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$$a) T(n) = \begin{cases} O(1), & n=0 \\ T(n-1) + O(1), & n \geq 1 \end{cases}$$

$$T(n) \leq T(n-1) + C \leq T(n-2) + C + C \leq T(n-3) + C + C + C \leq \dots \leq T(n-n) + C(n) = O(n)$$

$$b) T(n) = \begin{cases} O(1), & n \leq a, a > 1 \\ aT(n-a) + O(1), & n > a \end{cases}$$

$$\begin{aligned} T(n) &\leq aT(n-a) + C \leq a(aT(n-2a) + C) + C = a^2T(n-2a) + (a+1)C \\ &\leq a^2(aT(n-3a) + C) + (a+1)C = a^3T(n-3a) + (a^2+a+1)C \\ &\leq a^kT(n - \frac{na}{a}) + C \sum_{i=0}^{k-1} a^i \leq a^{\frac{n}{a}} \left(1 + \frac{a^{\frac{n}{a}} - 1}{a - 1} \right) C = O(a^{\frac{n}{a}}) \end{aligned}$$

$$g) T(n) = \begin{cases} O(1), & n \geq 1 \\ aT(\lceil n/a \rceil) + O(1), & n \geq 2, a \geq 2 \end{cases}$$

 $m \geq \log_a n$

$$T(n) \leq aT(\lceil n/a \rceil) + C = aT(a^{m-1}) + C \leq a^2T(a^{m-2}) + aC$$

$$+ C \leq a^3 T(a^{m-3}) + a^2 C + a C + C \leq a^m T(a^{n-m}) + C \sum_{i=0}^{m-1} a^i$$

$$= a^m C + C \frac{a^m - 1}{a - 1} \leq n C + \frac{n-1}{a-1} \cdot C = O(n)$$

$$h/T(n) = \begin{cases} O(1), & n=1 \\ \text{at}([n/a]) + O(h), & n \geq 2, a \geq 2 \end{cases}$$

$$m = \log_a n$$

$$\begin{aligned} T(n) &\leq a T(a^{m-1}) + a^m C \leq a^2 T(a^{m-2}) + a^{m-1} C + \\ &+ a^m C \leq a^2 T(a^{m-2}) + a^m C + a^m C \leq a^m T(a^{m-m}) + \\ &+ m a^m C = C(\log_a n \cdot n) = O(n \log n) \end{aligned}$$