SPH-Based Fluids

Xuan Wang

March 2023

1 Smoothed Interpolation

Suppose each particle j has a physical quantity A_j. We can estimate the quantity at a new location $\mathbf{x}_i$ as the weighted sum of all its neighbors within a kernel radius R:

$$A_i^{\text{smooth}} = \sum_j V_j A_j W_{ij} \quad \text{for } ||\mathbf{x}_i - \mathbf{x}_j|| < R \tag{1}$$

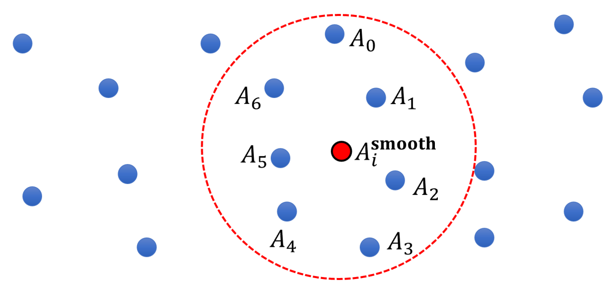

Figure 1: Smoothed Interpolation

Note that in $V_i = \frac{m_i}{\rho_i}$, ρ_i is also a smoothed interpolation of its neighboring particles:

$$\rho_i^{\text{smooth}} = \sum_j V_j \rho_j W_{ij} = \sum_j m_j W_{ij} \tag{2}$$

So finally, the solution is

$$A_i^{\text{smooth}} = \sum_j \frac{m_j}{\sum_k m_k W_{jk}} A_j W_{ij} \tag{3}$$

We can easily compute the derivatives of the smoothed interpolation. Simply assume that the motion of particle i would not affect its neighbors:

$$\text{gradient}: \quad \nabla A_i^{\text{smooth}} = \sum_j V_j A_j \nabla W_{ij} \tag{4}$$

$$\text{Laplacian}: \quad \nabla^2 A_i^{\text{smooth}} = \sum_j V_j A_j \nabla^2 W_{ij} \tag{5}$$

For example, the Spiky kernel is

$$W_{Spiky}(\vec{r}, h) = \begin{cases} K_{Spiky}(h-r)^3 & 0 \le r \le h \\ 0 & otherwise \end{cases} \tag{6}$$

where $r = |\vec{r}|$. In 3D, $K_{Spiky} = \frac{15}{\pi h^6}$.

$$\nabla W_{Spiky} = \frac{15}{\pi h^6} \nabla (h-r)^3 = -\vec{r} \frac{45}{\pi h^6 r}(h-r)^2 \tag{7}$$

The viscosity kernel is

$$W_{viscosity}(\vec{r}, h) = \begin{cases} K_{viscosity}(-\frac{r^3}{2h^3} + \frac{r^2}{h^2} + \frac{h}{2r} - 1) & 0 \le r \le h \\ 0 & otherwise \end{cases} \tag{8}$$

In 3D, $K_{viscosity} = \frac{15}{2\pi h^3}$.

$$\nabla^2 W_{viscosity} = \frac{45}{\pi h^6}(h-r) \tag{9}$$

2 Fluid Force

Gravity force:

$$\mathbf{F}_i^{\text{gravity}} = m_i \mathbf{g} \tag{10}$$

Pressure is related to the density by some empirical function:

$$P_i = k((\frac{\rho_i}{\rho_{\text{constant}}})^7 - 1) \tag{11}$$

Or

$$P_i = k(\rho_i - \rho_0) \tag{12}$$

Pressure force depends on the difference of pressure:

$$\mathbf{F}_i^{\text{pressure}} = -V_i \nabla_i P^{\text{smooth}} \tag{13}$$

where we assume the pressure is also smoothly represented:

$$P_i^{\text{smooth}} = \sum_j V_j P_j W_{ij} \tag{14}$$

Viscosity effect means that particles should move together in the same velocity, i.e. minimize the difference between the particle velocity and the velocities of its neighbors. Mathematically, it means:

$$\mathbf{F}_i^{\text{viscosity}} = -\mu \nabla_i^2 \mathbf{u}^{\text{smooth}} \tag{15}$$

where $\mathbf{u}$ is the relative velocity and it is also smoothly represented:

$$\mathbf{u}_i^{\text{smooth}} = \sum_j V_j \mathbf{u}_j W_{ij} \tag{16}$$

In this assignment, we also consider a boundary penalty force if a particle is intersecting with some implicit geometry ϕ (e.g., the container):

$$\mathbf{F}_i^{\text{boundary}} = k_s(\phi(\mathbf{c}_i) - r_i)(-\nabla\phi) \tag{17}$$

Comparing the above equation with the spring force $\mathbf{F}_i^s = k_s(|\mathbf{x}_j - \mathbf{x_i}| - l_0)\frac{\mathbf{x}_j - \mathbf{x_i}}{|\mathbf{x}_j - \mathbf{x_i}|}$ in a mass-spring model, we use the direction $-\nabla\phi$ pointing from the exterior to the interior of the object to replace the direction pointing from $\mathbf{x}_i$ to $\mathbf{x}_j$.

In the course note, we replace the mass of a particle with its density, e.g., $\mathbf{f}_i^{\text{gravity}} = \rho\mathbf{g}$. Therefore, the dimension of forces changes to $MT^{-2}L^{-2}$. But it seems that the boundary penalty force still has a dimension of $MT^{-2}L$? Maybe the dimension of k_s is changed accordingly?

3 Neighbor Search

Separate the space into cells, and each cell stores the particles in it: To find

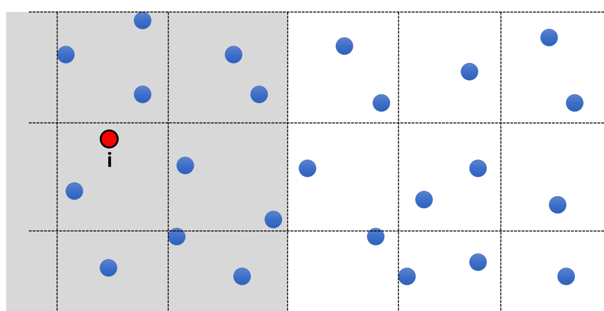

Figure 2: Neighbor Search

the neighbors of i, just look at the surrounding cells.

We use a spatial hashing algorithm, with the table key as the cell coordinate and table list as the particles contained in each cell.