Image Processing in the Frequency Domain

Pengolahan Citra

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Laksmita Rahadianti, Aniati Murni

Fakultas Ilmu Komputer Universitas Indonesia

Basic Concepts

Richard G. Lyons (2011). *Understanding Digital Signal Processing, 3rd edition.* Prentice Hall.

Allen B. Downey (2016). Think DSP: Digital Signal Processing in Python. O'Reilly Media.

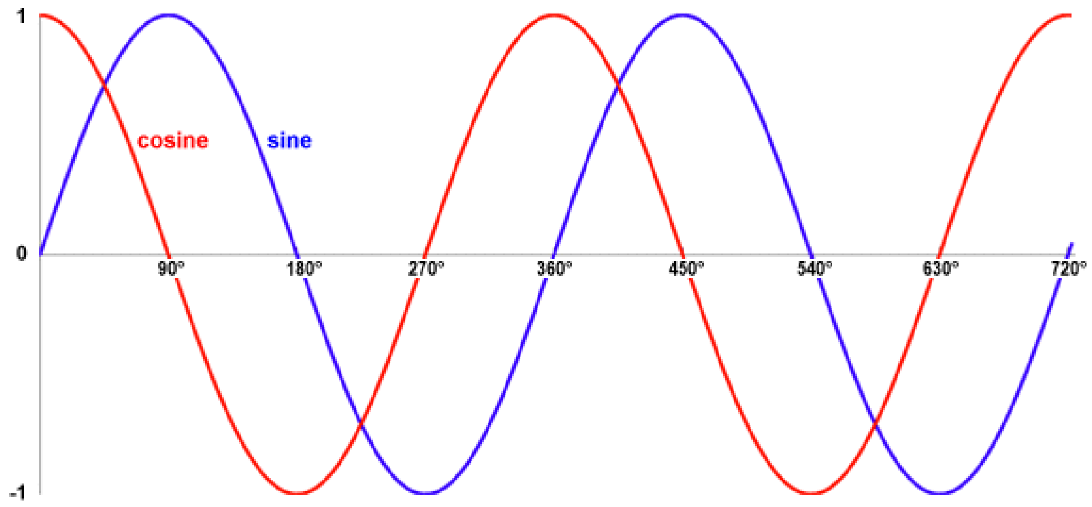
History

- Jean Baptiste Joseph Fourier (1768-1830)
  - Ahli matematika dan fisika dari Perancis
- Memulai penelitian mengenai *fourier series*
  - Dikembangkan menjadi *fourier transform* dan *fourier law*
- "*Théorie analytique de la chaleur* (*The Analytical Theory of Heat*)"
  - Fourier series untuk transfer kalor dan getaran
  - Greenhouse effect

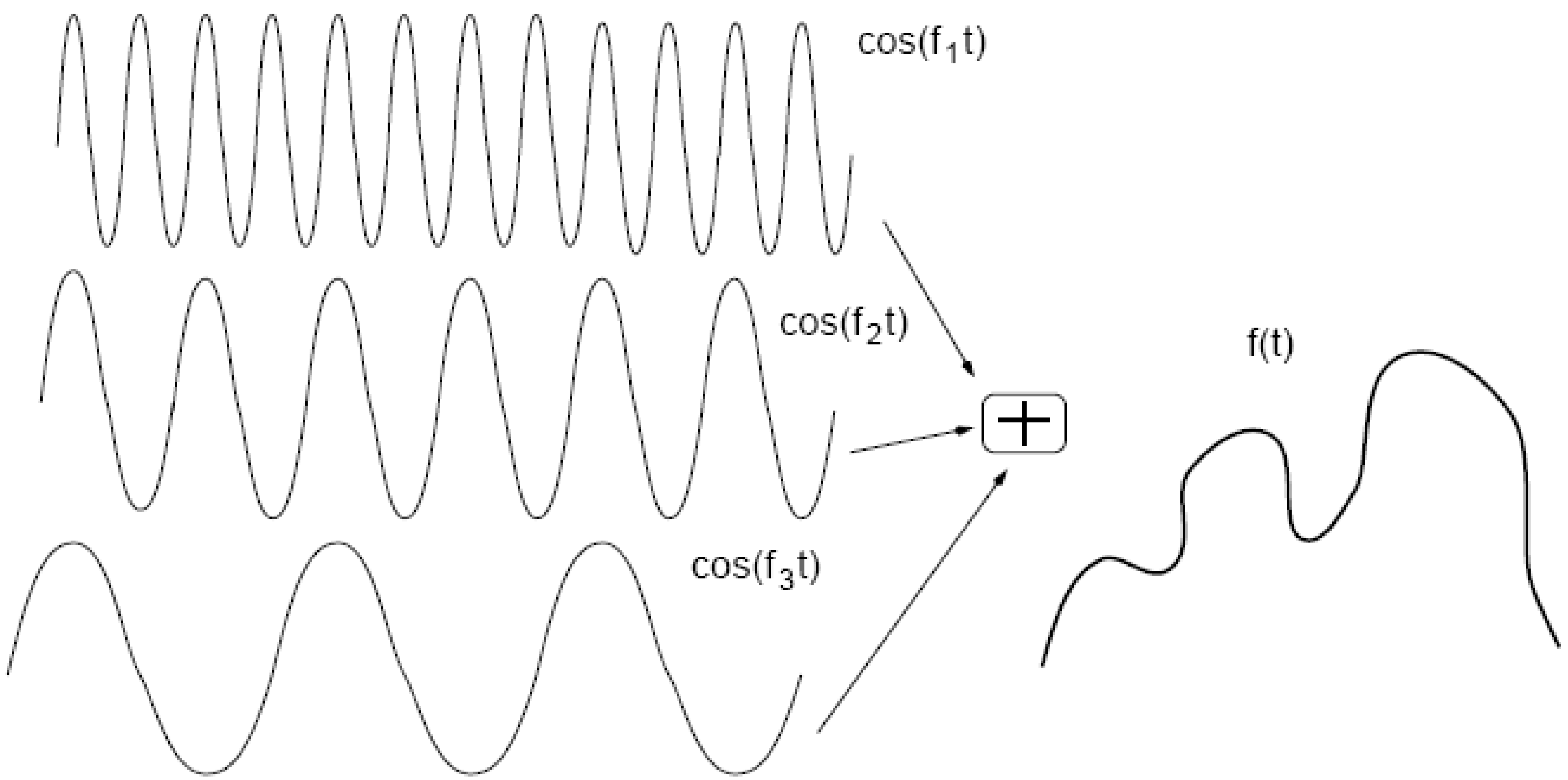
Sekilas tentang Seri Fourier

"Fungsi periodik manapun dapat dinyatakan sebagai jumlahan dari fungsi-fungsi sinus dan/atau kosinus.

- Fungsi Periodik:
  - Fungsi yang berulang pada interval tertentu.
- Fungsi sinus/kosinus



Visualisasi Seri Fourier



Seri Fourier

- Untuk suatu fungsi $f(t)$ pada variabel kontinyu t yang periodik dengan periode T.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

- c_n adalah koefisien pemberat untuk setiap fungsi sinus / cosinus, di mana

$$c_n = \frac{1}{\mathrm{T}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-j\frac{2\pi n}{T}t} dt, \qquad for\ n = 0, \pm 1, \pm 2, \ldots$$

The Fourier Transform

- Fungsi dapat dinyatakan sebagai integral dari fungsi sin dan/atau cosinus yang dikalikan dengan suatu fungsi pemberat (*weighting function*).
  - Tidak harus periodikpun, asalkan area di bawah kurvanya berhingga.

Fourier Transform gives us these weights.

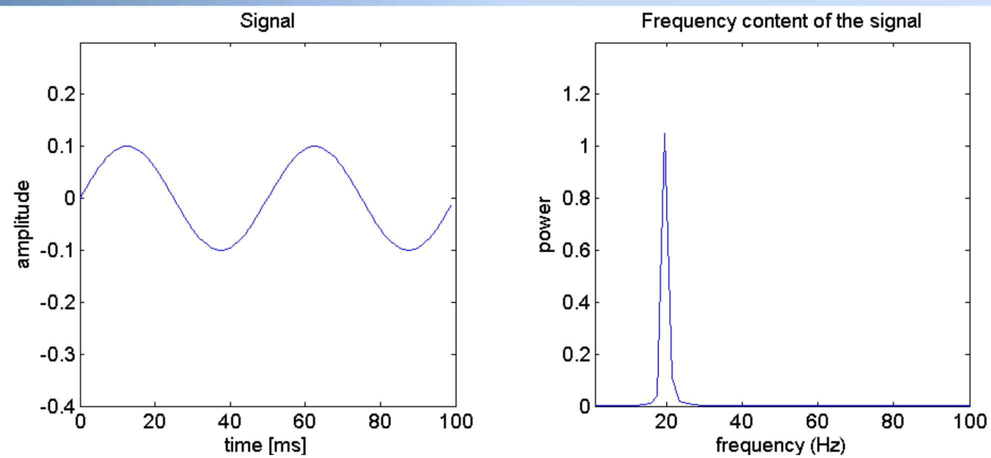
- Poin penting!
  - Seri dan transformasi fourier dapat dikembalikan ke bentuk asal menggunakan suatu proses inverse

Fourier Transform

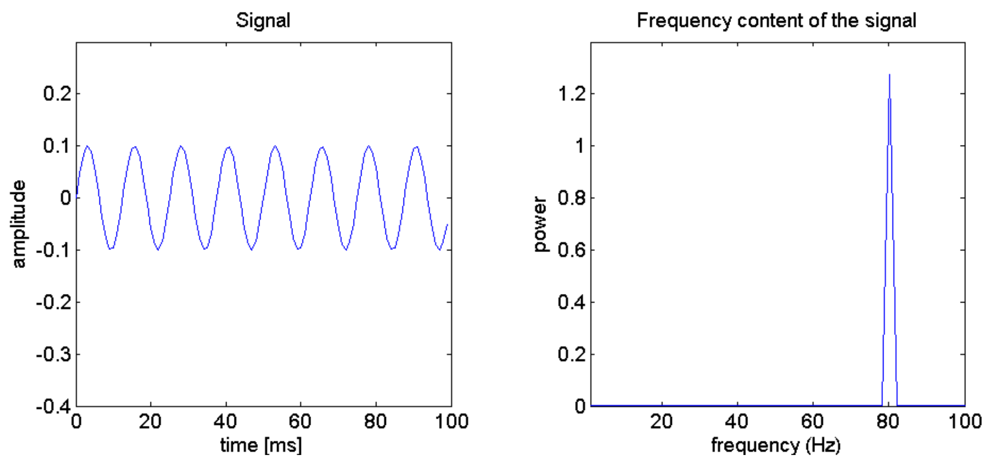
- Another way to look at it:

  Fourier Transform transforms signal from Time domain to Frequency domain.

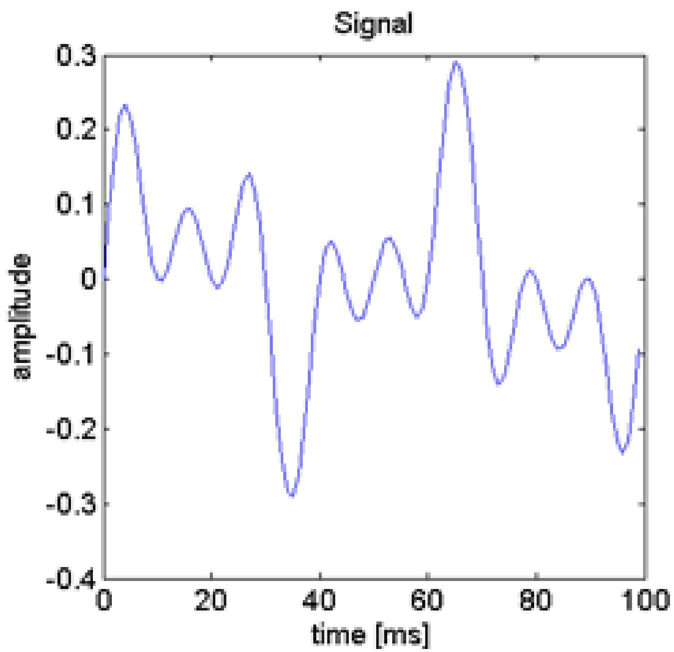
Time to Frequency Domain



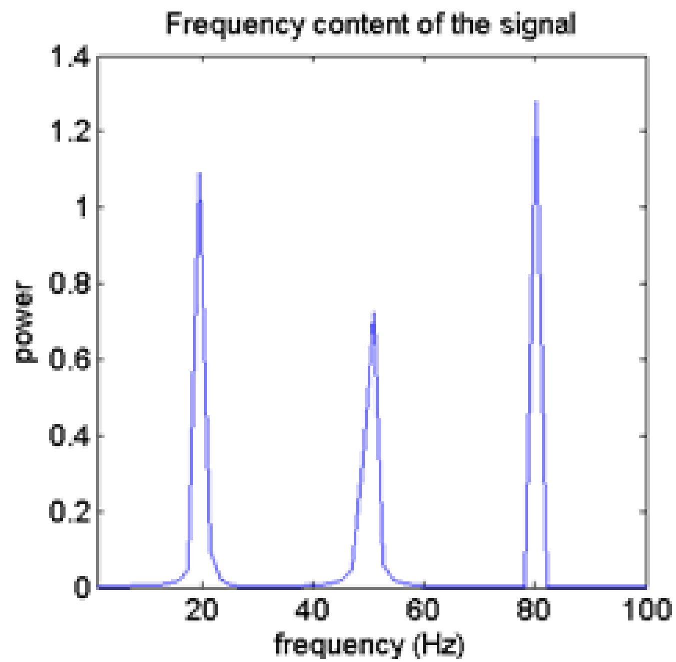
time domain   frequency domain



Time to Frequency Domain (2)



time domain



frequency domain

Fungsi Impulse

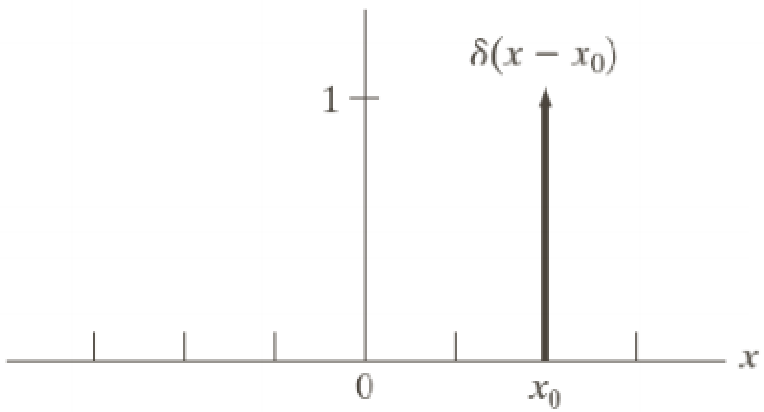
- The Dirac delta function, or $\delta(t)$ function, is zero for all values of t except when the t is equal to zero, and its integral over the parameter from $-\infty$ to ∞ is equal to one

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Fungsi Impulse (2)

- Generalized, we can define any discrete impulse where δ is zero everywhere except at $x = x_0$



Complex Numbers

$$C = R + jI$$

- Where R and I are real numbers, j is an imaginary number equal to the square of -1.
    - R denotes the *real part* of the complex number
    - I denotes the *imaginary part* of the complex number
- Can be plotted geometrically on a complex plane as a point (R, I)
    - The x-axis or the *real axis* for all the real values of the complex number
    - The y-axis or the *imaginary axis* for all the imaginary values of the complex number

The Derivation of Fourier Transform

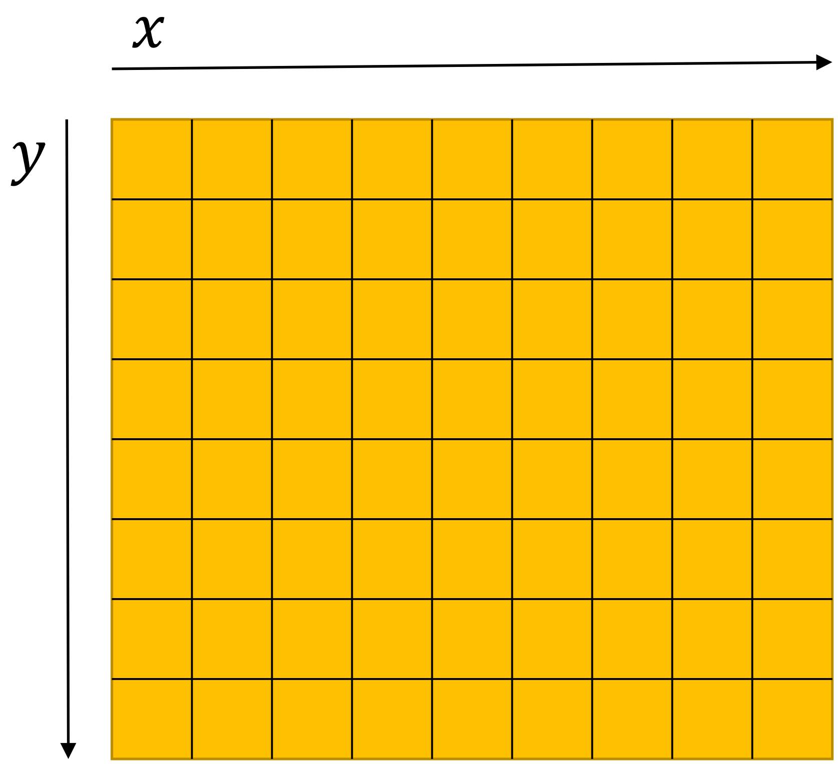
- Gonzales & Woods, Digital Image Processing, section 4.2-4.5.
  - Please read if you are interested
  - Not covered in this class
- We are more interested how we can use it for image processing.

Fourier Transform

Rahat Khan (2010). CIMET Course Slides. Laboratory Hurbert Curien, Universite Jean Monnet Saint Etienne.

Fourier Transform on Digital Images

- Discrete Values
- We need discrete fourier transform



1D Discrete Fourier Transform

Forward DFT $$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{j\frac{2\pi ux}{N}}, \qquad u = 0,1, \ldots . N-1$$

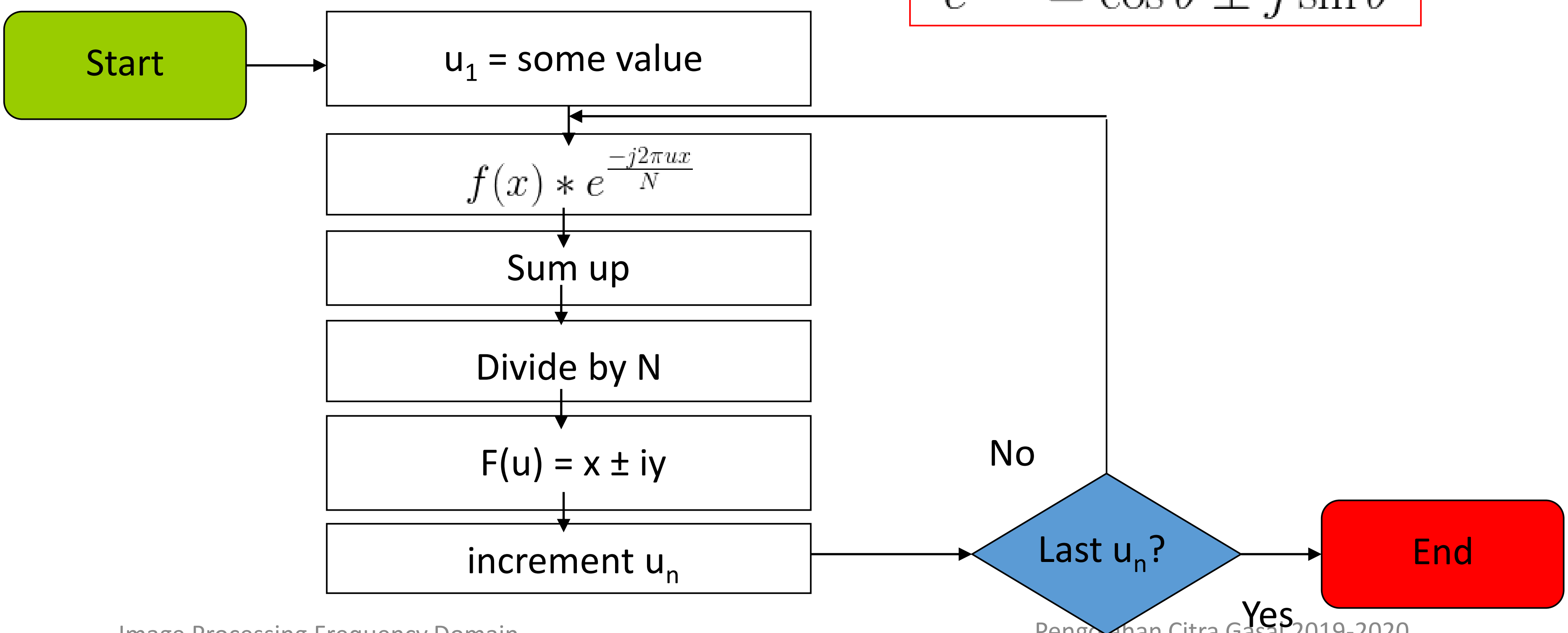
Inverse DFT $$f(x) = \sum_{u=0}^{N-1} F(u)e^{j\frac{2\pi ux}{N}}, \qquad x = 0,1, \ldots . N-1$$

1D Discrete Fourier Transform Process

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-j2\pi ux}{N}}$$

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\cos\frac{2\pi ux}{N} - \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, j\sin\frac{2\pi ux}{N}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$



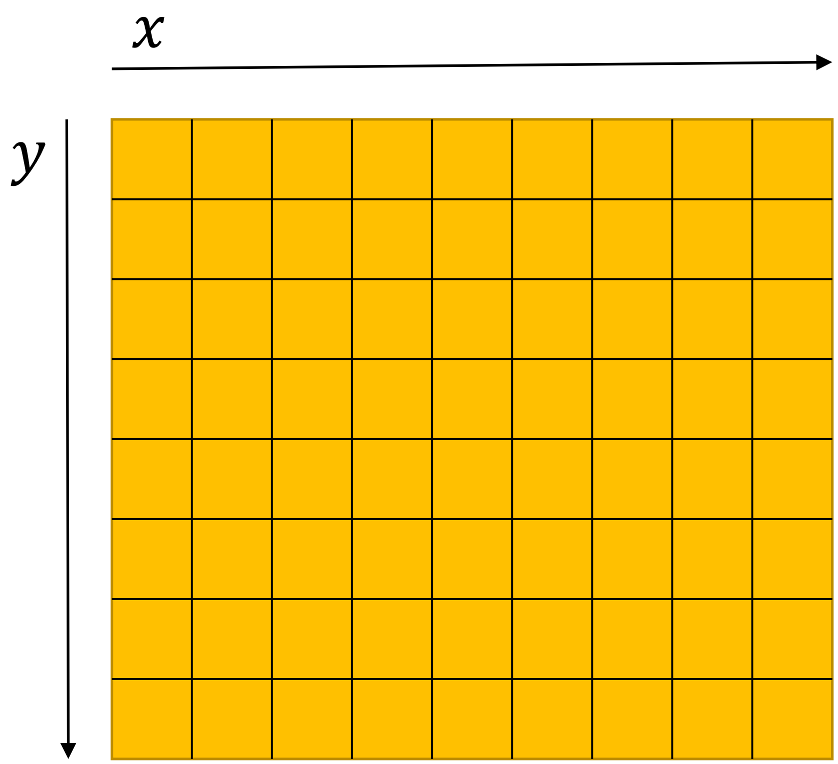
Properties of the Fourier Transform

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\cos\frac{2\pi ux}{N} - \frac{1}{N}\sum_{x=0}^{N-1} f(x) j \sin\frac{2\pi ux}{N}$$

- **F(u)** is a complex function: $F(u) = R(u) + jI(u)$
- Magnitude of FT (spectrum): $|F(u)| = \sqrt{R^2(u) + I^2(u)}$
- Phase of FT: $\phi(F(u)) = \tan^{-1}(\frac{I(u)}{R(u)})$
- Power of $f(x)$: **P(u)**=|F(u)|²=$R^2(u) + I^2(u)$

Fourier Transform on Digital Images

- Images have 2 dimensions, x, and y.
- We must apply the 1D fourier transform in both directions



2D Discrete Fourier Transform (DFT)

- Assume that f(x,y) is M x N image.

Forward DFT

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$u = 0,1, \dots . M-1, v = 0,1,2, \dots . , N-1$$

Inverse DFT

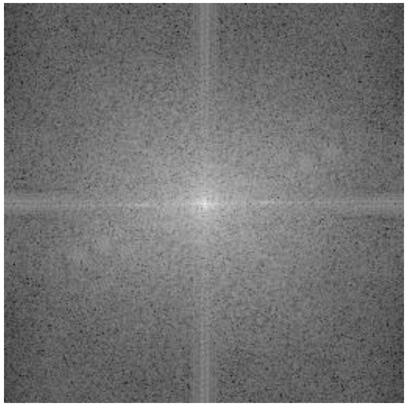
$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$x = 0,1, \dots . M-1, y = 0,1,2, \dots . , N-1$$

2D DFT (2)

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$f\ (x,y)$



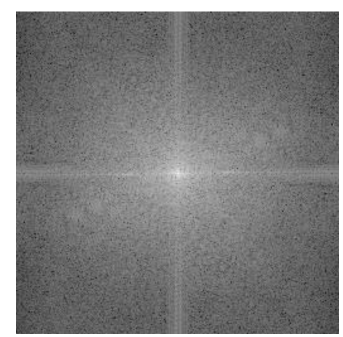
$|F\ (u,v)|$

Identical in Size!

2D DFT (3)

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \longrightarrow \underbrace{\text{Imaginary},\ \text{Real}}_{\text{complex valued matrix}}$$

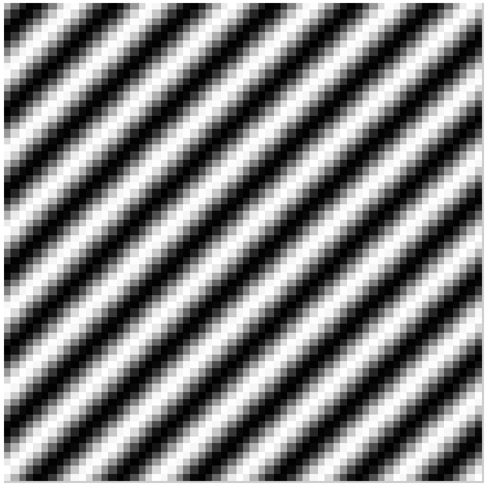
$$\sqrt{\text{Real}^2 + \text{Imaginary}^2} =$$



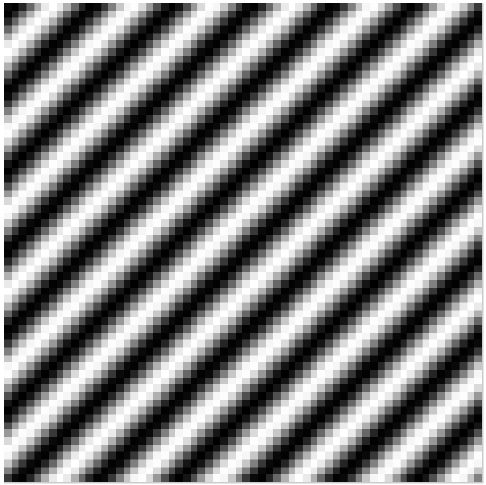
Real and Imaginary Results of DFT

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

M=N=64 (image size), u=6 AND v=6 (index)



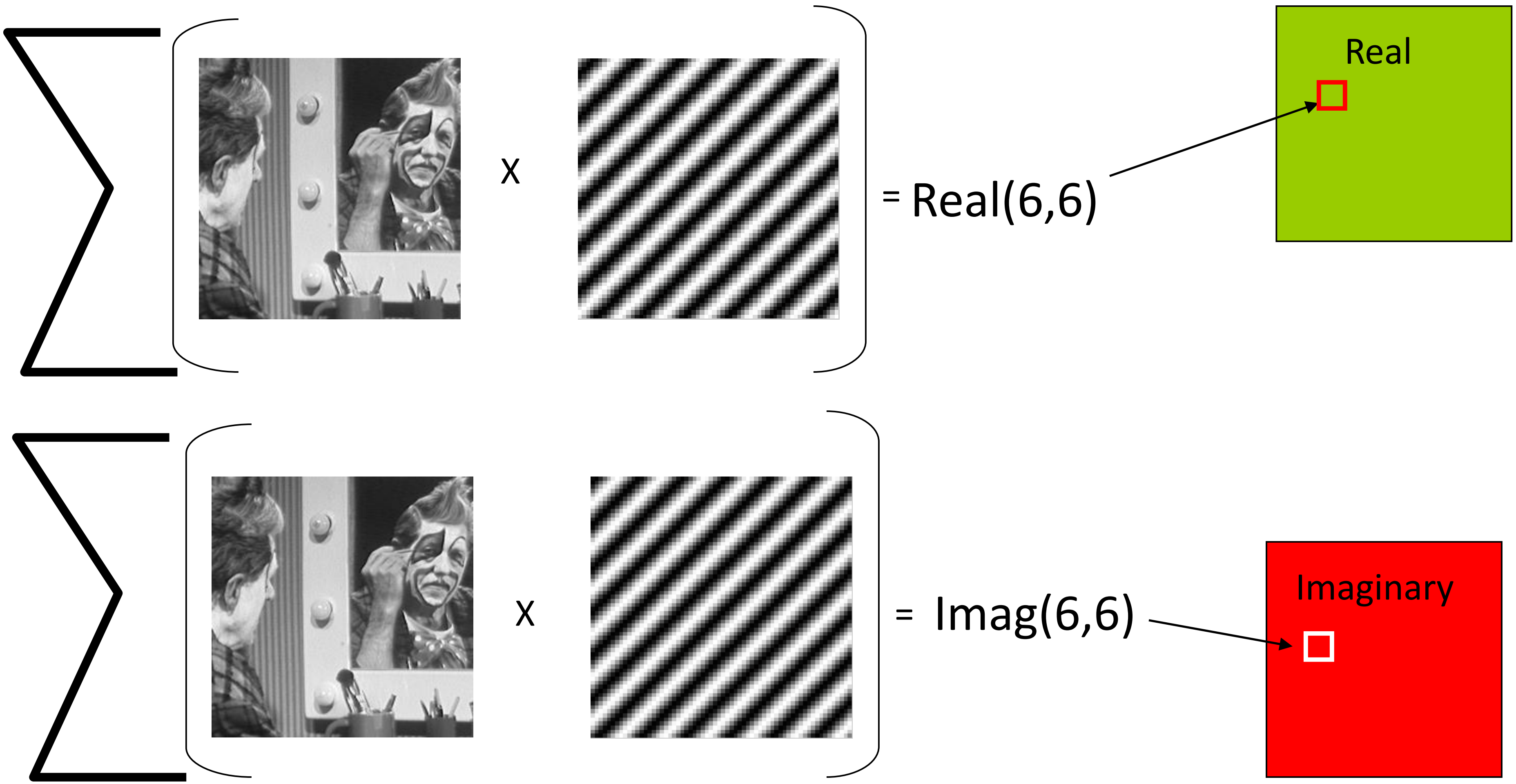
Real Part



Imaginary Part

These are known as Fourier Basis functions

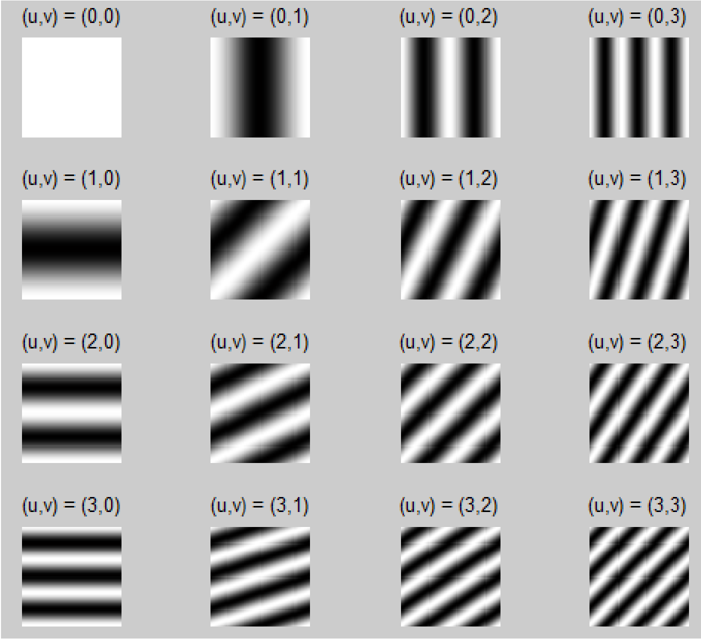
Fourier Basis Functions

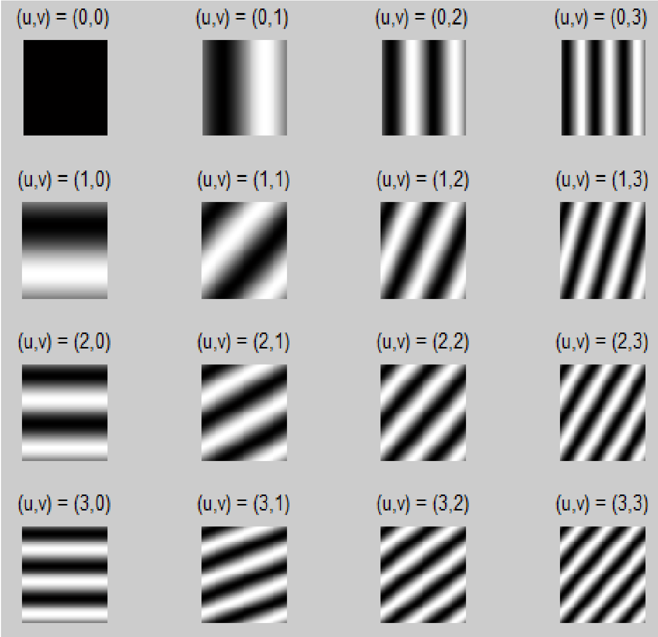


Fourier Basis Functions (2)

- In this example, we used a 64 X 64 image.
- The Fourier transform result has the same size.
- Thus, we have 64 X 64 = 4096 complex basis functions for this size of images.

Real Part

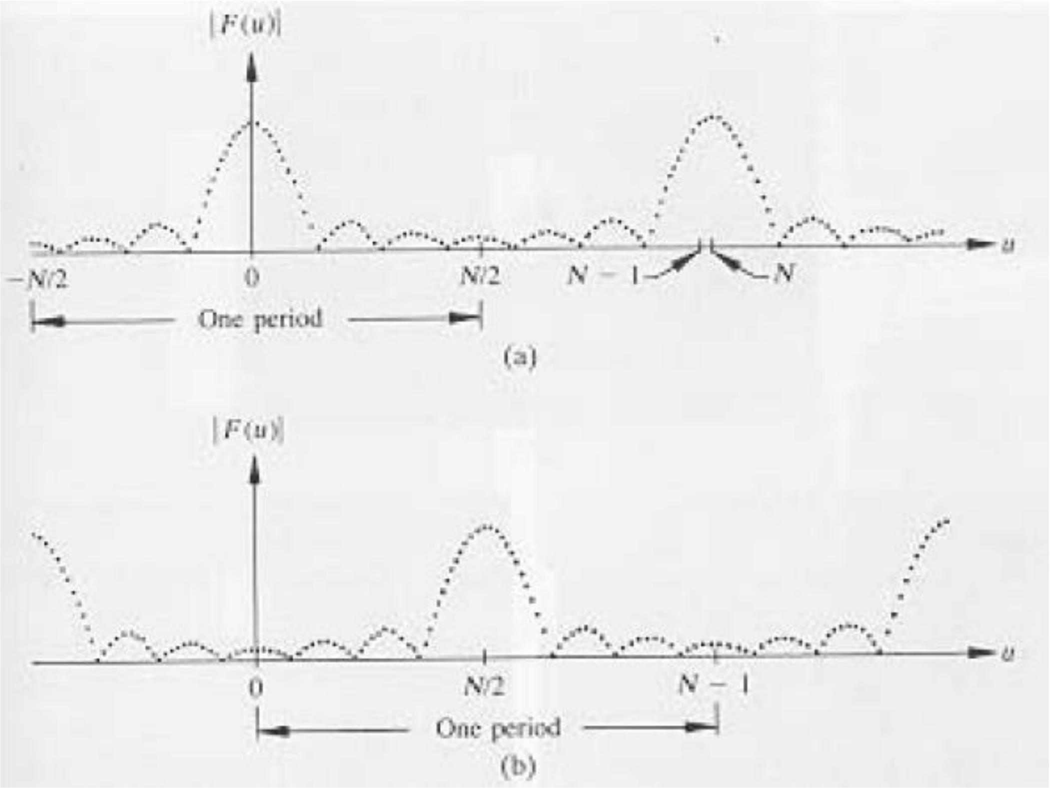




Imaginary Part

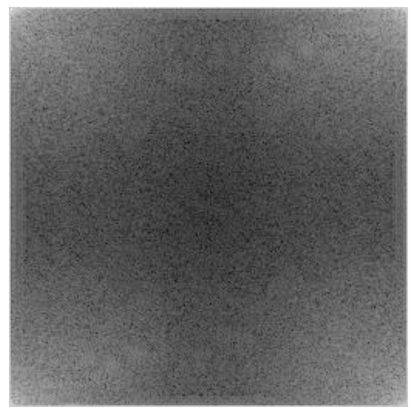
Visualizing 2D DFT – Shifting

- A Fourier Transform is well visualized with the Zero Frequency component in the center N/2 (N/2,N/2 in case of 2D)

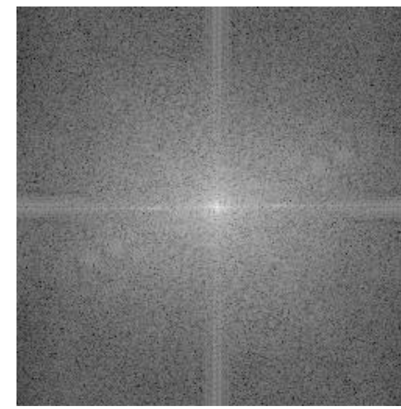


Visualizing 2D DFT – Shifting (2)

original image



before shifting



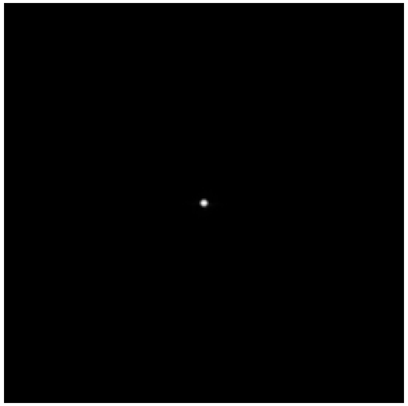
after shifting

Visualizing 2D DFT - Scaling

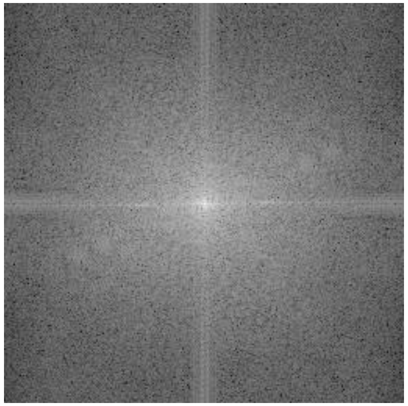
- The Dynamic range of Fourier spectra usually is much higher than the typical display device is able to reproduce faithfully. Therefore, we often use the logarithm function to perform the appropriate compression of the range.

$$D(u,v) = c\log(1+|F(u,v)|)$$

original image



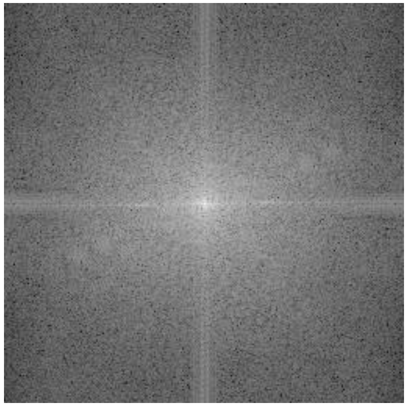
before scaling



after scaling

Visualizing 2D DFT

original image

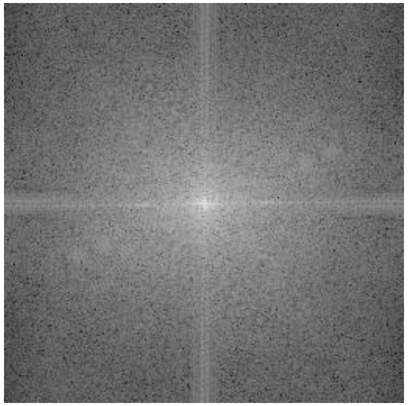


after scaling

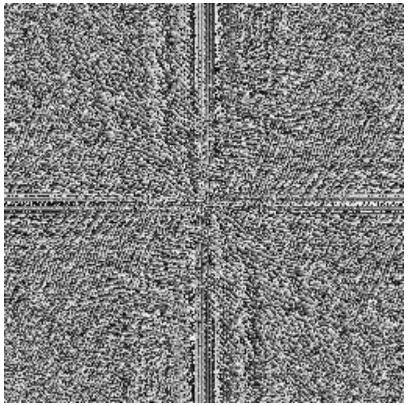
```
F = fft2( f );
Imshow  ( log (abs ( fftshift ( F ) ) + 1),  [ ] )
```

Visualizing 2D DFT (3)

f (x, y)



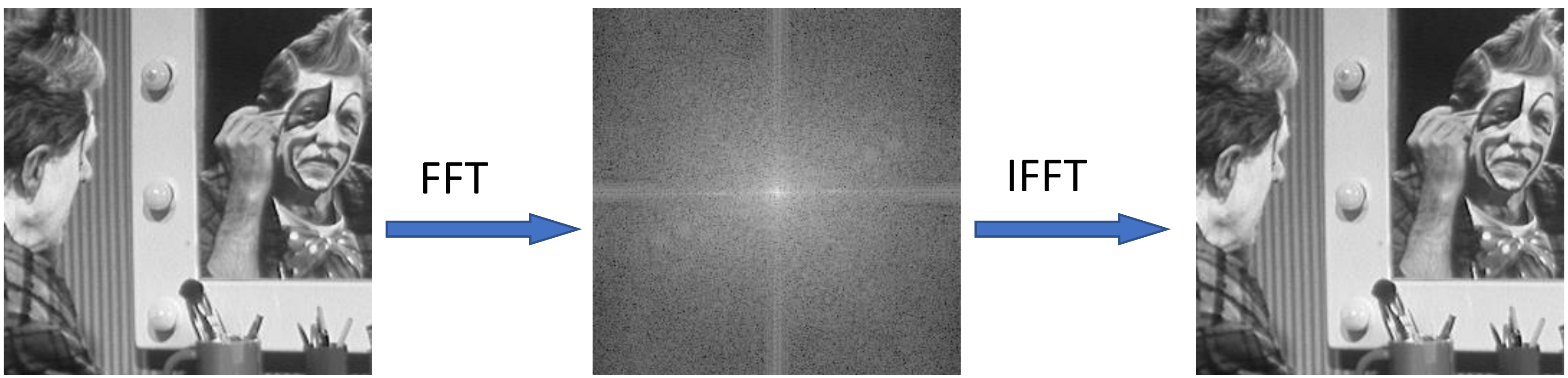
Magnitude
|F (u ,v)|



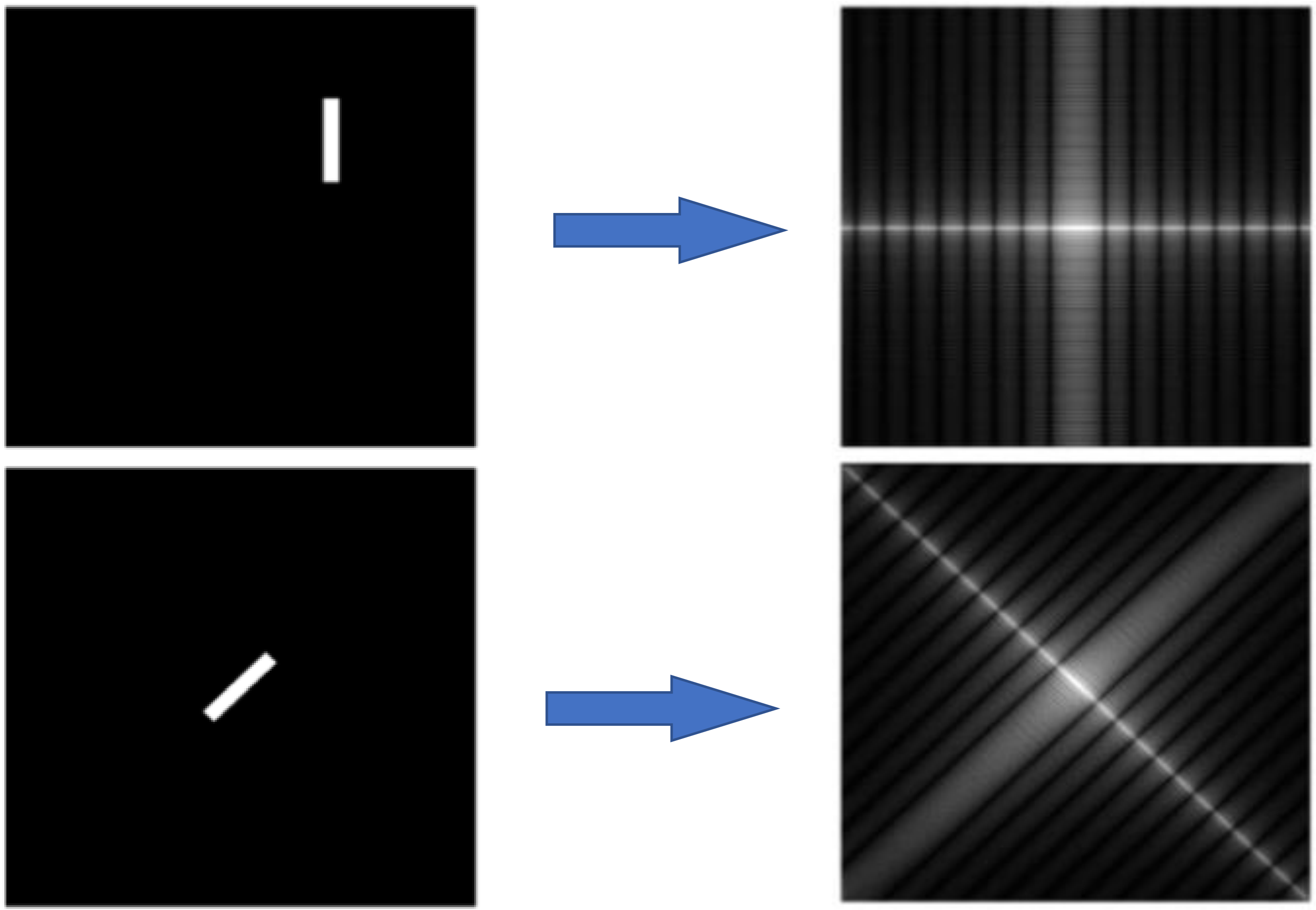
Phase
Φ (F (u, v))

Inverse Fourier Transform

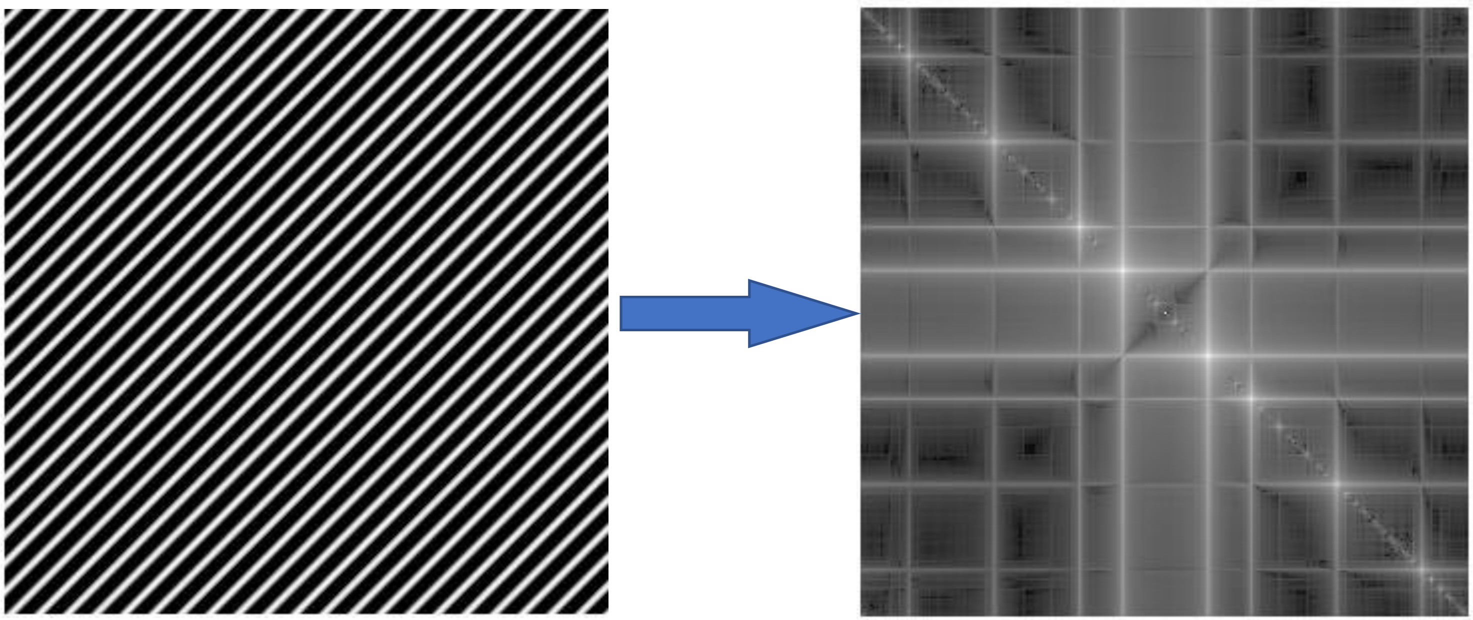
- Seri dan transformasi fourier dapat dikembalikan ke bentuk asal menggunakan suatu proses inverse
- Memudahkan untuk pindah domain, mengerjakan frequency processing, lalu kembali ke asal



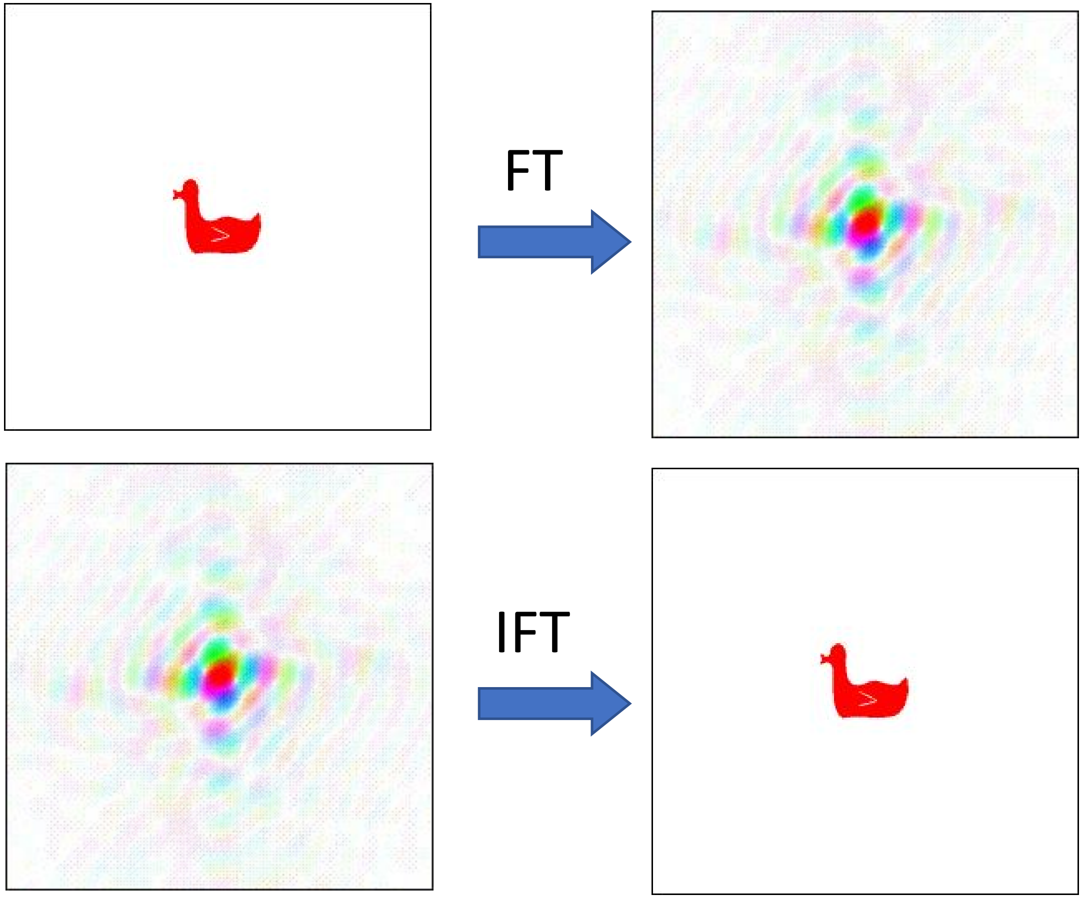
Fourier Transform of Basic Shapes

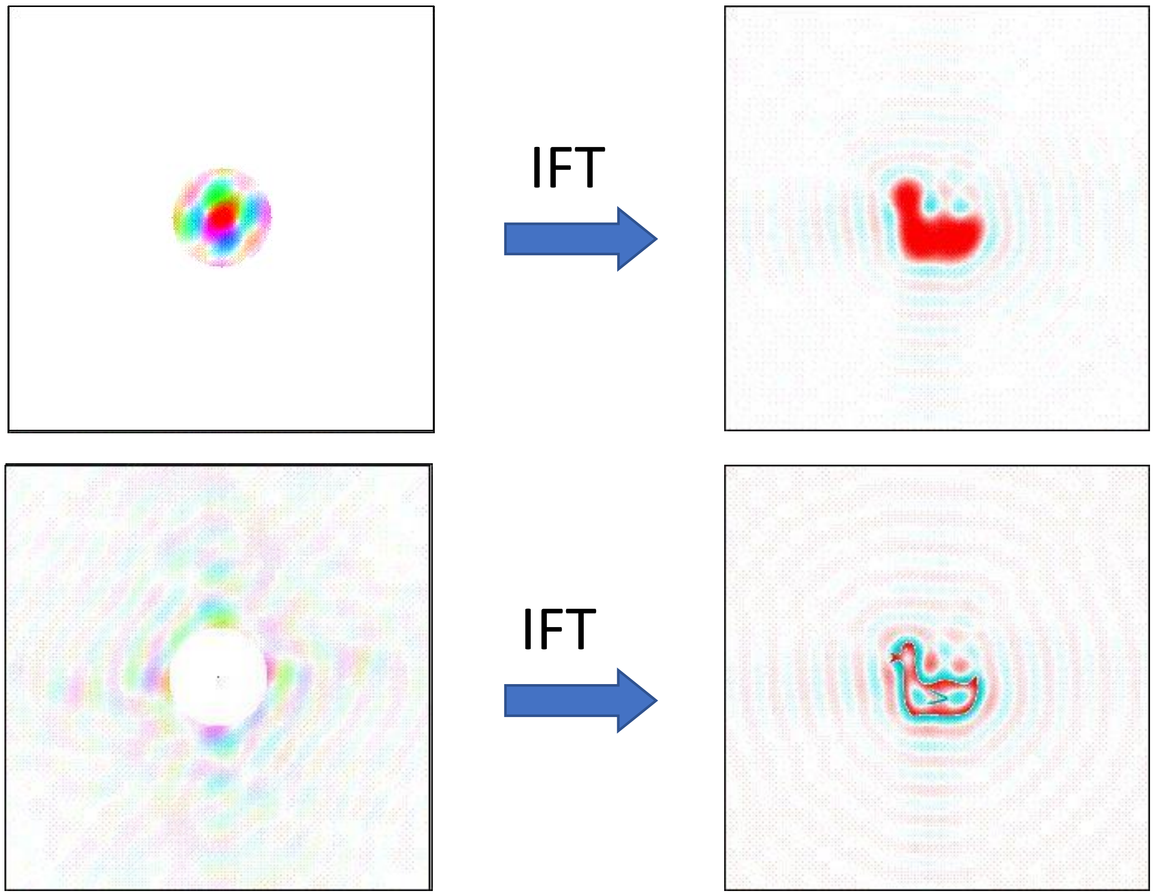


Fourier Transform of an Image with a Specific Frequency

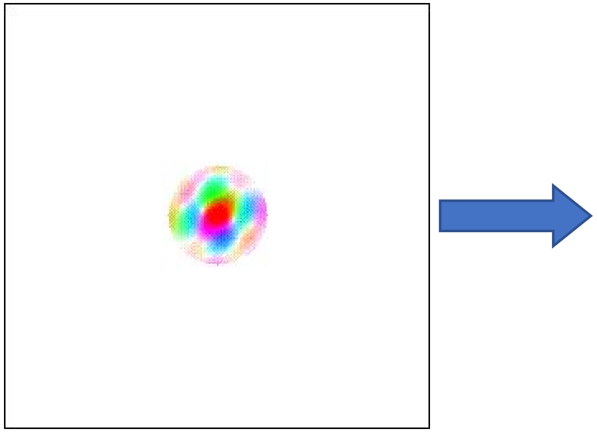


Some Applications

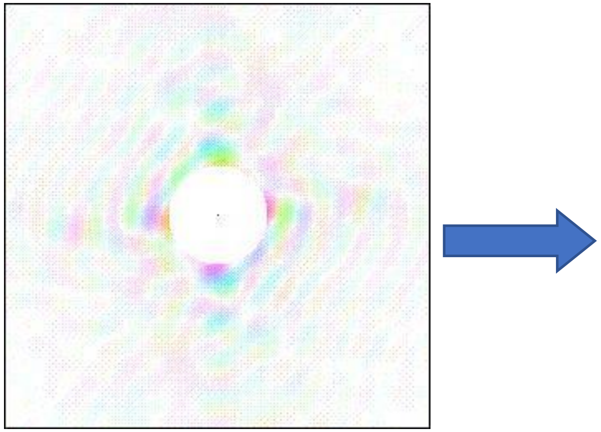



IFT
IFT

Different Frequencies in the FT



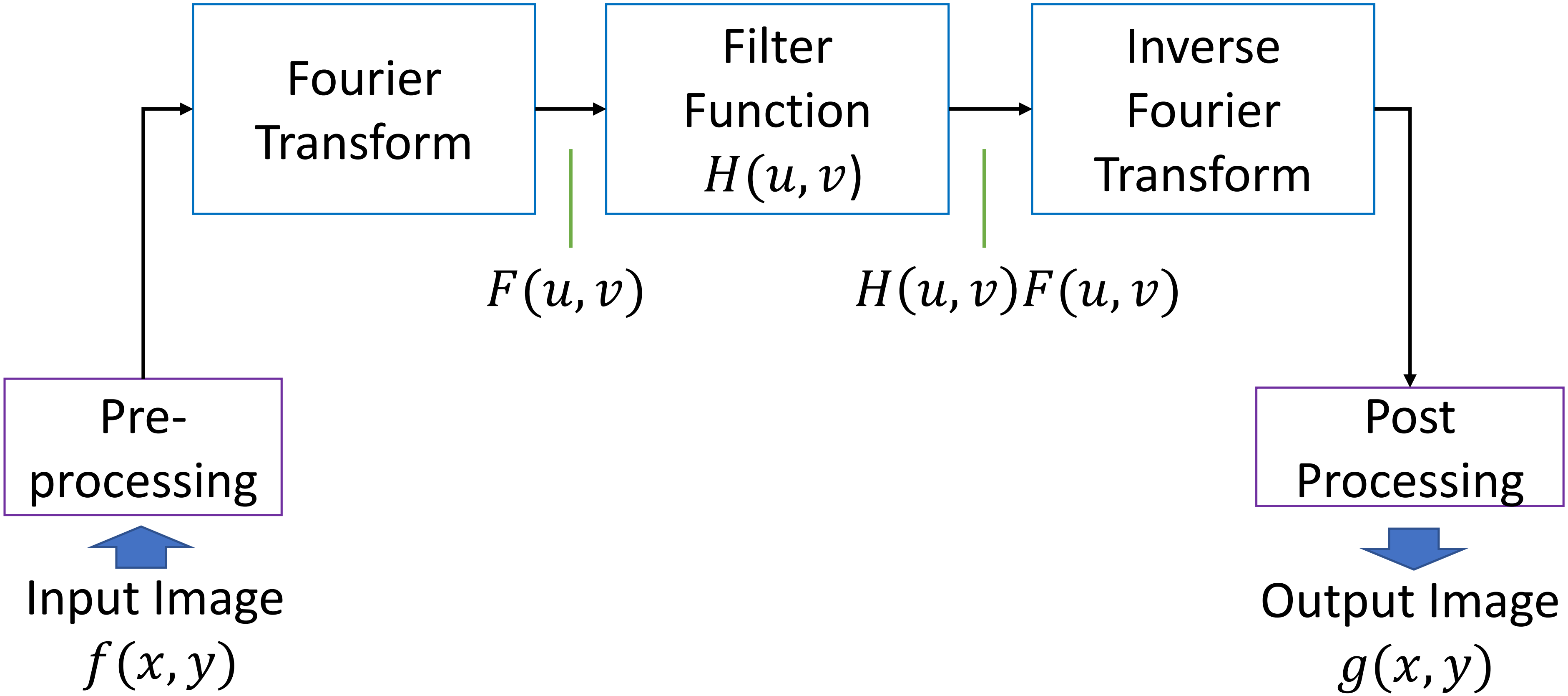
The central part of FT, i.e. the low frequency components are responsible for the general gray-level appearance of an image.



The high frequency components of FT are responsible for the detail information of an image.

Filtering in the Frequency Domain

Basic Steps for Filtering in the Frequency Domain



Filtering in the Frequency Domain

- Filtering is done in the frequency domain

$$G(u, v) = H(u, v)F(u, v)$$

- Band Pass Filters
  - Low Pass
  - High Pass
- Band Reject Filters
- Other

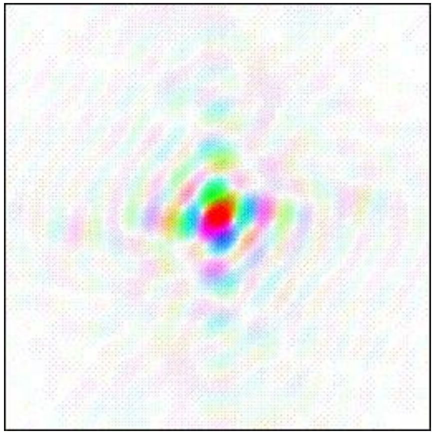
Low Pass Filters

- Yang diambil adalah frekuensi-frekuensi rendah yang ada pada citra.

Smoothing (low pass filter): informasi komponen frekuensi tinggi hilang.

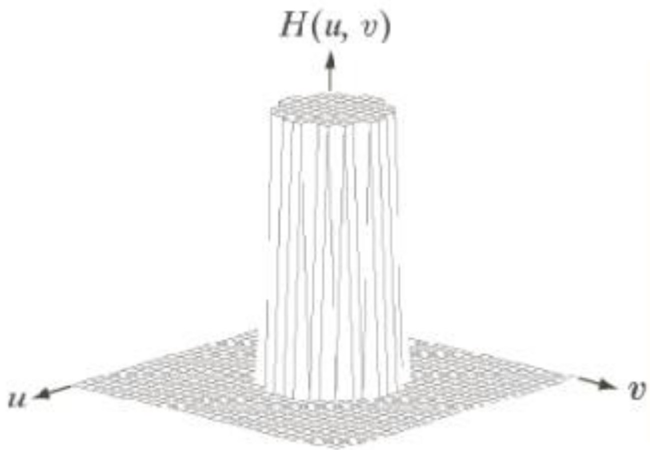
- Smoothing terjadi karena noise bersifat high frequency. Dengan *Low Pass Filter* kita bisa membuang info high frequency tersebut.
- *But!* Bisa kehilangan informasi detail pada citra.

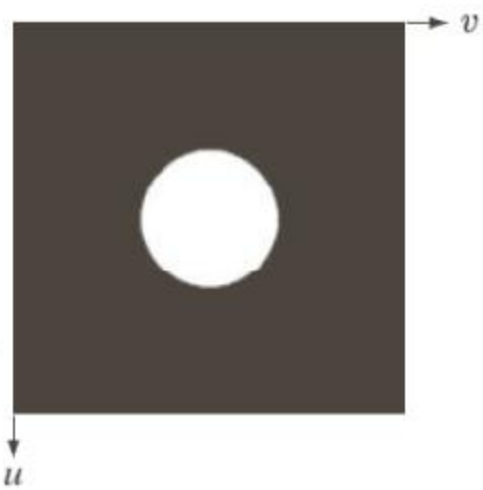
Low Pass Filters (2)

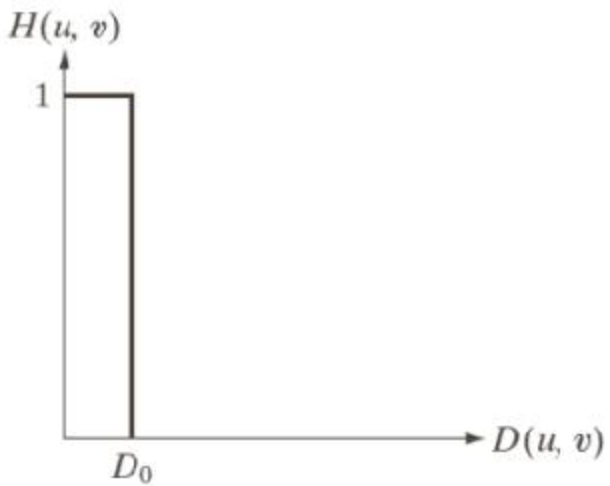


- $H(u,v) = 1$ if $D(u,v) \leq D_0$
  $= 0$ if $D(u,v) > D_0$
- D_0 adalah nilai ambang (cutoff frequency locus, nilainya > 0)
- $D(u,v)$ adalah jarak (u,v) terhadap titik origin. $D(u,v) = (u^2+v^2)^{1/2}$

Ideal Low Pass Filter





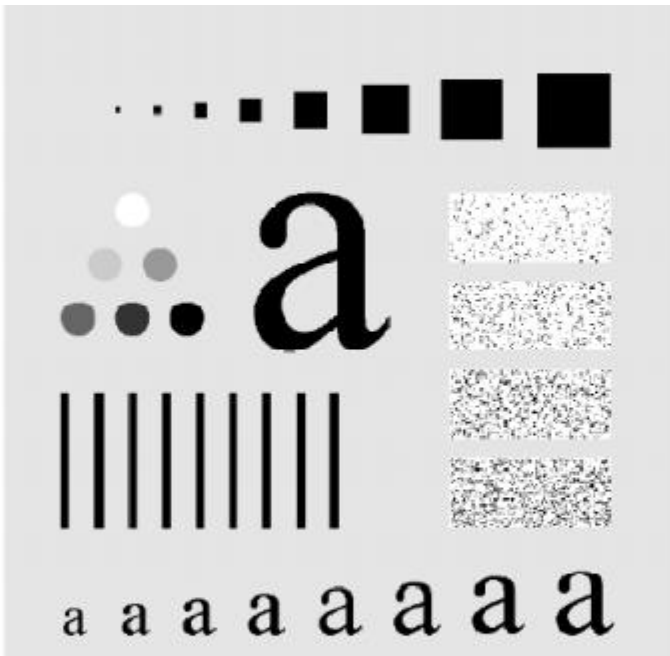


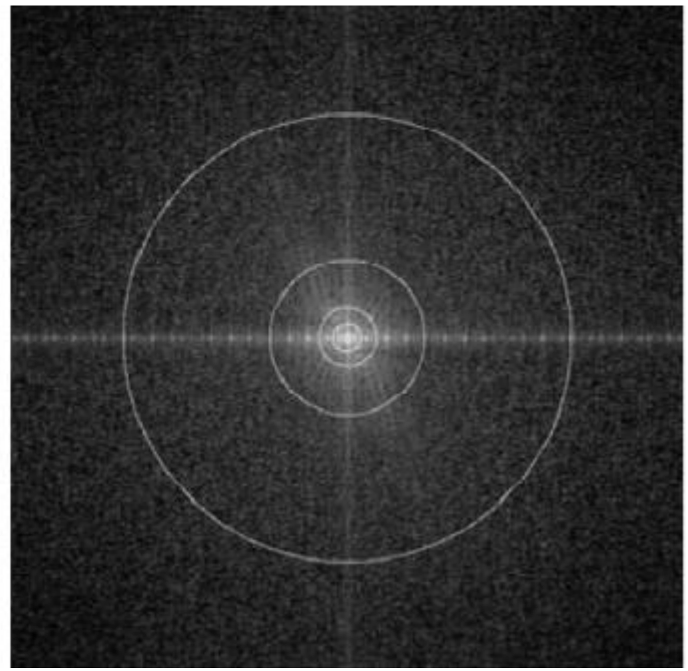
Perspective plot of ideal LPF

Filter as an image

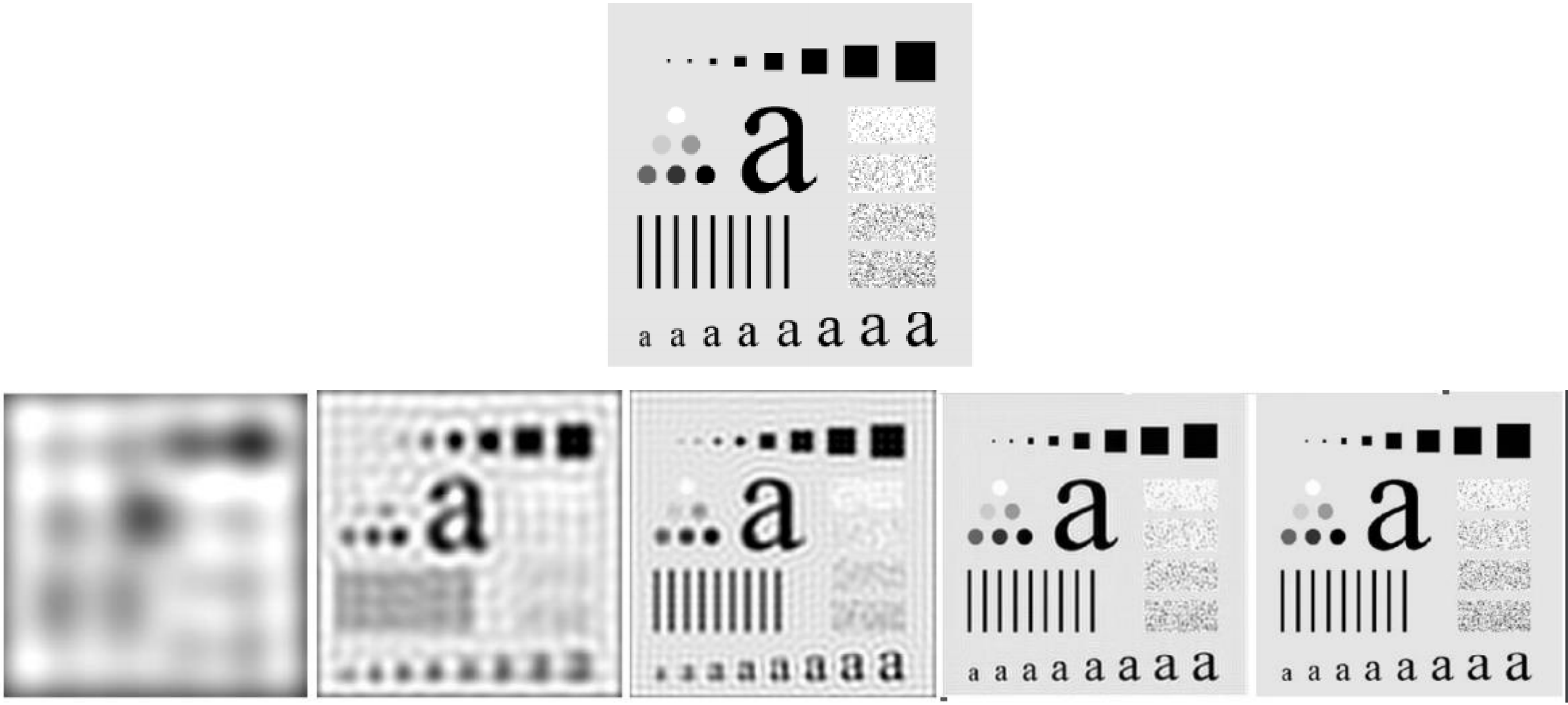
Radial Cross Section

Ideal Low Pass Filter Example



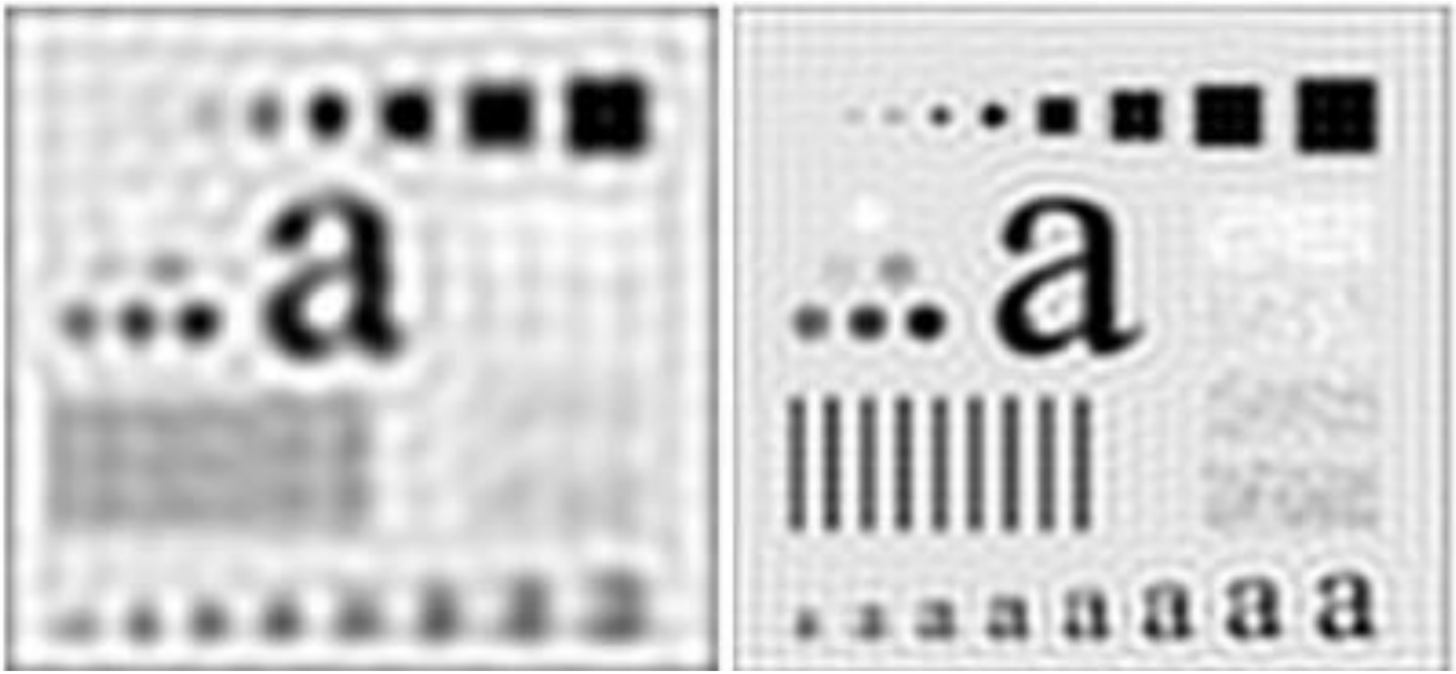


Ideal Low Pass Filter Example (2)

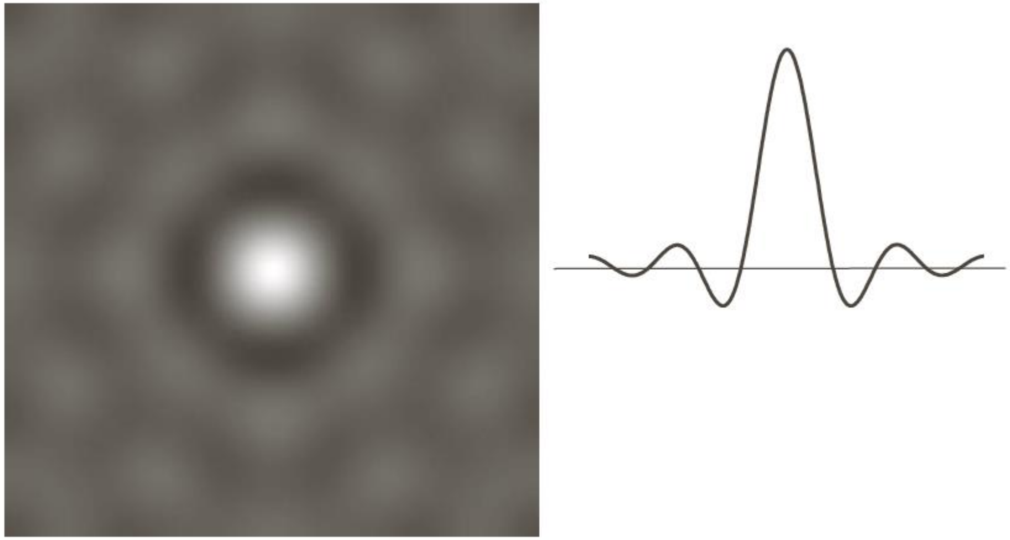


- Radius low pas filter: 10,30,60,160, 460

Ringing Effect



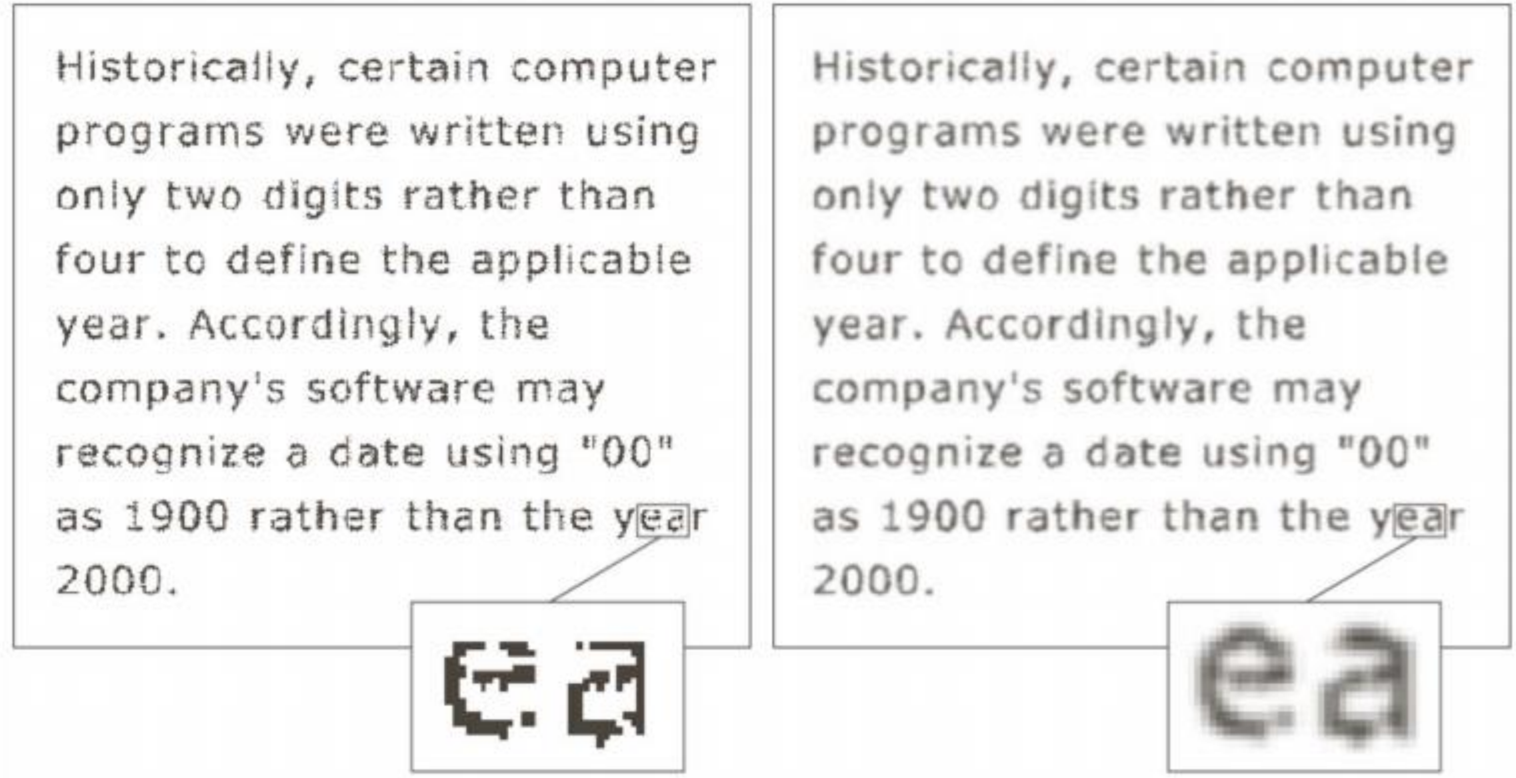
Why Ringing Effect?



- Visually, they appear as bands or "ghosts" near edges.
- The term "ringing" is because the output signal oscillates at a fading rate around a sharp transition in the input, similar to a bell after being struck

Potential Applications

- Machine perception and OCR



More on Low Pass Filtering

- Butterworth low pass filter
- Disadvantages of low pass filter
- Potential applications
  - Printing and Publishing Industry
  - Satellite and Aerial Images

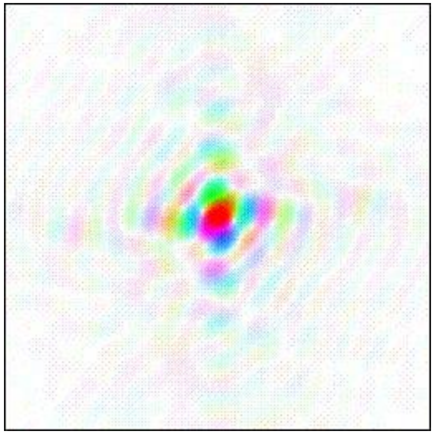
High Pass Filters

- Yang diambil adalah frekuensi-frekuensi tinggi yang ada pada citra.

Sharpening (high pass filter) : meloloskan komponen frekuensi tinggi

- Edges and abrupt changes in intensities are associated with high-frequency.
- Attenuates the low-frequency components without disturbing high-frequency information.

High Pass Filters (2)



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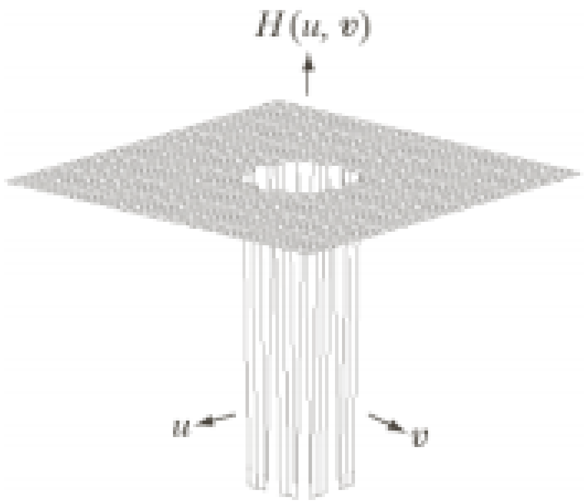
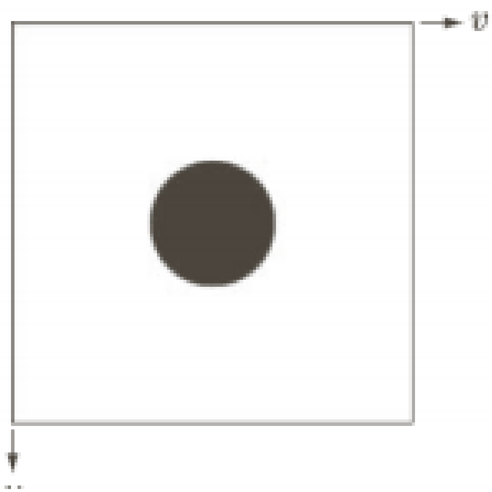
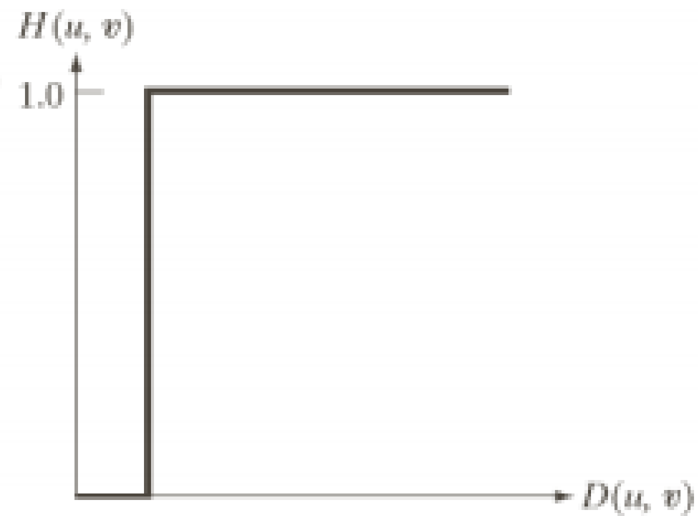
Ideal High Pass Filter



Perspective plot of ideal HPF



Filter as an image

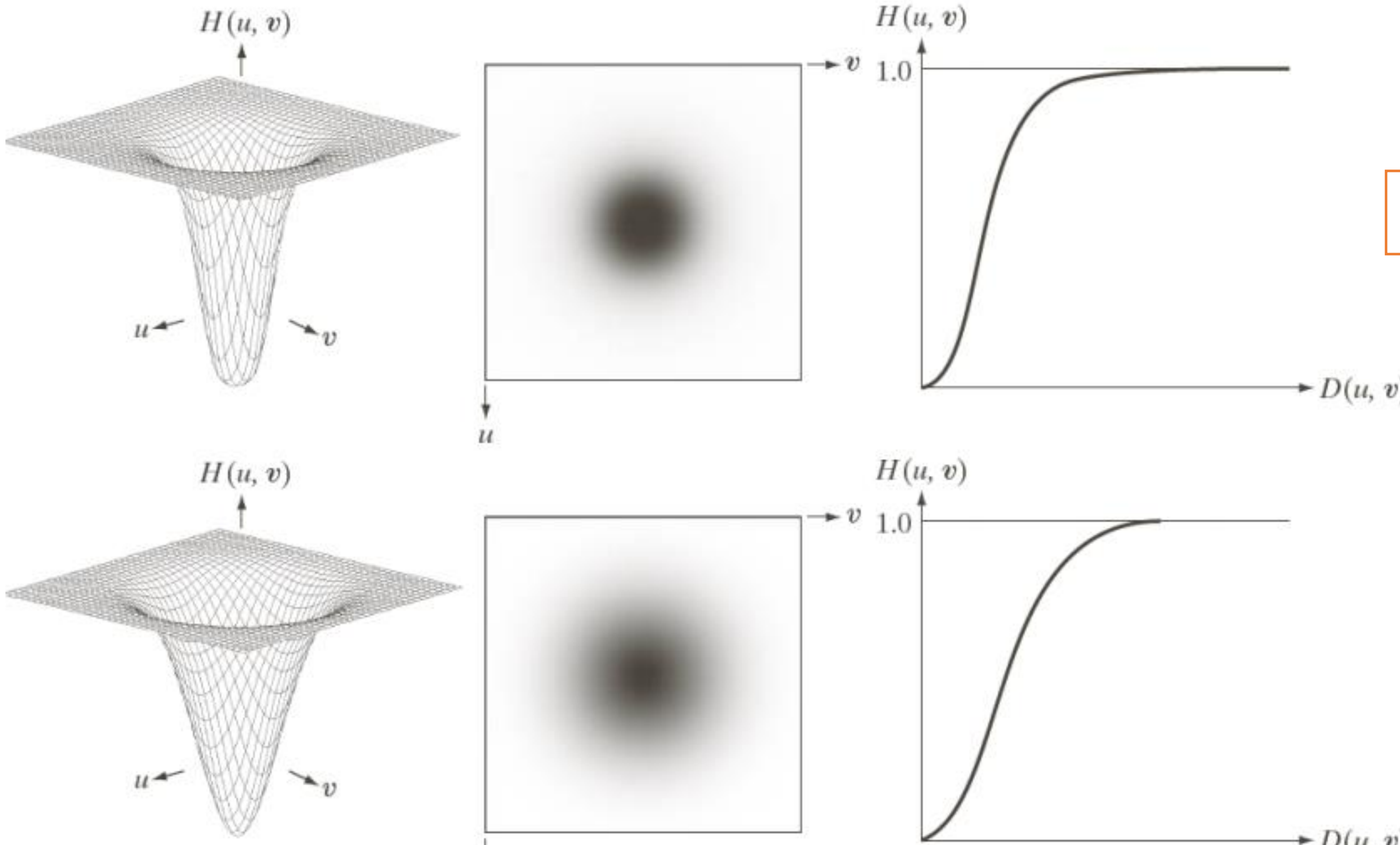


Radial Cross Section

Ideal High Pass Filter Example

- Ideal High Pass Filter results with $D_0 = 30, 60, \text{and } 160$

Butterworth and Gaussian High Pass Filter



Butterworth

Gaussian

Perspective plot of ideal HPF

Filter as an image

Radial Cross Section

Butterworth High Pass Filter Example

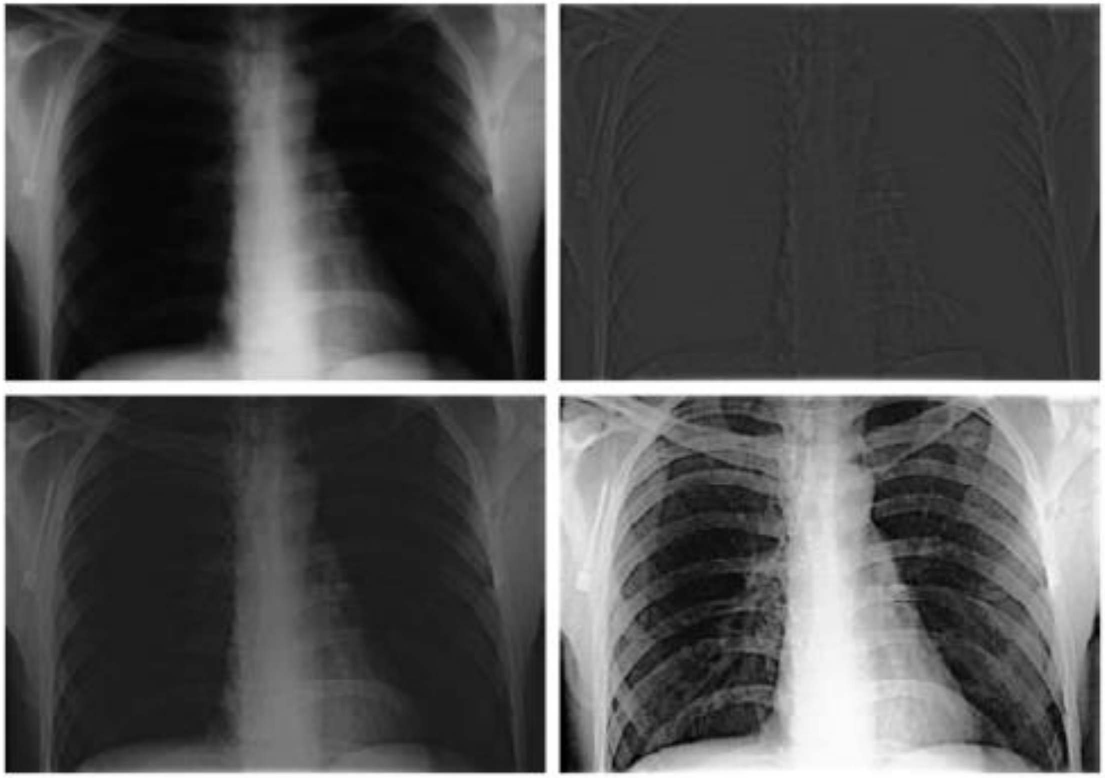
- Butterworth High Pass Filter results with D_0= 30,60, and 160

Gaussian High Pass Filter Example

- Gaussian High Pass Filter results with $D_0 = 30, 60, \text{and } 160$

Potential Applications

- Medical Field
  - HPF, HPF emphasis, and histogram equalization



More on High Pass Filtering

- Laplacian Filtering
- Homomorphic Filtering
- Potential Applications
  - PET Scans
  - Satelite Imagery of Space

Other Topics

- Selective Filtering (Band Reject Filters)