Signed Distance Field

Xuan Wang

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1 Definition

Assume we have a set Σ. Firstly, we define the unsigned distance function as the function that yields the distance from a point $\mathbf{p}$ to the closest point in Σ:

$$\mathrm{dist}_\Sigma(\mathbf{p}) = \inf_{\mathbf{x}\in\Sigma} ||\mathbf{x} - \mathbf{p}|| \tag{1}$$

Here, we are mainly interested in the signed distance function associated with a solid S. The sign is used to denote whether we are inside or outside S:

$$\mathrm{d}_S(\mathbf{p}) = \mathrm{sgn}(\mathbf{p}) \inf_{\mathbf{x}\in\partial S} ||\mathbf{x} - \mathbf{p}|| \tag{2}$$

where

$$\mathrm{sgn}(\mathbf{p}) = \begin{cases} -1 & \text{if } \mathbf{p} \in S \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

2 ∇d

An important property of the SDF is that

$$||\nabla d|| = 1 \tag{4}$$

almost everywhere, except for these points without a unique closest point (e.g., the center of a sphere). Intuitively, we can think ∇d as the speed of the expanding front. When the front moves at unit speed, then the accumulative result is just the distance from $\mathbf{p}$ to the closest point on the original front.

Equation 4 is an instance of the Eikonal equation. In fact, we can make a stronger claim than this: the solution to this Eikonal equation is just the signed distance function (since SDF is the viscosity solution to this Eikonal equation).

Furthermore, on the boundary, SDF satisfies

$$\nabla d(\mathbf{x}) = N(\mathbf{x}) \tag{5}$$

where N is the inward normal vector field.

A sketchy proof is as follows. We start by considering a small displacement $\delta\mathbf{x}$ along the surface from the point $\mathbf{x}$. Since $d(\mathbf{x}) = 0$ on the surface, we know that a small change in d along the surface should also be 0. So

$$\nabla d \cdot \delta\mathbf{x} = 0 \tag{6}$$

This equation tells us that ∇d is perpendicular to any displacement vector $\delta\mathbf{x}$ on the surface. In other words, ∇d is perpendicular to the surface, which is exactly the definition of the surface normal. Therefore, we can arrive at Equation 5.

3 Second Order Derivative

The second order derivatives contain information about the curvature of isosurfaces of the distance function. For example, the Hessian of SDF on the boundary gives the Weingarten map:

$$H = \begin{pmatrix} d_{xx} & d_{xy} & d_{xz} \\ d_{yx} & d_{yy} & d_{yz} \\ d_{zx} & d_{zy} & d_{zz} \end{pmatrix} \tag{7}$$