Investment, Finance and Asset Pricing
Group Assignment

Question 1.1

Write down the Bellman equation and derive the optimal investment decision condition. Define marginal Q and provide an economic interpretation. How is average Q related to marginal Q for this firm?

The Bellman equation is a dynamic programming approach to solve sequential optimization problems.

In sequential decision-making, the Bellman equation breaks down a complex problem into simpler, recursive subproblems. It is mainly used for problems where decisions at one point in time affect outcomes and decisions in the future. This dynamic approach finds the optimal strategy by working backward, often referred to as backward induction.

Firm is dealing with adjustment cost model and faces following sequential maximization problem:

$$max_{K_{t+1}} E_0 \sum_{t=o}^{\infty} \beta^t [\pi(\theta_t, K_t) - p\, I_t) - C(I_t, K_t)]$$

Subject to constraint:

$$K_{t+1} = K_t + I_t$$

Substituting the constraint, maximization problems becomes:

$$max_{K_{t+1}} E_0 \sum_{t=o}^{\infty} \beta^t [\pi(\theta_t, K_t) - p\,(K_{t+1} - (1 - \delta)K_t) - \frac{\gamma}{2}(K_{t+1} - (1 - \delta)K_t)^2]$$

As there is a theta in the maximization equation, which is a time varying stochastic process, we need to take into account the expectation for the next period value, as there is randomness generated by it.

In the firm's problem, time goes to infinity, which means that the future from today and future from tomorrow has the same length, therefore the firm has only 2 periods: current and next. Taking also that the depreciation rate is equal to 0, the recursive functional equation of the model written by Bellman equation is:

$$V(K) \ = \ max_{K',\,I}[\pi(K) \ - \ p\,I \ - \ \frac{\gamma}{2}I^2 \ + \ \beta\,E[V(\,K')]]$$

Where *K* is a state variable and represents the capital in the current period. I and K' are choice or control variables and are current period's investment and the next period's capital respectively.

Substituting the capital accumulation constraint we get:

$$V(K) \ = \ max_{K'}[\pi(K) \ - \ p\,(K' \ - \ K) \ - \ \frac{\gamma}{2}(K' \ - \ K)^2 \ + \ \beta\,E[V(\,K')]]$$

In order to find the optimal investment decision condition, we must calculate the first order condition(FOC) w.r.t the next capital:

$$\frac{\partial V(K)}{\partial K'} \ = \ 0$$

$$p \ + \ \Upsilon I \ = \ \beta\,E[\frac{\partial V(K')}{\partial K'}]$$

LHS: Marginal cost of capital accumulation, that consists of direct cost of new capital and marginal adjustment cost.

RHS: Expected discounted marginal benefit, that shows gains from the change of the firm value.

RHS is also the Marginal Q that represents the shadow price of an additional unit of capital of the firm and is denoted by q :

$$q \ = \ \beta\,E[\,V_{K'}(K')] \ = p \ + \ \Upsilon I$$

From the formula we get that in order to have optimal investment, marginal cost of the investment must be equal to expected discounted marginal benefit. In case when benefit is greater than marginal cost, the firm has the opportunity to invest more, else if cost is higher, it must invest less, since cost does not justify benefit.

From the FOC we get that optimal investment is:

$$I \ = \ \frac{\beta\, E[\, V_{K'}(K')] - p}{\Upsilon}$$

It states that in order to have positive investment, Marginal Q must be greater than the price of capital unit. If we take the price as standardized, meaning p=1, then Marginal Q must be greater than 1 to have optimal investment.

Derivative of value function w.r.t K using Envelope theorem:

$$\frac{\partial V(K)}{\partial K} \ = \frac{\partial \pi(K)}{\partial K} + \ p \ + \ \gamma(K' \ - \ K) \ = e^{\overline{A}}\, \theta\, K^{\theta - 1} \ + \ p \ + \ \gamma I$$

Taking derivative one period forward:

$$\frac{\partial V(K')}{\partial K'} \ = \frac{\partial \pi(K')}{\partial K'} + \ p \ + \ \gamma I'$$

Above we calculated that

$$q \ = \ \beta\, E[\, V_{K'}(K')] \ = p \ + \ \Upsilon I$$

Substituting back will lead to:

$$p \ + \ \Upsilon I \ = \ \beta\, E[\frac{\partial \pi(K')}{\partial K'} + \ p \ + \ \gamma I']$$

In terms of marginal Q:

$$q \ = \ \beta\, E\, [\frac{\partial \pi(K')}{\partial K'} + \ q']$$

Q represents the shadow value of an additional unit of invested capital and is equal to the expected discounted sum of additional marginal profits in the next period generated by one more unit of capital today and shadow price of capital in the next period. From the equation it can be implied that the future uncertainty represented by q’ also affects the current productivity.

Nevertheless, testing an investment model with marginal q can be difficult as it is not observable. According to Hayashi, if some conditions hold, marginal q can be replaced with average Q which is equal to the market value of the firm divided by firm capital stock.

Firstly the firm must be a price taker in the market. As in our case the firm is maximizing its value, it can be implied that the firm takes prices and is not able to control them.

Secondly production function of the firm must be linearly homogeneous, that is

$$\pi(\lambda K) = \lambda\ e^{\overline{A}} K^{\theta}$$

$$\pi(\lambda K) = e^{\overline{A}} (\lambda K_t)^{\theta} = \lambda^{\theta} e^{\overline{A}} K^{\theta}$$, profit function of the firm

So the production function is linearly homogeneous if and only if the elasticity of output with respect to capital is equal to 1. So generally we cannot assume that function is linearly homogeneous, as theta is a parameter and can change.

Lastly, cost function must be linearly homogeneous too.

$$C(\lambda I\ ,\ \lambda K\ ) = \lambda \frac{\gamma}{2} I^2$$

$$C(\lambda I\ ,\ \lambda K\ ) = \frac{\gamma}{2} (\lambda I)^2 = \lambda^2 \frac{\gamma}{2} I^2$$, cost function of the firm

Cost function is homogeneous of degree 2, so the third condition is not satisfied either.

Taking into account all the points, we can conclude that for the firm average Q is not equal to marginal Q. That inequality also can be seen from Figure 1. Mostly it is because of convex adjustment costs. Marginal Q contains additional cost of investment, while average (Q) neglects that, so the latter will be greater than former.

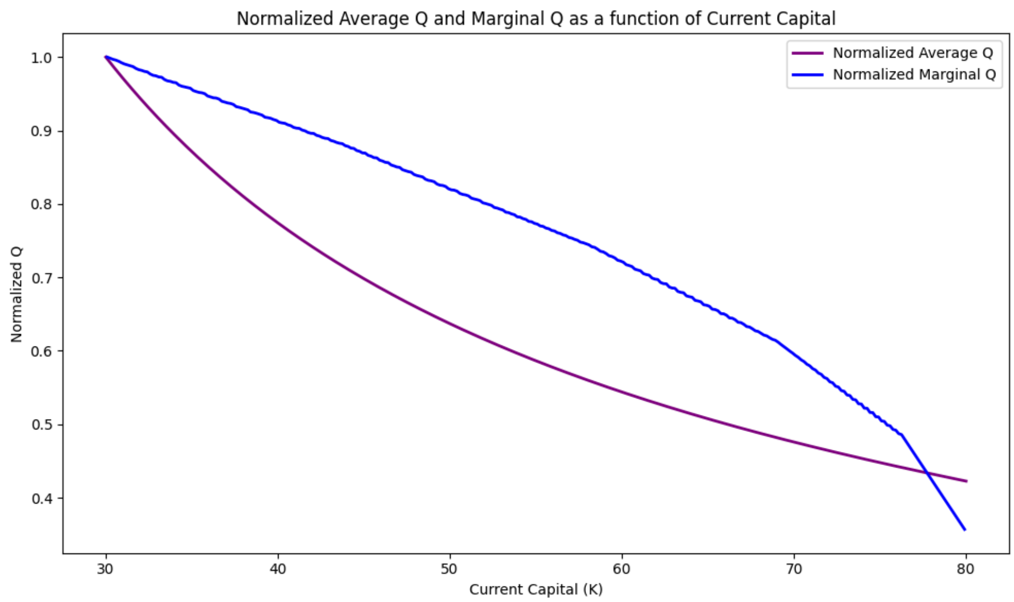

Figure 1. Normalized Marginal and Average Q

Question 1.2

Solve the Bellman equation using dynamic programming. You can calibrate model parameters as follows

- β = 0.95, θ = 0.7, γ = 0.2 and p = 1.2. Discretize capital grid with 301 uniformly spaced points in the interval [30,80]. Constant Productivity Level is $\overline{A}$ = 1.5.

Plot value function and investment policy functions. Interpret these graphs.

From the first question we get that Bellman equation for the value function of the firm is following:

$$V(K) \ = \ max_{K'}[\pi(K) - p\,(K' - K) \ - \ \frac{\gamma}{2}(K' - K)^2 \ + \ \beta\, E[V(\,K')]]$$

But since here the theta is given, there is no need for the expectation, as the randomness is eliminated. So the Bellman equation to solve will be:

$$V(K) \ = \pi(K) - p\,(K' - K) \ - \ \frac{\gamma}{2}(K' - K)^2 \ + \ \beta\, V(\,K')$$

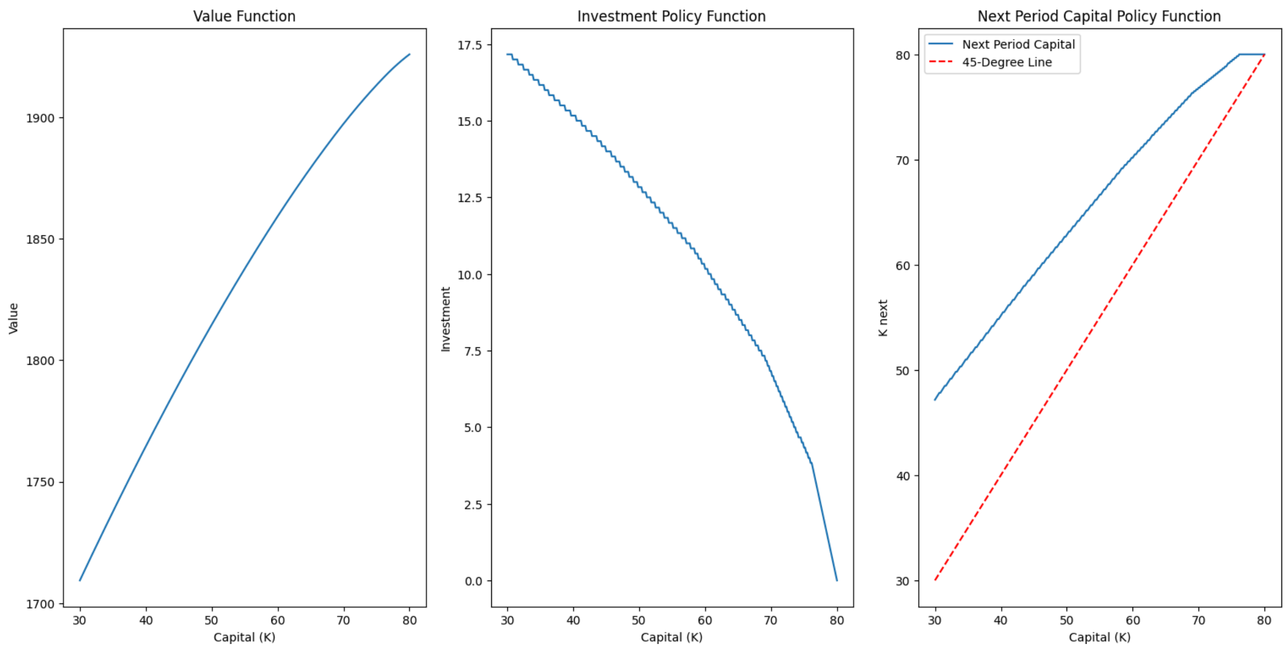

Figure 2. Value and policy functions

First plot of Figure 2 is a value function that shows maximum values of profits for each capital level. With the growth of capital, the value of the firm is also increasing, since the firm's profit function depends on K. Value function is concave which means that every additional unit of capital will contribute less to the overall value of the firm. This shows diminishing returns.

The investment policy function is the second plot from FIgure 2 and it is decreasing with the increase of K, which is because higher investment requires higher adjustment costs that outweighs gains from the investment. When a firm has a high capital level, the benefit from further investment is reduced as adjustment cost is quadratically proportional to investment, and therefore subtracting higher adjustment cost from the profit will lead to lower investment level.

Last plot of Figure 2 is the next period capital policy function is increasing, which is quite intuitive, as the higher current capital generally leads to the higher capital in the next period. It can also be seen from capital accumulation constraint:

$$K' = K + I$$

Assuming optimal and positive continuous investment.

As capital reaches higher levels, next period capital begins to approach current period capital, which is in line with decreasing investment and diminishing returns.

To conclude, as a firm has lower capital, it invests heavily. When capital reaches an optimal level, investment slows down. At a higher capital level, the firm invests only a little.

Explain how optimal investment responds to

- changes in the adjustment cost parameter.
- changes in how the firm manager values time.

From optimal investment:

$$I = \frac{\beta * V_{K'}(K') - p}{\Upsilon}$$

I is directly related to the discount factor and inversely related to the adjustment cost parameter. P is equal to 1.2 for the firm.

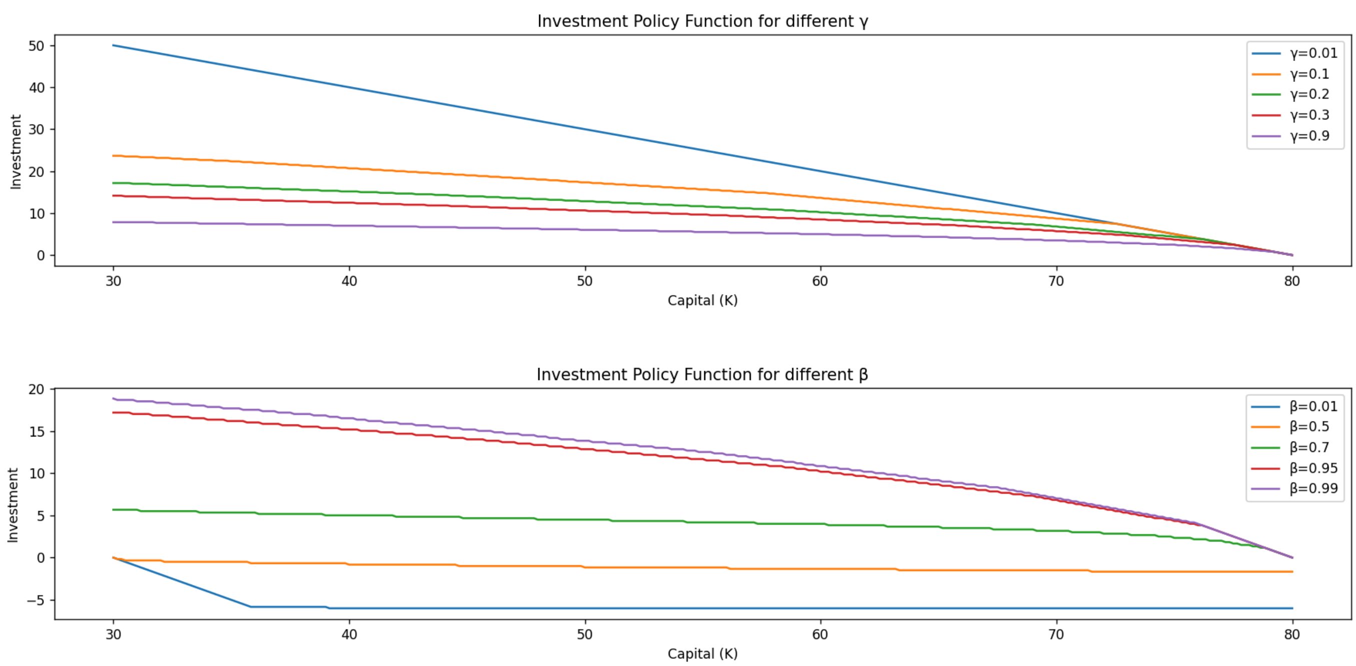

Figure 3. Investment policy function according to different adjustment costs and discount factors

The first plot of Figure 3 illustrates investment responses to five different levels of adjustment costs. As the adjustment cost increases the investment decreases, indicating how high adjustment costs can discourage investment. It means that higher adjustment costs make it more costly for the firm to change or add its capital. As a result, the firm makes less investment as capital is growing. With higher adjustment costs investment is getting flat even for the lower capital levels, remaining a positive sign though. For lower values of adjustment cost, investment function is steeper, which indicates that investment is more responsive to capital changes.

Value of time is represented by the discount factor, which shows how the manager values profit in the next period compared to the current period. Higher discount factor means future profit is priced more, so investment is encouraged and the firm is more forward looking. In case of lower factors current value is more prioritized, so investment is appreciated less. Second plot of Figure 3 depicts the direct relationship of discount factor and investment. High discount factors lead to steeper investment function. In case of low discount factors, investment flattens and even turns to be negative, which means selling is more preferable for the firm than investing, that is the firm is interested in immediate gains.

In conclusion, from the graphs it can be seen that the overall investment policy function's behavior is highly sensitive and dependent on adjustment cost and discounting factor.

Question 1.3

How does marginal Q and optimal investment decision condition change in 1.1 if we assume that investments instantaneously turn into capital, in other words there is no delay in investments becoming productive capital?

If investment becomes capital without delay formula for capital will be:

$$K' = K + I'$$

In 1.1 we have found that optimal decision is the following:

$$p + \Upsilon I = \beta E[V_{K'}(K')]$$

That is, the marginal cost of one additional unit of capital that has been invested will be equal to expected discounted marginal benefit that that investment will bring in the next period as we had a capital delay of one period.

In this case when there is no gap in investment becoming capital, marginal cost of the investment in the current period must be equal to marginal profit from that additional invested unit in the current period. Discounting, as well as expectation become unnecessary, as there is no delay. Since the investment immediately has an impact on the current period capital, there is no need to take future values into account, since they are irrelevant to the current period.

Value function will be:

$$V(K) = max_I [\pi(K + I) - p\, I - \frac{\gamma}{2} I^2]$$

Calculating FOC w.r.t investment lead to the following decision:

$$p + \Upsilon I = \frac{\partial \pi(K)}{\partial K} = e^{\overline{A}} \theta\, (K + I)^{\theta - 1}$$

So in this scenario the firm's main objective becomes to maximize the immediate profit, as there is no long-term capital accumulation to consider.

Marginal Q in the previous questions was defined as

$$q = \beta\, E\, [\frac{\partial \pi(K')}{\partial K'} + q']$$

and represented the shadow value of capital containing marginal profit from the next period and some future uncertainty represented by marginal q from the next period.

In the scenario where there is no investment delay there is no future uncertainty and discounting either. Hence, marginal Q will be defined as the marginal current period value of additional capital generated by investment. It becomes an immediate measure to value the increase of profit.

$$q = \frac{\partial \pi(K)}{\partial K} = e^{\overline{A}}\, \theta\, K^{\theta - 1}$$

This change would likely lead to more volatile investment behavior, since the firm can adjust its capital more quickly in response to changes in productivity or market conditions.

References

1. Adda, J., & Cooper, R. Dynamic Economics: Quantitative Methods and Applications. MIT Press, 2003.
2. Ramanan, Sisir. Lecture Notes on Investment, Finance & Asset Pricing. University of Glasgow, 2024.
3. Investment, Finance & Asset Pricing: Lab 3 materials. University of Glasgow, 2024.
4. Hayashi, Fumio. "Tobin's Marginal q and Average q: A Neoclassical Interpretation." Econometrica, vol. 50, no. 1, 1982, pp. 213-224.

Appendix: Python Implementation

```
import numpy as np
import matplotlib.pyplot as plt


### Function calculates and returns
# converged value function and optimal capital level policy
def get_V_and_policy(adj_cost, discount, max_iter=1000, tolerance=1e-6):
    # Initial guesses
    V_new = np.zeros(K_points)
    V_old = np.zeros(K_points)
    policy_K_next = np.zeros(K_points)

    exp_prod_level = np.exp(prod_level)

    for i in range(max_iter):
        for j in range(K_points):
            investment = K_grid - K_grid[j]
            values = exp_prod_level * K_grid[j]**out_elasticity -
k_unit_price*investment - (adj_cost/2)*investment**2 + discount*V_old
            V_max = np.max(values)
            V_new_index = np.argmax(values)
            V_new[j] = V_max
            policy_K_next[j] = K_grid[V_new_index]

        if np.max(np.abs(V_old - V_new)) < tolerance:
            print(f"Converged after {i+1} iterations")
            break

        V_old = V_new.copy()

    return V_new, policy_K_next


### Function plots value function, investment and next period policies
# in the subplots with initial capital grid as the x-axis
def plot_V_policies(K_grid, V, policy_investment, policy_K_next):
    plt.figure(figsize=(15, 10))
```

```
    plt.subplot(1, 3, 1)
    plt.plot(K_grid, V)
    plt.title('Value Function')
    plt.xlabel('Capital (K)')
    plt.ylabel('Value')

    plt.subplot(1, 3, 2)
    plt.plot(K_grid, policy_investment)
    plt.title('Investment Policy Function')
    plt.xlabel('Capital (K)')
    plt.ylabel('Investment')

    plt.subplot(1, 3, 3)
    plt.plot(K_grid, policy_K_next, label="Next Period Capital")
    plt.plot(K_grid, K_grid, 'r--', label="45-Degree Line")
    plt.title("Next Period Capital Policy Function")
    plt.xlabel('Capital (K)')
    plt.ylabel("K next")
    plt.legend()

    plt.tight_layout()
    plt.show()


### Function calculates and returns investment policy
# with corresponding changed parameters
def analyze_sensitivity(param_name, param_values):
    results = []

    for value in param_values:
        if param_name == 'adj_cost':
            adj_cost_local, discount_local = value, discount
        elif param_name == 'discount':
            discount_local, adj_cost_local = value, adj_cost

        _, policy_K_next = get_V_and_policy(adj_cost_local,
discount_local)

        investment = policy_K_next - K_grid
        results.append((value, investment))
```

```
    return results


### Function analyzes and plots investment policy sensitivity
# to different adjustment costs and discount factors
def plot_sensitivities():
    adj_cost_values = [0.01, 0.1, 0.2, 0.3, 0.9]
    adj_cost_results = analyze_sensitivity('adj_cost', adj_cost_values)

    plt.subplot(2, 1, 1)
    for value, inv in adj_cost_results:
        plt.plot(K_grid, inv, label=f'γ={value}')
    plt.title('Investment Policy Function for different γ')
    plt.xlabel('Capital (K)')
    plt.ylabel('Investment')
    plt.legend()

    discount_values = [0.01, 0.5, 0.7, 0.95, 0.99]
    discount_results = analyze_sensitivity('discount', discount_values)

    plt.subplot(2, 1, 2)
    for value, inv in discount_results:
        plt.plot(K_grid, inv, label=f'β={value}')
    plt.title('Investment Policy Function for different β')
    plt.xlabel('Capital (K)')
    plt.ylabel('Investment')
    plt.legend()

    plt.tight_layout()
    plt.show()


### Function calculates and plots marginal and average Q
def calc_plot_Q():
    # Calculating marginal Q with the use of derivative definition
    # f'(x) = lim Δx->0 ((f(x+Δx) - f(x)) / Δx)
    marginal_Q = discount * (np.diff(V) / np.diff(K_grid))

    # Aligning marginal Q with capital interval midpoints for plotting
```

```
    K_midpoints = (K_grid[:-1] + K_grid[1:]) / 2

    average_Q = V / K_grid

    # Normalizing to plot in the same axis for comparison
    average_Q_normalized = average_Q / np.max(average_Q)
    marginal_Q_normalized = marginal_Q / np.max(marginal_Q)

    plt.figure(figsize=(10, 6))

    plt.plot(K_grid, average_Q_normalized, label="Normalized Average Q",
color="purple", linewidth=2)
    plt.plot(K_midpoints, marginal_Q_normalized, label="Normalized
Marginal Q", color="blue", linewidth=2)

    plt.xlabel("Current Capital (K)")
    plt.ylabel("Normalized Q")
    plt.title("Normalized Average Q and Marginal Q as a function of
Current Capital")
    plt.legend()

    plt.tight_layout()
    plt.show()


### Program entry point
### While running the next plot will not appear
# unless the current one is closed
if __name__ == '__main__':
    # Defining initial parameters for the model
    discount = 0.95
    out_elasticity = 0.7
    adj_cost = 0.2
    k_unit_price = 1.2
    prod_level = 1.5

    # Constructing  capital grid
    K_min, K_max = 30, 80
    K_points = 301
    K_grid = np.linspace(K_min, K_max, K_points)
```

```
# Number of maximum iterations and convergence tolerance
max_iter = 1000
tolerance = 10**-8

V, policy = get_V_and_policy(adj_cost, discount, max_iter, tolerance)
investment = policy - K_grid

plot_V_policies(K_grid, V, investment, policy)

calc_plot_Q()

plot_sensitivities()
```