle C, $\vec{zc_1}$ makes an angle at least $\pi/2 - \arcsin((2/K)(1 + O(\varepsilon)))$ with the plane of C. The angle at which B_1 and B_2 intersect is greater than this angle. □

Lemma 12 and 13 say that, for sufficiently large K and small ε, one can find an angle $\theta > 0$ so that the neighbor balls in $\mathcal{B}_I$ and $\mathcal{B}_O$ intersect at an angle less than θ whereas a ball from $\mathcal{B}_I$ intersects a ball from $\mathcal{B}_O$ at an angle larger than θ. This becomes the basis of separating the inner balls from the outer ones.

5.2 Algorithm

Now we have all ingredients to design an algorithm that computes a surface homeomorphic to Σ. The algorithm first chooses Delaunay balls whose radius is bigger than a constant (k_3) times the distance between any sample point p on its boundary and the κth nearest sample point of p. In practice, we take $\kappa = 3$. Then, we start walking from an infinite Delaunay ball circumscribing an infinite tetrahedron formed by a convex hull triangle and a point at infinity. This Delaunay ball is outer. We interpret the angle of intersection between an infinite Delaunay ball and other Delaunay balls intersecting it properly taking infinity into account. We continue to collect all big balls that intersect a ball already marked *outer* at an angle more than a threshold angle θ. The boundary of the union of these outer balls, or the remaining inner big balls can be output as the approximated surface. Alternatively, one can compute a *skin* surface [8] out of these balls that approximates the boundary with a C^2-smooth surface.

6. INTERPOLATING SURFACE

In this section we extend the previous algorithm to produce a piecewise linear surface interpolating through the sample points residing on the inner (or outer) balls. For the proofs we will use $\kappa = 1$ in this extension though $\kappa = 3$ works well in practice. We compute the restricted Delaunay triangulation $\mathrm{Del}\, P_I|_{S_I}$ or equivalently $\mathrm{Del}\, P_O|_{S_O}$. In what follows we argue for $\mathrm{Del}\, P_I|_{S_I}$.

Our goal is to show that the $\mathrm{Del}\, P_I|_{S_I}$ is homeomorphic to Σ. Thanks to Theorem 1, this can be established by showing that the restricted Voronoi diagram $\mathrm{Vor}\, P_I|_{S_I}$ satisfies the *topological ball property* according to a result of Edelsbrunner and Shah [15]. This means, in $\mathrm{Vor}\, P_I$, each Voronoi edge intersecting S_I should intersect it in a single point. Each Voronoi facet intersecting S_I should intersect it in a topological 1-ball, that is, in a single open curve. Each Voronoi cell should intersect S_I in a topological disk.

We can show all these three conditions using the technique used to prove the topological ball property for skin surfaces [8]. Although this surface is smooth whereas S_I is not, we identify the main ingredients in the proofs of Cheng et al. [8] which can be recast in the context of S_I and thus the rest of the proof can be carried through.

The main three ingredients in the proof of Cheng et al. are
(i) Each point on the surface in a Voronoi cell V_p is close to p with respect to the local feature size at p.
(ii) Each Voronoi facet $F \in V_p$ intersecting the surface is parallel to the normal at p.
(iii) Each Voronoi edge $e \in V_p$ intersecting the surface is almost parallel to the normal at p.

The next lemma is the counterpart of (i) for S_I.

LEMMA 14. *Let $p \in P_I$ be a sample point in S_I and x be any point in $V_p \cap S_I$. We have $\|p - x\| \le \varepsilon_9 f(\tilde{x})$ where $\varepsilon_9 = \varepsilon_3 + \varepsilon_5 + \varepsilon_8$.*

PROOF. Let y be the closest point of x on $\Sigma_{-\varepsilon}$. It follows from Lemma 5 that y has a sample point, say p', within $\varepsilon_5 f(\tilde{x})$ distance on S_I. Also, $\|x - y\| \le \|x - \tilde{x}\| + \|\tilde{x} - y\|$. Lemma 9 gives $\|x - \tilde{x}\| \le \varepsilon_8 f(\tilde{x})$ and Lemma 3 gives $\|\tilde{x} - y\| \le \varepsilon_3 f(\tilde{x})$. Therefore,

$$\|x - p\| \le \|x - p'\| \le \|x - y\| + \|y - p'\| \le \varepsilon_9 f(\tilde{x}).$$

□

Next lemma prepares for Lemma 16 that proves the counterpart of condition (ii) for S_I.

LEMMA 15. *Let $p \in P_I$ be a sample point in $B \in \mathcal{B}_I$. Let $B' = B_{r,c}$ be the maximal ball tangent to B at p and whose interior does not contain any sample point from P_I within $2\varepsilon_9 f(\tilde{p})$ distance from p. Then, there is a constant $k_4 > 0$ so that $r > k_4\sqrt{K}\varepsilon f(\tilde{p})$.*

PROOF. Consider a ball B' tangent to B at p with the same radius as of B. If this ball satisfies the condition of the lemma we are done since B has a radius more than $K\varepsilon f(\tilde{p})$ by Observation 1. Otherwise, shrink B' keeping p on its boundary and moving its center c towards p along the direction $p - c$. Stop when all sample point(s) meeting B' within $2\varepsilon_9 f(\tilde{p})$ distance from p lie on its boundary. Let $B' = B_{r,c}$ at this moment and q be any such sample point. Let q lie on a ball $B'' = B_{r',c'}$ in $\mathcal{B}_I$. We claim that the angle $\angle \vec{cq}, \vec{cp} = O(\sqrt{1/K})$.

Consider the case where B' becomes tangent to B'' at a point further than q from p when we shrink B' further. In this case $\theta = \angle \vec{cq}, \vec{cp} \le \angle \vec{qc'}, \vec{cp}$. The vector $\vec{cp}$ is normal to the boundary of the ball B at p. Since $\|p-q\| \le 2\varepsilon_9 f(\tilde{p})$, one can apply Lemma 6 to claim that the normals to B and B'' at p and q respectively make an angle $\angle \vec{qc'}, \vec{cp} = O(\sqrt{1/K})$. Therefore, the radius r of B' satisfies $r \sin\frac{\theta}{2} = \frac{\|p-q\|}{2}$. Since $\|p-q\| \ge k_2 \varepsilon f(\tilde{p})$ by the sampling condition (iii) when $\kappa = 1$, we have $r > k_4\sqrt{K}\varepsilon f(\tilde{p})$ for some suitable constant $k_4 > 0$.

In the other case when B' becomes tangent to B'' at a point, say x, after shrinking where x is closer to p than q, we have $\angle \vec{cq}, \vec{cp} \le 2\angle \vec{xc'}, \vec{cp}$. Again appealing to Lemma 6 we get $\angle \vec{xc'}, \vec{cp} = O(\sqrt{1/K})$ which gives $\angle \vec{cq}, \vec{cp} = O(\sqrt{1/K})$. Now we can apply the same argument as in the previous case to claim the desired bound on r. □

LEMMA 16. *Let pq be an edge in the restricted Delaunay triangulation $\mathrm{Del}\,\mathcal{P}_{\mathcal{I}}|_{\mathcal{S}_{\mathcal{I}}}$ where p is on the boundary of $B_{r,c} \in \mathcal{B}_I$. Then, $\angle \vec{pq}, \vec{pc} = \frac{\pi}{2} - O(\frac{1}{\sqrt{K}})$.*

PROOF. Consider a maximal ball $B' = B_{r',c'}$ tangent to $B = B_{r,c}$ at p which does not have any sample inside within $2\varepsilon_9 f(\tilde{p})$ distance. According to Lemma 15, r' is at least $k_4\sqrt{K}\varepsilon f(\tilde{p})$ for some constant k_4. This is also a lower bound on r. Since pq is at most $2\varepsilon_9 f(\tilde{p})$ long and both B and B' do not have any sample point within $2\varepsilon_9 f(\tilde{p})$ distance from p, its intersection with either of B or B' cannot be any longer. This implies that $\vec{pq}$ makes an angle at most $O(arcsin(1/\sqrt{K})$ angle with the tangent plane to B at p. The claim of the lemma follows. □

Lemma 16 above implies that the dual Voronoi facet of pq is almost parallel to the normal at p on the boundary of a big Delaunay ball in $\mathcal{B}_I$. This is the counterpart of the condition (ii) as listed before.

Using similar argument we can prove that each restricted Delaunay triangle in $\mathrm{Del}\,\mathcal{P}_{\mathcal{I}}|_{\mathcal{S}_{\mathcal{I}}}$ also lies flat to S_I. Precisely we can show the following.

LEMMA 17. *Let pqr be a restricted Delaunay triangle in* $\operatorname{Del}\mathcal{P}_{\mathcal{I}}|_{S_{\mathcal{I}}}$. *Let p be contained in the boundary of* $B_{r,c} \in \mathcal{B}_I$. *Then the normal to the triangle pqr makes an angle at most* $O(\frac{1}{\sqrt{K}})$ *with the vector* $\vec{pc}$.

Lemma 17 implies that the dual Voronoi edge of pqr is almost parallel to the normal at p on the boundary of a big Delaunay ball in $\mathcal{B}_I$. This is the counterpart of condition (iii) listed before.

Using the above lemmas one can carry out the proof that $\operatorname{Vor} P_I|_{S_I}$ satisfies the topological ball property. We have the next theorem.

THEOREM 2. *For sufficiently small* $\varepsilon > 0$, $\operatorname{Del} P_I|_{S_I}$ *is homeomorphic to* Σ. *Further, each point x in* $\operatorname{Del} P_I|_{S_I}$ *has a point in* Σ *within* $O(\varepsilon)f(\tilde{x})$ *distance, and conversely, each point x in* Σ *has a point in* $\operatorname{Del} P_I|_{S_I}$ *within* $O(\varepsilon)f(x)$ *distance.*

6.1 Algorithm and Implementation

We collect the points on the outer (or inner) balls as we described before and then compute the restricted Delaunay triangulation of the filtered point set with respect to the boundary of the union of the outer (or inner) Delaunay balls. Naming after the COCONE algorithm for noiseless point clouds, we call this algorithm ROBUST COCONE.

We implemented ROBUST COCONE using CGAL [22] and the output is shown in Figure 4. We took $k_3 = 0.5$, $\kappa = 3$ for choosing the big Delaunay balls and $\theta = 15^\circ$ for separating outer balls from the inner ones.

We compare our results with the output of TIGHT COCONE [21] that is known to tolerate some amount of noise. Our *new algorithm* performs much better on noisy models where noise is reasonably high. One aspect of the algorithm is that it tends to produce much less non-manifold vertices and edges as depicted in the BUNNY model. Also, the algorithm is able to reconstruct the surface where the point set samples the volume as depicted in the FEMUR model.

7. CONCLUSIONS

In this paper we presented a provable algorithm for surface reconstruction from noisy point cloud data. The noise model is reasonable. The input point set has to be dense with respect to the local feature sizes to capture the features of the shape. Its scatter around the surface has to be bounded to achieve any sort of theoretical guarantees. Furthermore, as points may collaborate to form arbitrary patterns such as a dense sampling of a spurious surface, some sort of locally uniform sampling condition is necessary. We have incorporated these three conditions into our noise model. In order to have a less restrictive noise model, it seems that some sacrifice in the guarantees has to be made. It would be interesting to see what kind of trade offs can be achieved between the guarantees and the noise models.

Our algorithm requires that the sampled surface have no boundary. It is not clear how the algorithm should be adapted for surfaces with boundary. Reconstruction of surfaces with boundaries from noiseless point samples have been addressed [11]. However, noise together with boundaries pose a difficult challenge.

Although the output of ROBUST COCONE has the exact topology and approximate geometry of the sampled surface, our experiments show that, sometimes it contains undesirable undulations. In most applications this surface needs to be smoothed. There are several mesh smoothing techniques known in graphics. Currently we are experimenting with the idea of merging one such smoothing technique [18] with the concepts used in ROBUST COCONE [12].

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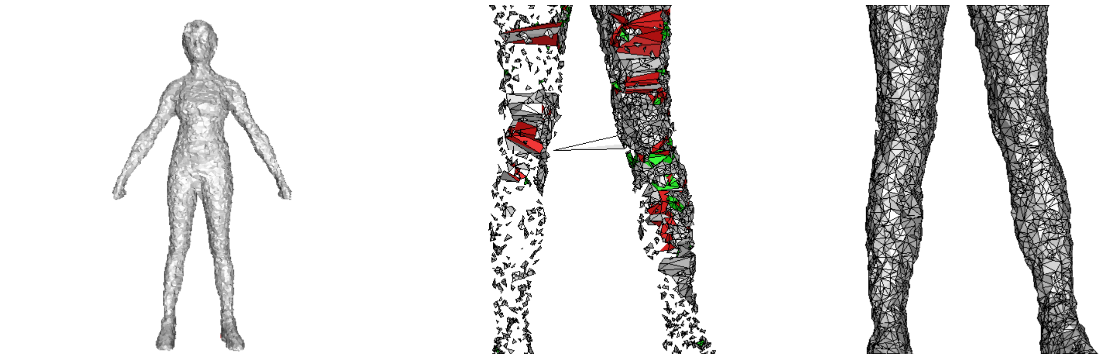
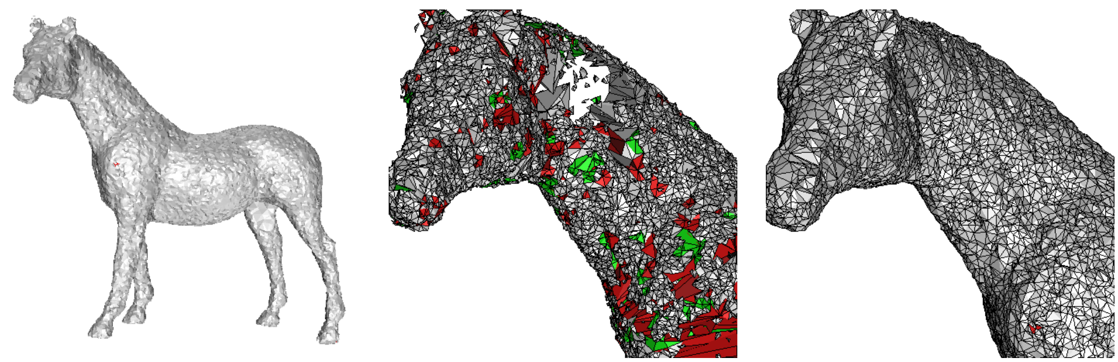
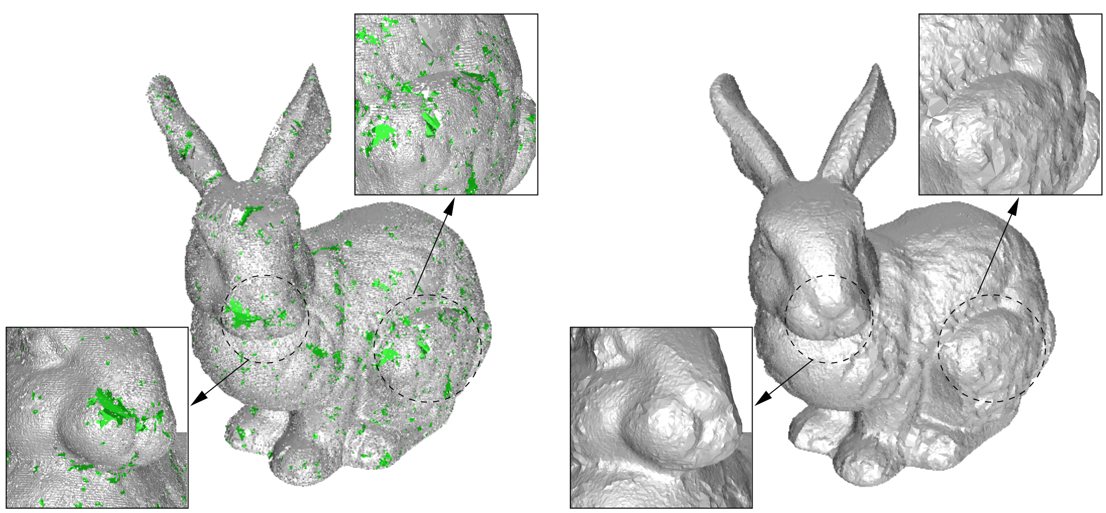


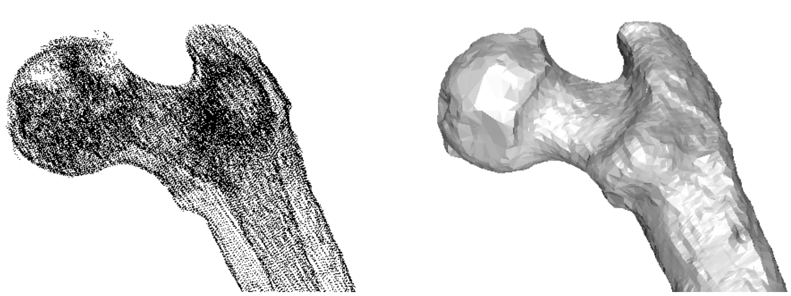
Noisy Female (Robust Cocone) Leg (Tight Cocone) Leg (Robust Cocone)



Noisy Horse (Robust Cocone) Neck (Tight Cocone) Neck (Robust Cocone)



Noisy Bunny (Tight Cocone) Robust Cocone



Femur volume pointset Robust Cocone

Figure 4: Results: Reconstruction by Robust Cocone **and its comparison to** Tight Cocone.