Mass-Spring Model

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1 Newton's Second Law

The spring force $\vec{f}^s$ depends on the positions only:

$$\vec{f}_i^s = k_s(|\vec{x}_j - \vec{x}_i| - l_0)\frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|} \tag{1}$$

The damping force $\vec{f}^d$ depends on both positions and velocities:

$$\vec{f}_i^d = k_d((\vec{v}_j - \vec{v}_i) \cdot \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|}) \cdot \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|} \tag{2}$$

The total force is:

$$\vec{f}_i^{total} = \vec{f}_i^s + \vec{f}_i^d + m_i\vec{g} \tag{3}$$

The Newton's second law $m\ddot{q} = -kq$ is a second-order ODE. By introducing velocity $\dot{q} = v$, we get the first order ODE:

$$\frac{d}{dt}\begin{pmatrix} \vec{x}_i \\ \vec{v}_i \end{pmatrix} = \begin{pmatrix} \vec{v}_i \\ M_i^{-1}\vec{f}_i \end{pmatrix} \tag{4}$$

where $M_i = \begin{pmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{pmatrix}$

2 Time Integration

Replace time derivative with finite difference:

$$\frac{\vec{x}_{n+1} - \vec{x}_n}{\Delta t} = \vec{v} \tag{5}$$

$$\frac{\vec{v}_{n+1} - \vec{v}_n}{\Delta t} = M^{-1}\vec{f}(\vec{x}, \vec{v}) \tag{6}$$

Different evaluation on the right-hand side leads to two types of schemes.

2.1 Explicit Euler

At current time step:

$$\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n \tag{7}$$

$$\vec{v}^{n+1} = \vec{v}^n + M^{-1}\vec{f}(\vec{x}^n, \vec{v}^n) \tag{8}$$

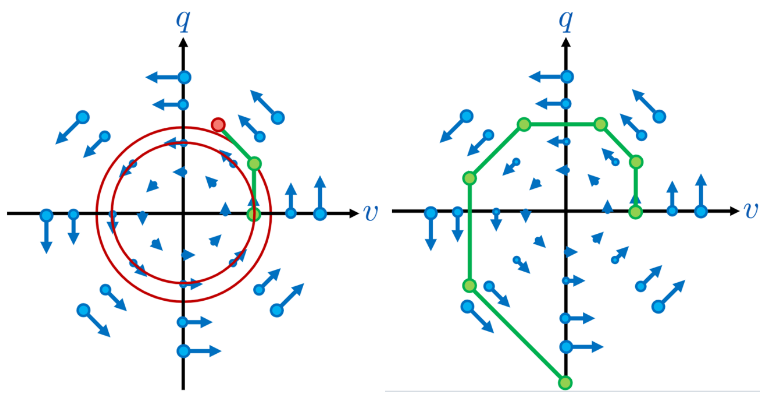

Figure 1: Explicit Euler

2.2 Implicit Euler

From the future:

$$\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^{n+1} \tag{9}$$

$$\vec{v}^{n+1} = \vec{v}^n + M^{-1}\vec{f}(\vec{x}^{n+1}, \vec{v}^{n+1}) \tag{10}$$

Expand $\vec{f}$:

$$\vec{f}(\vec{x}^{n+1}, \vec{v}^{n+1}) \approx \vec{f}(\vec{x}^n, \vec{v}^n) + \Delta\vec{x}\frac{\partial \vec{f}(\vec{x}^n, \vec{v}^n)}{\partial \vec{x}^n} + \Delta\vec{v}\frac{\partial \vec{f}(\vec{x}^n, \vec{v}^n)}{\partial \vec{v}^n} \tag{11}$$

Note that we approximate $\Delta\vec{x} = \Delta t \vec{v}^{n+1}$ and $\Delta\vec{v} = \vec{v}^{n+1} - \vec{v}^n$. Finally move all $\vec{v}^{n+1}$ to the left-hand side:

$$(M - \Delta t\frac{\partial \vec{f}(\vec{x}^n, \vec{v}^n)}{\partial \vec{v}^n} - \Delta t^2\frac{\partial \vec{f}(\vec{x}^n, \vec{v}^n)}{\partial \vec{x}^n})\vec{v}^{n+1} = M\vec{v}^n + \Delta t\vec{f}^n - \Delta t\frac{\partial \vec{f}(\vec{x}^n, \vec{v}^n)}{\partial \vec{v}^n}\vec{v}^n \tag{12}$$

Next we compute the two Jacobians with the help of basic matrix derivation, such as $\frac{\partial}{\partial \vec{x}}|\vec{x}| = \frac{\vec{x}}{|\vec{x}|}$ and $\frac{\partial}{\partial \vec{x}}(\frac{\vec{x}}{|\vec{x}|}) = \frac{1}{|\vec{x}|}I - \frac{1}{|\vec{x}|^3}\vec{x}\vec{x}^\top$:

$$\frac{\partial \vec{f}_i}{\partial \vec{x}_i} = k_s((\frac{l_0}{|\vec{x}_j - \vec{x}_i|} - 1)I - \frac{l_0}{|\vec{x}_j - \vec{x}_i|^3}(\vec{x}_j - \vec{x}_i)(\vec{x}_j - \vec{x}_i)^\top) \tag{13}$$

Note that we ignore the term $\frac{\partial f_i^d}{\partial \vec{x}_i}$. Define $K_{ij} = \frac{\partial \vec{f}_i}{\partial \vec{x}_j}$, then $K_{ii} = K_{jj} = -K_{ij} = -Kji$.

$$\frac{\partial \vec{f}_i}{\partial \vec{v}_i} = -k_d \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|} \cdot (\frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|})^\top \tag{14}$$

Define $D_{ij} = \frac{\partial \vec{f}_i}{\partial \vec{v}_j}$, then $D_{ii} = D_{jj} = -D_{ij} = -Dji$.
Finally we can change Equation 12 into the matrix form:

$$(M - \Delta t D - \Delta t^2 K)\vec{v}^{n+1} = M\vec{v}^n + \Delta t \vec{f}^n - \Delta t D \vec{v}^n \tag{15}$$

This is linear as $ku = b$, so we can use any linear sparse solver such as conjugate gradient. Use $\vec{v}^n$ as the initial guess for $\vec{v}^{n+1}$ when solving iteratively.

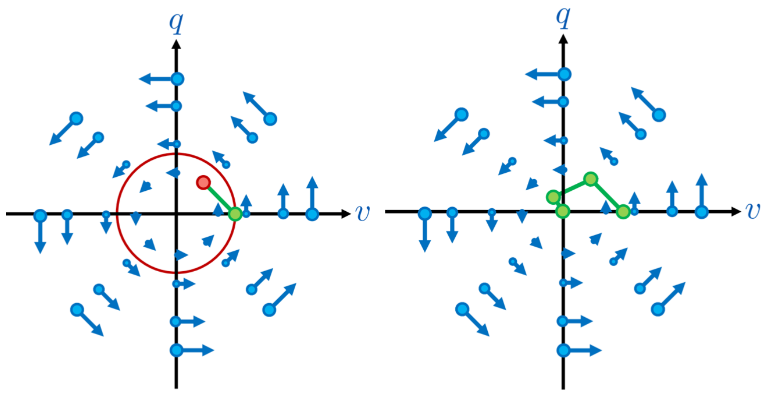

Figure 2: Implicit Euler

3 Implementation Detail

- Matrix Calculus for calculating matrix derivation. But this article may offer a better understanding to it.
- Notice how we assemble all n particles into a $3n \times 3n$ matrix and fill elements to it carefully.
- Compare the different results between Explicit and Implicit Euler.