Numerical Methods AA Student ID:

Midterm Exam Name:

2015.05.05

1. Let the function f be defined by $f(x) = -x^3, x^2, \sqrt{x}$ on $(-\infty, 0], (0, 1], (1, \infty)$, respectively.
   (a) (10 pts) Write an R function that inputs x and returns $f(x)$.

   (b) (10 pts) Using R built-in functions **`seq`**, **`sapply`**, **`plot`**, and the above function, write a simple program to plot the graph of f over a suitably chosen interval.

2. Suppose we represent Pascal's triangle as a list, where item n is row n of the triangle. For example, Pascal's triangle to depth four would be given by

```
list(c(1), c(1, 1), c(1, 2, 1), c(1, 3, 3, 1))
```

The n-th row can be obtained from row $n - 1$ by adding all adjacent pairs of numbers, then prefixing and suffixing a 1.

(a) (10 pts) Complete the following R function that inputs a Pascal's triangle to depth n and returns a Pascal's triangle to depth $n + 1$.

```
pascal_increment <- function(pt) {
  # adds a row to Pascal's triangle
  # pt is assumed to be a list of the form
  # {{1}, {1, 1}, {1, 2, 1}, {1, 3, 3, 1}, ...}
  # with length at least 1
  n <- length(pt)
  if (n == 1) return(list(1, c(1, 1)))

  # R codes to be completed below





  return(pt)
 }
```

(b) (10 pts) Using a while/for loop that calls `pascal_increment` repeatedly, complete the following R function that inputs an integer n and returns a Pascal's triangle to depth n.

```
pascal <- function(n) {
  # returns Pascal's triangle to depth n
  n <- max(1, floor(n))
  pt <- list(1)

  # R codes to be completed below





  return(pt)
 }
```

3. (Choose one from questions 3 and 4) Consider the problem of finding the numerical integral of $f(x) = x^3$ over the interval [1, 3] using the Simpson's rule that divides the interval into 2 subintervals.

   (a) (15 pts) Approximate the integrand f over the interval [1,3] by an interpolation polynomial of degree two in the Lagrange form. Simplify this polynomial $p(x)$ into the form $a_2 x^2 + a_1 x + a_0$.

   (b) (10 pts) Find the numerical integral of $f(x) = x^3$ over the interval [1, 3] using the abovementioned Simpson's rule by evaluating the definite integral $\int_1^3 p(x)dx$

   (c) (5 pts) Find the exact value of the definite integral of $f(x) = x^3$ over the interval [1, 3] and determine the absolute and relative errors of the numerical integral obtained in (b).

4. (Choose one from questions 3 and 4) Let p be the quadratic $p(x) = c_0 + c_1 x + c_2 x^2$. Simpson's rule uses a quadratic to approximate a given function f over two adjacent intervals, then uses the integral of the quadratic to approximate the integral of the function.

   (a) (5 pts) Show that $\int_{-h}^{h} p(x)\,dx = 2hc_0 + \frac{2}{3}c_2 h^3$.

   (b) (15 pts) Write down three equations that constrain the quadratic to pass through the points $(-h, f(-h))$, $(0, f(0))$, and $(h, f(h))$, then solve them for c_0 and c_2.

   (c) (10 pts) Show that $\int_{-h}^{h} p(x)\,dx = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right)$.

5. You have learned in the class how to numerically integrate a definite integral using trapezoidal rule, Simpson's rule and adaptive quadrature rule.
   (a) (15 pts) When using the Simpson's rule and the adaptive quadrature rule, we need some rule for choosing the size of the partition or the length of the subintervals which results in reasonably accurate approximation. Let $\varepsilon > 0$ be the desired tolerance. Describe the stopping conditions used in the Simpson's rule and the adaptive quadrature rule, respectively.
   (b) (15 pts) Describe some methods of change of variables to transform integrals over infinite intervals, such as $[a, \infty)$, $(-\infty, a]$, and $(-\infty, \infty)$, to definite integrals over finite intervals. Briefly justify these methods.
   (c) (10 pts) Describe how to approximate a definite integral by the method of Monte Carlo integration.

6. (10 pts) The following R code evaluates the integral of $1.5\sqrt{x}$ over the interval [0, 1] using the adaptive quadrature rule.

```
> ftn <- function(x) return(1.5*sqrt(x))
> quadrature(ftn, 0, 1, tol = 1e-3, trace = TRUE)
integral over [0, 0.0009765625] is 3.005339e-05 (at level 11)
integral over [0.0009765625, 0.001953125] is 5.579888e-05 (at level 11)
integral over [0.001953125, 0.00390625] is 0.0001578231 (at level 10)
integral over [0.00390625, 0.0078125] is 0.000446391 (at level 9)
integral over [0.0078125, 0.015625] is 0.001262585 (at level 8)
integral over [0.015625, 0.03125] is 0.003571128 (at level 7)
integral over [0.03125, 0.0625] is 0.01010068 (at level 6)
integral over [0.0625, 0.125] is 0.02856903 (at level 5)
integral over [0.125, 0.25] is 0.08080541 (at level 4)
integral over [0.25, 0.5] is 0.2285522 (at level 3)
integral over [0.5, 1] is 0.6464433 (at level 2)
final value is 0.9999944 in 45 function evaluations
[1]  0.9999944 45.0000000
```

Explain why the integral over [0.5, 1] is 0.6464433 (at level 2), and why the integral over [0.25, 0.5] is 0.2285522 (at level 3).