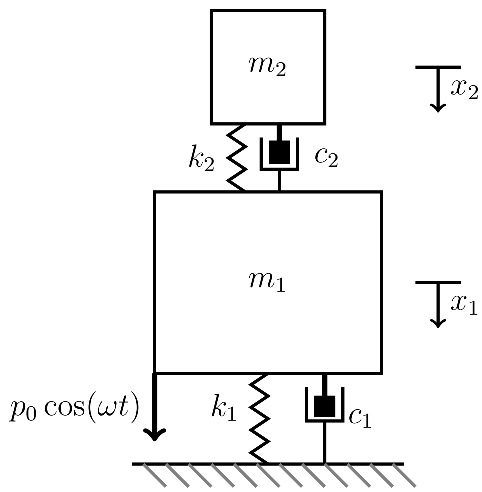
Tuned Mass Dampers

A tuned mass damper is a system for damping the amplitude in one oscillator by coupling it to a second oscillator. If tuned properly the maximum amplitude of the first oscillator in response to a periodic driver will be lowered and much of the vibration will be 'transferred' to the second oscillator.

This is used, for example in tall buildings to limit the swaying of the building in the wind. People are sensitive to this swaying, so by adding a tuned mass damper the building sways less and the damper, which no one can feel, vibrates instead.

I found a nice powerpoint presentation at
http://www14.informatik.tu-muenchen.de/konferenzen/Jass06/courses/4/Stroscher/Stroscher.ppt.



In the figure, the spring system m_1, k_1, c_1 is the oscillator to be damped (say a building) and m_2, k_2, c_2 is the damping oscillator. (say a reasonably large mass attached to the building).

Note: x_2 is the absolute position of m_2. This is often replaced by the relative position of m_2 with respect to m_1, i.e., with what we would call $x_2 - x_1$.

Assuming that the damping force is proportional to velocity and there is a periodic force $p_0 \cos(\omega t)$ on m_1 it is easy to work out the differential equations governing the motion of the system.

We simplify slightly by letting $c_1 = 0$ and get the following equations. (x_1' is the time derivative of x_1.)

$$\begin{array}{lllllllll} m_1 x_1'' & + & k_1 x_1 & + & k_2(x_1 - x_2) & + & c_2(x_1' - x_2') & = & p_0 \cos(\omega t) \\ m_2 x_2'' & + & & & k_2(x_2 - x_1) & + & c_2(x_2' - x_1') & = & 0 \end{array}$$

Our goal is to grind through the calculations and find explicit expressions for the periodic solutions to these equations. The computation is similar to what we'd see if we converted to a 4-by-4 system of first order equations, complexified and looked for a periodic solution of the form $e^{i\omega t}\overrightarrow{\mathbf{K}}$ for some 4-vector $\overrightarrow{\mathbf{K}}$.

Replacing the first equation by the sum of the two equations gives

$$\begin{aligned} m_1 x_1'' + k_1 x_1 + m_2 x_2'' &= p_0 \cos(\omega t) \\ m_2 x_2'' + k_2(x_2 - x_1) + c_2(x_2' - x_1') &= 0 \end{aligned}$$

Now, we find the periodic solution in the form

$$\begin{aligned} x_1 &= a\cos(\omega t) + b\sin(\omega t) \\ x_2 &= c\cos(\omega t) + d\sin(\omega t) \end{aligned}$$

Substituting these into the differential equations gives the following algebraic system of equations.

$$\begin{pmatrix} k_1 - m_1\omega^2 & 0 & -m_2\omega^2 & 0 \\ 0 & k_1 - m_1\omega^2 & 0 & -m_2\omega^2 \\ -k_2 & -c_2\omega & k_2 - m_2 w^2 & c_2\omega \\ c_2\omega & -k_2 & -c_2\omega & k_2 - m_2\omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Call the coefficient matrix M. We can write and invert M in block form.
Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where
$A = r_1 I, \quad B = r_2 I, \quad C = r_3 I - s_1 W, \quad D = r_4 I + s_1 W,$
$r_1 = k_1 - m_1\omega^2, \quad r_2 = -m_2\omega^2, \quad r_3 = -k_2, \quad r_4 = k_2 - m_2\omega^2, \quad s_1 = c_2\omega.$
Since A and B commute with everything in sight we get

$$M^{-1} = \begin{pmatrix} (AD - BC)^{-1} & 0 \\ 0 & (AD - BC)^{-1} \end{pmatrix} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

Now define r and s by, $AD - BC = (r_1 r_4 - r_2 r_3)I + s_1(r_1 + r_2)W = rI + sW$ $\Rightarrow\ (AD - BC)^{-1} = \dfrac{1}{r^2 + s^2}(rI - sW)$.
Putting this together we get

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{p_o}{r^2 + s^2} \begin{pmatrix} r r_4 + s s_1 \\ -r s_1 + s r_4 \\ -r r_3 + s s_1 \\ -r s_1 - s r_3 \end{pmatrix}$$

The amplitude of x_1 is $A_1 = \sqrt{a^2 + b^2}$ and that of x_2 is $A_2 = \sqrt{c^2 + d^2}$. We see that

$$A_1 = \frac{p_0}{r^2 + s^2}(r_4^2 + s_1^2) \quad A_2 = \frac{p_0}{r^2 + s^2}(r_3^2 + s_1^2)$$

We write out A_1^2 and A_2^2 explicitly in terms of the parameters.

$$
\begin{aligned}
A_1^2 &= p_0^2 \frac{c_2^2\omega^2 + (k_2 - m_2\omega^2)^2}{[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2]^2 + c_2^2\omega^2(k_1 - m_1\omega^2 - m_2\omega^2)^2} \\
A_2^2 &= p_0^2 \frac{c_2^2\omega^2 + k_2^2}{[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2]^2 + c_2^2\omega^2(k_1 - m_1\omega^2 - m_2\omega^2)^2}
\end{aligned}
$$

It seems fairly standard to write this in terms of the following constants (the names are from the Stroscher powerpoint)

eigenfrequencies: $\omega_1^2 = \dfrac{k_1}{m_1}, \quad \omega_2^2 = \dfrac{k_2}{m_2}$

mass ratio: $\mu = \dfrac{m_2}{m_1}$

damping ratio: $\xi = \dfrac{c_2}{2m_2\omega_2}$

static deformation: $u_{1,\text{stat}} = \dfrac{p_0}{k_1}$

It is easy enough to write the amplitudes in terms of these parameters. We get

$$\frac{A_1}{u_{1,stat}} = \frac{w_1^2\sqrt{4\xi^2\omega_2^2\omega^2+(\omega_2^2-\omega^2)^2}}{\sqrt{(\omega_1^2-\omega^2)^2(\omega_2^2-\omega^2)^2+\mu^2\omega_2^4\omega^4-2\mu\omega_2^2\omega^2(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)+4\xi^2\omega_2^2\omega^2((\omega_1^2-\omega^2)^2+\mu^2\omega^4-2\mu\omega^2(\omega_1^2-\omega^2))}}$$