The Perceptron

A **linear classifier** inspired by the human **neuron**

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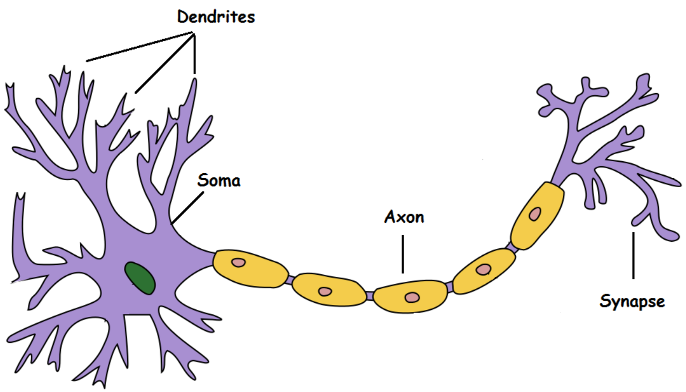
The Biological Neuron

- Electrical signals arrive at the neuron's **dendrites** (which can be considered *inputs*).
- The signals cause electrical potential to be accumulated in the neuron's *body* (the **soma**).
- When the potential reaches a certain *threshold*, a pulse is transmitted down the neuron's **axon** (the *output*).
- *Synapses* are connections from the *axon* to the *dendrites* of other neurons.



The Artificial Neuron

- The first artificial model of the neuron was proposed by **Warren MuCulloch** and **Walter Pitts** in 1943.
- It had *Boolean* inputs (either *present* or *not present*) and could be either *excitatory* or *inhibitory*.
    - If the number of present excitatory inputs were greater than the number of present inhibitory inputs by some threshold, then the neuron would fire.

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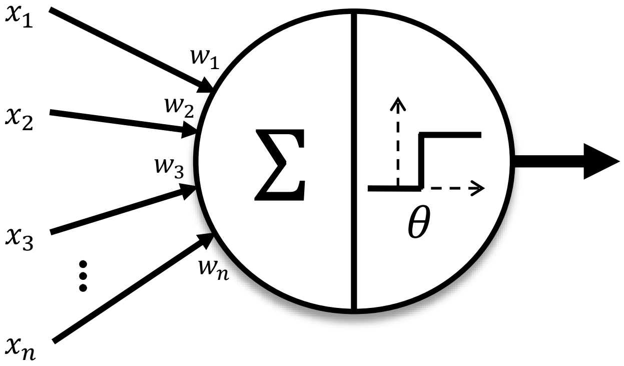
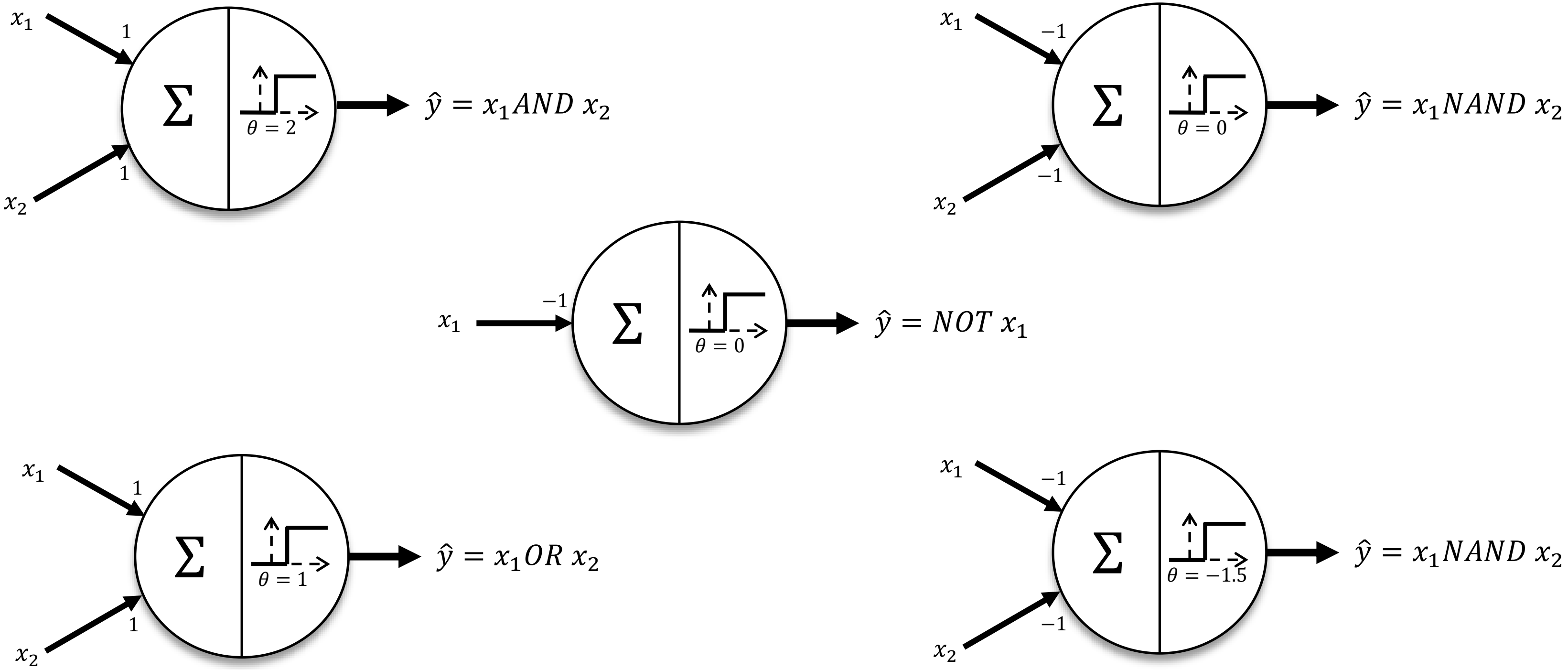
- It had *Boolean* inputs (either *present* or *not present*) and could be either *excitatory* or *inhibitory*.
  - If the number of present excitatory inputs were greater than the number of present inhibitory inputs by some threshold, then the neuron would fire.
- Mathematically, the neuron's output is 1 if the weighted sum of binary inputs would exceed a threshold and 0 otherwise.

$$x_1, x_2, \dots, x_n \in \{0,1\}, \qquad w_1, w_2, \dots, w_n \in \{-1,1\}, \qquad \theta \in \mathbb{R}, \qquad \hat{y} \in \{0,1\}$$



$$\hat{y} = \begin{cases} 1 & \text{if} \sum_{i=1}^{n} w_i x_i \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

The Artificial Neuron



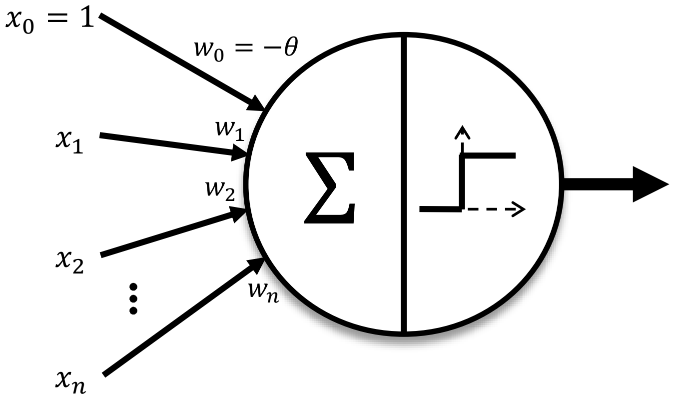
Hebb’s Learning Rule

- **Hebb’s rule**, postulated by *Donald Hebb* (psychologist) in 1949, states that the connections between two (biological) neurons might be strengthened, if a neuron often takes part in firing the other.
    - *“When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased”.*
    - Often summarized as "*Cells that fire together wire together.*"
- It gave rise to two ideas concerning the *MuCulloch-Pitts* artificial neuron model:
    - Inputs are not just *excitatory* or *inhibitory*, but have a *synaptic strength* (some connections between neurons are stronger than others).
    - It gave an indication of a method to update the synaptic strength (increase the strength of connection which is active when a neuron *should* fire).

The Perceptron

- Based on the *MuCulloch-Pitts* model and with Hebb's ideas in mind, **Frank Rosenblatt** invented, in 1957, a machine and an associated learning algorithm, which he called **"The Perceptron",** designed for image recognition.
    - It was intended to be a physical machine (with photocells as inputs, potentiometers as weights and electric motors which performed weight updates), although the first implementation was in software.

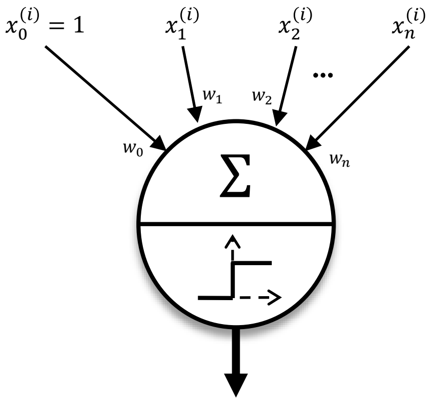
$$x_0 = 1, \qquad x_1, x_2, \dots, x_n \in \{0,1\}, \qquad w_0, w_1, w_2, \dots, w_n \in \mathbb{R}, \qquad \hat{y} \in \{0,1\}$$



$$\hat{y} = \begin{cases} 1 & \text{if} \sum_{i=0}^{n} w_i x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Perceptron Learning Algorithm

- Training Set: $E = \{(\vec{x}^{(1)}, y^{(1)}), \ldots, (\vec{x}^{(m)}, y^{(m)})\},\ \vec{x}^{(i)} \in \{0,1\}^{n+1} \left(x_0^{(i)} = 1, \forall i\right),\ y^{(i)} \in \{0,1\}$



$$\hat{y}^{(i)} = \begin{cases} 1 & \text{if } \sum_{j=0}^{n} w_j x_j^{(i)} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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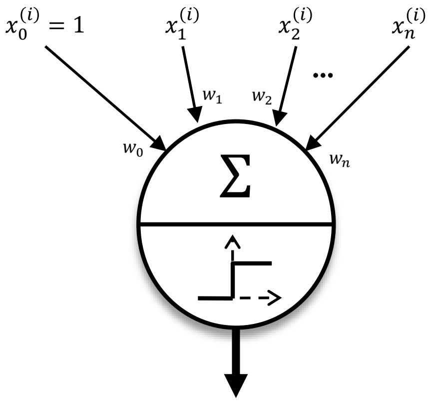
- Weights w_j start off as 0.
- For every example $(\vec{x}^{(i)}, y^{(i)})$ in the dataset:
  - If $\hat{y}^{(i)} == y^{(i)}$ we don't change anything



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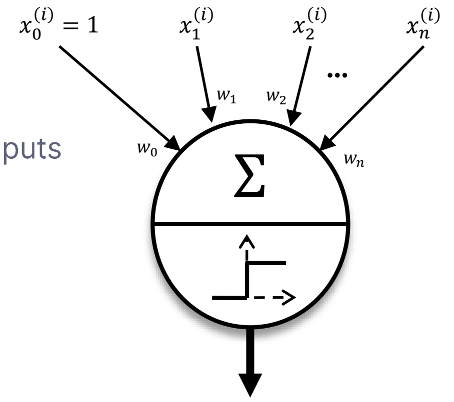
  - If $\hat{y}^{(i)} == 0$ and $y^{(i)} == 1$ we need to increase the strength of all active inputs



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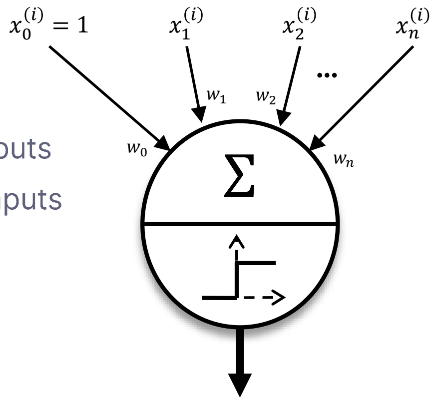
  - If $\hat{y}^{(i)} == 1$ and $y^{(i)} == 0$ we need to decrease the strength of all active inputs



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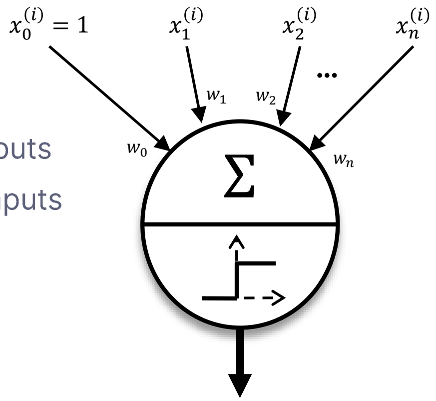
  - If $\hat{y}^{(i)} == 1$ and $y^{(i)} == 0$ we need to decrease the strength of all active inputs
- Perceptron update rule:

$$\boldsymbol{\Delta w_j = \eta\left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}}$$

Amount by which the strength is changed (“**learning rate**”).

Sets the update direction.

Selects active inputs.



$$\hat{y}^{(i)} = \begin{cases} 1 & \text{if } \sum_{j=0}^{n} w_j x_j^{(i)} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Perceptron Learning Algorithm

```
def perceptron(X, y, n_epochs, η):
    M, n = X.shape # number of samples, number of inputs
    for j in range(n):
        $w_j = 0$
    for epoch in range(n_epochs): # an “epoch” is a run through all training data.
        for i in range(m):        # a “training step” is one of update of the weights.
            $\hat{y}^{(i)} = \text{unit\_step\_function}\left(\sum_{j=0}^{n} w_j x_j^{(i)}\right)$
            for j in range(n):
                $w_j \mathrel{+}= \eta\left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}$
```

Perceptron Learning Algorithm

- An alternative formulation is to consider inputs and outputs to be ± 1, instead of binary.

$$\vec{x}^{(i)} \in \{-1,1\}^{n+1}, \qquad y^{(i)} \in \{-1,1\}$$

- Prediction:

$$\hat{y}^{(i)} = \operatorname{sign}\left(\sum_{j=0}^{n} w_j x_j^{(i)}\right)$$

- Update rule:

$$\Delta w_j = \eta \cdot y^{(i)} \cdot x_j^{(i)}$$

We only apply the update rule if $y^{(i)} \neq \hat{y}^{(i)}$

Example – Learning the OR function

	x_0	x_1	x_2
$\vec{x}^{(1)}$	1	0	0
$\vec{x}^{(2)}$	1	0	1
$\vec{x}^{(3)}$	1	1	0
$\vec{x}^{(4)}$	1	1	1

y
0
1
1
1

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$\vec{x}^{(4)}$	1	1	1

y
0
1
1
1

$\mathbf{w}_0$	$\mathbf{w}_1$	$\mathbf{w}_2$
0	0	0

Accuracy: 25%

$\hat{y}$
0
0
0
0

$\eta = 1$

$$\Delta w_j = \left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}$$

Example – Learning the OR function

	x_0	x_1	x_2
$\vec{x}^{(1)}$	1	0	0
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y
0
1
1
1

w_0	w_1	w_2
0	0	0

Accuracy: 25%

$\hat{y}$
0
0
0
0

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	0	0	0
$\vec{x}^{(2)}$			
$\vec{x}^{(3)}$			
$\vec{x}^{(4)}$			

Epoch 1

$\eta = 1$

$\Delta w_j = \left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}$

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$\vec{x}^{(4)}$	1	1	1

y
0
1
1
1

w_0	w_1	w_2
1	0	1

Accuracy: 75%

$\hat{y}$
1
1
1
1

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	0	0	0
$\vec{x}^{(2)}$	1	0	1
$\vec{x}^{(3)}$			
$\vec{x}^{(4)}$			

Epoch 1

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y
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1
1
1

w_0	w_1	w_2
1	0	1

Accuracy: 75%

$\hat{y}$
1
1
1
1

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	0	0	0
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w_0	w_1	w_2
1	0	1

Accuracy: 75%

$\hat{y}$
1
1
1
1

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	0	0	0
$\vec{x}^{(2)}$	1	0	1
$\vec{x}^{(3)}$	0	0	0
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w_0	w_1	w_2
1	0	1

Accuracy: 75%

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1
1
1
1

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	-1	0	0
$\vec{x}^{(2)}$			
$\vec{x}^{(3)}$			
$\vec{x}^{(4)}$			

Epoch 2

$\eta = 1$

$$\Delta w_j = \left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}$$

Example – Learning the OR function

	x_0	x_1	x_2
$\vec{x}^{(1)}$	1	0	0
$\vec{x}^{(2)}$	1	0	1
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1
0
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	Δw_0	Δw_1	Δw_2
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0
1

	Δw_0	Δw_1	Δw_2
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Accuracy: 75%

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y
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1	1	1

Accuracy: 75%

$\hat{y}$
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1
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1

	Δw_0	Δw_1	Δw_2
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$\vec{x}^{(2)}$	0	0	0
$\vec{x}^{(3)}$	1	1	0
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Epoch 2

$\eta = 1$

$\Delta w_j = \left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}$

Example – Learning the OR function

	x_0	x_1	x_2
$\vec{x}^{(1)}$	1	0	0
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y
0
1
1
1

w_0	w_1	w_2
1	1	1

Accuracy: 75%

$\hat{y}$
1
1
1
1

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	-1	0	0
$\vec{x}^{(2)}$			
$\vec{x}^{(3)}$			
$\vec{x}^{(4)}$			

Epoch 3

$\eta = 1$

$$\Delta w_j = \left(y^{(i)} - \hat{y}^{(i)}\right) \cdot x_j^{(i)}$$

Example – Learning the OR function

	x_0	x_1	x_2
$\vec{x}^{(1)}$	1	0	0
$\vec{x}^{(2)}$	1	0	1
$\vec{x}^{(3)}$	1	1	0
$\vec{x}^{(4)}$	1	1	1

y
0
1
1
1

w_0	w_1	w_2
0	1	1

Accuracy: 100%

$\hat{y}$
0
1
1
1

	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	-1	0	0
$\vec{x}^{(2)}$			
$\vec{x}^{(3)}$			
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Epoch 3

$\eta = 1$

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Accuracy: 100%

$\hat{y}$
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1
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	Δw_0	Δw_1	Δw_2
$\vec{x}^{(1)}$	-1	0	0
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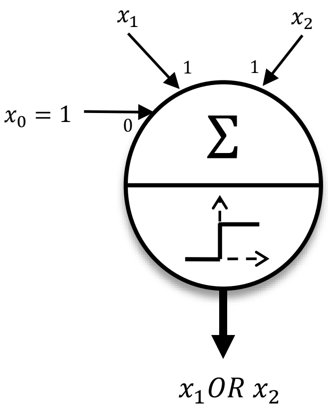
y
0
1
1
1

w_0	w_1	w_2
0	1	1

Accuracy: 100%

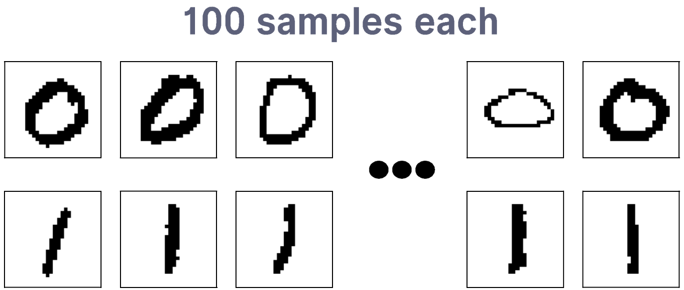
$\hat{y}$
0
1
1
1

The *Perceptron* learned the OR function in 3 epochs.



Example – Recognizing Digits

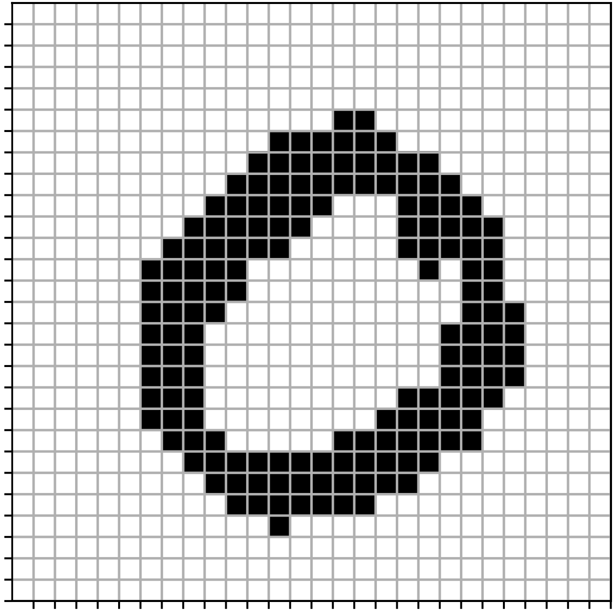
Can a *perceptron* be trained to distinguish handwritten pictures of 0s from pictures of 1s?



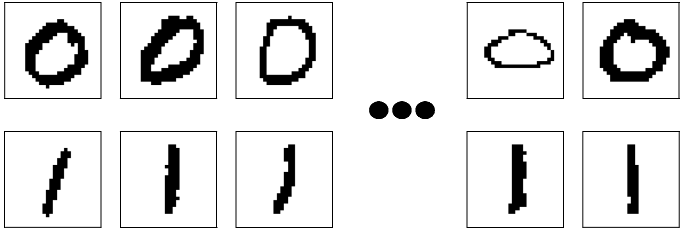
Example – Recognizing Digits

Can a *perceptron* be trained to distinguish handwritten pictures of 0s from pictures of 1s?

28x28 pixels



100 samples each



Example – Recognizing Digits

Can a *perceptron* be trained to distinguish handwritten pictures of 0s from pictures of 1s?

28x28 binary features

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0
0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0
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0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0
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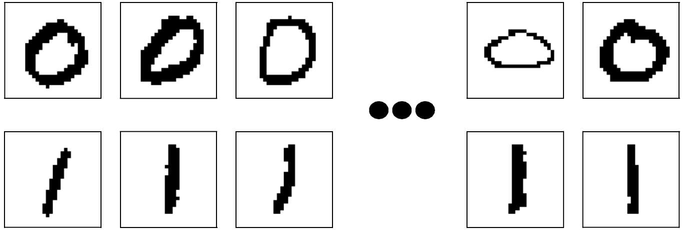
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

100 samples each



Example – Recognizing Digits

Can a *perceptron* be trained to distinguish handwritten pictures of 0s from pictures of 1s?

28x28 binary features

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
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0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0
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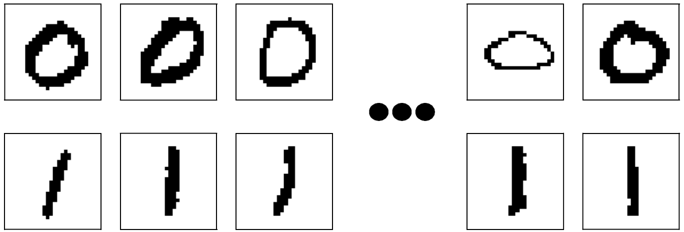
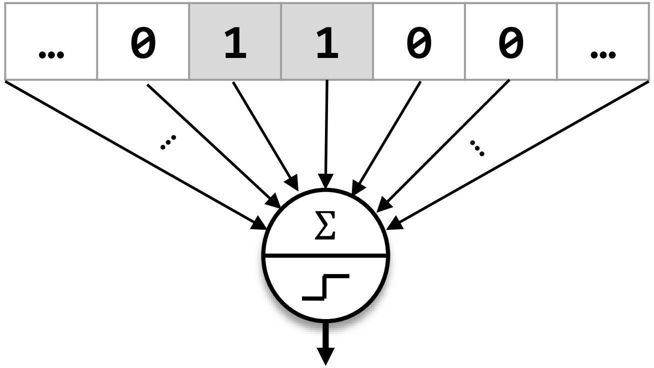
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

100 samples each

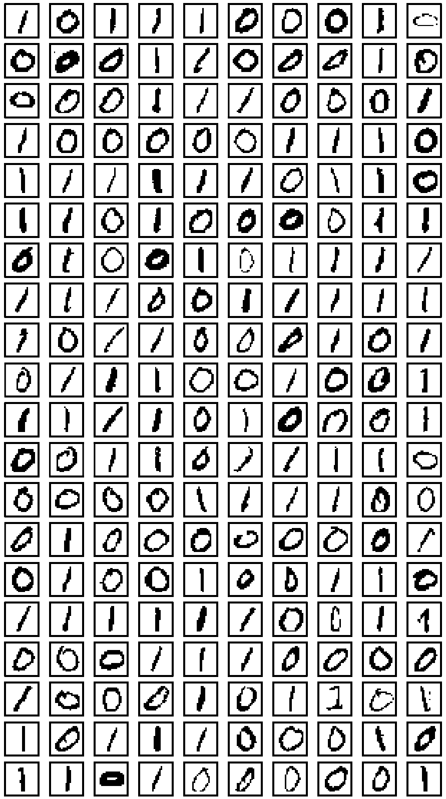


Flatten

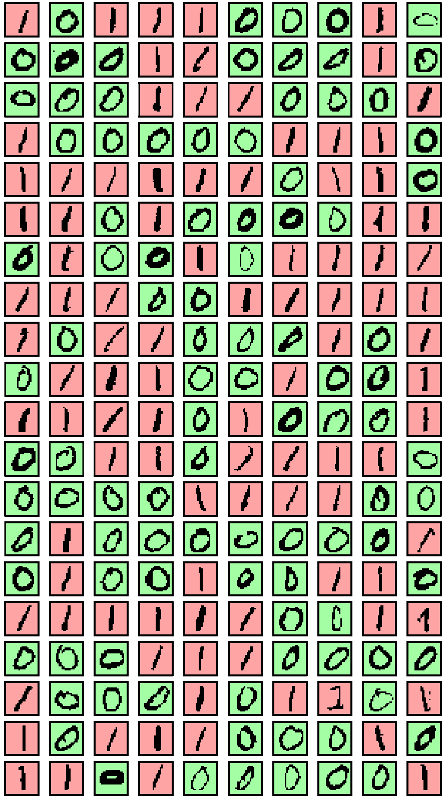
724 features (inputs)



Example – Recognizing Digits



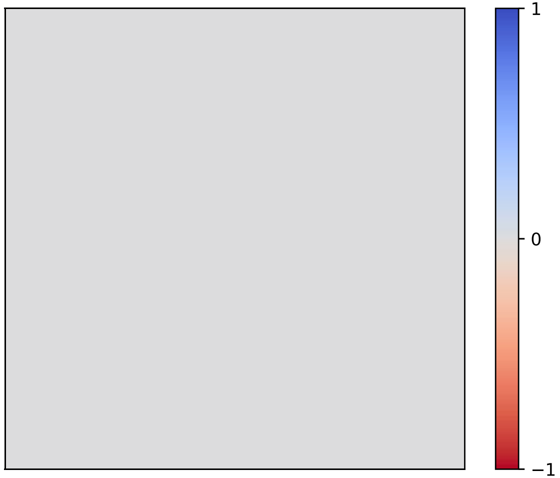
Example – Recognizing Digits



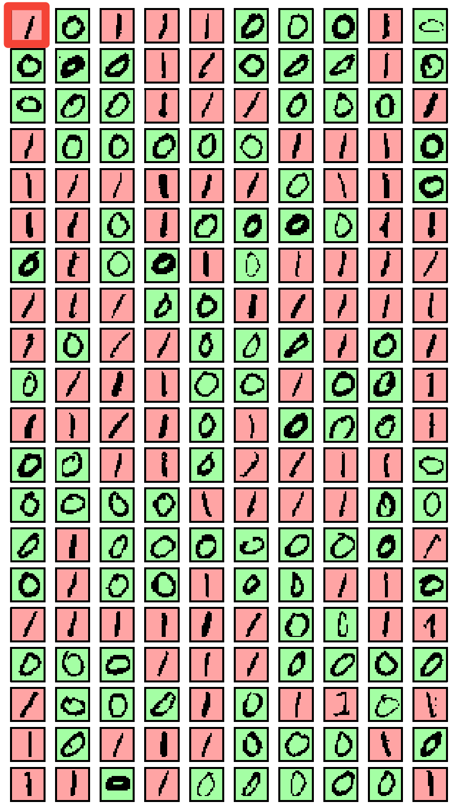
Training step 0

50% Accuracy

Weights as a 28x28 picture



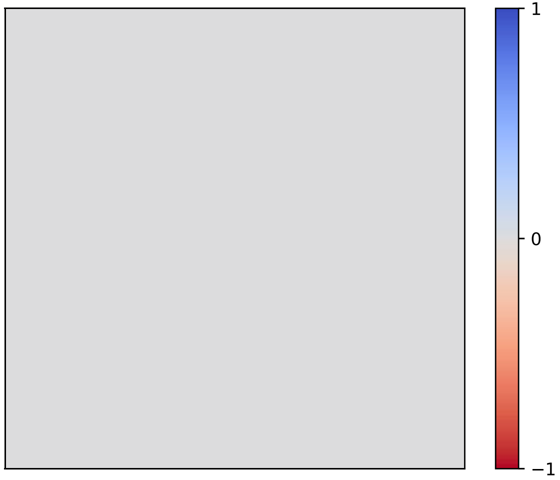
Example – Recognizing Digits



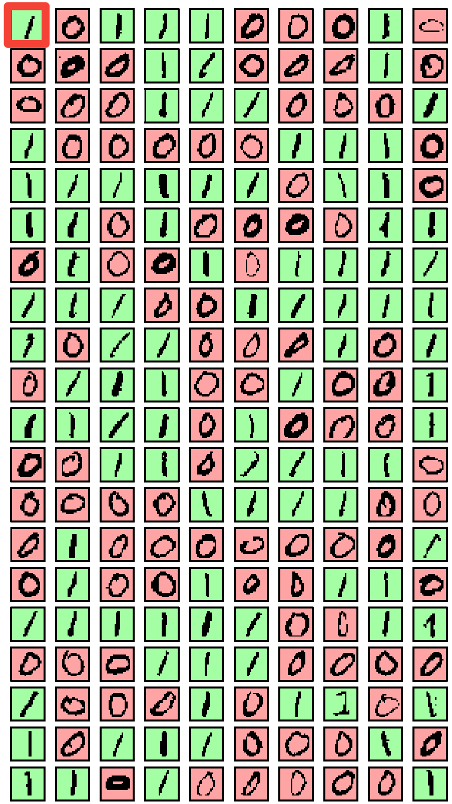
Training step 1

50% Accuracy

Weights as a 28x28 picture



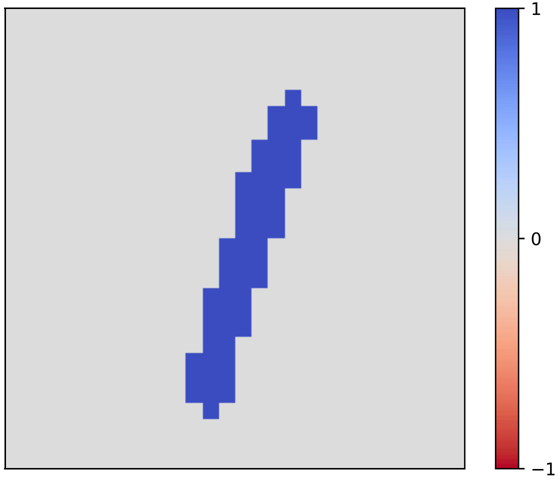
Example – Recognizing Digits



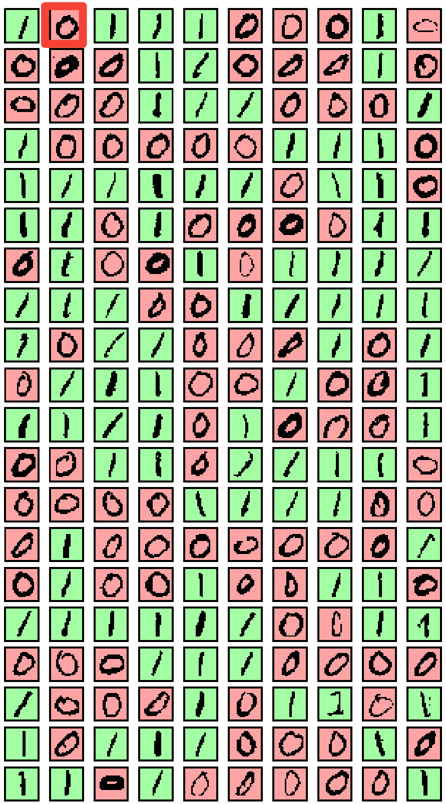
Training step 1

50% Accuracy

Weights as a 28x28 picture



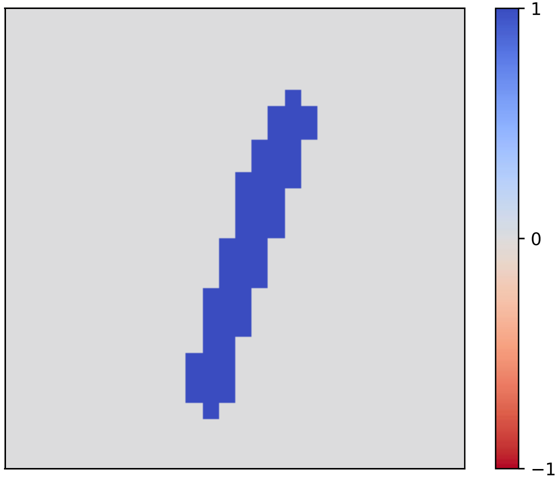
Example – Recognizing Digits



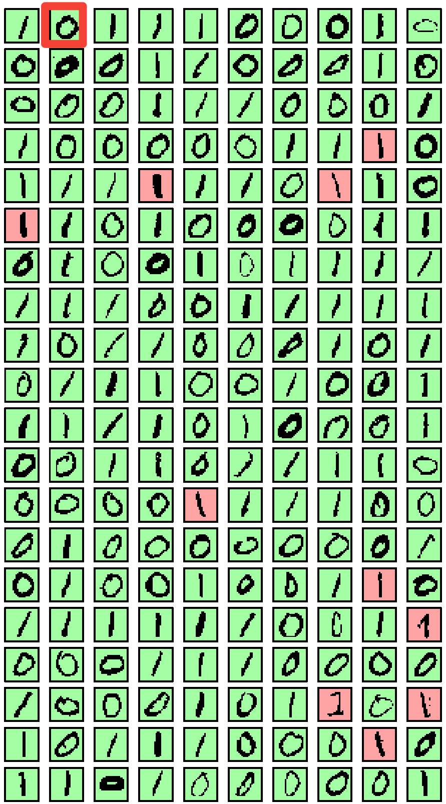
Training step 2

50% Accuracy

Weights as a 28x28 picture



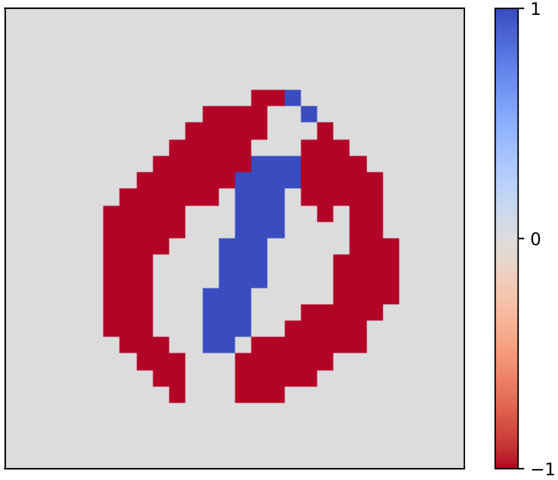
Example – Recognizing Digits



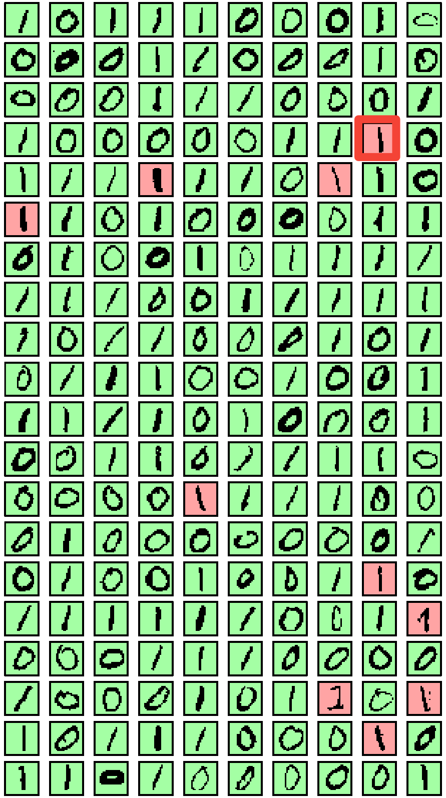
Training step 2

95% Accuracy

Weights as a 28x28 picture



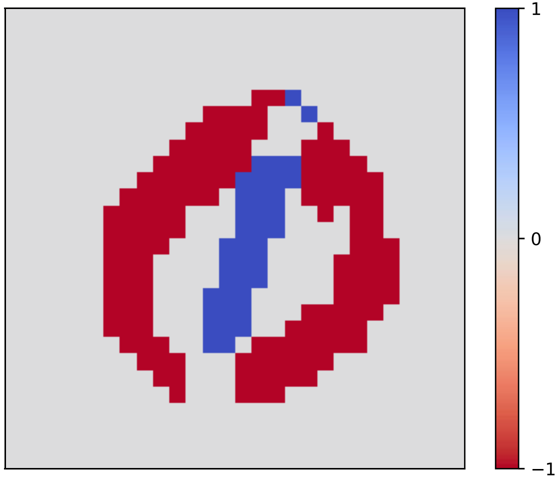
Example – Recognizing Digits



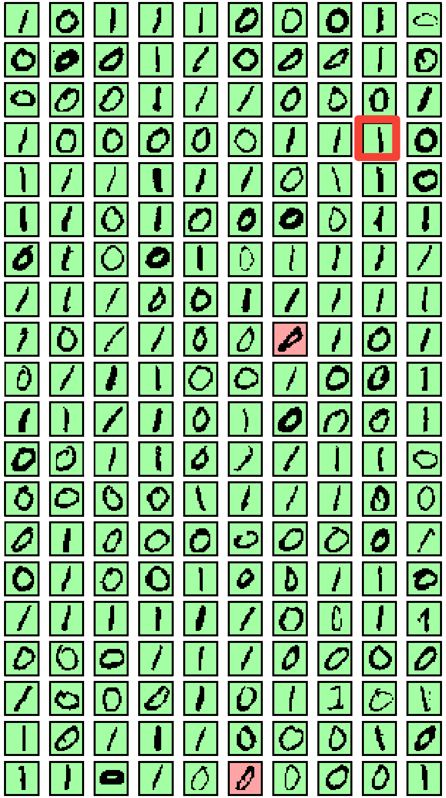
Training step 39

95% Accuracy

Weights as a 28x28 picture



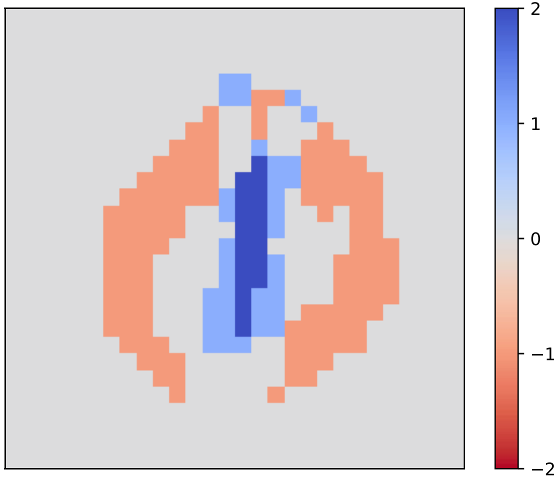
Example – Recognizing Digits



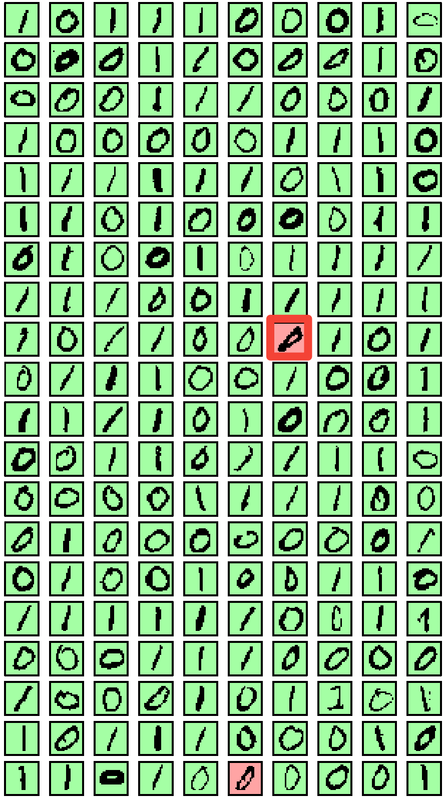
Training step 39

99% Accuracy

Weights as a 28x28 picture

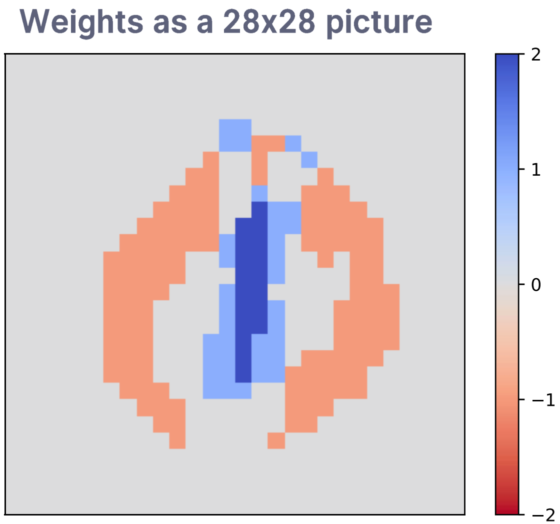


Example – Recognizing Digits

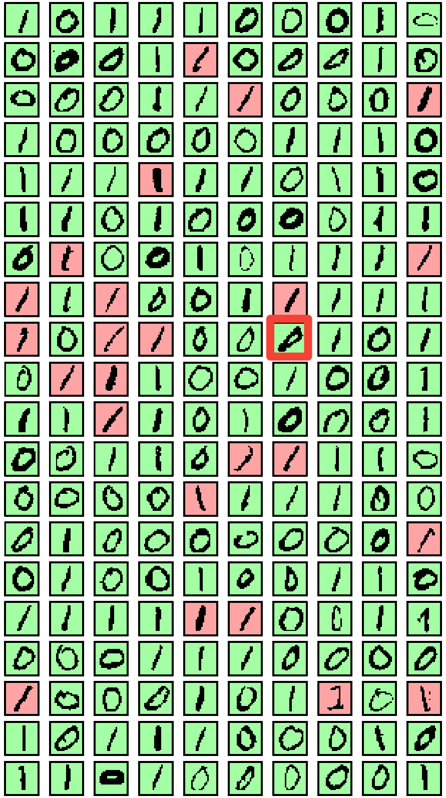


Training step 87

99% Accuracy



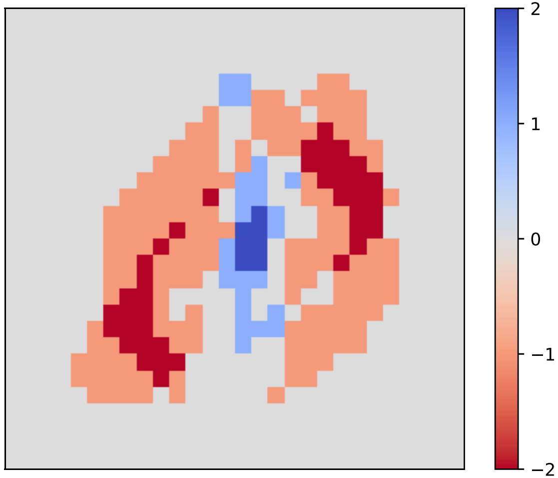
Example – Recognizing Digits



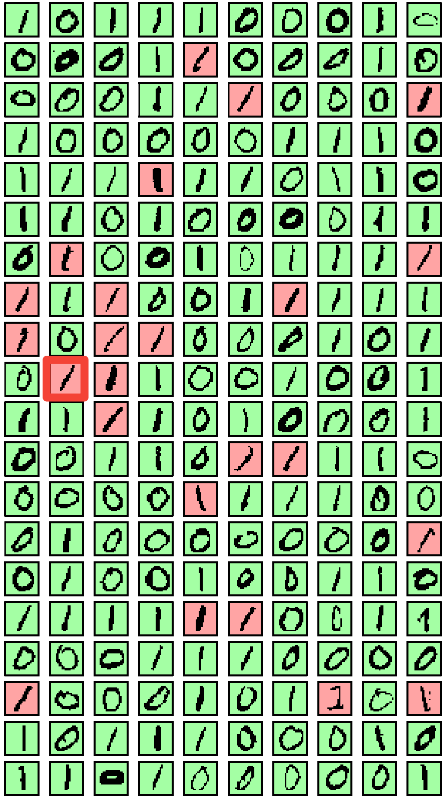
Training step 87

88% Accuracy

Weights as a 28x28 picture



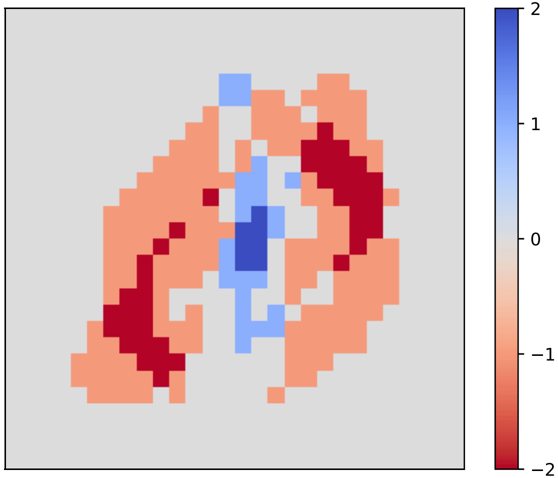
Example – Recognizing Digits



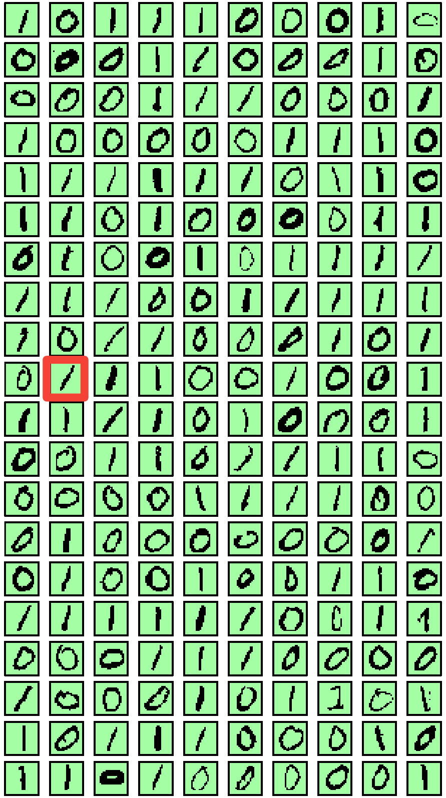
Training step 92

88% Accuracy

Weights as a 28x28 picture



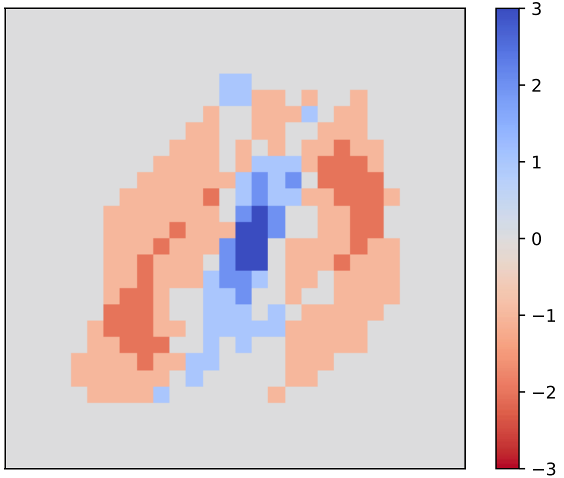
Example – Recognizing Digits



Training step 92

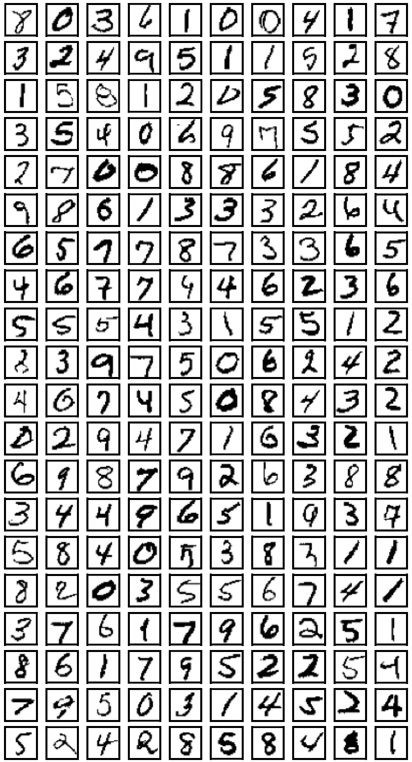
100% Accuracy

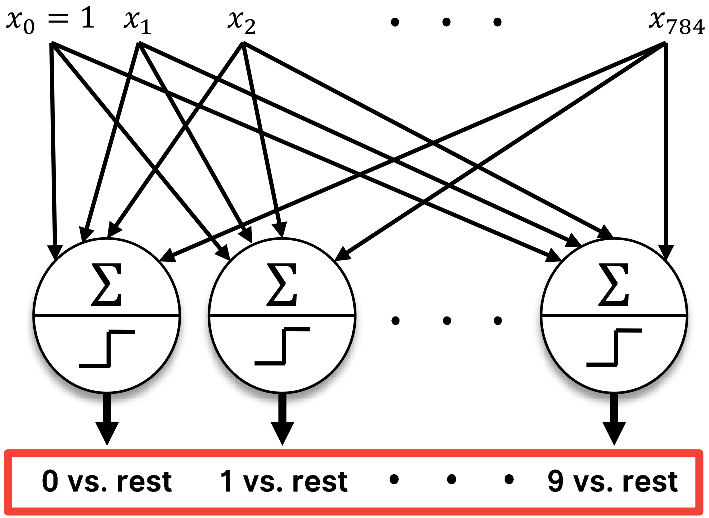
Weights as a 28x28 picture



Example – Recognizing All Digits

100 samples of each digit





One-hot encoding of predicted digit

- The perceptron can predict multiple classes by using a separate neuron for each class.
- It easily achives ~99% accuracy.
- Pretty impressive, right? Rosenblatt certainly thought so… ☺

Overconfidence in the Perceptron

Excerpts from **The New York Times**, July 8, 1958

NEW NAVY DEVICE LEARNS BY DOING

Psychologist Shows Embryo of Computer Designed to Read and Grow Wiser

WASHINGTON, July 7 (UPI) —The Navy revealed the embryo of an electronic computer today that it expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence.

The service said it would use this principle to build the first of its Perceptron thinking machines that will be able to read and write. It is expected to be finished in about a year at a cost of $100,000.

Dr. Frank Rosenblatt, designer of the Perceptron, conducted the demonstration. He said the machine would be the first device to think as the human brain. As do human beings, Perceptron will make mistakes at first, but will grow wiser as it gains experience, he said.

The Navy said the perceptron would be the first non-living mechanism "capable of receiving, recognizing and identifying its surroundings without any human training or control."

The "brain" is designed to remember images and information it has perceived itself. Ordinary computers remember only what is fed into them on punch cards or magnetic tape.

Later Perceptrons will be able to recognize people and call out their names and instantly translate speech in one language to speech or writing in another language, it was predicted.

- The article was published after statements made by Frank Rosenblatt in press conference held by the US Navy.
- http://jcblackmon.com/wp-content/uploads/2018/01/MBC-Rosenblatt-Perceptron-NYT-article.jpg.pdf

Recap

- **MuCulloch-Pitts model** – In 1943, *MuCulloch* and *Pitts* proposed a model for an *artificial neuron* which mimicked the behavior of the *biological neuron*.
  - The biological neuron has dendrites, a soma (body) and an axon.
  - The neuron “fires” (sends a pulse through its axon) if enough electrical potential has accumulated in the soma though its dendrites.
- **Hebb’s rule** – In 1949, *Donald Hebb* published the idea that connections between neurons have an associated *strength* and that strength changes based on how often one neuron is involved in firing the other.
- **The Perceptron** – In 1957, *Frank Rosenblatt* proposed an algorithm (and a machine) which was based on the *MuCulloch-Pitts model* and *Hebb’s rule* which could learn to recognize images.

Perceptron Limitations

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- It turns out that the *Perceptron* had no trouble learning AND/OR and recognizing digits because they are not very hard problems.

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- Let's consider the **XOR** (exclusive OR) function:

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

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For a Perceptron to learn XOR, it needs:

$$w_0 + 0 \cdot w_1 + 0 \cdot w_2 < 0$$
$$w_0 + 0 \cdot w_1 + 1 \cdot w_2 \geq 0$$
$$w_0 + 1 \cdot w_1 + 0 \cdot w_2 \geq 0$$
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$$w_0 + 0 \cdot w_1 + 0 \cdot w_2 < 0 \Longrightarrow w_0 < 0$$
$$w_0 + 0 \cdot w_1 + 1 \cdot w_2 \geq 0 \Longrightarrow w_0 \geq -w_2$$
$$w_0 + 1 \cdot w_1 + 0 \cdot w_2 \geq 0 \Longrightarrow w_0 \geq -w_1$$
$$w_0 + 1 \cdot w_1 + 1 \cdot w_2 < 0 \Longrightarrow w_0 < -w_1 - w_2$$

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$$w_0 + 0 \cdot w_1 + 0 \cdot w_2 < 0 \Rightarrow w_0 < 0$$

$$\left.\begin{aligned} w_0 + 0 \cdot w_1 + 1 \cdot w_2 \geq 0 \Rightarrow w_0 \geq -w_2 \\ w_0 + 1 \cdot w_1 + 0 \cdot w_2 \geq 0 \Rightarrow w_0 \geq -w_1 \end{aligned}\right\} 2w_0 \geq -w_1 - w_2$$

$$w_0 + 1 \cdot w_1 + 1 \cdot w_2 < 0 \Rightarrow w_0 < -w_1 - w_2$$

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1	1	0

For a Perceptron to learn XOR, it needs:

$$w_0 + 0 \cdot w_1 + 0 \cdot w_2 < 0 \Longrightarrow w_0 < 0$$

$$w_0 + 0 \cdot w_1 + 1 \cdot w_2 \geq 0 \Longrightarrow w_0 \geq -w_2$$

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$$w_0 + 1 \cdot w_1 + 1 \cdot w_2 < 0 \Longrightarrow w_0 < -w_1 - w_2$$

$$2w_0 \geq -w_1 - w_2$$

$$2w_0 > w_0$$

Perceptron Limitations

- It turns out that the *Perceptron* had no trouble learning AND/OR and recognizing digits because they are not very hard problems.
- Let's consider the **XOR** (exclusive OR) function:

x_1	x_2	y
0	0	0
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Contradiction

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$\Longrightarrow$ The Perceptron cannot possibly learn this function, no matter how many training steps it takes.

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- In fact, the *Perceptron* can only learn the class of problems known as **linearly separable**.

Perceptron Convergence Theorem

- If the training set E is *linearly separable* with a margin γ, the perceptron algorithm is **guaranteed to converge** to a state in which in makes no training mistakes in a **finite number of steps**.
- The number of steps k is at most ${}^{R^2}/_{\gamma^2}$, where R is the radius of the sphere enclosing all training examples.

$$k \leq \frac{R^2}{\gamma^2}$$

"On convergence proofs for perceptrons", A. Novikoff, 1962

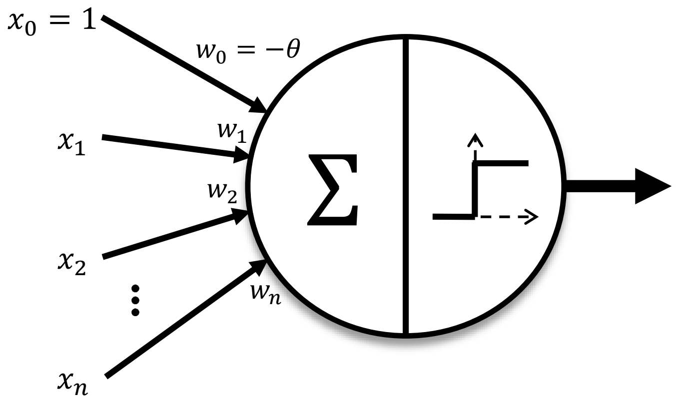
- However, the statement is that it will reach *some solution*, not necessarily a *good one* (like the *SVM*) and, if the data is not *linearly separable*, it will not converge to an "*approximate*" solution.

AI Winter

- Early 50s to mid 60s were a period of very strong optimism in the field of AI.
  - Distinguished figures, such as **Allan Turing, Claude Shannon, Frank Rosenblatt**, talked about human-level AI within a few years.
- The promised breakthroughs did not come as quickly as promised, so mid to late 60s saw an abrupt decline in interest and funding for AI research.
  - **Marvin Minsky** and **Seymour Papert**'s 1969 book **"Perceptrons: an introduction to computational geometry"**, which discussed the limitations of perceptrons in detail, is regarded as the main cause for the lack of funding in the field (especially for neural networks) for over a decade.
  - Mathematician **Sir James Lighthill** also criticized the utter failure of AI to achieve its "grandiose objectives.", mentioning that many of AI's supposedly successful algorithms were only suitable for „toy" problems.
- The period from late 60s to late 80s is known as the (first) **AI Winter**.

Improving the Perceptron

Simplifying Notation



$$\hat{y} = \begin{cases} 1 & \text{if} \sum_{i=0}^{n} w_i x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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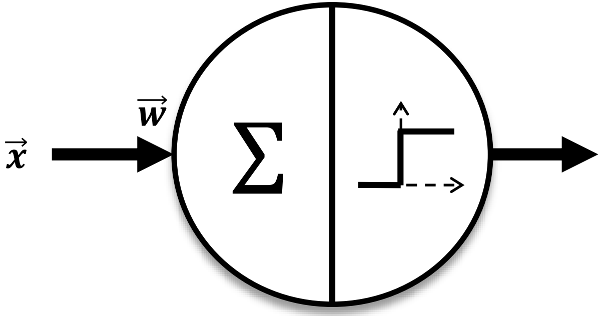
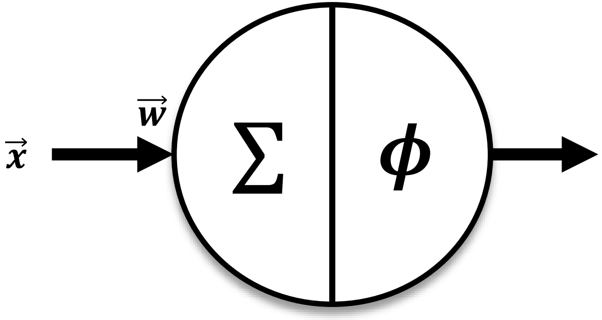


$$\hat{y} = \begin{cases} 1 & \text{if} \sum_{i=0}^{n} w_i x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Inputs and weights
in vector format.

Simplifying Notation



$$\hat{y} = \phi\left(\sum_{i=0}^{n} w_i x_i\right)$$

Inputs and weights in vector format.

Denote the *unit step function* by ϕ (and call it an “**activation function**”)

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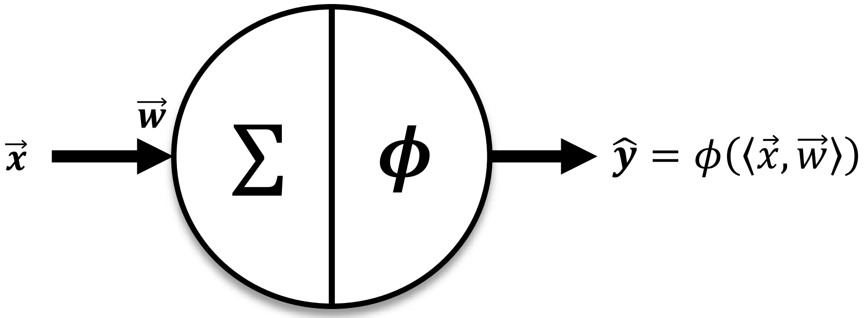

Inputs and weights in vector format.

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Weighted sum of inputs as *dot product*.

Better Justification for the Learning Rule

- Perceptron output:

$$\hat{y} = \phi(\langle \vec{x}, \vec{w} \rangle)$$

- *Squared-error* of the perceptron (for one example):

$$E(\vec{x}) = \frac{1}{2}(y - \hat{y})^2$$

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- **Gradient descent** (*Cauchy*, 1847) is a method of finding the minimum of a function by taking steps in the direction of the negative gradient.

$$\frac{\partial E(\vec{x})}{\partial w_j} =$$

We want to find the minimum of the error w.r.t. weights.

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E is function of $\hat{y}$ and $\hat{y}$ is a function of $w_j \Longrightarrow$ we apply **chain rule**.

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Chain rule again.

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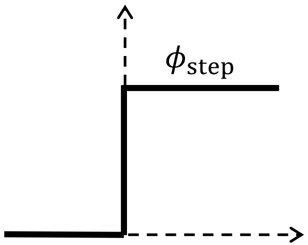
$$\boldsymbol{\Delta w_j = \eta \cdot (y - \hat{y}) \cdot \phi'(\langle \vec{x}, \vec{w} \rangle) \cdot x_j}$$

Delta rule

Applying the Delta Rule to the Perceptron

- Delta rule: $\Delta w_j = \eta \cdot (y - \hat{y}) \cdot \phi'(\langle \vec{x}, \vec{w} \rangle) \cdot x_j$



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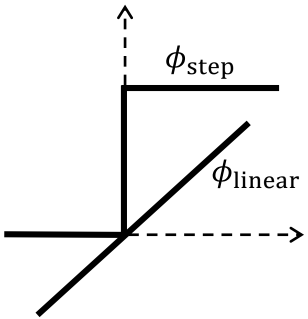

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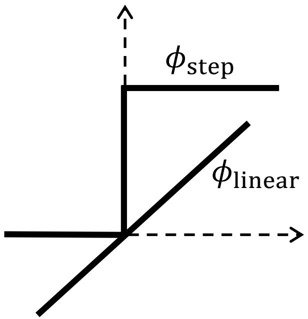

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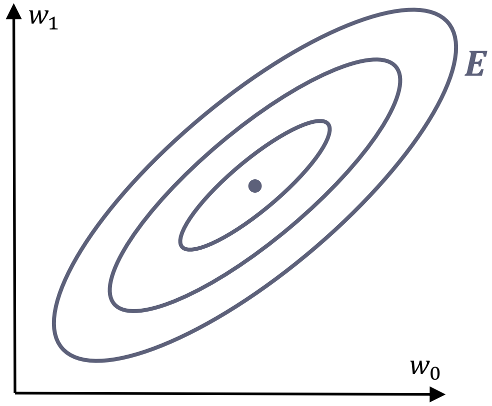
Very similar to the Perceptron update rule, but $\hat{y}$ is now real and unbounded.

Batch Update

- We are interested in finding weights to minimize the error over *all training samples*.
- In the original perceptron algorithm, each misclassified training sample caused an update in the weights.
    - Changing the weights in the direction indicated by a single example might not be optimal in terms of the whole training set.



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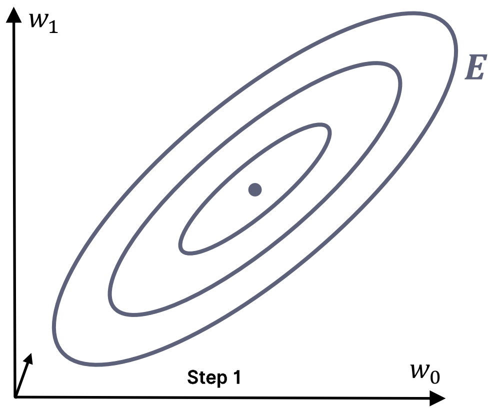

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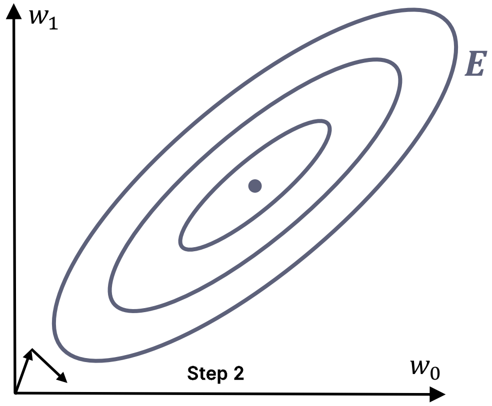

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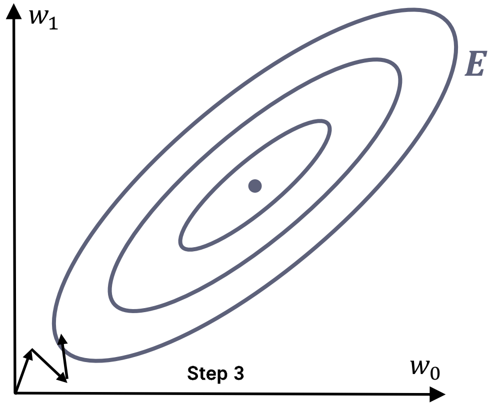

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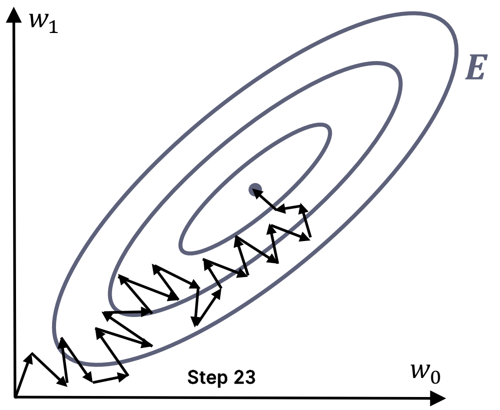

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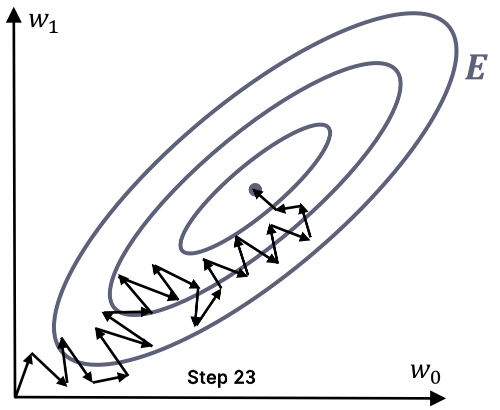
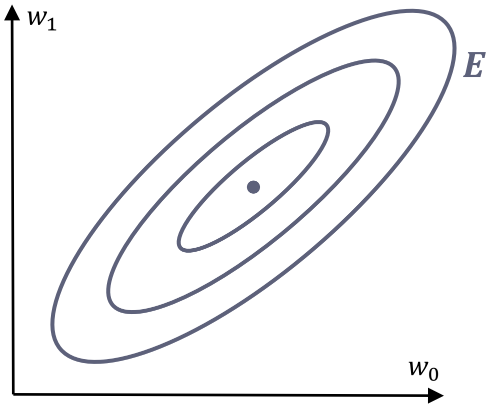
$$\Delta w_j = \frac{\eta}{b} \sum_{i_b=0}^{b-1} \left(y^{(i+i_b)} - \hat{y}^{(i+i_b)}\right) \cdot x_j^{(i+i_b)}$$

b – Batch Size
i_b - Batch Iterator





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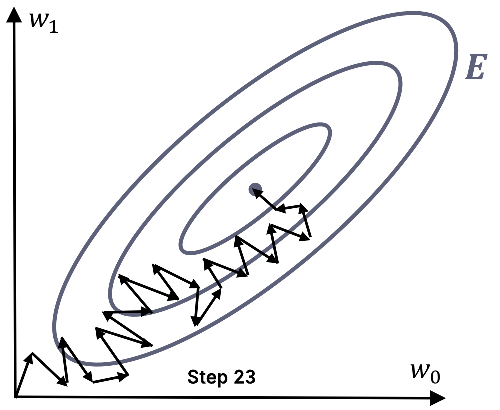

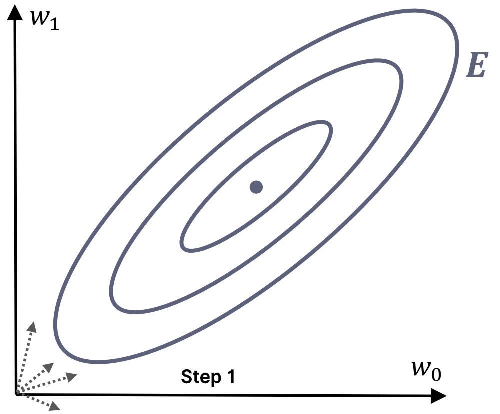

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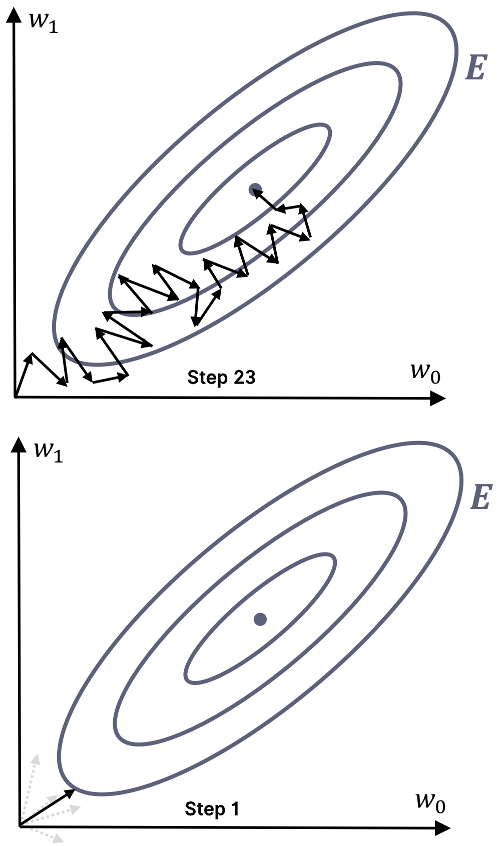

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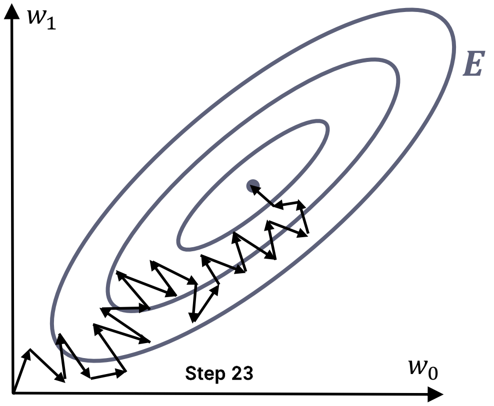

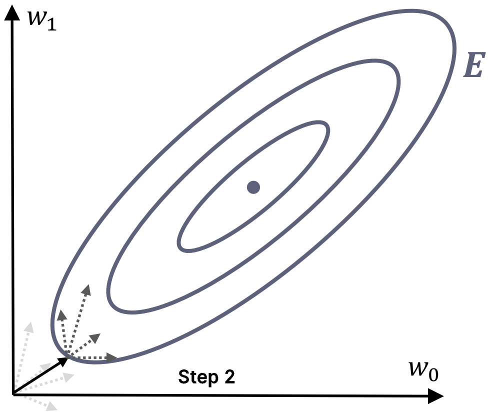

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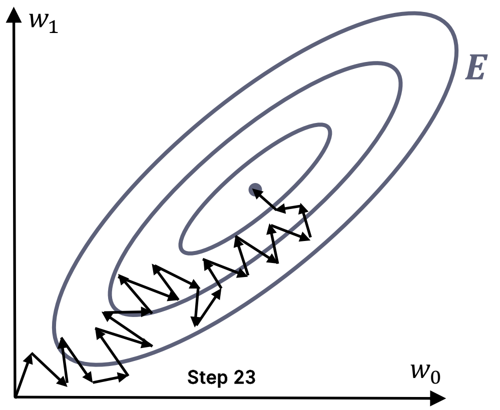
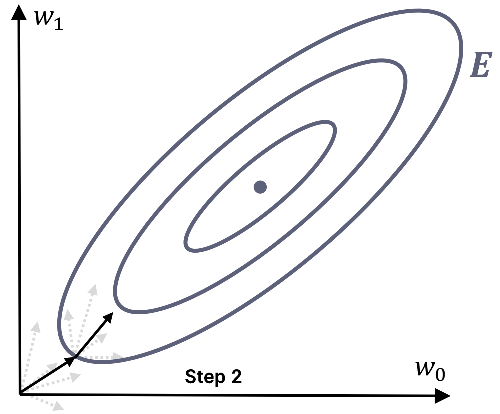
b – Batch Size
i_b - Batch Iterator



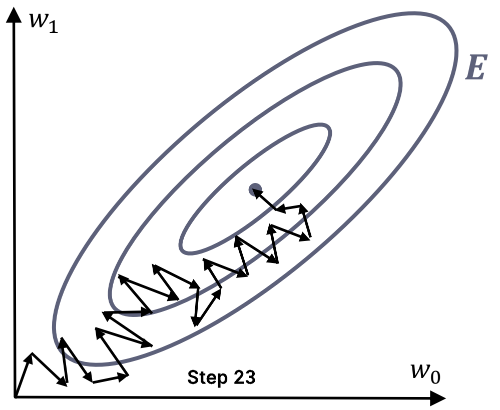


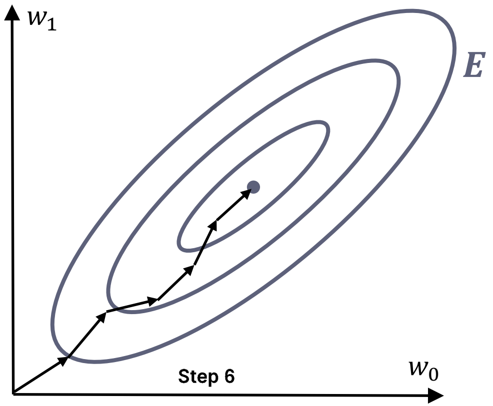
Batch Update

- We are interested in finding weights to minimize the error over *all training samples*.
- In the original perceptron algorithm, each misclassified training sample caused an update in the weights.
    - Changing the weights in the direction indicated by a single example might not be optimal in terms of the whole training set.
- **Batch update** means updating the weights only once with an average gradient over multiple training samples.

$$\Delta w_j = \frac{\eta}{b} \sum_{i_b=0}^{b-1} \left(y^{(i+i_b)} - \hat{y}^{(i+i_b)} \right) \cdot x_j^{(i+i_b)}$$

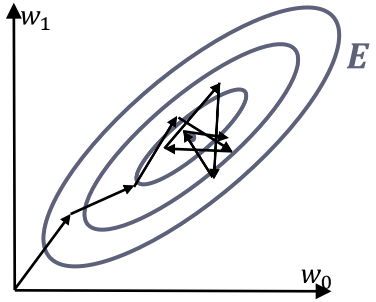
b – Batch Size
i_b - Batch Iterator



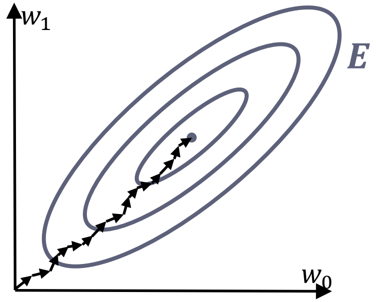


Influence of Learning Rate

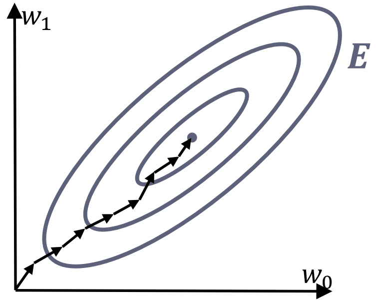
Learning rate too large
(It may fail to converge)



Learning rate too small
(It can take very long to converge)



Proper learning rate
(Compromise between speed and convergence)



Learning rate decay
(Start with a larger value and decrease it as training progresses)