Time Derivative of Rotation Matrix

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March 2023

1 Rotation Matrix

The velocity can be expressed with the cross product of the angular velocity and poistion:

$$\vec{v} = \vec{w} \times \vec{x} \tag{1}$$

Note that the cross product can be represented in a matrix form:

$$\vec{v} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix} \vec{x} \tag{2}$$

So Equation 1 is actually in the form of a linear ODE:

$$\frac{d\vec{x}}{dt} = [\vec{w}]\vec{x} \tag{3}$$

which has analytical solution:

$$\vec{x}(t) = \exp([\vec{w}]t)\vec{x} \tag{4}$$

where exp refers to the matrix exponential

$$\exp(\mathbf{X}) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{X}^k \tag{5}$$

With $\vec{x}(t) = R(t)\vec{x}$, we can express the rotation matrix R as the matrix exponential:

$$R(t) = \exp([\vec{w}]t) \tag{6}$$

Next we are to use Rodrigues' rotation formula to simplify Equation 3. First we express $\vec{w}t$ as $\frac{\vec{w}}{\|\vec{w}\|}\|\vec{w}\|t$, where $\frac{\vec{w}}{\|\vec{w}\|}$ is the rotation axis $\vec{a}$ and $\|\vec{w}\|t$ is the rotation angle θ. Then $R(t)$ can be calculated as

$$R(t) = \mathbf{I} + \sin(\theta)[\vec{a}] + (1 - cos(\theta))[\vec{a}][\vec{a}] \tag{7}$$

where $[\vec{a}]$ is the cross product matrix.

2 Time Derivative of $R(t)$

Take the time derivative:

$$\dot{\vec{x}}(t) = R\dot{(t)}\vec{x} \tag{8}$$

And with Equation 3, we have

$$\dot{R} = [\vec{w}] \tag{9}$$

Here comes the tricky part:

$$\vec{x}(t + \Delta t) = R(\Delta t)R(t)\vec{x}_0 \tag{10}$$

where t is fixed while Δt is the variable. So after taking the time derivative w.r.t. Δt, we get

$$\vec{v} = \frac{d\vec{x}}{d\Delta t} = \dot{R}(\Delta t)R(t)\vec{x}_0 \tag{11}$$

And Equation 9 tells that

$$\vec{v} = [\vec{w}]R\vec{x}_0 \tag{12}$$

We then need some transformation

$$\begin{aligned} \vec{v} &= \vec{w} \times R\vec{x}_0 \\ &= -(R\vec{x}_0) \times \vec{w} \\ &= -R(\vec{x}_0 \times R^T\vec{w}) \\ &= -R([\vec{x}_0]R^T\vec{w}) \end{aligned} \tag{13}$$

where $[\vec{x}_0]$ is a skew-symmetric matrix. Thus we can throw the negative sign and arrive at our final fomula:

$$\vec{v} = R[\vec{x}_0]^T R^T\vec{w} \tag{14}$$