Two Dimensional Steady State Conduction

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Initial conditions:

$$\textit{Rectangular Cross Section} \tag{i}$$
$$L = 100\ mm \tag{ii}$$
$$2w = 30\ mm \tag{iii}$$
$$k = 5\ W/m \cdot K \tag{iv}$$
$$\textit{Right end of fin},\ T_0 = 150\ °C \tag{v}$$
$$T_\infty = 25\ °C \tag{vi}$$
$$h = 500\ W/m^2 \cdot K \tag{vii}$$
$$0 \le x \le L \tag{viii}$$
$$0 \le y \le w \tag{ix}$$

Known boundary conditions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$
$$T(L, y) = T_0$$
$$\frac{\partial T}{\partial y}\Big|_{y=0} = 0 \tag{2}$$
$$k\frac{\partial T}{\partial x}\Big|_{x=0} = h \cdot [T(0, y) - T_\infty] \tag{3}$$
$$-k\frac{\partial T}{\partial y}\Big|_{y=w} = h \cdot [T(x, w) - T_\infty] \tag{4}$$

By applying the change of variables to *Equation 5*, *Equations 6* through *9* are obtained.

$$\theta(x, y) = \frac{T - T_\infty}{T_0 - T_\infty} \tag{5}$$
$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \tag{6}$$
$$\theta(L, y) = 1$$
$$\frac{\partial \theta}{\partial y}\Big|_{y=0} = 0 \tag{7}$$
$$k\frac{\partial \theta}{\partial x}\Big|_{x=0} = h \cdot \theta(0, y) \tag{8}$$
$$-k\frac{\partial \theta}{\partial y}\Big|_{y=w} = h \cdot \theta(x, w) \tag{9}$$

Solution

By setting $\theta(x, y) = f(x) \cdot g(y)$, the heat diffusion equation following this separation of variables reduces to *Equation 10,* where λ^2 is a real positive number.

$$\frac{1}{f(x)}\frac{d^2 f}{dx^2} = -\frac{1}{g(y)}\frac{d^2 g}{dy^2} = \lambda^2 \tag{10}$$

1. By explicit substitution show that *Equation 11* is a solution to the heat diffusion equation, *Equation 6.*

$$\theta(x,y) = [A^{'}cosh(\lambda x) + B^{'}sinh(\lambda x)] \cdot [C^{'}cos(\lambda y) + D^{'}sin(\lambda y)] \tag{11}$$

Proof:

$$\frac{\partial^2\theta}{\partial x^2} =$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[A^{'}cosh(\lambda x) + B^{'}sinh(\lambda x)] \cdot [C^{'}cos(\lambda y) + D^{'}sin(\lambda y)]\right]$$
$$= \frac{\partial}{\partial x}\left[[\lambda A^{'}sinh(\lambda x) + \lambda B^{'}cosh(\lambda x)] \cdot [C^{'}cos(\lambda y) + D^{'}sin(\lambda y)]\right]$$
$$= [\lambda^2 A^{'}cosh(\lambda x) + \lambda^2 B^{'}sinh(\lambda x)] \cdot [C^{'}cos(\lambda y) + D^{'}sin(\lambda y)]$$

Similarly,

$$\frac{\partial^2\theta}{\partial y^2} =$$

$$= [A^{'}cosh(\lambda x) + B^{'}sinh(\lambda x)] \cdot [-\lambda^2 C^{'}cos(\lambda y) - \lambda^2 D^{'}sin(\lambda y)]$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} =$$

$$= [\lambda^2 A^{'}cosh(\lambda x) + \lambda^2 B^{'}sinh(\lambda x)] \cdot [C^{'}cos(\lambda y) + D^{'}sin(\lambda y)] +$$
$$[A^{'}cosh(\lambda x) + B^{'}sinh(\lambda x)] \cdot [-\lambda^2 C^{'}cos(\lambda y) - \lambda^2 D^{'}sin(\lambda y)]$$
$$= 0$$

Let, $\quad \text{AB} = A^{'}cosh(\lambda x) + B^{'}sinh(\lambda x)$

$\text{CD} = C^{'}cos(\lambda y) + D^{'}sin(\lambda y)$

Then,

$$\lambda^2 \cdot (ABCD) - \lambda^2 \cdot (ABCD) = 0$$

2. Using the boundary condition at $y = 0$ show that $D^{'} = 0$ and that the solution reduces to *Equation 12*.

$$\theta(x,y) = [Acosh(\lambda x) + Bsinh(\lambda x)] \cdot cos(\lambda y) \tag{12}$$

Proof:

$$\theta(x,y) = [A^{'}cosh(\lambda x) + B^{'}sinh(\lambda x)] \cdot [C^{'}cos(\lambda y) + D^{'}sin(\lambda y)]$$

Using *Equation 7:*

$$\frac{\partial\theta}{\partial y}\Big|_{y=0} = 0$$

$$\frac{\partial}{\partial y}\left[[A'cosh(\lambda x) + B'sinh(\lambda x)] \cdot [C'cos(\lambda y) + D'sin(\lambda y)]\right] = 0$$

$$[A'cosh(\lambda x) + B'sinh(\lambda x)] \cdot [-\lambda C'sin(\lambda y) + \lambda D'cos(\lambda y)] = 0$$

But, $y = 0$

So,

$$[A'cosh(\lambda x) + B'sinh(\lambda x)] \cdot [0 + \lambda D'] = 0$$

Since $0 \le x \le L$,

Then, $D' = 0$

Let, $A = \frac{A'}{C'}$

$B = \frac{B'}{C'}$

Then,

$$\theta(x, y) = [Acosh(\lambda x) + Bsinh(\lambda x)] \cdot cos(\lambda y)$$

3. Using the boundary condition at $y = w$ show that λ can take on a number of values satisfying *Equation 13.*

$$\lambda_n \cdot sin(\lambda_n w) = Bi \cdot cos(\lambda_n w) \tag{13}$$

Where,

$$Bi = \frac{h}{k}$$

Proof:

$$\frac{\partial\theta}{\partial y} = [A'cosh(\lambda x) + B'sinh(\lambda x)] \cdot [-\lambda C'sin(\lambda y) + \lambda D'cos(\lambda y)]$$

Using *Equation 9:*

$$-k\frac{\partial\theta}{\partial y}\Big|_{y=w} = h \cdot \theta(x, w)$$

$$-k \cdot \left[[A'cosh(\lambda x) + B'sinh(\lambda x)] \cdot [-\lambda C'sin(\lambda w) + \lambda D'cos(\lambda w)]\right]$$

$$= h \cdot [Acosh(\lambda x) + Bsinh(\lambda x)] \cdot cos(\lambda w)$$

$$\Rightarrow k \cdot \left[[Acosh(\lambda x) + Bsinh(\lambda x)] \cdot [\lambda sin(\lambda w) + \lambda D'cos(\lambda w)]\right]$$

$$= h \cdot [Acosh(\lambda x) + Bsinh(\lambda x)] \cdot cos(\lambda w)$$

$$\Rightarrow k \cdot [\lambda sin(\lambda w) = hcos(\lambda w)]$$

But,

$Bi = \frac{h}{k}$

So,

$$\lambda_n \cdot sin(\lambda_n w) = Bi \cdot cos(\lambda_n w)$$

As a result of the last question, the general solution to the heat diffusion equation, *Equation 6,* is a series expansion of the form:

$$\theta(x, y) = \sum_{n=1}^{\infty} [A_n cosh(\lambda_n x) + B_n sinh(\lambda_n x)]\, cos(\lambda_n y) \tag{14}$$

4. Using the boundary condition at $x = 0$ show that for all n values

$$B_n = \frac{A_n}{\lambda_n} Bi \tag{15}$$

Proof:

Using *Equation 8:*

$$k \frac{\partial\theta}{\partial x}\Big|_{x=0} = h \cdot \theta(0, y)$$

$$\frac{\partial\theta}{\partial x} = [\lambda A' sinh(\lambda x) + \lambda B' cosh(\lambda x)] \cdot [C' cos(\lambda y) + D' sin(\lambda y)]$$

But, $x = 0$

So,

$$k \cdot \big[[\lambda B'] \cdot [C' cos(\lambda y) + D' sin(\lambda y)]\big] = h \cdot A'[C' cos(\lambda y)]$$

But,

$D' = 0$

$A = \frac{A'}{C'}$

$B = \frac{B'}{C'}$

So,

$$B = \frac{h}{k\lambda} \cdot A \frac{cos(\lambda y)}{cos(\lambda y)}$$

For any *n,*

$$B_n = \frac{A_n}{\lambda_n} Bi$$

As a result of the last question, the general solution to the heat diffusion equation, *Equation 6,* is a series expansion of the form

$$\theta(x,y) = \sum_{n=1}^{\infty} A_n cos(\lambda_n y)\left(cosh(\lambda_n x) + \frac{Bi}{\lambda_n} sinh(\lambda_n x)\right) \quad (16)$$

5. Show that if $\lambda_n \neq \lambda_m$ then

$$\int_0^w cos(\lambda_n y) cos(\lambda_m y) dy = 0 \quad (17)$$

Proof:

Using *Equation 13* and trigonometric identities:

$$\lambda_n \cdot sin(\lambda_n w) = Bi \cdot cos(\lambda_n w)$$

$$2cos(\theta)cos(\gamma) = cos(\theta - \gamma) + cos(\theta + \gamma)$$

$$sin(u \pm v) = sin(u)cos(v) \pm cos(u)sin(v)$$

$$cos(\lambda_n y)cos(\lambda_m y) = \tfrac{1}{2} \cdot cos((\lambda_n - \lambda_m)y) + cos((\lambda_n + \lambda_m)y)$$

$$\Rightarrow$$

$$\int_0^w cos(\lambda_n y)cos(\lambda_m y)dy = \tfrac{1}{2}\int_0^w cos((\lambda_n - \lambda_m)y)dy + \tfrac{1}{2}\int_0^w cos((\lambda_n + \lambda_m)y)dy$$

$$= \frac{1}{2(\lambda_n - \lambda_m)}\,[sin((\lambda_n - \lambda_m)w)] + \frac{1}{2(\lambda_n + \lambda_m)}\,[sin((\lambda_n + \lambda_m)w)]$$

$$= \frac{1}{2(\lambda_n - \lambda_m)}\,[sin(\lambda_n w - \lambda_m w)] + \frac{1}{2(\lambda_n + \lambda_m)}\,[sin(\lambda_n w + \lambda_m w)]$$

$$= \frac{\lambda_n + \lambda_m}{2(\lambda_n - \lambda_m)(\lambda_n + \lambda_m)}\,[sin(\lambda_n w - \lambda_m w)] + \frac{\lambda_n - \lambda_m}{2(\lambda_n + \lambda_m)(\lambda_n - \lambda_m)}\,[sin(\lambda_n w + \lambda_m w)]$$

$$= \frac{1}{2(\lambda_n^2 - \lambda_m^2)}[(\lambda_n + \lambda_m) \cdot sin(\lambda_n w - \lambda_m w) + (\lambda_n - \lambda_m) \cdot sin(\lambda_n w + \lambda_m w)]$$

$$= \frac{1}{2(\lambda_n^2 - \lambda_m^2)}\,[(\lambda_n + \lambda_m) \cdot [sin(\lambda_n w)cos(\lambda_m w) - cos(\lambda_n w)sin(\lambda_m w)]$$
$$+ (\lambda_n - \lambda_m) \cdot [sin(\lambda_n w)cos(\lambda_m w) + cos(\lambda_n w)sin(\lambda_m w)]$$

$$= \frac{1}{2(\lambda_n^2 - \lambda_m^2)}[\lambda_n sin(\lambda_n w)cos(\lambda_m w) + \lambda_m sin(\lambda_n w)cos(\lambda_m w) - \lambda_n cos(\lambda_n w)sin(\lambda_m w) -$$
$$\lambda_m cos(\lambda_n w)sin(\lambda_m w) + \lambda_n sin(\lambda_n w)cos(\lambda_m w) + \lambda_n cos(\lambda_n w)sin(\lambda_m w) -$$
$$\lambda_m sin(\lambda_n w)cos(\lambda_m w) - \lambda_m cos(\lambda_n w)sin(\lambda_m w)]$$

$$= \frac{1}{2(\lambda_n^2 - \lambda_m^2)}[2 \cdot \lambda_n sin(\lambda_n w)cos(\lambda_m w) - 2 \cdot \lambda_m cos(\lambda_n w)sin(\lambda_m w)]$$

But,

$$\lambda_n \cdot sin(\lambda_n w) = Bi \cdot cos(\lambda_n w)$$
$$\lambda_m \cdot sin(\lambda_m w) = Bi \cdot cos(\lambda_m w)$$

So,

$$= \frac{1}{(\lambda_n^2 - \lambda_m^2)}[Bi \cdot cos(\lambda_n w)cos(\lambda_m w) - Bi \cdot cos(\lambda_m w)cos(\lambda_n w)]$$
$$= \frac{1}{(\lambda_n^2 - \lambda_m^2)}[Bi \cdot cos(\lambda_n w)cos(\lambda_m w) - Bi \cdot cos(\lambda_n w)cos(\lambda_m w)]$$
$$\int_0^w cos(\lambda_n y)cos(\lambda_m y)dy = 0$$

6. Using the last boundary condition at $x = L$ show that

$$A_n = \frac{-2(-1)^n \cdot Bi\sqrt{\lambda_n^2 + Bi^2}}{[\lambda_n^2 w + Bi \cdot (1 + Bi \cdot w] \cdot [\lambda_n cosh(\lambda_n L) + Bi \cdot sinh(\lambda_n L)]} \tag{18}$$

Proof:

$$\theta(x, y) = \sum_{n=1}^{\infty} A_n cos(\lambda_n y)\left(cosh(\lambda_n x) + \frac{Bi}{\lambda_n} sinh(\lambda_n x)\right)$$

But, $x = L$

$\theta(L, y) = 1$

So,

$$1 = A_n cos(\lambda_n y)\left(cosh(\lambda_n L) + \frac{Bi}{\lambda_n} sinh(\lambda_n L)\right)$$

$$\int_0^w cos(\lambda_n y) = A_n \left[\int_0^w cos^2(\lambda_n y)\right]\left[cosh(\lambda_n L) + \frac{Bi}{\lambda_n} sinh(\lambda_n L)\right]$$

$$A_n = \frac{1}{\left[cosh(\lambda_n L) + \frac{Bi}{\lambda_n} sinh(\lambda_n L)\right]} \cdot \frac{\lambda_n}{\lambda_n} \cdot \left(\frac{\int_0^w cos(\lambda_n y)}{\int_0^w cos^2(\lambda_n y)}\right)$$

$$= \frac{\lambda_n}{[\lambda_n cosh(\lambda_n L) + Bi \cdot sinh(\lambda_n L)]}\left(\frac{\int_0^w cos(\lambda_n y)}{\int_0^w cos^2(\lambda_n y)}\right)$$

Let,

$$k = [\lambda_n cosh(\lambda_n L) + Bi \cdot sinh(\lambda_n L)]$$

Then,

$$= \frac{\lambda_n}{k} \left(\frac{\lambda_n^{-1}[sin(\lambda_n w) - 0]}{\frac{1}{2}\int_0^w [1 + cos(2\lambda_n y)]} \right)$$

$$= \frac{2}{k} \frac{sin(\lambda_n w)}{w + \frac{1}{2\lambda_n} sin(2\lambda_n w)}$$

$$= \frac{2}{k} \frac{sin(\lambda_n w)}{w + \frac{1}{2\cdot\lambda_n} 2\cdot sin(\lambda_n w) cos(\lambda_n w)} \cdot \frac{\lambda_n^2}{\lambda_n^2}$$

From *Equation 13,*

$$\lambda_n \cdot sin(\lambda_n w) = Bi \cdot cos(\lambda_n w)$$

$$= \frac{2\cdot\lambda_n^2 \cdot sin(\lambda_n w)}{k[w\cdot\lambda_n^2 + Bi\cdot cos^2(\lambda_n w)]}$$

$$= \frac{(2\cdot\lambda_n)\cdot(\lambda_n \cdot sin(\lambda_n w)}{k\left[w\cdot\lambda_n^2 + Bi\cdot cos^2(\lambda_n w)\right]}$$

$$= \frac{(2\cdot\lambda_n)\cdot Bi\cdot cos(\lambda_n w)}{k\left[w\cdot\lambda_n^2 + Bi\cdot cos^2(\lambda_n w)\right]}$$

But,

$$\lambda_n \cdot sin(\lambda_n w) = Bi \cdot cos(\lambda_n w)$$

$$\lambda_n \cdot sin^2(\lambda_n w) = Bi \cdot cos(\lambda_n w) \cdot sin(\lambda_n w)$$

$$sin^2(\lambda_n w) = \frac{Bi}{\lambda_n} \cdot cos(\lambda_n w) \cdot sin(\lambda_n w) \qquad \textit{Equation a}$$

Multiply *Equation a* by $cos(\lambda_n w)$

$$\lambda_n \cdot cos(\lambda_n w) \cdot sin(\lambda_n w) = Bi \cdot cos^2(\lambda_n w)$$

$$cos^2(\lambda_n w) = \frac{\lambda_n}{Bi} \cdot cos(\lambda_n w) \cdot sin(\lambda_n w) \qquad \textit{Equation b}$$

Add *Equation a* and *Equation b*

Since, $1 = Cos^2 x + Sin^2 x$

$$1 = \left(\frac{Bi}{\lambda_n} + \frac{\lambda_n}{Bi}\right) \cdot cos(\lambda_n w) \cdot sin(\lambda_n w)$$

$$\frac{Bi\cdot\lambda_n}{Bi^2 + \lambda_n^2} = cos(\lambda_n w) \cdot sin(\lambda_n w)$$

$$\frac{Bi\cdot\lambda_n}{Bi^2+\lambda_n^2} = cos(\lambda_n w)\cdot\frac{Bi}{\lambda_n}\cdot cos(\lambda_n w)$$

$$\frac{\lambda_n^2}{Bi^2+\lambda_n^2} = cos^2(\lambda_n w)$$

So,

$$cos(\lambda_n w) = \frac{\pm\lambda_n}{\sqrt{Bi^2+\lambda_n^2}} \qquad \textit{Equation c}$$

$$A_n = \frac{2\cdot\lambda_n\cdot Bi\left[\frac{\pm\lambda_n}{\sqrt{Bi^2+\lambda_n^2}}\right]}{k\left[w\lambda_n^2+Bi\cdot\left(\frac{\lambda_n}{\sqrt{Bi^2+\lambda_n^2}}\right)^2\right]}$$

$$= \frac{2\cdot\lambda_n\cdot Bi\left[\frac{\pm\lambda_n}{\sqrt{Bi^2+\lambda_n^2}}\right]}{k\left[w\lambda_n^2+Bi\cdot\left(\frac{\lambda_n^2}{\lambda_n^2+Bi^2}\right)\right]}\cdot\frac{\lambda_n^{-1}}{\lambda_n^{-1}}$$

$$= \frac{2\cdot\lambda_n\cdot Bi\left[\frac{\pm1}{\sqrt{Bi^2+\lambda_n^2}}\right]}{k\left[w\cdot\lambda_n+Bi\cdot\left(\frac{\lambda_n}{\lambda_n^2+Bi^2}\right)\right]}\cdot\frac{\lambda_n^2+Bi^2}{\lambda_n^2+Bi^2}$$

But,

$$\frac{\lambda_n^2+Bi^2}{\sqrt{Bi^2+\lambda_n^2}} = \frac{\pm(\lambda_n^2+Bi^2)\cdot\sqrt{Bi^2+\lambda_n^2}}{(\lambda_n^2+Bi^2)}$$

So,

$$A_n = \frac{\pm2\cdot\lambda_n\cdot Bi\left[\frac{\lambda_n^2+Bi^2}{\sqrt{Bi^2+\lambda_n^2}}\right]}{k\left[w\cdot\lambda_n\cdot(\lambda_n^2+Bi^2)+Bi\cdot\lambda_n\right]}$$

$$= \frac{\pm2\cdot Bi\cdot\sqrt{\lambda_n^2+Bi^2}}{k\left[w\cdot\lambda_n^2+w\cdot Bi^2+Bi\right]}$$

But,

$$k = [\lambda_n\cdot cosh(\lambda_n L) + Bi\cdot sinh(\lambda_n L)]$$

So,

$$A_n = \frac{\pm2\cdot Bi\cdot\sqrt{\lambda_n^2+Bi^2}}{\left[w\cdot\lambda_n^2+Bi(1+wBi)\right]\cdot[\lambda_n\cdot cosh(\lambda_n L)+Bi\cdot sinh(\lambda_n L)]}$$

At $n=1$ we get a positive value of A_n.
At $n=2$ we get a negative value of A_n.
This trend continues infinitely.