- Linear Systems

Linear Algebra provides a way to complactly represent Linear Systems

$$4x_1 - 5x_2 = -13$$
$$-2x_1 + 3x_2 = 9$$

$\xrightarrow{\text{rewrite}}$ $Ax = b$

$$A = \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} -13 \\ 9 \end{pmatrix}$$

$$A \in \mathbb{R}^{m,n}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m$$

- Inner Products

if $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ then it's inner product is

$$c^T x = \sum_{i=1}^{n} c_i x_i \in \mathbb{R}$$

e.g. $x = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ $c = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$ $\rightarrow c^T x = -5 + 6 = 1$

- Outer Products

$cx^T \in \mathbb{R}^{n,n}$ is an outer product

$$cx^T = \begin{pmatrix} c_1x_1 & c_2x_1 & c_3x_1 \\ c_1x_2 & c_2x_2 & c_3x_2 \\ c_1x_3 & c_2x_3 & c_3x_3 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 2 \\ 0 & -5 & -2 \\ 0 & 15 & 6 \end{pmatrix}$$

- Matrix Multiplication

$$C = AB \quad A \in \mathbb{R}^{m,n}, \; B \in \mathbb{R}^{n,p} \Rightarrow C \in \mathbb{R}^{m,p}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

at least 3 different ways of thinking about matrix multiplication

1. w/ inner products

$$C = AB = \begin{pmatrix} — a_1^T — \\ — a_2^T — \\ \vdots \\ — a_m^T — \end{pmatrix} \begin{pmatrix} | & | & & | \\ b^1 & b^2 & \cdots & b^p \\ | & | & & | \end{pmatrix} = \begin{pmatrix} a_1^T b^1 & a_1^T b^2 & \cdots & a_1^T b^p \\ a_2^T b^1 & a_2^T b^2 & \cdots & a_2^T b^p \\ \vdots & \vdots & \ddots & \\ a_m^T b^1 & a_m^T b^2 & & a_m^T b^p \end{pmatrix}$$

2. w/ outer products

$$C = AB = \begin{pmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{pmatrix} \begin{pmatrix} — b_1^T — \\ — b_2^T — \\ \vdots \\ — b_n^T — \end{pmatrix} = \sum_{i=1}^{n} a^i b_i^T$$

3. matrix-vector products

$$C = AB = A \begin{pmatrix} | & | & & | \\ b^1 & b^2 & \cdots & b^p \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Ab^1 & Ab^2 & \cdots & Ab^p \\ | & | & & | \end{pmatrix}$$

- Trace

$$Tr(A) = \sum_{i=1}^{n} A_{ii}$$

exercise: check that $Tr(AB) = Tr(BA)$ $\quad A \in \mathbb{R}^{m,n}, B \in \mathbb{R}^{n,m}$

$$Tr(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} B_{ji} \right)$$

$$= \sum_{i=1}^{m} A_{ij} B_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} B_{ji} A_{ij} \right) = \sum_{j=1}^{n} (BA)_{jj} = Tr(BA)$$

Note: $AB \in \mathbb{R}^{m,m}$

$BA \in \mathbb{R}^{n,n}$

- Norms : informally a measure of length of a vector

More formally, a norm is any fn $f: \mathbb{R}^n \mapsto \mathbb{R}$ that satisfies the following 4 properties

1. $\forall x \in \mathbb{R}^n$, $f(x) \geq 0$ (non-negativity)
2. $f(x) = 0 \Leftrightarrow x = 0$ (definiteness)
3. $\forall x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t| f(x)$ (homogeneity)
4. $\forall x, y \in \mathbb{R}^n$, $f(x+y) \leq f(x) + f(y)$ (triangle inequality)

examples

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

note that $\|x\|_2^2 = x^T x$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_i |x_i|$$

Cauchy-Schwartz

$$u^T v \leq \|u\|_2 \|v\|_2$$