RANKING AND RISK-TAKING: THE CASE OF HEDGE FUNDS

Abstract. This paper models risk taking by hedge fund managers facing both absolute and rank-based performance incentives. Managers are free to choose return distributions subject only to a value conservation constraint determined by managerial ability. The analysis shows that rank-based rewards, typically provided by funds flow/management fees–ranking relationship, engender a *win small* incentive that counters the gambling incentives generated by the high-water mark compensation. The endogenously determined distribution of returns matches the qualitative features of observed hedge fund returns, e.g., returns are positively skewed. Increasing the weight the fund manager places on rank relative to absolute performance increases the welfare of expected utility maximizing hedge fund investors. In contrast, if investors have behavioral preferences over returns (Mitton and Vorkink, 2007) consistent with the underdiversification characterizing hedge fund investing, investor welfare is maximized when managers are motivated by a mixture of rank and absolute performance rewards. In multiperiod competitions, the possibility that the fund might close after a series of relatively poor performance realizations increases the manager's concern for ranking and leads to derisking.

Hedge funds are characterized by two distinguishing features, their (i) fee structure and (ii) unrestricted menu of investment options. The fee structure of a typical hedge fund contains both management fees and performance fees. Like mutual fund managers, hedge funds managers receive management fees equal to a percentage of assets under management (AUM). In contrast to mutual funds, which are restricted to offering symmetric performance fees, hedge funds also typically offer managers asymmetric performance compensation based on returns topping a high-water mark. Although the stereotypical hedge fund management compensation package is "2 and 20," a 2% fee on AUM and 20% of returns in excess of the high-water mark, currently average management fees are typically closer to 1.5% and performance fees 18%.[1]

In contrast to mutual funds, whose asset allocations are constrained by regulation (in the U.S. by the Investment Company Act), hedge funds are free to form portfolios consisting of any mixture of assets, e.g., stocks, bonds, derivatives, and structured products. This freedom permits hedge funds to generate return distributions quite distinct from the typical return distributions associated with mutual funds.

The compensation structure of hedge fund managers and the lack of restrictions on their portfolio allocations shapes fund manager behavior. Management fees produce an incentive to maximize funds under management. Good current performance increases funds under management both by increasing the flow of new investment and by enlarging the funds asset base. In fact, Lim et al. (2015) shows that the funds-flow effect on hedge fund manager

Date: October 30, 2017.

[1] `https://www.hedgefundresearch.com/sites/default/files/articles/pr_20160317.pdf`

rewards is at least 1.4 times larger than the effect of the absolute performance incentives. Because AUM depends on the flow of funds into and out of the hedge fund and because flows are positively related to the fund's performance ranking relative to other funds, relative performance ranking matters to managers.[2] At the same time, unrestricted portfolio choice and asymmetric performance rewards provide hedge fund managers with both the incentive and the ability to generate risky asymmetric return distributions. Such gambling strategies will reap upside performance fee rewards without incurring compensating downside performance fee penalties.

We develop a model of hedge fund competition that incorporates these two distinguishing features hedge fund investing: Managers are endowed with skill. Skill provides managers with the capacity to earn returns that, on average, exceed risk-adjusted market returns. A manager's problem is to form an investment portfolio, subject to this capacity constraint, that maximizes her welfare. Given that hedge fund managers are able to trade in all classes of financial instruments and given that financial markets are reasonably complete, a good approximation for the manager's feasible set is the set of all portfolios whose value satisfies the manager's capacity and wealth endowment constraints. For this reason, we develop our model of hedge-fund asset allocation under a complete market assumption: a manager can attain any distribution of returns she can afford given her capacity.

In this setting, we first show that performance fees based on absolute levels of upside performance and rank-based management fees generate very different incentives. As is well known, performance fees reward high-risk portfolios. In order to capture performance fees, the manager must realize high returns. Because of the capacity constraint on average returns, these high returns must be balanced with low returns. Because all returns below the high-water mark fail to capture performance fees, and because reducing returns below the benchmark permits the manager to increase either the realized level of returns above the benchmark or increase the probability of beating the benchmark, performance fees encourage the adoption of "lose-big/win-big" strategies, involving extreme risk taking.

Rank-based fees, in contrast, provide an incentive to *win small*. As long as a manager's performance tops a rival's performance, no matter by how small a margin, the manager's ranking will top that rival's ranking. Thus, rank-based incentives encourage managers to adopt win-small strategies, choosing return performance levels that just top rivals' performance. Since the "ranking game" itself is a constant-sum game, each manager also has a complementary incentive to deprive rival managers of win-small opportunities. Such opportunities arise whenever a manager follows a predictable strategy that places large probability weight on a narrow range of performance, thereby permitting other managers to greatly increase their probability of outranking the manager by barely topping this narrow range. These incentives generate equilibrium probabilities of attaining rankings fairly "flat" in fund

[2]Past studies in hedge funds have documented that higher relative performance leads to more money inflows, thus more AUM-based fees. See Ding et al. (2009), Basak and Makarov (2014), Hamdani et al. (2016), etc.

performance, with the marginal gain from increasing performance proportional to capacity. Because capacity and the reward for rank are both bounded, this leads to bounded non-extremal risk taking strategies even when managers are risk neutral.

In order to identify the specific effect of convex performance rewards on hedge fund manager behavior, we consider two types of performance-based types: a fixed bonus for beating the benchmark and a call option with a strike price equal to the benchmark. Fixed bonus compensation provides asymmetric but non-convex performance rewards. Option rewards correspond to the high-water mark compensation typically provided to hedge fund managers. We assume that the benchmark is random, which captures, in stylized fashion, the dependence of the benchmark on market conditions and past performance. In the baseline analysis, the relative weight on ranking versus performance incentives is fixed.

Under both bonus and option-based performance compensation, the introduction of rank-based incentives has a significant effect on hedge-fund managers' behavior. First, because rank-based compensation produces an incentive to deprive rival managers of cheap win-small opportunities, even when the weight on absolute performance is high, hedge fund managers opt for performance distributions that do not concentrate returns on the expected level of benchmark. Instead, they spread returns off to cover a range of realized gains. However, as the weight on absolute performance increases, the weight placed on extreme ends of the performance distribution increases. Under option-based compensation, the weight on absolute compensation must be restricted to avoid unbounded solutions.

Under bonus compensation, when the weight on absolute performance is sufficiently great, the support of the manager's performance distribution can be disconnected. In the lower region, topping the benchmark is very unlikely and thus the motivation for increasing returns is largely provided by rank-based incentives. In the upper region, performance-based rewards are likely and these rewards motivate very costly, in terms of capacity, generation of high-return realizations. Realizing returns in the intermediate range between the two regions is suboptimal: The high capacity cost of topping the upper endpoint of the lower region makes topping suboptimal based on purely rank-based incentives. At the same time, the probability of beating the benchmark over the intermediate region is also small, and thus provides insufficient performance-based rewards to counter the suboptimality of topping based on rank-based incentives. The disconnected performance distribution implies that returns are not unimodal.

In contrast, under the option-like compensation typical of the hedge fund industry, the support of the manager's performance distribution is connected. However, above a threshold value, the distribution of returns flattens out and covers a region surrounding the expected benchmark. The elongation of the support is positively associated with the weight placed on absolute performance. Thus, in contrast to bonus fees, option-based performance fees lead to highly positively-skewed equilibrium performance characteristic of actual hedge fund returns (e.g., Gregoriou et al., 2005; Ding and Shawky, 2007). Positive skewness is produced endogenously by our model without imposing any restrictions on distributional form of hedge-fund returns. The difference between equilibrium return distributions under bonus and option compensation suggests that the observed positive skewness of hedge fund returns is not *per se* the product of asymmetric performance compensation but rather a product of specific convex, option-based compensation provided to typical hedge fund managers.

The welfare effects of hedge fund competition depend on the preferences of hedge fund investors. If these investors are risk-averse expected utility maximizers, as Strack (2016) shows, trading based on rank-based incentives can produce investor welfare losses relative to simple buy-and-hold strategies. However, trading motivated by absolute performance incentives, because it motivates gambling, also produce welfare losses relative to simple buy-and-hold strategies. We show that the welfare losses from rank-motivated trading are smaller than the welfare losses from absolute-performance motivated trading, i.e., increasing the weight on relative performance always increases the welfare of risk averse expected utility maximizing investors. The costs of rank-based trading to such investors are smaller for two reasons. First, the win-small incentive produced by rank concerns makes gambling for rank-based rewards less extreme than gambling to reach high-water mark. Second, when high-water mark is reached performance fee reduces the fraction of gains captured by investors.

However, the assumption that hedge fund investors are risk-averse expected utility maximizers is itself problematic. Such investors' portfolios are likely to be underdiversified relative to optimal level of diversification for a rational expected utility maximizing investor (Boyer et al., 2010). As Mitton and Vorkink (2007) show, underdiversification can be rationalized by behavioral preferences that place a positive weight on portfolio skewness. Thus, we examine the effect of rank-based versus absolute performance based incentives on investor welfare under Mitton and Vorkink specification for investor preferences. Under this specification, we find that a roughly equal mix between rank based and absolute performance based incentives maximizes investor welfare. The intuition for this result is that absolute performance incentives are required to induce hedge fund managers to select the positively skewed return distributions which investors prefer. However, absolute performance rewards also create an incentive to increase portfolio variance, which is disliked by investors. The high-variance portfolios, portfolios that "win-big" in some states but frequently "lose-big," are costly for managers as the returns on such portfolios will frequently be outranked by rival manager returns. The "lose-small" incentive from rank-based competition thus constrains portfolio variance and renders a mixture of rank and absolute performance based incentives optimal for investors. Thus, for such investors, given the association of rank-based incentives with management fees and absolute performance incentives with high-water mark compensation, the mixed rank and performance based compensation structures observed in the hedge fund industry are optimal.

Extending the analysis to a multiperiod setting in which the relative importance of rank versus absolute performance is determined endogenously by the fund's AUM provides additional insights. The effects of dynamic competition are investigated in simple setting where poor relative performance leads to fund outflows and, when outflows are sufficiently large, the fund is liquidated. Although closed-form results are not possible in this setting, numerical results indicate that the risk of liquidation, which deprives the fund of future rents, leads funds that have performed poorly in the past to become more focused on rank and less focused on absolute performance. Such funds avoid using up capacity to meet challenging benchmark targets and instead focus performance distribution on avoiding being outranked. This effect leads to derisking. Thus, rank-based incentives reverse the standard "gambling-on-resurrection" intuition that approaching distress always increases risk taking. Extreme risk-taking strategies use up capacity and thus imply lower expected ranking. When ranking is key to survival, managers avoid such strategies.

Institutional background. Hedge funds are investment funds structured as partnerships or limited liability companies. The fund is administered by a professional management firm, acting as the general partner. Only "sophisticated investors" are permitted to invest in hedge funds, holding shares or limited-liability partnership stakes. In contrast to private equity funds, hedge funds tend to invest in liquid assets and profit by employing complex trading strategies. Unlike mutual funds, hedge funds are characterized by having complex and sometimes opaque asset portfolios. Moreover, their compensation to fund managers typically includes an option-like "carried-interest" component tied to besting ambitious performance targets.

Thanks to the interest of sophisticated private investors and institutions, assets under hedge fund management have ballooned over the past decade to approximately two trillion dollars. Nevertheless, recently, hedge funds suffered the largest quarterly asset outflow after crisis in 2015, and continued to shrink during the first quarter of the year 2016.[3] Outflows appear to have been generated by investors shifting from active fund managers to passive index trackers. According to Daniel Loeb, a leading hedge fund manager, 2015-2016 was "one of the most catastrophic periods of hedge fund performance since the inception of the fund" (Financial Times, May 2016).[4] However, despite the turmoil, the structure of hedge fund manager compensation has not significantly changed: the average management fee held at 1.5 percent and the average performance fee fell just 0.1 percent.[5]

The ranking incentives gain importance especially in an overall unpromising environment, where beating benchmark is difficult. If one manager takes extreme risk, it is cheap for her rival to top her in the market decline, by buying cheap option insurance on the seemingly unlikely scenarios. This strategy is actually proved to be rather profitable, by Mr Spitznagel,

[3] `https://www.hedgefundresearch.com/sites/default/files/articles/PR_20160420.pdf`

[4] "Hedge funds: overpriced, underperforming", Financial Times, 25 May 2016. For the fund outflow phenomenon, see, for example: "Active fund managers 'growing too big' ", Financial Times, 11 May 2016; "Shake-up coming to asset manager sector", Financial Times, 9 June 2016; "Outflows from US active funds speed up", Financial Times, 20 July 2016.

[5] `https://www.hedgefundresearch.com/sites/default/files/articles/pr_20160616.pdf`

who runs a hedge fund profiting from beating tail risk crashes during outsized decline (Financial Times, May 2016).[6] Funds of this kind, usually referred to as "Black Swans", are gaining popularity.

Related literature. Hedge funds, and alternative asset funds (e.g., venture capital, private equity) in general have been the subject of an extensive body of research. Some of this research has aimed to rationalize the highly non-linear high-powered nature of fund manager compensation. A number of researchers have proposed that high-powered compensation is a response to the problem of "fly-by-night" managers (Axelson et al., 2009; Grossman and Horn, 1988). In a world characterized by asymmetric information regarding managers' talents and where there is an unlimited supply of potential fly-by-night fund managers, agents who have no particular skill managing funds but are willing to manage money require some separating mechanism to deter the formation of fly-by-night funds. High-powered performance fees, because the expected payoff from such fees is much higher for skilled managers, provide an effective separation mechanism.[7] In this paper, we take the compensation structure of the hedge fund industry as given and investigate its consequences for fund risk taking.

As pointed out by Goetzmann et al. (2003), convex high water-mark incentives encourage risk taking. In fact, as Carpenter (2000) and Ross (2004) show, absent risk aversion, convex rewards generate an unbounded appetite for risk. For this reason, models of hedge fund risk taking that ignore rank-based incentives typically use risk aversion to bound risk-taking strategies. In fact, most of the performance-compensation based literature models fund manager behavior under the assumptions that mangers have CARA utility and that AUM evolves as a geometric Brownian motion. This approach permits closed-form solutions for the optimal portfolio allocations using the Hamilton-Jacobi-Bellman equation (Janecek and Sîrbu, 2012; Lan et al., 2013; Drechsler, 2014). Panageas and Westerfield (2009) and Guasoni and Obłój (2013) take a different approach and model the risk of liquidation in a long-horizon setting. They show that long-horizon effects can also bound risk taking even by risk neutral managers.

In contrast to much of the performance fee literature, this paper does not impose log-normality and instead allows for general fund performance distributions. In fact, our model endogenously generates the observed positive skewness of hedge fund returns. Also, we do not use managerial risk aversion or liquidation costs to bound risk taking. Furthermore, because the high-water mark is usually adjusted according to the hedge fund's own past performance, this paper allows for a random benchmark. When we shut down the effects of rank-based competition, our model delivers predictions identical to the predictions of the above models: hedge fund managers prefer to gamble using "win-or-lose-all" strategies. However, our paper

[6] "Libertarian fund manager beats tail risk drum", Financial Times, 17 May 2016.

[7] There exists literature considering, theoretically and empirically, the optimality of fund manager compensation. Starks (1987) concludes that "symmetric" fees dominate the "bonus" fees from an agency theoretic view. In contrast, Christoffersen et al. (2013) conclude that convex high-water mark compensation improves efficiency.

also shows that, even in a single-period setting, the introduction of rank-based incentives bounds the risk taking incentives of risk-neutral managers.

Brown et al. (2001) and Aragon and Nanda (2012) empirically document the dependence of rank-based "tournament" incentives on funds flow and fees income.[8] In fact, Lan et al. (2013) show that management fees, correlated with ranking, represent about 75% of the total hedge-fund manager compensation. Rank-based "tournament" models have been developed by a number of authors and used to model many economic situations (Lazear and Rosen, 1981; Taylor, 2003; Seel and Strack, 2013; Roussanov, 2010; Krasny, 2011; Chen et al., 2012). Recently, theoretical research on investment management has also considered the effect of rank-based incentives of fund managers. For example, Basak and Makarov (2014) develop a model of rank-based compensation in which trading strategies are restricted by log normality assumptions and the rank concerns of managers are based on nonlinearities in their payoff function produced by social-status concerns.

This paper's approach is very different and relies instead on the complete-markets approach developed by Strack (2016). Strack (2016) provides conditions on self-financing trading strategies under which markets are dynamically *randomly complete*.[9] We use the random completeness approach to rationalize our model of hedge fund trading behavior. Like Strack (2016) we consider the effect of rank-based competition on fund trading strategy. In contrast to Strack (2016), we model the effect of mixed incentives, both performance and rank-based, on managerial behavior.

When the conditions for randomly complete markets are satisfied, absent performance-based incentives, the hedge-fund manager competition game is equivalent to a pure rank-based, risk-taking contest. For an arbitrary number of contestants and arbitrary prize schedules, Fang and Noe (2015) develop complete characterization of the comparative statics of the dispersion and skewness of contestant strategies. When the effect of performance incentives is shut down, our paper delivers predictions consistent with the ones in Fang and Noe (2015).

Combining rank-based and absolute performance based incentives both permits rationalizing the structure of hedge-fund returns and provides new insights into the welfare effects of managerial compensation. For example, neither performance nor rank-based incentives alone can generate the unimodal positively skewed pattern of hedge fund returns documented in the literature. However, in an otherwise very parsimonious modeling framework, combined rank- and performance-based incentives can. The combined effect also has interesting welfare implications. Strack (2016) shows that the random strategies followed by fund managers

[8]Massive empirical studies addressing the ranking–performance relation in mutual funds include Brown et al. (1996), Busse (2001), Kempf and Ruenzi (2008). Experimental evidence for the effects of rank concerns on economic behavior is also provided in Dijk et al. (2014).

[9]Random completeness is a stronger condition than complete markets, its satisfaction requires expanding the set of feasible trading strategies to ensure that managers can generate conditionally independent performance distributions. Intuitively, this assumption ensures the space of trading strategy rich enough to support optimal random strategies in rank-based competition games. For the detailed definition, see Section 2.2 in Strack (2016). A closely related approach to modeling rank-based contest games is the stopping time approach (Seel and Strack, 2013; Feng and Hobson, 2014; Feng et al., 2016).

generate welfare losses for underdiversified fund investors. This paper shows that the risk-taking strategies induced by convex performance rewards generate even larger welfare losses and that increasing the focus on rank-based rewards reduces welfare losses. Moreover, in our dynamic extension of the baseline model, having two sorts of incentives, permits the path of hedge fund returns to endogenously shift the weights on the two types of incentives. These shifts generate path-dependent predictions about hedge-fund performance distributions. Notably, the prediction that a series of realized losses that brings the fund's value close to the liquidation boundary will lead to derisking. These results are consistent with Drechsler (2014), which shows that, when management fees are high relative to performance fees, managers follow low risk strategies. In contrast to Drechsler (2014), we develop this result in a setting where managers' asset bases and thus their weight on rank-based incentives, are endogenously determined and where feasible performance distributions are unrestricted.

Furthermore, our results are applicable within the fund contexts possessing similar compensation structures. For example, Metrick and Yasuda (2010) and Sorensen et al. (2014) point out that both management fees and incentive fees contribute to the GP's compensation. Robinson and Sensoy (2013) examine that during fundraising booms, the compensation components shift towards the fixed fee (management fee) and away from the variable fee (carried-interest). Moreover, the modeling in this paper is consistent with the all-pay auction literature with given expenditure.

The remainder of the paper is organized as follows. Section 1 presents the set up and results of the bonus-contract model. In Section 2, we tailor the benchmark incentive to the option-contract compensation used in the hedge fund industry. Here we re-examine the interaction between the combined rank–performance incentives and compare the results with the Section 1 results. In Section 3, we introduce the multiperiod dynamic analysis. Section 4 checks the robustness of the models in the general parametric settings as well as in a multi-manager setting. Section 5 concludes and considers the possible applications outside the hedge fund context. Tables and proofs are in the appendices.

1. Fixed-Bonus based Performance Fees

1.1. Assumptions and model setup

We first build a bonus-contract model of investment manager behavior, in which performance fees take the form of a fixed payment received whenever a manager's performance exceeds a "benchmark". In the current section, we restrict attention to competition between two hedge fund managers, X and Y. At date 0, these managers choose asset allocations. These asset allocations generate random *performance*, i.e. portfolio values, at date 1. Managers are risk neutral. The managers' utility is determined both by their expected performance-fee income and by their expected relative performance. Thus, managers have two incentives: an "absolute" performance benchmark incentive associated with their performance relative to the benchmark and a relative performance "contest" incentive associated with the probability that the manager's performance tops her rival's. The weight placed on benchmark versus relative performance incentives is fixed. For manager i, the weight on benchmark incentives equals ι_i and the weight on relative performance incentives equals $1 - \iota_i$, $\iota_i \in [0, 1]$.

Following Strack (2016) we assume that the financial market is randomly complete and we also assume that asset prices equal expected asset payoffs. Thus, through replication, managers can generate any pattern of returns they choose subject to value conservation. Given our assumption of risk-neutral pricing this amounts to assuming that, subject only to an expected value constraint, managers choose return distributions. At date 0, each manager is endowed with one dollar. The expected value constraint depends on the manager skill or *capacity* to generate excess expected returns. Equivalently, in our setting a manager's *capacity* represents the manager's ability to construct portfolios with a given expected performance level.

The benchmark itself is random to equal $\tilde{\varepsilon}_m$, where $\tilde{\varepsilon}_m$ is a random variable representing uncertainty. In order to obtain tractable closed-form solutions to our problem we assume that $\tilde{\varepsilon}_m \overset{d}{\sim} G$, where

$$G(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{3}{4}x^2 - \dfrac{1}{4}x^3 & \text{if } x \in [0, 2], \\ 1 & \text{if } x > 2. \end{cases} \tag{1.1}$$

G is a symmetric, absolutely continuous, unimodal distribution centered at $x = 1$. $\tilde{\varepsilon}_m$ can be interpreted either as an evolution of a market based parameter or as representing uncertain trading costs. Under the second interpretation, the returns generated by the manager should be interpreted as gross returns which will be netted against trading costs. The key requirement we impose on $\tilde{\varepsilon}_m$ is that it is common to both managers and thus does not affect relative ranking.

The benchmark-based bonus is received by a manager if the realized value of the manager's portfolio exceeds $\tilde{\varepsilon}_m$. The specific functional form of G does not have significant qualitative effects on our analysis. However, some uncertainty is required to avoid the trivial openness problem caused by a non-stochastic benchmark. If the benchmark were non-stochastic, any performance exceeding this non-stochastic benchmark would capture the bonus and performance exactly equaling the benchmark would not. Thus, ignoring tournament incentives, a solution to the managers' optimal portfolio problem would not exist.

Unimodality is also essential to our characterization of optimal benchmark besting strategies. As will be apparent from the subsequent analysis, ignoring contest incentives, unimodality leads to the optimality, under the bonus incentive structure assumed in this section, of lose-big, win-small strategies. This amounts to say that unimodality bounds risk taking. By conventional wisdom, hedge funds do not design portfolios aiming to produce unbounded risk. The literature on hedge fund manager imposes some conditions to rule out such extreme gambling, typically by imposing the CARA and occasionally other utility assumption on the manager (Carpenter, 2000; Ross, 2004; Basak et al., 2007; Hodder and Jackwerth, 2007). In reality, this no doubt reflects aversion of too much leverage, reputational concerns, endowed stake of managers, all of which will mitigate the managers' risk-taking. Capturing this constraint on risk taking by uncertainty regarding the level of gross returns required to top the benchmark seems as reasonable as the utility-based assumptions made in the extant literature. The assumption, implied by the definition of G, that spread of the support of $\tilde{\varepsilon}_m$ covers a large region is not required but it simplifies the derivations by eliminating the need to solve the portfolio problem for cases where the realized performance is on, above, and below the support. However, the analysis can easily be extended to allow for unimodal distributions of $\tilde{\varepsilon}_m$ which have much thinner tails.

Fixed payments conditioned on besting a benchmark are not typical forms of compensation in the hedge-fund industry. However, we consider fixed benchmarks for three reasons: (i) fixed-bonus compensation is commonly used in many other asset management settings, (ii) the fixed-bonus compensation model developed in this section is more tractable and thus more transparently highlights the effects of mixed tournament and absolute performance incentives than the more complex convex performance fee setting developed in the next section, and (iii) the fixed bonus setting provides us with a point of reference which helps in answering the question of the extent to which the equilibrium distribution of hedge fund returns is a product of the specific "2 and 20" compensation typical for the industry versus being simply a product of mixed relative and absolute performance incentives.

Denote the *portfolio value* (or gross return, which equals gross return in the static case considered due to the one dollar endowment assumption) under manager X and Y by the random variables X and Y respectively. We do not impose any normality restriction on these two variables, which gives us the great freedom to capture the highly-skewed feature in the distribution of hedge fund values.

Each manager chooses a distribution of realized date 1 portfolio value, or more succinctly, a *performance distribution*. Because, abstracting from different parameter assignments, the two rivals are identical, their decision problem is the same. So, we develop the problem for manager X. Manager X's problem is to choose a *challenge distribution*, F_X, for performance given the absolute and relative incentives.

Thus, the payoff to manager X is given by

$$\iota_X\,\mathbb{P}(X \geq \tilde{\varepsilon}_m) + (1-\iota_X)\,\mathbb{P}(X \geq Y) = \iota_X \int_0^\infty G(x)\,\mathrm{d}F_X(x) + (1-\iota_X)\int_0^\infty F_Y(x)\,\mathrm{d}F_X(x).$$

The capacity constraint is produced by requiring that the performance of manager satisfies

$$\int_0^\infty x\,\mathrm{d}F_X(x) \leq \mu_X. \tag{1.2}$$

By the construction of $G(\cdot)$, we assume that the managers' capacities, μ_X and μ_Y, are both less than $3/2$, which amounts to assuming that managerial capacity for performance is not so high that the managers can beat the benchmark with high probability without risking low return realizations. X maximizes her payoff subject to three constraints: (i) the capacity constraint on the expected value of F_X given by equation (1.2), (ii) the requirement that F_X is a cumulative probability distribution, and (iii) the requirement that $X \geq 0$. The problem faced by manager Y is the same except that the capacity for Y and the weight placed on the absolute versus relative performance might be different. An equilibrium is a choice of distributions, F_X and F_Y, satisfying the condition that each manager's chosen distribution is a best reply given the choice of the other manager. In cases where the managers are *ex ante* identical, i.e., have the same capacity and place the same weight on the benchmark incentive, we focus on symmetric equilibria.

Hence, the manager's problem can be reformulated as the following optimization problem:

$$\begin{aligned}
&\max_{\mathrm{d}F_X \geq 0} \ \iota_X \int_0^\infty G(x)\,\mathrm{d}F_X(x) + (1-\iota_X)\int_0^\infty F_Y(x)\,\mathrm{d}F_X(x); \\
&\text{s.t.} \ \ \mathbb{E}[X] = \int_0^\infty x\,\mathrm{d}F_X(x) = \int_0^\infty \Big(1 - F_X(x)\Big)\,\mathrm{d}x \leq \mu_X, \\
&\qquad \int_0^\infty \mathrm{d}F_X(x) = 1, \\
&\qquad X \geq 0.
\end{aligned} \tag{1.3}$$

Before solving the optimization problem we derive some basic and important features of managerial behavior when relative performance matters.

Lemma 1 *As long as some weight is placed on relative performance, i.e., $\iota_X, \iota_Y < 1$,*

(i) the equilibrium performance distribution is atomless except perhaps at $x = 0$, and

(ii) the probability that the performance of manager X equals the performance of manager Y is zero.

(iii) In a symmetric equilibrium, the performance distribution cannot have an atom at 0 and thus the equilibrium is atomless.

Lemma 1 is quite intuitive. If player X did choose a distribution placing mass on some discrete point unequal to 0, then it is easy for her rival Y to put some weight infinitesimally above this point, which only costs Y an infinitesimally more in terms of capacity, but increases her probability of besting X by a non-infinitesimal amount. So Y will never put weight on manager X's mass point. In fact, by the same logic, manager Y will never place weight on points in a sufficiently small neighborhood just below the mass point. Thus, Manager X can lower the mass point without lowering her chance of winning. Lowering the mass point conserves capacity. Conserved capacity could then be applied by manager X to compete more aggressively around other performance levels, thereby increasing X's probability of winning. Thus, putting weight at the mass point greater than 0 is never a best reply for manager X. In a symmetric equilibrium, one manager placing probability mass on zero would imply that the other manager was also placing mass on zero. By part (ii) performance ties do not occur with positive probability in equilibrium. Thus, in symmetric equilibria, the performance distribution of both managers is purely atomless. The basic insight from Lemma 1 is thus that, as long as the managers place some weight on relative performance, equilibrium performance distributions will be locally "spread out" to avoid providing rivals with low-cost win-small opportunities.

1.2. Pure contest and pure benchmark incentives

Pure risk-taking contest. In order to isolate the effects of absolute and relative performance incentives, we initiate the analysis of equilibrium strategies by considering the extreme cases where managerial incentives are provided by relative performance or the benchmark alone. When incentives are provided by relative performance alone, competition reduces to a pure risk-taking contest. In which case, equilibria are characterized by standard results from the risk-taking contest and all-pay auction literature.

Lemma 2 (Pure risk-taking contest) *Assume $\iota_X = \iota_Y = 0$, i.e., benchmark incentives are absent. Let the capacities of the managers be $\mu_X \geq \mu_Y > 0$. Then the unique equilibrium can be characterized as*

$$F_X^* = U(0, 2\mu_X),$$

and

$$F_Y^* = \Big(1 - \frac{\mu_Y}{\mu_X}\Big)\mathbb{1}_0 + \frac{\mu_Y}{\mu_X}\, U(0, 2\mu_X).^{10}$$

Both managers' capacity constraints bind, i.e.,

$$\mathbb{E}\Big[\frac{X}{\mu_X}\Big] = \mathbb{E}\Big[\frac{Y}{\mu_Y}\Big] = 1.$$

In the special case where both managers have the same capacity, μ, the unique equilibrium is a symmetric one in which both managers pick a uniform distribution *over the support* $[0, 2\mu]$.

The intuition for this result is that, the pure contest game is a two-player constant sum game. Thus, each manager is choosing a distribution that minimizes the opponent's probability of topping her performance.[11] The manager must concede to her rival the payoff from randomizing between the two end points of her equilibrium distribution. Under a linear performance distribution, she never gives her rival better opportunities than provided by this bound. Any deviations from linearity offer the rival the prospect of increasing her probability of winning with a smaller utilization of capacity. Thus, the equilibrium performance distributions for both managers are linear. For a formal proof, see the appendix.

The manager with more capacity (in the lemma, by assumption, manager X) submits uniformly distributed performance; in contrast, the weaker manager sometimes "gives up" by placing some probability mass at zero, and uses the remaining probability mass to "mimic" the distribution played by her stronger rival. In the symmetric case, both managers choose uniform performance distributions and both win with probability 1/2. Clearly, for any challenge distribution F_X chosen by manager X, a uniform distribution $F_Y(x) = \min[x/(2\mu), 1]$

[10] $\mathbb{1}_0$ stands for the degenerated distribution of point mass at zero, with probability $1 - \mu_Y/\mu_X$. $F_Y = U$ with probability μ_Y/μ_X, where the notation $U(0, x)$ stands for the uniform distribution on $[0, x]$.

[11] In a two-player constant/zero sum game, the minimax rule applies.

over the support $\mathrm{supp}\{F_X\} = [0, 2\mu]$ implies

$$\int_{\mathrm{supp}\{F_X\}} F_Y(x)\,\mathrm{d}F_X(x) = \int_{\mathrm{supp}\{F_X\}} \frac{x}{2\mu}\,\mathrm{d}F_X(x) = \frac{1}{2\mu}\int_{\mathrm{supp}\{F_X\}} x\,\mathrm{d}F_X(x) = \frac{1}{2}.$$

In the symmetric case, the performance distribution of the two rivals is uniform and thus has zero skewness. In the asymmetric case, the performance distribution of the manager with greater capacity is uniform and thus has zero skewness; the weaker rival's performance distribution is a mixture between a point mass at zero and a uniform distribution and thus is negatively skewed. As Fang and Noe (2015) show, strictly unimodal performance distributions are never optimal because they can be bested either by playing safe or a "win-small/lose-big" strategy.

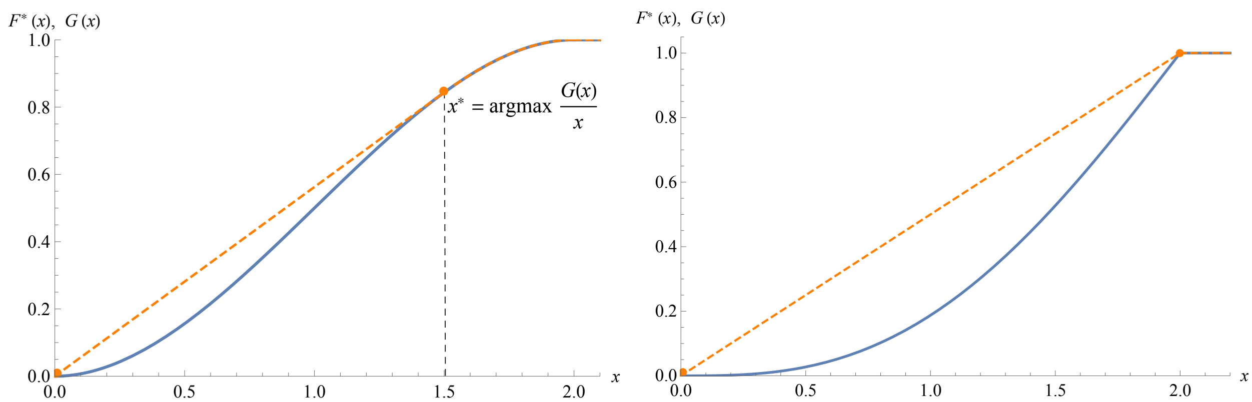
Pure bonus-driven competition. Now we consider pure benchmark competition. Because the benchmark distribution is unimodal, absent relative performance incentives, the managers optimize their payoffs by choosing win-small/lose-big strategy of randomizing between $x = 0$ and x^*, where x^* maximizes the ratio between the probability of besting the benchmark $G(x)$ and the capacity used to best that benchmark x. This leads to the following characterization of pure benchmark based competition.

Lemma 3 (**Pure bonus-driven competition**) *If $\iota_X = \iota_Y = 1$, i.e., only benchmark incentives matter, then both X and Y will choose equilibrium performance distributions that place all weight on either $x = 0$ and $x = 3/2$. The probability weights on these points will be fixed by the binding capacity constraint.*

The intuition for Lemma 3 is provided by Figure 1. Panel A of the figure illustrates the unimodal case assumed in this paper. Panel B illustrates the case of a convex benchmark distribution. Our result in fact brings in a useful concept, *concave envelope*,[12] which many studies use to analyze the hedge fund contracts (e.g., Carpenter, 2000; Basak et al., 2007; Buraschi et al., 2014). These studies basically suggest that the solution to the constrained optimization will be taken at the levels where the objective function meets its concave envelope, which is indeed true in our case. The manager's problem is to choose a distribution of target performance that maximizes the probability of besting the benchmark subject to the capacity constraint. Placing weight on values in the convex region $x \in (0, 1)$ of the benchmark distribution is suboptimal because convexity ensures that randomizing between $x = 0$ and $x = 1$ always produces higher probability of beating the benchmark. For $x \geq 1$, the trade-off faced by the manager is that targeting a higher upside asset value x requires placing more weight on 0 through the capacity constraint. Since all capacity is used for attaining the upside asset value, x^*, the capacity constraint requires that the probability of upside performance p satisfies $p\,x^* = \mu_X$. Thus, the probability of beating the benchmark

[12]The concave envelope is the smallest concave function greater than or equal to the original objective function. For the precise definition see the proof of Lemma 3 in the appendices.

will equal $p\,G(x^*) = \mu_X(G(x^*)/x^*)$. As long as the ratio $G(x)/x$ is increasing, the probability of besting the benchmark is increasing. So the manager will choose x^* to maximize the "bang-for-the-buck," i.e., maximize this ratio. $G(x)/x$ is maximized at $x = 3/2$, and the manager's probability of besting the benchmark equals $\mu_X \times G(3/2)/(3/2) = \mu_X\,(9/16)$. We see that, under the assumed unimodality of G, managers will not opt for the extreme risk taking strategy of placing all weight on the upper and lower end points of the support of the benchmark distribution. In Panel B, which features a convex distribution function for the benchmark, the ratio $G(x)/x$ is increasing over the entire support of the benchmark distribution. Thus, when the benchmark distribution is convex, the manager will opt for extreme risk taking. In short, under pure benchmark incentives, unimodal benchmark distributions lead to extreme downside risk taking but limited upside risk taking, i.e., to negatively skewed performance.



(A) Convex-then-concave case

(B) Convex case

FIGURE 1. The figure shows the optimal solution when beating a convex-then-concave (Panel A) or a convex (Panel B) CDF. We plot the target distributions (solid line), the concave envelopes (dashed line), and the optimal solution levels.

1.3. Equilibrium solution with mixed incentives

We now turn to the case where managers have mixed incentives which place positive weight on both absolute and relative performance. Mixed incentives, combined with the capacity constraint, produce rather complex trade-offs: The rewards associated with besting the benchmark provide an incentive to select high performance realizations. However, high performance has to be balanced with low performance in order to ensure that the capacity constraint is satisfied. Low performance, in turn, increases the probability that the manager will be outperformed by the rival. Moreover, even at very high performance levels — performance levels too high to be justified purely by the relative performance incentive — the manager still aims to avoid offering her rival cheap win-small opportunities; thus the manager will spread out performance over a range of returns rather than targeting the benchmark.

We initially focus on the symmetric case where $\iota_X = \iota_Y = \iota$ and the managers have the same capacity. Within these baseline model sections, we assume that the capacity equals the expected market performance, which, by our choice of G in (1.1), equals 1. Later, in Section 4.1, we will extend the results to a more general capacity setting. To solve for the equilibrium, we first must solve the individual manager's optimization problem. We begin by considering the Lagrangian of the optimization problem specified by (1.3):

$$\mathcal{L}(\mathrm{d}F_X, \alpha, \beta) = \int_0^\infty \Big(\iota\, G(x) + (1-\iota)\, F_Y(x)\Big)\, \mathrm{d}F_X(x) - \alpha\bigg(\int_0^\infty \mathrm{d}F_X(x) - 1\bigg) - \beta\bigg(\int_0^\infty x\, \mathrm{d}F_X(x) - 1\bigg),$$

where α, $\beta \geq 0$ are the Lagrangian multipliers. Since this problem is a linear program, is feasible and satisfies the weak form of Slater's condition (see, e.g., Boyd and Vandenberghe (2004)), the strong duality holds. The dual problem can be expressed as follows:

$$\min_{\alpha,\beta\geq 0} \sup_{\mathrm{d}F_X \geq 0} \mathcal{L}(\mathrm{d}F_X, \alpha, \beta) = \min_{\alpha,\beta\geq 0} \sup_{\mathrm{d}F_X \geq 0} \int_0^\infty \Big(\iota\, G(x) + (1-\iota)\, F_Y(x) - \alpha - \beta x\Big)\, \mathrm{d}F_X(x) + \alpha + \beta.$$

Note that, due to the benchmark setting, the players would never place probability mass on performance above 2, the upper bound of the support of the benchmark. Considering Lemma 2, it is not hard to show that the support of the optimal F will not exceed $[0, 2]$. Hence, to solve the dual problem, we must have

$$\iota\, G(x) + (1-\iota)\, F_Y(x) - \alpha - \beta x \leq 0, \qquad \forall x \in [0, 2],$$

otherwise $\sup_{\mathrm{d}F_X} \mathcal{L}(\mathrm{d}F_X, \alpha, \beta)$ goes to infinity. *Ex ante* the two players choose their distributions independently. However, the same capacity constraint, weight and initial endowment will yield at least one symmetric solution to the game, in which both managers choose the same equilibrium performance distribution $F_X(\cdot) = F_Y(\cdot) = F(\cdot)$ for an equilibrium. By solving the optimization system we in fact prove that it is the unique symmetric equilibrium. According to the previous analysis, as long as $1 - \iota \neq 0$ (i.e., $\iota \neq 1$), in equilibrium distribution F should be atomless, and thus intersect the origin. The Lagrangian conditions ensure that the optimal performance measure associated with the optimal performance distribution always concentrates on the points at which the upper support line, $\alpha + \beta x$, meets the sum of the two incentive functions. Formally, the optimal dual variables, α and β, must satisfy the following conditions:

$$\begin{aligned} &\iota\, G(x) + (1-\iota)\, F(x) - \alpha - \beta x \leq 0, \qquad \forall x \in [0, 2]; \\ &\mathrm{d}F\{x \in [0, 2] :\ \iota\, G(x) + (1-\iota)\, F(x) - \alpha - \beta x < 0\} = 0. \end{aligned} \tag{1.4}$$

We will show that the whole system becomes a solvable programming of the parameters (α, β). We have already demonstrated that the symmetric equilibrium distribution intersects the origin, mathematically, $\alpha = 0$. This means that placing positive mass at zero level would not help the players win the game. Also, the equilibrium capacity constraint should bind, i.e., managers use all their capacity to generate performance. These two conjectures will be proved to be true when we prove the equilibrium solution.

The symmetric solution to the game is presented in the following proposition. The results are separated into two subcases divided based on the strength of the benchmark incentive.

Proposition 1 (**Symmetric equilibrium**) *For any $\iota \in [0,1]$, in the equilibrium we have that the capacity constraint binds. The optimal challenge distribution $F^*(\cdot)$ chosen by the managers is described separately in the following two cases.*

(i) When managers are more focused on relative performance, i.e., $\iota \in (0, 2/3]$: F^* is given by

$$F^*(x) \equiv T^*(x), \quad x \in [0,2],$$

where

$$T^*(x) := \frac{\alpha^* + \beta^* x - \iota G(x)}{1-\iota}, \tag{1.5}$$

and the optimal Lagrangian multipliers are given by $\alpha^* \equiv 0$, $\beta^* \equiv 0.5$.

(ii) When managers are more focused on the benchmark, i.e., $\iota \in (2/3, 1)$: F^* is a piecewise function given by

$$F^*(x) = \min\big[\min_{y \geq x} T^*(y), 1\big], \quad x \in [0,2],$$

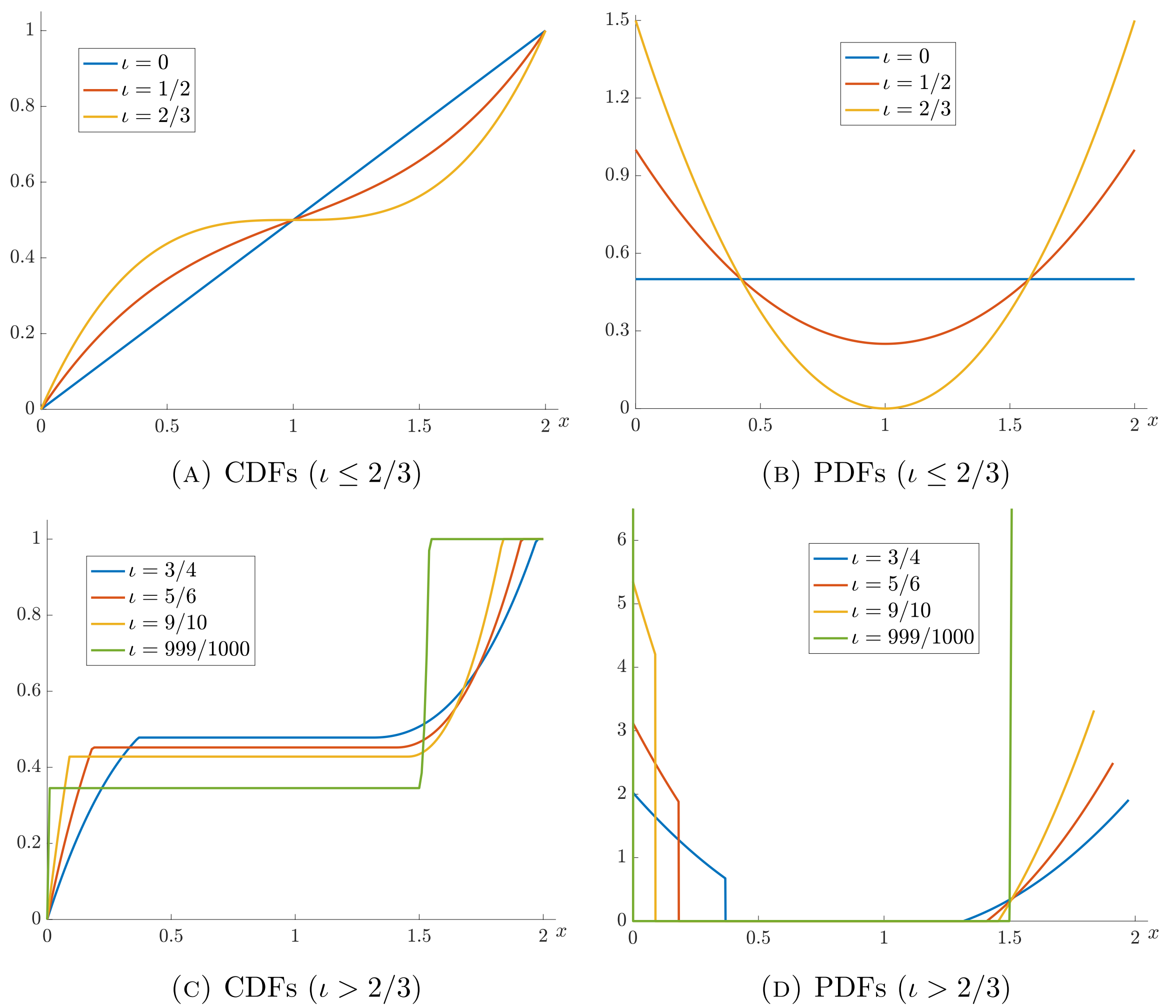
where $T^*(\cdot)$ is as defined in (1.5), $\alpha^* \equiv 0$ and β^* takes the value which ensures that the capacity constraint binds.

Note that as ι converges to its extreme values, 0 and 1, and thus either absolute or relative performance maximization, our solution for mixed incentives, provided by Proposition 1, converges to the solution for pure benchmark or pure rank-based competition described in Lemmas 2 and 3.

1.4. Analysis of equilibrium solution

The intuition for Proposition 1 is revealed by contrasting the effects of relative performance contest incentives and absolute performance benchmark-based incentives. Relative performance incentives militate against gaps in the support of the equilibrium performance distribution. A gap in the support of the rival manager's performance distribution implies that any probability weight placed by the manager near the upper bound of the gap can be moved downward, conserving capacity, without reducing the manager's probability of topping the rival. Thus, both managers, as well as having an incentive to spread out their performance distribution to deprive rivals of cheap win-small opportunities, also have an incentive to "stay-close". However intermediate performance levels, levels that use significant capacity yet are very unlikely to capture the bonus, are very suboptimal from the perspective of the benchmark-payoff maximization. Thus, benchmark incentives militate against placing any probability weight at intermediate performance levels. As long as the weight on benchmark incentives is not too high, case (i) where $\iota < 2/3$, relative performance incentives are sufficient to counter the benchmark incentive and thus the support of the equilibrium distribution is connected. Mathematically, this is reflected by $T^*(\cdot)$ being nondecreasing. This implies that the optimal solution is given by $F^*(\cdot) \equiv T^*(\cdot)$ for $T^* \leq 1$. This property reduces the problem to a simple linear programming. Moreover, our previous conjectures that $\alpha^* = 0$ and $\mathbb{E}[X] = 1$ are verified. Our optimal solution gives a concave-then-convex distribution, see, the cases where $\iota = 0$ (the pure contest), $1/2$ (the two incentives are equally weighted), and $2/3$ (the critical point) in Panel A and B of Figure 2.

We see that, even in case (i), where benchmark incentives do not dominate relative performance incentives and thus the support of the performance distribution remains connected, benchmark incentives affect equilibrium performance distribution by pulling mass away from intermediate performance levels. This is best seen by comparing the result of the mixed incentives case in Proposition 1 and pure relative performance based competition ($\iota = 0$) case analyzed in Lemma 2. Lemma 2 shows, when contestants are symmetric, the equilibrium performance distribution is uniform. As ι increases from zero, the managers put more weight at tails of the performance distribution and less weight at intermediate levels.



(A) CDFs ($\iota \leq 2/3$)

(B) PDFs ($\iota \leq 2/3$)

(C) CDFs ($\iota > 2/3$)

(D) PDFs ($\iota > 2/3$)

Figure 2. Panel A and B in the figure present the optimal challenge CDFs and PDFs of $F^*(\cdot)$ respectively, when ι is within the range $[0, 2/3]$. The representative levels we choose are $\iota = 0$, $1/2$, and $2/3$. Panel C and D plot the optimal CDFs and PDFs of $F^*(\cdot)$ respectively, when ι is within the range $(2/3, 1)$. For a clear look we only plot the cases when $\iota = 3/4$, $5/6$, $9/10$, and $999/1000$. For the detailed results one can refer to Panel A of Table 1.

In part (ii), where $\iota > 2/3$, and thus managerial competition is more focused on benchmark rather than relative performance, the effect of the suboptimality of intermediate performance levels for attaining the benchmark becomes so large that the support of the performance distribution "breaks" into segments, as illustrated by Figure 3. From this figure we see that

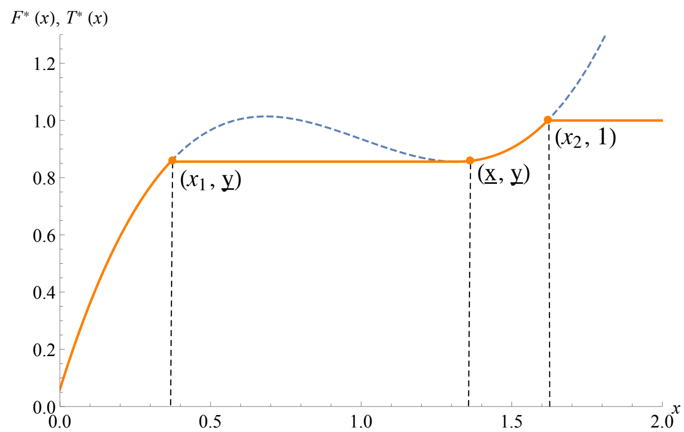


FIGURE 3. The figure illustrates the shape of the optimal challenge distribution F^*(solid line) and its relationship with function T^*(dashed line) in the case where managers are more focused on absolute benchmark incentives than relative rank-based incentives, i.e., $\iota \in (2/3, 1)$. Note that $(\underline{x}, \underline{y})$ is the local minimizer of T^*, and the parameters x_i $(i = 1, 2)$ are some functions of (α, β) specified in the detailed proof. x_2 takes value within $[3/2, 2]$, and decreases gradually to $3/2$ as ι gets larger and closer to 1.

T^*, is not uniformly increasing. The non-monotonicity of T^* makes the optimization system more complex than it was in case (i). For a given performance level x, $T^*(x)$ measures the value of performance at level x, taking into account the shadow price of capacity. Thus, placing probability mass on the region where T^* is decreasing, reduces the manager's payoff. Because the cumulative distribution of performance cannot decrease, the optimal distribution is no longer T^* but rather the increasing lower envelope of T^*. This implies that the optimal performance distribution is flat over the decreasing segment of T^*, i.e., managers place no probability weight on performance levels in this segment, $[x_1, \underline{x}]$ in the figure, where $\underline{x}$ is the local minimizer of $T^*(\cdot)$, and x_1 satisfies $T^*(x_1) = T^*(\underline{x})$.

As ι increases further, the upper support point x_2 moves down and thus becomes closer to the expected benchmark. Concentrating mass around expected benchmark performance is suboptimal based on relative performance incentives alone as it provides the rival with fairly cheap win small opportunities. However, as ι increases, the importance of relative performance incentives declines. Lemma 3 shows that in the case where capacity is symmetric and equal to 1, the benchmark-only optimal performance distribution places weight only on 0 and $3/2$. Thus, performance in excess of $3/2 \in [\underline{x}, x_2]$ is inefficient from the benchmark perspective and inefficiency increases as the distance from $3/2$ increases. Thus, as ι increases the upper region contracts to the point $3/2$. In fact, When ι approaches 1, the optimal performance distribution, F^*, converges to a discrete two-point distribution characterizing pure benchmark competition (for a clear illustration of convergence, see Figure 11 in the appendices). These observations are illustrated by Panel C and D of Figure 2.

1.5. Comparative statics and welfare analysis

We now study the welfare effect under equilibrium by looking from the customers' perspective. Assume there exists a continuum of risk-averse investors. The net-of-fee wealth of investors, denoted by W, can be expressed as a function of $(X, \tilde{\varepsilon}_m)$:

$$W(X, \tilde{\varepsilon}_m) = a\,X - b\,\mathbb{1}_{X \geq \tilde{\varepsilon}_m},$$

where parameters a, b are chosen with flexibility, in order to capture the different structures of AUM, including the case where the managers have their own share investment.[13] For example, in the classical case where there is no managerial stake, we take the values of the parameters as $a = 0.98$ and $b = 0.2$.

Now we figure out the distribution of the wealth W, under the equilibrium choices of the managers. Note that we still allow for a random benchmark, $\tilde{\varepsilon}_m$, taken from the distribution as stated in (1.1). To analyze the welfare effect of investors, we have the following proposition.

Proposition 2 *Given the equilibrium performance distribution F^* provided in Proposition 1,*

(i) the expectation of investors' wealth can be calculated from $F^*(\cdot)$ as

$$\mathbb{E}[W] = a - b\int_0^2 G(x)\,\mathrm{d}F^*(x),$$

where G is the distribution of the benchmark as stated in (1.1);

(ii) the cumulative distribution of investors' wealth, denoted by $F_W(\cdot)$, can be calculated from F^* as

$$F_W(w) = \begin{cases} F^*\left(\dfrac{w}{a}\right) + \displaystyle\int_{\frac{w}{a}}^{\frac{w+b}{a}} G(x)\,\mathrm{d}F^*(x) & \text{for } w \in [0, 2a-b), \\ F^*\left(\dfrac{w}{a}\right) + \displaystyle\int_{\frac{w}{a}}^{2} G(x)\,\mathrm{d}F^*(x) & \text{for } w \in [2a-b, 2a), \\ 1 & \text{for } w \in [2a, 2]. \end{cases}$$

The CDFs and PDFs of investor wealth for different choices of the weight the managers place on absolute performance, ι, are provided in Figure 4. The figure illustrates that the

[13] Denote the fractions of managerial stake, management fees and performance fees by ω_s, ω_m and ω_p, respectively. The net-of-fee wealth of investors becomes

$$W = (1-\omega_s)X - (1-\omega_s)(\omega_m X + \omega_p \mathbb{1}_{X \geq \tilde{\varepsilon}_m}).$$

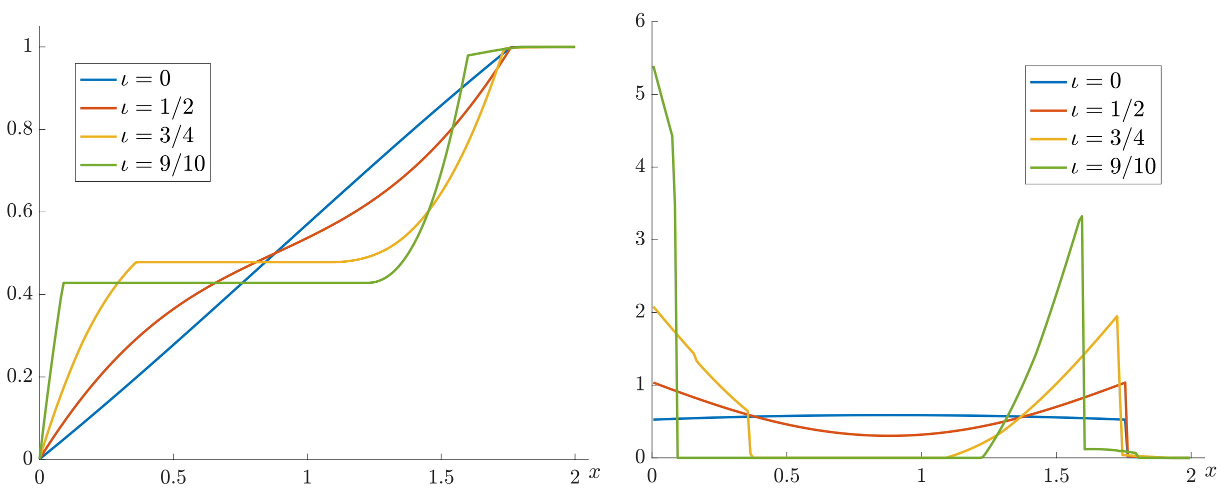
Hence the parameters satisfy

$$a = (1-\omega_s)(1-\omega_m), \quad b = (1-\omega_s)\omega_p.$$

This holds also for the option-contract in the next section. The generality of our model makes the case with managerial share only differ in the values of parameters. Thus in the following analysis we mainly consider the case without the managers' own investment, i.e., $\omega_s = 0$.

mean of the investors' wealth decreases as ι increases. The values of mean and variance of investor wealth under different weights ι are shown in Panel B of Table 1. To compare the certainty equivalent of investor wealth under various choices of ι, we assume that investors have CARA utility, i.e., $u(W) = (W^{1-\gamma}-1)/(1-\gamma)$, and assume that $\gamma = 3$, the conventional choice. The corresponding values of certainty equivalent are given in the same table. It can be seen that as ι increases towards 1, the value of certainty equivalent decreases as well. Therefore, as the benchmark incentive grows larger, the risk premium required by the CARA investors for holding the portfolio grows larger.

Comparing Figure 4, which illustrates fund performance net of fees to Figure 2, which illustrates gross performance, we can see that (i) due to the existence of tournament, the distribution of investors' wealth shifts toward the left; (ii) investors bear almost all the losses under poor performance; and (iii) in the profitable region, a large portion of the profit is captured by the managers, as demonstrated by the plummeting in the PDFs when investors' wealth comes to the right end of the support. These welfare results for the investors come directly from the compensation structure.



(A) CDFs (B) PDFs

Figure 4. This figure plots the CDFs (Panel A) and PDFs (Panel B) of the equilibrium investor wealth $F_W(\cdot)$ for some parameters $\iota = 0$, $1/2$, $3/4$ and $9/10$. For more detailed results, see Panel B of Table 1.

2. Option-Contract based High-Water Mark Fees

2.1. Model setup

In this section we tailor our model of compensation to match compensation norms in the hedge-fund industry. Performance fees are typically not fixed bonuses but rather a fixed fraction of gains in excess of the fund's benchmark or "high-water mark". This design yields an option-like convex compensation structure.

Previously, the bonus condition was an indicator function for whether the hedge fund manager is able to beat the benchmark. Now, we modify the objective, by replacing the probability of beating the benchmark by the expected excess of performance relative to the benchmark. Under this assumption, managers care about how much their performance exceeds the benchmark. As in the previous section, the benchmark is a random variable distributed $G(\cdot)$, where $G(\cdot)$ is defined by (1.1). Under this specification, our objective function becomes

$$\iota\,\mathbb{E}[(X-\tilde{\varepsilon}_m)^+] + (1-\iota)\,\mathbb{P}(X \geq Y) = \iota \int_0^\infty \widehat{G}(x)\,\mathrm{d}F_X(x) + (1-\iota)\int_0^\infty F_Y(x)\,\mathrm{d}F_X(x),^{14}$$

where $\widehat{G}(\cdot)$ is a continuous and piecewise function defined as follows:

$$\widehat{G}(x) := \begin{cases} \dfrac{x^3}{4} - \dfrac{x^4}{16} & \text{for } x \in [0,2), \\ x-1 & \text{for } x \in [2,\infty). \end{cases} \tag{2.1}$$

It is not hard to see that $\widehat{G}(\cdot)$ is a purely convex function, whereas the $G(\cdot)$ defined in (1.1) in the previous model is a convex-then-concave function. Convexity will change the solution to the manager's problem a great deal even though the formulation of the problem is quite similar to the bonus formulation. Convexity implies that, when benchmark incentives dominate, i.e., $\iota > 1/2$, and managers focus on the convex incentives produced by the high-water mark rather than relative performance, managerial risk appetite is unlimited. In this case, increasing the level of upside returns at the cost of a reduced probability of attaining this level is always attractive to managers. Thus a solution to the managers' optimization problem does not exist. Using the function $\widehat{G}(\cdot)$ allows us to express the optimization problem under convex performance fees in a fashion similar in formulation to (1.3) in the bonus-contract model:

$$\begin{aligned} &\max_{\mathrm{d}F_X \geq 0}\ \iota \int_0^\infty \widehat{G}(x)\,\mathrm{d}F_X(x) + (1-\iota)\int_0^\infty F_Y(x)\,\mathrm{d}F_X(x); \\ &\text{s.t.}\ \ \mathbb{E}[X] = \int_0^\infty x\,\mathrm{d}F_X(x) = \int_0^\infty \Big(1 - F_X(x)\Big)\,\mathrm{d}x \leq 1, \\ &\qquad \int_0^\infty \mathrm{d}F_X(x) = 1, \\ &\qquad X \geq 0. \end{aligned} \tag{2.2}$$

We solve this problem using the Lagrangian method as in Section 1.

[14] Assume $X \overset{d}{\sim} F(\cdot)$, then we have

$$\begin{aligned} &\mathbb{E}[(X-\tilde{\varepsilon}_m)^+] = \mathbb{E}\big[\mathbb{E}[(X-\tilde{\varepsilon}_m)^+ \mid \tilde{\varepsilon}_m]\big] \\ &= \int_0^2 \Big(\int_0^\infty (x-\tilde{\varepsilon}_m)^+\,\mathrm{d}F(x)\Big)\,\mathrm{d}G(\tilde{\varepsilon}_m) = \int_0^\infty \Big(\int_0^{x\wedge 2} (x-\tilde{\varepsilon}_m)\,\mathrm{d}G(\tilde{\varepsilon}_m)\Big)\,\mathrm{d}F(x) \\ &= \int_0^2 \Big(\frac{x^3}{4} - \frac{x^4}{16}\Big)\,\mathrm{d}F(x) + \int_2^\infty (x-1)\,\mathrm{d}F(x). \end{aligned}$$

2.2. Equilibrium solution

We first find the boundary condition for the optimal solution to exist, as shown in Proposition 3.

Proposition 3 *The boundary condition for the optimal solution to exist is $\iota < 0.5$ and $\beta \geq \iota$.*

Thus, it is only meaningful to model the case where $\iota < 0.5$. Note that we have already mentioned that the support will change as ι changes. Hence, in contrast to the bonus-fee case, the dependence of the solution on the multipliers for the dual program, (α, β), is rather complex when performance fees are convex. In fact, we will show that the solution imposes non-linear restrictions on these parameters, which differ from the linear constraints imposed in the bonus-contract case. This will be shown when we verify the equilibrium solution. The main result about the symmetric equilibrium under convex performance fees is given in Proposition 4.

Proposition 4 (**Symmetric equilibrium**) *When $\iota < 0.5$, the symmetric equilibrium solution to the optimization problem* (2.2)*, denoted by $F^{**}(\cdot)$, is a piecewise function with a support* $\mathrm{supp}\{F^{**}\} = [0, \hat{x}_2]$*, where $\hat{x}_2 = (1-2\iota)/(\beta^{**} - \iota)$:*

$$F^{**}(x) = \begin{cases} \widehat{T}^{**}(x) & \text{for } x \in \Big[0, \dfrac{1-2\iota}{\beta^{**}-\iota}\Big), \\ 1 & \text{for } x \in \Big[\dfrac{1-2\iota}{\beta^{**}-\iota}, \infty\Big); \end{cases} \tag{2.3}$$

where $\widehat{T}^{**}(\cdot)$ is defined by

$$\widehat{T}^{**}(x) = \frac{\alpha^{**} + \beta^{**}x - \iota\widehat{G}(x)}{1-\iota};$$

and $\widehat{G}$ is as stated in (2.1)*. The optimal parameters are:*

$$\alpha^{**} = 0\,, \qquad \beta^{**} = \frac{4\iota^2 - 10\iota + 5}{2(5-8\iota)}\,.$$

Moreover, the equilibrium distribution implies a binding capacity constraint, i.e.,

$$\mathbb{E}[X] = \int_0^\infty \Big(1 - F^{**}(x)\Big)\,\mathrm{d}x = 1.$$

The results in Proposition 4 that capacity constraint binds and $\alpha^{**} \equiv 0$ are consistent with our conjectures. With the optimal F^{**}, the measure (with respect to $\mathrm{d}F^{**}$) of the points where $F^{**}(\cdot)$ and $\widehat{T}^{**}(\cdot)$ differ is zero. Intuitively, the optimal piecewise F^{**} and its relationship with $\widehat{T}^{**}$ are shown in Figure 5.

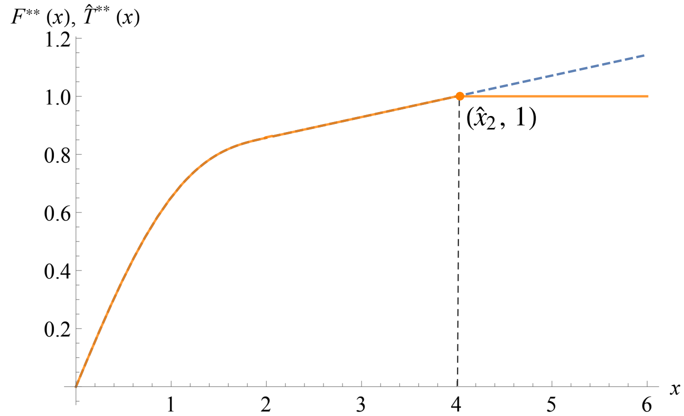
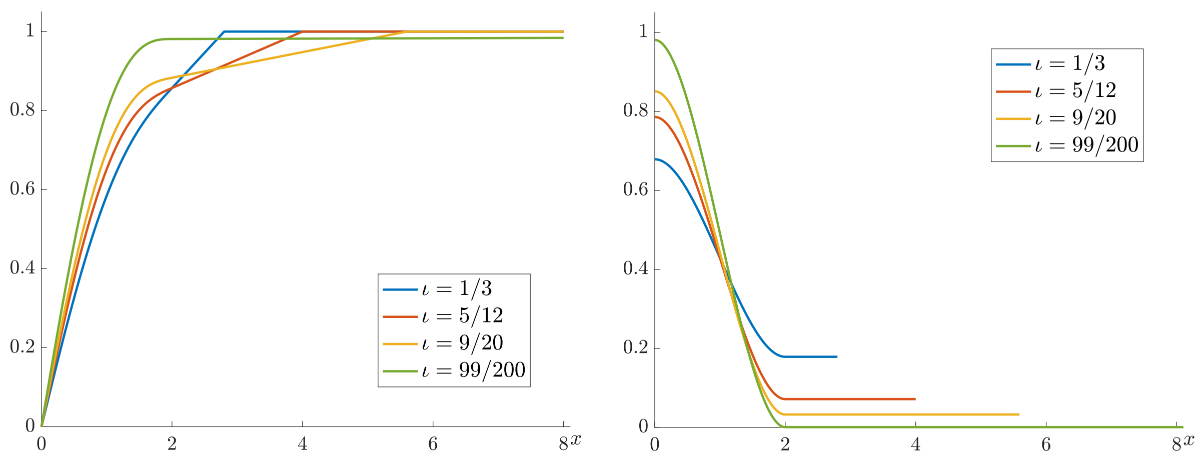


FIGURE 5. The figure illustrates the shape of the optimal performance distribution F^{**}(solid line) and its relationship with function $\widehat{T}^{**}$(dashed line).

Now the equilibrium performance distribution chosen by the hedge fund managers has a larger support than in the bonus-contract model, but the most probability weight is still concentrated in the range $[0, 2]$. We plot the equilibrium F^{**} (Panel A) and the corresponding density functions (Panel B) as ι increases in Figure 5. Note that when $\iota = 99/200$, the right end of the support of F^{**} already grows to approximately 40, although, for the sake of making the plot interpretable over the region that contains most of the probability mass, we only plot over the range $[0, 8]$.



(A) CDFs

(B) PDFs

FIGURE 6. This figure plots the optimal CDFs (Panel A) and PDFs (Panel B) of $F^{**}(\cdot)$. For a clear look we only plot the cases when $\iota = 1/3$, $5/12$, $9/20$, and $99/200$. For the detailed results one can refer to Panel A of Table 2.

Comparing Figure 6 for convex fees to Figure 2 for the fixed-bonus case, we can see the difference in the PDFs of performance. The gap within range $[0, 2]$ now disappears, instead the managers place significant probability weight on this region. Also weight is placed on points greater than 2. Weighting such points is no longer simply a waste of capacity as these points will provide huge carried-interest rewards. However, placing weight on these points is also quite expensive in terms of capacity and thus the PDF of the performance distribution is fairly flat and decreasing over this region. As ι increases, the probability of very low performance realizations increases, represented by the increase of the intercept at zero in the PDFs. This occurs because the managers need to save more capacity for the extremely beneficial and risky performance levels when the high-water mark incentive is strong. Hence, as ι grows larger and closer to 0.5, they take extra risk by placing small weights on the levels which could help them win a lot, bearing huge downside risk to meet the capacity constraint. The right limit of the support, denoted by $\hat{x}_2$, which can be expressed as

$$\hat{x}_2 = \frac{1-2\iota}{\beta^{**}-\iota} = \frac{5-8\iota}{5(1/2-\iota)},$$

which explodes as $\iota \to 0.5$, with the same polynomial order as $(1/2-\iota)^{-1}$. On the other hand, as $\iota \to 0$, the support shrinks back to $[0, 2]$, and the PDF converges to uniform distribution, as was the case in the bonus-contract model.

W provided the detailed numerical results in Panel A of Table 2 in the appendix. In summary: (i) When ι is close to zero, the optimal solution is very much like the one in the bonus-contract case; (ii) When ι grows bigger, the managers actually take riskier performance than in the bonus-contract model, by choosing a more concave, i.e. more positively skewed, cumulative distribution function and a wider support; (iii) The length of the support grows as ι increases, and explodes to infinity as $\iota \to 0.5$; (iv) When $\iota \geq 0.5$, no solution exists because the managers' appetite for increasing risk is unbounded.

2.3. Analysis of equilibrium solution

The following theorem shows that risk-taking increases as ι increases. The binding capacity constraint implies the same mean for any two performance distributions with different ι. We can then verify the *single crossing property* to demonstrate the following stochastic dominance results.

Theorem 1 *If $\iota_2 > \iota_1$, the equilibrium performance distribution under ι_2 is a* mean preserving spread *of the performance distribution under ι_1, i.e., ι_2 is riskier in the sense of* second-order stochastic dominance.[15]

Hence, when a manager cares more about winning the carried-interests as opposed to capturing funds flow through superior relative performance, the manager takes more risk in the sense of the convex stochastic order. If we had restricted manager to choosing symmetric unimodal distributions, increased risk would have been expressed by moving probability mass from center of the distribution towards its tails. However, our analysis shows that when hedge fund managers are free to choose performance distributions, an increase in the

[15] For more information about these stochastic orders, one can see, for example, Rothschild and Stiglitz (1970).

benchmark incentive weight ι, optimal risk taking typically takes the form of elongating the right tail of the distribution by increasing skewness. This can be shown by calculating the skewness coefficient, as shown in Panel A of Table 2. In fact, taking any two performance distributions with different ι, we can numerically show that they are ordered in the sense of *skewness ordering*, which, for random variables with the same mean, is an even stronger partial order than the convex order used to characterize risk-taking in Theorem 1. Skewness ordering is defined as follows: F_2 weakly dominates F_1 in the skewness order if $F_2^{-1}\big(F_1(x)\big)$ is convex on the set $\{x : 0 < F_1(x) < 1\}$.[16] We can show numerically that for any $\iota_2 > \iota_1$, the corresponding equilibrium distributions have the skewness ordering relationship that $F_2^{-1}\big(F_1(x)\big)$ is convex. For example, the following figure gives us some examples of the property that the managers with stronger benchmark incentive pick more positively skewed distributions. Moreover, the convexity is stronger when the difference between ι_2 and ι_1 becomes larger.

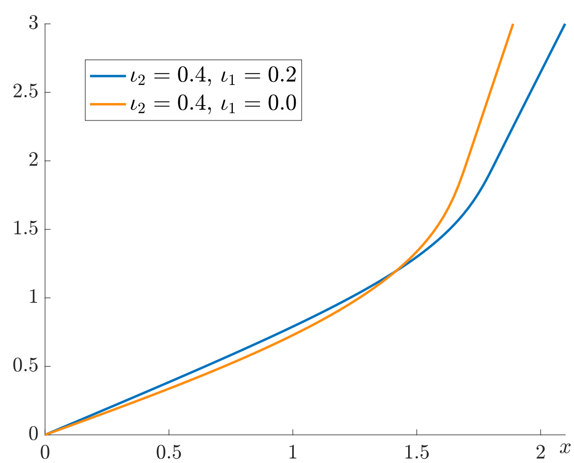


FIGURE 7. This figure plots $F_2^{-1}\big(F_1(x)\big)$ for different pairs of (ι_2, ι_1) to test the stochastic ordering. Note that our main concern is the shape of $F_2^{-1}\big(F_1(x)\big)$ within $[0, 2]$ and around the connecting point 2. In the region $(2, \infty)$ both equilibrium distributions are linear, thus $F_2^{-1}\big(F_1(x)\big)$ is linear.

Thus, in contrast to the bonus reward, which typically leads to bimodal equilibrium performance distributions and sometimes to performance distributions with disconnected supports, the convex performance reward model leads to weakly unimodal performance distributions with connected supports and long-right tails. These characterizations much more closely track observed hedge fund returns. Thus, combining the insights of previous section, we can see that the distinctive structure of hedge fund returns is not simply the result of a mixture of relative and absolute performance incentives, but rather the specific consequence of the managerial compensation in the industry. The relative dominance of the "2%" funds flow

[16]Here "convex" includes the possibility of linearity. Skewness ordering is first proposed by van Zwet (1964) and slightly modified by Oja (1981).

management fees over the convex "20%" performance rewards, produces sufficient relative performance incentives to bound risk taking. At the same time, the convexity of performance rewards encourages positively skewed performance.

2.4. Comparative statics and welfare analysis

The net-of-fee wealth of investors in the option-contract model, denoted by $\widehat{W}$ to distinguish from the indicator model, is a function of $(X, \tilde{\varepsilon}_m)$:

$$\widehat{W}(X, \tilde{\varepsilon}_m) = a\,X - b\,(X - \tilde{\varepsilon}_m)^+.$$

We conclude the properties of wealth in the following proposition.

Proposition 5 *Given the equilibrium performance distribution F^{**} in Proposition 4,*

(i) the expectation of investors' wealth is given by

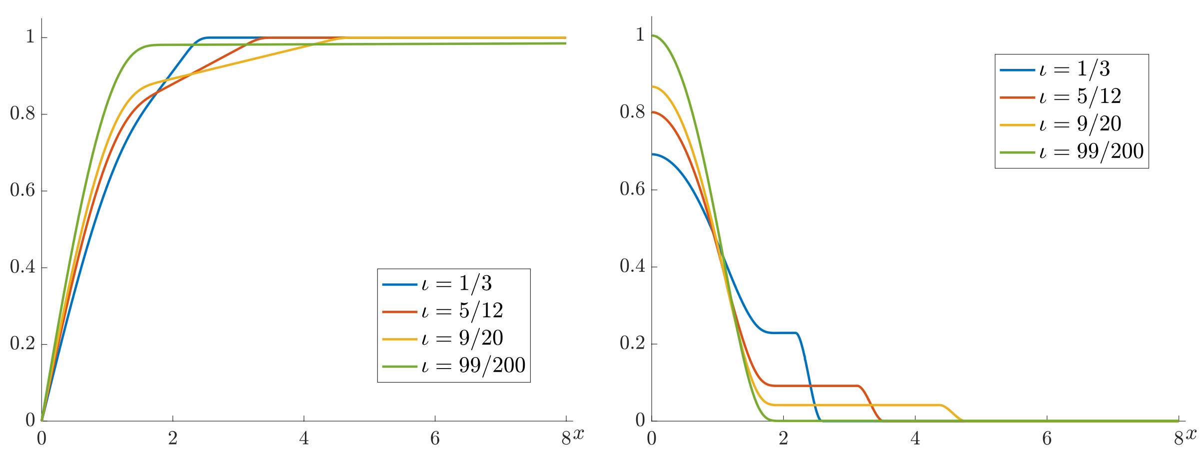
$$\mathbb{E}[\widehat{W}] = a - b\int_0^\infty \widehat{G}(x)\,\mathrm{d}F^{**}(x),$$

where $\widehat{G}$ is defined as in (2.1)*;*

(ii) the cumulative distribution of investors' wealth, denoted by $F_{\widehat{W}}(\cdot)$, can be calculated from F^{**} as

$$F_{\widehat{W}}(w) = \begin{cases} F^{**}(\frac{w}{a}) + \int_{\frac{w}{a}}^{\frac{w}{a-b}} G\Big(\frac{b-a}{b}\Big(x - \frac{w}{a-b}\Big)\Big)\,\mathrm{d}F^{**}(x) & \text{for } w \in [0, 2a), \\ F^{**}(\frac{w-2b}{a-b}) + \int_{\frac{w-2b}{a-b}}^{\frac{w}{a-b}} G\Big(\frac{b-a}{b}\Big(x - \frac{w}{a-b}\Big)\Big)\,\mathrm{d}F^{**}(x) & \text{for } w \in [2a, \infty); \end{cases}$$

where G is the distribution of the benchmark as stated in (1.1).



(A) CDFs (B) PDFs

FIGURE 8. This figure plots the CDFs (Panel A) and PDFs (Panel B) of equilibrium investor wealth $F_{\widehat{W}}(\cdot)$ for some parameters $\iota = 1/3$, $5/12$, $9/20$, and $99/200$. For more detailed results one can see Panel B of Table 2.

The CDFs and PDFs of investors' wealth with respect to different ι are shown in Figure 8. Comparing to Figure 6, we get the similar results to the bonus model, i.e., investors bear almost all the losses when performance is poor but share the gains with the managers when performance is good. As shown by the next theorem, investors also bear more risk when the managers are more focused for the performance fees as opposed to relative rank. These stochastic order results are summarized in Theorem 2.

Theorem 2 *Increasing benchmark incentives relative to rank-based incentives both lowers mean investor returns and increases investor's risk exposure in the sense of* second-order stochastic dominance.

Under the same parameter choices developed in Section 1, values of mean, variance, and certainty equivalent of wealth with respect to the weight placed on absolute performance, ι, are shown in Panel B of Table 2. As with the bonus contract, the risk premium required by the CARA investors for holding the portfolio becomes higher as the option-contract part of the incentive grows stronger.

However, the assumption that hedge fund investors are risk-averse expected utility maximizers with CARA utility functions it itself problematic. The minimum investment in a hedge fund is fairly large. Risk averse expected utility maximizing investors should hold highly diversified portfolios. Under the assumption that investors have no private information about the future performance of the fund, the vary fact that these investors are placing such a large portion of their wealth in the fund is likely to be inconsistent with the assumption that the investors have preference characterized by risk averse expected utility maximization. Thus, to assess welfare effects, we should consider a framework in which rationalizes the underdiversification that typically characterizes hedge fund investing. Boyer et al. (2010) shows that a preference for idiosyncratic risk can be rationalized by the Mitton and Vorkink (2007) model of skewness preference. Hence, we investigate investor welfare under these preferences.

The skewness preference utility function in Mitton and Vorkink (2007), denoted as "M-V utility" $U_{\text{M-V}}$, is defined as follows:

$$U_{\text{M-V}}(W) = \mathbb{E}[W] - \frac{1}{2\tau}\text{Var}(W) + \frac{1}{3\phi}\text{Skew}(W),$$

To calibrate the utility function, we use the standard measures of variance and skewness, and the preference coefficients $\tau = \phi = 2.5$ in their paper. As shown in the last column of Table 2, M-V utility is a concave function of ι and the investor-optimal weight on performance compensation equals $\iota = 0.4925$. Thus, roughly speaking, balanced weights on rank and performance incentives maximize the welfare of investors with M-V preferences.

3. Dynamic Hedge Fund Competition

In this section we extend the analysis to explicitly account for the effect of relative performance on fund flows. In an infinite date setting, we model competition between two hedge funds, X and Y. Both funds operate under the same compensation fee contract. The fee structure consists of two components, a non-performance based management fee related to assets under management and a convex performance-based high watermark fee based on the gross returns of the funds relative to the benchmark. *Ex ante* the two funds are identical. We capture the effect of relative performance on fund flows by assuming that when, in a given period, one fund's performance is better than the other's, the capacity of the better performing fund increases by $\delta > 0$ and the capacity of the worse performing fund falls by $\delta > 0$. Note that performance is the realized value of the portfolio and capacity determines the expected realized value, and expected realized value equals assets under management multiplied by the expected return on the hedge fund portfolio. Thus, capacity shifts, capture the idea, albeit in a stylized fashion, that outranking rivals leads to fund inflows and being outranked leads to fund outflows. If a fund's capacity falls to zero, the fund becomes insolvent and is liquidated. In this event, the surviving solvent fund has no competitor. Thus, surviving solvent fund's capacity is no longer affected by relative performance. Hence the surviving fund's incentives are purely generated by high-water mark.

Denote the *continuation value* of manager X as $V_X(\mu_X, \mu_Y)$, where we have two state variables, μ_i ($i = X, Y$), representing the capacities of the two funds. We use dynamic programming to solve the system backwards. In each period, the manager X's objective is:

$$\max_X V_X(\mu_X, \mu_Y) = \Pi_X + \rho \left[\mathbb{P}\Big(\frac{X}{\mu_X} > \frac{Y}{\mu_Y}\Big) V_X(\mu_X + \delta, \mu_Y - \delta) + \mathbb{P}\Big(\frac{X}{\mu_X} < \frac{Y}{\mu_Y}\Big) V_X(\mu_X - \delta, \mu_Y + \delta) \right],$$

where Π_X is the current reward of the manager:

$$\Pi_X = a\,\mu_X + b\,\mathbb{E}[(X - \mu_X \tilde{\varepsilon}_m)^+],$$

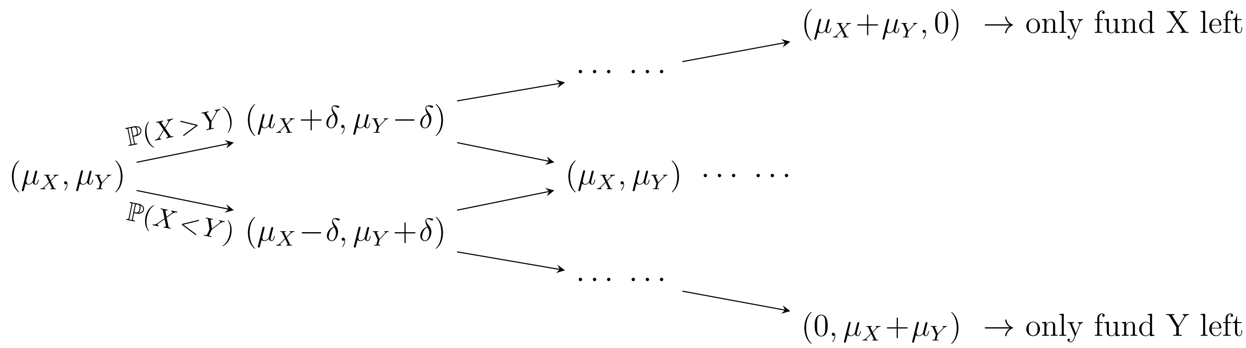
with a the proportion of management fees, b the proportion of performance fees, and $\tilde{\varepsilon}_m$ as defined in (1.1). In the function of continuation value, $\rho < 1$ can be regarded as the exogenous rate of liquidation, the diseconomy of scale, or a discount rate. X and Y are still defined to be the random performance of fund X and Y. As our previous assumption, both managers have an initial endowment of one dollar. Moreover, they still face a capacity constraint, respectively:

$$\mathbb{E}[X] \le \mu_X\,, \quad \mathbb{E}[Y] \le \mu_Y.$$

In essence, the constraint says that there is no difference between the capabilities for both managers to generate expected gross returns, and such ability remains unchanged throughout the game.

Hence the game, with each node represented by (μ_X, μ_Y), can be formulated by the following tree structure:[17]

[17]Here we mainly consider the continuous challenge distributions. Hence, we assume that X and Y are two continuous and independent random variables, which means $\mathbb{P}(X = Y) = 0$. If not, we could deal with the discrete case by assuming that each manager gets half of the payoff when they tie. Then the winning probability would be modified to $\mathbb{P}(X > Y) + 0.5\,\mathbb{P}(X = Y)$, with the rest following the same rationale.



As an example, we assume a fixed total capacity, 2, due to the restricted share of market. Take $\delta = 0.5$. By symmetry, we assume $V_X(\mu, 2-\mu) = V_Y(2-\mu, \mu)$ for every $\mu \in [0, 2]$. Then we only need to figure out the programming for manager X. To simplify notation, we use V and Π instead of V_X and Π_X in the following analysis.

Then we are able to solve the dynamic programming starting from $t = 2$. At node $(2, 0)$, manager X has won in tournament and will continue with only the high-water mark incentive. Thus we already know her strategy in the future periods — the "win-or-die" aggressive strategy. Therefore, the continuation value of the nodes at $t = 2$ are, respectively:

$$
\begin{aligned}
&(2,0) : V(2,0) = 2a + b\,\mathbb{E}[(X - 2\tilde{\varepsilon}_m)^+] + \sum_{t=1}^{\infty} \rho^t \Big(2a + b\,\mathbb{E}[(X - 2\tilde{\varepsilon}_m)^+]\Big);\\
&(1,1) : V(1,1) = V(1,1);\\
&(0,2) : V(0,2) = 0.
\end{aligned}
$$

By continuity and independence, we have $\mathbb{P}(Y/\mu_Y > X/\mu_X) = 1 - \mathbb{P}(X/\mu_X > Y/\mu_Y)$. Solving backwards, we find that in period $t = 1$, the managers are facing a trade-off between higher current rewards or greater probability to continue in the game, due to their limited capacity. The continuation values of the nodes at $t = 1$ after simplification are, respectively:

$$
\begin{aligned}
&\Big(\frac{3}{2}, \frac{1}{2}\Big) : V\Big(\frac{3}{2}, \frac{1}{2}\Big) = \frac{3}{2}a + \rho V(1,1) + b\,\mathbb{E}\Big[\Big(X - \frac{3\tilde{\varepsilon}_m}{2}\Big)^+\Big] + \rho\mathbb{P}(X > 3Y)\Big(V(2,0) - V(1,1)\Big);\\
&\Big(\frac{1}{2}, \frac{3}{2}\Big) : V\Big(\frac{1}{2}, \frac{3}{2}\Big) = \frac{1}{2}a + b\,\mathbb{E}\Big[\Big(X - \frac{\tilde{\varepsilon}_m}{2}\Big)^+\Big] + \rho\mathbb{P}(3X > Y)\,V(1,1).
\end{aligned}
$$

Since we have $V_X(1/2, 3/2) = V_Y(3/2, 1/2)$, the above functions are the objectives which the managers want to maximize in the state $(3/2, 1/2)$. After proper choice of the parameters, the parts without the random variable reduce to some determined values. Hence, we are essentially finding the equilibrium solution of the following game:

$$
\begin{aligned}
&\text{Fund X}: \max_{F_X} b\mathbb{E}\Big[\Big(X - \frac{3\tilde{\varepsilon}_m}{2}\Big)^+\Big] + (\tilde{\rho} - c)\,\mathbb{P}(X \geq 3Y), \quad \text{s.t. } \mathbb{E}[X] \leq \frac{3}{2};\\
&\text{Fund Y}: \max_{F_Y} b\mathbb{E}\Big[\Big(Y - \frac{\tilde{\varepsilon}_m}{2}\Big)^+\Big] + c\,\mathbb{P}(3Y \geq X), \qquad\qquad \text{s.t. } \mathbb{E}[Y] \leq \frac{1}{2};
\end{aligned}
$$

where $\tilde{\rho} := \rho V(2,0)$, $c := \rho V(1,1)$. Hence, after some change of variables, the system becomes similar to that in (2.2) of the one-period option-contract model. The only difference is that the previous parameter ι becomes different for the two managers and is endogenously determined by the dynamic programming process.

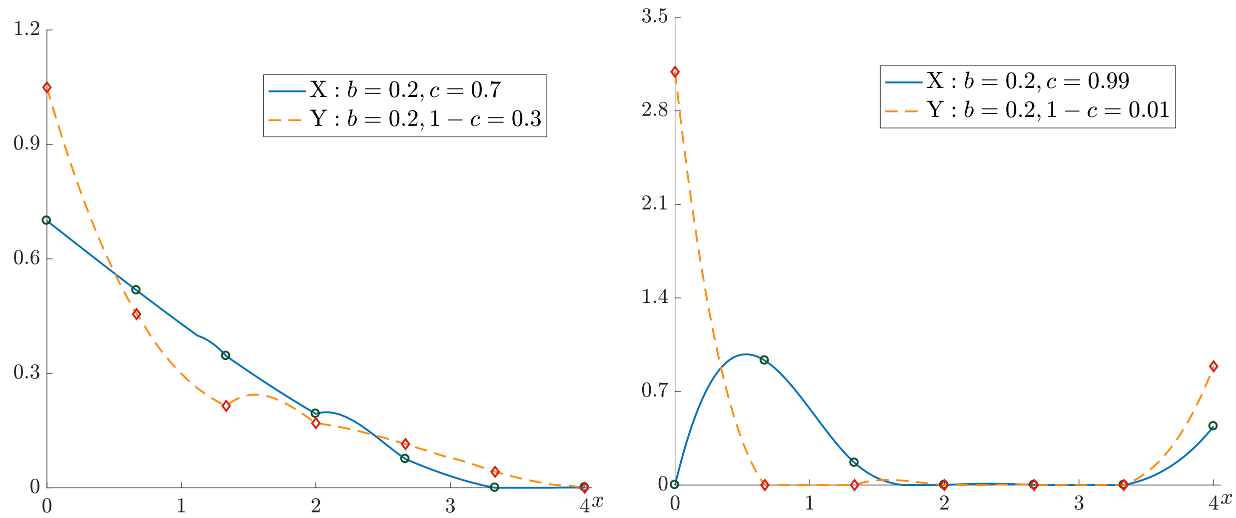
The equilibrium solutions are hard to find without the symmetric property of the objective functions, and might not be unique. But we can characterize the equilibria numerically to identify some dynamic effects. Using the *contraction mapping* theorem, for a given pair of initial densities, we iterate the optimization system to find the fixed point, which should be one of the equilibria. Then we could prove the equilibria numerically by examining the colinearity where $\mathrm{d}F^* > 0$ on the upper support line (as that in (1.4)), and test the stability with respect to the choice of starting value. We give an example as follows.

Example 1 We take $\rho = 0.8$ and the typical "2 and 20" compensation structure. Then by our parameter choice, $a = 0.02$, $b = 0.2$ and $\rho V(2,0) \approx 1$. In order to characterize the equilibrium challenge distributions numerically, we add some reasonable assumptions: (i) both managers are playing over the range $[0,4]$; (ii) in each optimization of the whole process, the density function satisfies some continuous and smooth properties so that we can use cubic spline interpolation.[18] Instead of discretizing the whole function, cubic spline makes the programming more efficient by allowing us to maximize the objective with respect to a lower dimensional vector. This assumption is acceptable also in that our previous equilibrium density functions are in the form of cubic polynomials.

Since the tournament incentives of the two managers add up to a constant and all the other parts are symmetric, we only consider the case where $c > 0.5$. Within the error tolerance of cubic spline method, we get the following equilibrium density functions for the two managers under different possible c, as shown in Figure 9 (we demonstrate in the appendices that in the numerical sense, these are indeed, respectively, one of the corresponding equilibria).[19] For a pair of tournament weights $(0.7, 0.3)$, as shown in Panel A, the results share some similarity with that in the single-period option-contract model. The one with less relative performance incentive behaves more aggressively, by bearing more downside risk to save capacity for more possible profits at top level. This is exactly the property we have shown in the previous theoretical model. For a more extreme division, $(0.99, 0.01)$ in Panel B, the player with 0.01 weight on contest almost gambles, whereas the other still faces a trade-off between current rewards and resurrection, thus is relatively conservative.

[18]Note that for a density function, we require more than the function generated by cubic spline: (i) we trim the fitting function to be nonnegative; (ii) the integration of the fitting function is one.

[19]In order for a clear comparison, we plot the scaled performance of both managers, i.e., X/μ_X and Y/μ_Y.



(A) $b = 0.2, c = 0.7$

(B) $b = 0.2, c = 0.99$

FIGURE 9. The figure plots the numerical equilibria under different parameter choices. It presents the equilibrium challenge density functions of fund X (solid line) and Y (dashed line). In both cases we take a benchmark weight $b = 0.2$. In Panel A the two players have different tournament weights, with a ratio between the two 3/7, whereas in Panel B the difference is more aggressive, with the ratio 1/99.

As in the example, for different choices of c, our results are robust and stable to the choice of the initial distribution F_Y. From the patterns we conclude that most of the properties in the single-period models maintain here. Furthermore, in our single-period option-contract model, we conclude that the managers will start to gamble if $\iota \geq 0.5$. However, this happens rarely in reality. Now in the multiperiod analysis we find the evidence. Even if plugging in the "2 and 20" weight on the relative and absolute performance incentives (in which case $\iota > 0.5$), the managers would not choose gambling, due to the existence of a third incentive — "survival" incentive in continuing the game. This is consistent with the results in Lan et al. (2013), in which the authors claim that facing a trade-off between alpha-generating strategies and going-concern in a long-horizon dynamic framework, the risk-neutral hedge fund managers become endogenously risk averse and take precautionary managerial behavior. Based on the above analysis, we conclude that when ranking is the key to survival, hedge fund managers are more conservative and mitigate their risk taking more significantly than usual.

4. Robustness

In this section we explore the robustness of our results with specific parameter choices made in the earlier analysis, as well as the robustness when we have more than two hedge fund managers. We illustrate the robustness of earlier results primarily in the fixed-bonus fee set-up, the results are similar for the convex performance fees. In the earlier analysis the capacity constraint for the rival managers was assumed to equal 1. Section 4.1 explores the effect of alternative specifications of capacity. Section 4.2 shows that our model is applicable to a multi-manager setting. In the earlier analysis we also assumed a specific functional form for the benchmark distribution, G. In Section 4.3 we illustrate how our results might be extended to encompass large class of benchmark distributions.

4.1. General capacity constraint

Our analysis can allow for a capacity μ unequal to the average market performance, 1, provided that μ lies in the interval $[1, 3/2]$. The assumption that $\mu \geq 1$, simply ensures that the hedge funds can at least attain market average performance levels. Absent this assumption, it is hard to justify investment in hedge funds. The assumption that $\mu \leq 3/2$ ensures that hedge funds cannot top the benchmark without taking risk. When managers can attain performance levels that exceed the benchmark with probability 1, the risk-taking incentives associated with fixed-bonus competition vanish and the character of the problem changes dramatically. However, as long as $\mu \in [1, 3/2]$, equilibrium distribution can be determined with the same methods used to derive Proposition 1. The qualitative properties of the resulting equilibrium performance distribution are also quite similar to the qualitative properties derived in Proposition 1. The most significant effect of introducing variation in capacity is that the support of the equilibrium performance becomes dependent on capacity. This dependence is suggested by the fact that, even when incentives are purely based on relative performance, the support of the equilibrium performance distribution is $[0, 2\mu]$ and thus dependent on μ. In addition, in the general mixed incentives case, ι_0 — the threshold separating connected and disconnected equilibrium performance distributions — becomes dependent on capacity.

More specifically, when $\mu \in [1, 3/2]$, the equilibrium performance distribution can be determined using the Lagrangian method employed in the earlier analysis, the dual problem now becomes:

$$\min_{\alpha,\, \beta \geq 0} \sup_{\mathrm{d}F_X \geq 0} \int_0^\infty \Big(\iota\, G(x) + (1-\iota)\, F_Y(x) - \alpha - \beta x\Big)\, \mathrm{d}F_X(x) + \alpha + \beta\mu.$$

Solving for the equilibrium performance distribution yields the following simple extension of the earlier analysis.

Proposition 6 (**Symmetric equilibrium for general capacity**) *Assume $\mu \in [1, 3/2]$. For any $\iota \in [0, 1]$, in the equilibrium we have that the capacity constraint binds. The optimal challenge distribution $F^*(\cdot)$ chosen by the managers is described separately in the following two cases.*

(i) When managers are more relative performance focused, i.e., $\iota \in [0, \iota_0]$: $F^*(\cdot) \equiv T^*(\cdot)$ before T^* reaches 1 as stated in (i) of Proposition 1, with the same T^* as defined in (1.5)*. The optimal Lagrangian multipliers are*

$$\alpha^* \equiv 0, \qquad \beta^* = 1/2(\iota + \mu - \mu\iota).$$

The support of F^* now varies with μ, becoming $[0, 2(\iota + \mu - \mu\iota)]$.

(ii) When managers are more benchmark focused, i.e., $\iota \in (\iota_0, 1]$:

(a) For $\iota < 1$, the results are similar to those stated in part (ii) of Proposition 1, with $F^*(\cdot)$ being a piecewise function with the same formulation.

(b) For $\iota = 1$, i.e., the extreme case when there is only benchmark incentive, F^* is a discrete two-point distribution, with the support remaining the same as the support identified in Proposition 1; the weight on the two support points changes in fashion which ensures that the capacity constraint binds, i.e.,

$$F^*(x) = \begin{cases} 0 & \text{with probability } 1 - \dfrac{2\mu}{3}, \\ \dfrac{3}{2} & \text{with probability } \dfrac{2\mu}{3}. \end{cases}$$

In both cases, ι_0 is a function of μ, given by:

$$\iota_0 = \begin{cases} \dfrac{2}{3} & \text{for } \mu = 1, \\ \dfrac{3\mu - \sqrt{9\mu^2 - 24\mu + 24}}{6(\mu - 1)} & \text{for } \mu \in (1, 3/2]. \end{cases}$$

As is apparent from Proposition 6, ι_0 decreases as capacity μ increases (for the detailed pattern, see Figure 12 in the appendices). This implies that increasing capacity leads to disconnected supports for equilibrium performance distributions at lower values of ι. The intuition for this result is that increasing capacity lowers the shadow price of placing weight on high returns around the benchmark. As beating the benchmark becomes relatively easier, both managers move probability weight out of the intermediate performance region where besting the benchmark is unlikely and into higher performance levels at which besting the benchmark is fairly probable.

4.2. Multi-manager symmetric equilibrium

Regarding the symmetric equilibrium, in which each manager weighs the incentive structure identically and has the same capacity limit, we can extend our results to n managers ($n \geq 3$). Moreover, in essence the equilibrium for more than two players is in the same fashion as in our previous models.

Assume there are n hedge fund managers ($n \geq 3$). We arbitrarily pick one manager, namely X, to analyze her risk-taking strategy; whilst denote her rivals as Y_1, Y_2, ..., Y_{n-1} (note that there is no difference between the situations X or Y_i's are in, the notations are just for distinguishing objects). Each manager makes her decision independently. By the idea of order statistics, the probability that X performs relatively best in a static game is

$$\mathbb{P}\big(X \geq \max_{1 \leq i \leq n-1} Y_i\big) = \int_0^\infty \mathbb{P}\big(\max_{1 \leq i \leq n-1} Y_i \leq x\big)\,\mathrm{d}F_X(x) = \int_0^\infty \prod_{i=1}^{n-1} F_{Y_i}(x)\,\mathrm{d}F_X(x),$$

where the notations of performance distributions are similar as before. Because the benchmark part remains unchanged, now the objective function of manager X, rather than in (1.3), becomes

$$\iota_X \int_0^\infty G(x)\,\mathrm{d}F_X(x) + (1-\iota_X)\int_0^\infty \prod_{i=1}^{n-1} F_{Y_i}(x)\,\mathrm{d}F_X(x).$$

Assume all the managers have a same ι and $\mu \in [1, 3/2]$. *A posteriori* in a symmetric equilibrium they all pick an identical challenge distribution. Using the same Euler-Lagrangian method, we can characterize the equilibrium performance as follows.

Proposition 7 (**Symmetric equilibrium for** $n \geq 3$ **managers**) *For any* $\iota \in [0, 1]$*, in the equilibrium we have that the capacity constraint binds. The optimal challenge distribution* $F^*(\cdot)$ *chosen by the managers is described separately in the following two cases.*

(i) *When managers are more relative performance focused, i.e.,* $\iota \in [0, \iota_0]$*:* $\operatorname{supp}\{F^*\} = [0, 1/\beta^*]$*, and the equilibrium performance distribution is given by*

$$F^*(x) = \Big(\frac{\alpha^* + \beta^* x - \iota G(x)}{1-\iota}\Big)^{\frac{1}{n-1}} := \big(T^*(x)\big)^{\frac{1}{n-1}},$$

where $T^*(\cdot)$ *is as defined in* (1.5)*,* $\alpha^* \equiv 0$*, and* β^* *takes the value that binds the following expression:*

$$\int_0^{\frac{1}{\beta^*}} \left(1 - \Big(\frac{\alpha^* + \beta^* x - \iota G(x)}{1-\iota}\Big)^{\frac{1}{n-1}}\right) \mathrm{d}x = \mu.$$

Specifically, for $\iota = 0$*, i.e., the extreme case when there is only rank-order incentive, the optimal* F^* *reduces to a power function with* $\beta^* = 1/n\mu$*:*

$$F^*(x) = \Big(\frac{x}{n\mu}\Big)^{\frac{1}{n-1}}.$$

(ii) *When managers are more benchmark focused, i.e.,* $\iota \in (\iota_0, 1]$*:*

(a) *For* $\iota \in (\iota_0, 1)$*, we have* $\operatorname{supp}\{F^*\} = [0, x_2]$*, and* F^* *is a piecewise function given by*

$$F^*(x) = \begin{cases} \big(T^*(x)\big)^{\frac{1}{n-1}} & \text{for } x \in [0, x_1), \\ \big(T^*(\underline{x})\big)^{\frac{1}{n-1}} & \text{for } x \in [x_1, \underline{x}), \\ \big(T^*(x)\big)^{\frac{1}{n-1}} & \text{for } x \in [\underline{x}, x_2), \\ 1 & \text{for } x \in [x_2, 2], \end{cases}$$

where $T^*(\cdot)$ *is as defined in* (1.5)*,* $\alpha^* \equiv 0$*, and* β^* *takes the value which binds the capacity constraint.* $\underline{x}$ *remains the same as in Proposition 1, and* x_i $(i = 1, 2)$ *changes accordingly (details in the proof).*

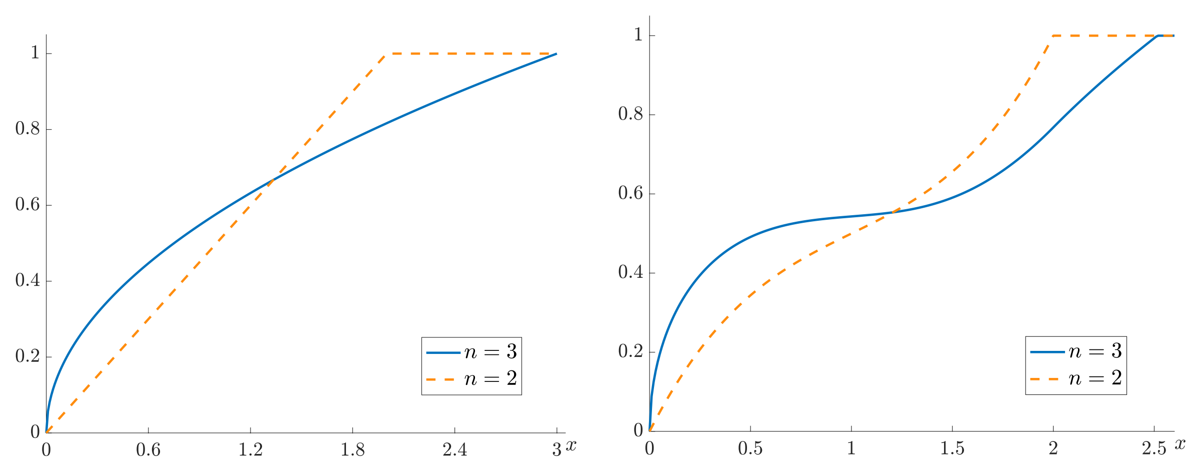
(b) *For* $\iota = 1$*, i.e., the extreme case when there is only benchmark incentive,* F^* *reduces to the discrete distribution in Proposition 6.*

In both cases, ι_0 *takes the value that solves*

$$\iota_0 = \frac{4\beta^*}{3}.$$

Let us give some intuition by comparing this equilibrium to the one with only two players. The larger the number of managers, the heavier the *selectivity* of the game. Thus with $n \geq 3$ managers it is harder for one to achieve the winning position. Therefore, the managers stretch out the final distribution to compete more at higher levels, showed by a wider support. Take the extreme case where $\iota = 0$ as an example. As can be seen from Panel A in Figure 10, the equilibrium challenge distribution varies from uniform to power function as n increases from 2. This amounts to say that to avoid being topped up by others, one need to take riskier strategy, by enlarging the distribution support and placing heavier weight on zero level. In the general case where $\iota > 0$, because the power function will not change the general shape and monotonicity of $T(\cdot)$, our equilibrium performance distribution $F^*(\cdot)$ still traces a similar function T, but in a more riskier way — baring higher downside risk and chasing more at higher levels. Mathematically, as n increases, the distribution function pointwise converges to one, and to ensure the expectation unchanged due to capacity constraint, the support will grow wider. Intuitively the concavity of the challenge distribution picked by the managers increases. In other words, the equilibrium performance distribution "lifts up" and "stretches out" as the number of managers increases.

To analyze the change brought by the increasing selectivity, we take a typical scenario where there are three managers ($n = 3$), all of whom have a capacity equivalent to the market average ($\mu = 1$). For comparison, we also display the old results when there are only two contestants, as shown in Figure 10. Moreover, keeping μ unchanged, a natural theorem follows as well, concerning the change of riskiness when we have more managers.



(A) Symmetric Equilibrium when $\iota = 0$ (B) Symmetric Equilibrium when $\iota = 1/2$

FIGURE 10. The figure plots the symmetric equilibrium performance distributions picked by the managers. To capture the idea we show the typical cases where there is only relative performance incentive ($\iota = 0$) and where the managers weigh the two incentives identically ($\iota = 1/2$). If there is no relative performance concern, our model yields the same results as before regardless of the number of managers. For comparison, the dashed line shows our previous result when there are only two contestants, whereas the solid line stands for the model with three managers.

Theorem 3 *The equilibrium performance distribution in a game with larger $n \in \mathbb{N}^+$ is a* mean preserving spread *of the one with less contestants. This is sufficient to say, the equilibrium performance distribution with more contestants is riskier in the sense of* second-order stochastic dominance.

Note that even though increasing contest selectivity increases risk taking, increasing the weight on rank-based rewards still reduces risk taking. In fact, one can show that, no matter how large the number of contestants, the return distribution under pure rank-based rewards is always bounded. This bounded risk taking regardless of number of players is in fact consistent with the all-pay auction literature, e.g., Baye et al. (1996). Therefore, the qualitative implications of our analysis are robust to increasing the number of competing funds.

4.3. General distribution of benchmark and quantile method

In this section, we show that even if the benchmark distribution, $G(\cdot)$, does not take the form specified in (1.1), the optimization problem that determines the equilibrium performance distribution is solvable and of a similar character to our baseline results, provided that the benchmark distribution is unimodal and its support is a subset of the non-negative real line. The solution to the problem in the more general case can be facilitated by using the recently introduced *quantile method* (He and Zhou, 2011; He and Kou, 2014) for optimizing portfolio performance.

More specifically, assume that the distribution of the benchmark, $G(\cdot)$, is a unimodal distribution with nonnegative but not necessarily finite support. Besides this more general specification of G, our optimization problem remains exactly the same as stated in (1.3).

An unbounded support makes it difficult to apply the Lagrangian method to solve the optimization problem defining the equilibrium performance distribution. However, using the quantile method, we can transform the problem in a way ensuring that the range of integration is the bounded interval, $[0, 1]$. Furthermore, the quantile transformation helps us to solve a pointwise optimization. Consider the *quantile* function of $F_X(\cdot)$, denoted by $Q(\cdot)$, defined by

$$Q(t) \coloneqq F_X^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\} = \sup\{x \in \mathbb{R} : F(x) < t\}, \quad t \in [0, 1],$$

where $Q(\cdot)$ is nondecreasing, left continuous, and $Q(1) \coloneqq Q(1-)$. Even though X and $Q(\cdot)$ might be two different random variables, they are identical in distribution. Hence, it can be proved that solving (1.3) is equivalent to solving the following transformation with respect to the quantile function:

$$\begin{aligned} &\max_{Q(\cdot)} \iota \int_0^1 G\big(Q(t)\big)\,\mathrm{d}t + (1-\iota)\int_0^1 F_Y\big(Q(t)\big)\,\mathrm{d}t; \\ &\text{s.t.} \quad \int_0^1 Q(t)\,\mathrm{d}t \leq \mu, \\ &\qquad G(0+) \geq 0. \end{aligned} \tag{4.1}$$

The relation between the original and quantile optimization problem is detailed in the following Lemma.

Lemma 4 *The relation between the quantile and optimization problem given by expression* (4.1) *and the original optimization problem given by expression* (1.3) *is characterized as follows:*

(i) Problem (1.3) *is feasible if and only if the quantile formulation* (4.1) *is feasible;*[20]

(ii) The existence (uniqueness) of the optimal solution to (1.3) *is equivalent to the existence (uniqueness) of the optimal solution to the quantile formulation* (4.1).

It is possible for us to ignore the $G(0+) \geq 0$ constraint first, since it can be recovered later without violating the optimal form of the solution. Consider the Lagrangian problem,

$$\max_{Q(\cdot)} \; \mathcal{L}\big(Q(\cdot), \beta\big) := \int_0^1 \Big(\iota G\big(Q(t)\big) + (1-\iota)\, F_Y\big(Q(t)\big) - \beta\, Q(t)\Big)\,\mathrm{d}t + \beta\mu, \tag{4.2}$$

where β is the Lagrangian multiplier.

Using calculus of variations, we can see that to find the maximizer of (4.2) is sufficient to consider the pointwise maximization, thus the following ODE as specified in Proposition 8. Note that we *a posteriori* know $F_X(\cdot) = F_Y(\cdot)$, following the similar symmetric equilibrium analysis as before.

Proposition 8 *The maximizer of the Lagrangian problem* (4.2) *satisfies the following ODE, for* $t \in [0,1]$*:*

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \frac{1-\iota}{\beta - \iota G'\big(Q(t)\big)}.$$

Solving the ODE yields

$$\beta\, Q(t) - \iota G\big(Q(t)\big) = (1-\iota)\, t + C,$$

where C is some constant determined by the boundary condition.

Theoretically, solving the ODE with all the properties of quantile function hold, we could get the optimal solution of the Lagrangian problem (4.2), as a function of β, denoted by $Q^*(\beta, t)$. Then we can find $\beta^* \in \mathbb{R}^+$ which binds the capacity constraint:

$$\int_0^1 Q^*(\beta, t)\,\mathrm{d}t = 1.$$

Usually the uniqueness of β^* can be guaranteed by examining the monotonicity of the left hand side. In the following proposition, we show that $Q^*(\beta^*, t)$ is indeed the equilibrium solution of the quantile problem (4.1). Then from Lemma 4, we obtain the equilibrium solution of problem (1.3).

[20] An optimization problem is *feasible* if, it admits at least one solution that satisfies all the constraints imposed.

Proposition 9 *By proving the strong duality, one can show that $Q^*(\beta^*, \cdot)$ is the optimal solution to the problem* (4.1)*. Moreover, in the symmetric equilibrium, provided that*

$$\frac{1-\iota}{\beta - \iota G'\big(Q(t)\big)},$$

is a Lipschitz function, the solution is unique.

However, we do not use this method in our main sections, since it introduces subtle difficulties when the distribution is not of a standard shape: (i) it is hard to define properly the differentiation and inverse of Q, and (ii) the solution of the ODE might not be unique. We need to rule out the infeasible solutions by the implied requirements from a quantile function, which might be hard to examine. Therefore, the practical applications of this technique for obtaining closed-form solutions might be limited.

5. Conclusions

In this paper, we explore the influence of incentive structure on the risk-taking behavior of hedge fund managers. We model the interaction between absolute-performance benchmark incentives generated by performance fees and relative performance incentives generated by order flow. We characterize the equilibrium distribution of fund value in both static and dynamic contests. When we examine the interaction between the two incentives, we find that, for both convex and fixed-bonus absolute performance incentives, the gambling behavior motivated by performance fees is moderated by relative performance incentives. Relative performance incentives do encourage risk-taking, but, because of the win-small incentive inherent in relative performance contests, gambles are confined to a relatively small interval of realized returns. Although the qualitative features of risk taking under fixed-bonus model and the convex bonus high-water mark compensation are similar, the convex bonus model generates performance distributions that more closely resemble the positively skewed performance distributions typically produced by hedge funds.

The benchmark incentive, especially the convex high-water mark performance compensation, leads managers to chase high returns even when such return chasing generates dramatic downside risks. Relative performance incentives, which are generated by the management fee component of hedge fund competition, moderates this incentive for extreme risk taking. In periods of market decline, attaining the high-water mark becomes more difficult, and thus our analysis suggests that, in such periods, managers will derisk their portfolios. Thus, relative performance incentives are socially beneficial.

From the perspective of risk-averse, expected-utility maximizing hedge fund investors, increasing the weight managers place on relative performance incentives is always welfare improving. In contrast, when hedge fund investor preferences are given by the behavioral utility function developed in Mitton and Vorkink (2007), which in contrast to the expected utility preferences, is consistent with the underdiversification characterizing hedge fund investing, investor welfare is maximized by a roughly equal balance between rank-based and performance based manager incentives.

Dynamic competition between managers adds a new incentive into the hedge fund managers' strategy formulation — the survival incentive. Managers will sacrifice current period benchmark rewards in order to wipe out or avoid being wiped out by their rivals. Moreover, as the competition proceeds, the results of past competitions change assets under management and thus the relative weight placed on survival. Because survival is tied to relative performance, the weight placed on relative performance becomes endogenous and path dependent.

The results suggest several avenues for future work. One is to find additional empirical support for our predictions about the risk taking behavior of hedge funds. Another is to relate specific hedge fund strategies to the equilibrium strategies identified in this paper. Another avenue of extension is to apply our results on mixed incentives to other economic and financial environments where relative performance is important. Finally, the closed-form solution of the discrete-time dynamic problem remains to be discovered.

References

Aragon, George O and Vikram Nanda, "Tournament behavior in hedge funds: High-water marks, fund liquidation, and managerial stake," *Review of Financial Studies*, 2012, *25* (3), 937–974.

Axelson, Ulf, Per Strömberg, and Michael S Weisbach, "Why are buyouts levered? The financial structure of private equity funds," *The Journal of Finance*, 2009, *64* (4), 1549–1582.

Basak, Suleyman and Dmitry Makarov, "Strategic asset allocation in money management," *The Journal of Finance*, 2014, *69* (1), 179–217.

_______, Anna Pavlova, and Alexander Shapiro, "Optimal asset allocation and risk shifting in money management," *Review of Financial Studies*, 2007, *20* (5), 1583–1621.

Baye, Michael R, Dan Kovenock, and Casper G De Vries, "The all-pay auction with complete information," *Economic Theory*, 1996, *8* (2), 291–305.

Bonnans, J Frederic and Alexander Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Science & Business Media, 2013.

Boyd, Stephen and Lieven Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

Boyer, Brian, Todd Mitton, and Keith Vorkink, "Expected idiosyncratic skewness," *Review of Financial Studies*, 2010, *23* (1), 169–202.

Brown, Keith C, W Van Harlow, and Laura T Starks, "Of tournaments and temptations: An analysis of managerial incentives in the mutual fund industry," *The Journal of Finance*, 1996, *51* (1), 85–110.

Brown, Stephen J, William N Goetzmann, and James Park, "Careers and survival: Competition and risk in the hedge fund and CTA industry," *The Journal of Finance*, 2001, *56* (5), 1869–1886.

Buraschi, Andrea, Robert Kosowski, and Worrawat Sritrakul, "Incentives and endogenous risk taking: A structural view on hedge fund alphas," *The Journal of Finance*, 2014, *69* (6), 2819–2870.

Busse, Jeffrey A, "Another look at mutual fund tournaments," *Journal of Financial and Quantitative Analysis*, 2001, *36* (01), 53–73.

Carpenter, Jennifer N, "Does option compensation increase managerial risk appetite?," *The Journal of Finance*, 2000, *55* (5), 2311–2331.

Chen, Joseph, Eric N Hughson, and Neal Stoughton, "Strategic mutual fund tournaments," *Available at SSRN 2023805*, 2012.

Christoffersen, Susan Kerr, David K Musto, and Bilge Yilmaz, "High water marks in competitive capital markets," in "EFA 2009 Bergen Meetings Paper" 2013.

Dijk, Oege, Martin Holmen, and Michael Kirchler, "Rank matters—The impact of social competition on portfolio choice," *European Economic Review*, 2014, *66*, 97–110.

Ding, Bill and Hany A Shawky, "The performance of hedge fund strategies and the asymmetry of return distributions," *European Financial Management*, 2007, *13* (2), 309–331.

______, Mila Getmansky, Bing Liang, and Russ Wermers, "Share restrictions and investor flows in the hedge fund industry," *SSRN eLibrary*, 2009, *111.*

Drechsler, Itamar, "Risk choice under high-water marks," *Review of Financial Studies*, 2014, *27* (7), 2052–2096.

Fang, Dawei and Thomas H Noe, "Skewing the odds: Strategic risk taking in contests," *Available at SSRN 2339212*, 2015.

Feng, Han and David Hobson, "Gambling in contests with regret," *Mathematical Finance*, 2014.

______, ______ et al., "Gambling in contests with random initial law," *The Annals of Applied Probability*, 2016, *26* (1), 186–215.

Goetzmann, William N, Jonathan E Ingersoll, and Stephen A Ross, "High-water marks and hedge fund management contracts," *The Journal of Finance*, 2003, *58* (4), 1685–1718.

Gregoriou, Greg N, Komlan Sedzro, and Joe Zhu, "Hedge fund performance appraisal using data envelopment analysis," *European Journal of Operational Research*, 2005, *164* (2), 555–571.

Grossman, Gene M and Henrik Horn, "Infant-industry protection reconsidered: The case of informational barriers to entry," *The Quarterly Journal of Economics*, 1988, *103* (4), 767–787.

Guasoni, Paolo and Jan Obłój, "The incentives of hedge fund fees and high-water marks," *Mathematical Finance*, 2013.

Hamdani, Assaf, Eugene Kandel, Yevgeny Mugerman, and Yishay Yafeh, "Incentive fees and competition in pension funds: Evidence from a regulatory experiment," Technical Report, National Bureau of Economic Research 2016.

He, Xue Dong and Steven Kou, "Profit sharing in hedge funds," *Available at SSRN 1929174*, 2014.

______ and Xun Yu Zhou, "Portfolio choice via quantiles," *Mathematical Finance*, 2011, *21* (2), 203–231.

Hodder, James E and Jens Carsten Jackwerth, "Incentive contracts and hedge fund management," *Journal of Financial and Quantitative Analysis*, 2007, *42* (04), 811–826.

Janecek, Karel and Mihai Sîrbu, "Optimal investment with high-watermark performance fee," *SIAM Journal on Control and Optimization*, 2012, *50* (2), 790–819.

Kempf, Alexander and Stefan Ruenzi, "Tournaments in mutual-fund families," *Review of Financial Studies*, 2008, *21* (2), 1013–1036.

Krasny, Yoel, "Asset pricing with status risk," *The Quarterly Journal of Finance*, 2011, *1* (03), 495–549.

Lan, Yingcong, Neng Wang, and Jinqiang Yang, "The economics of hedge funds," *Journal of Financial Economics*, 2013, *110* (2), 300–323.

Lazear, Edward P and Sherwin Rosen, "Rank-order tournaments as optimum labor contracts," *Journal of Political Economy*, 1981, *89* (5), 841–864.

Lim, Jongha, Berk A Sensoy, and Michael S Weisbach, "Indirect incentives of hedge fund managers," *The Journal of Finance*, 2015.

Metrick, Andrew and Ayako Yasuda, "The economics of private equity funds," *Review of Financial Studies*, 2010, *23* (6), 2303–2341.

Milgrom, Paul and Ilya Segal, "Envelope theorems for arbitrary choice sets," *Econometrica*, 2002, *70* (2), 583–601.

Mitton, Todd and Keith Vorkink, "Equilibrium underdiversification and the preference for skewness," *Review of Financial Studies*, 2007, *20* (4), 1255–1288.

Oja, Hannu, "On location, scale, skewness and kurtosis of univariate distributions," *Scandinavian Journal of Statistics*, 1981, pp. 154–168.

Panageas, Stavros and Mark M Westerfield, "High-water marks: High risk appetites? Convex compensation, long horizons, and portfolio choice," *The Journal of Finance*, 2009, *64* (1), 1–36.

Robinson, David T and Berk A Sensoy, "Do private equity fund managers earn their fees? Compensation, ownership, and cash flow performance," *Review of Financial Studies*, 2013, *26* (11), 2760–2797.

Ross, Stephen A, "Compensation, incentives, and the duality of risk aversion and riskiness," *The Journal of Finance*, 2004, *59* (1), 207–225.

Rothschild, Michael and Joseph E Stiglitz, "Increasing risk: I. A definition," *Journal of Economic Theory*, 1970, *2* (3), 225–243.

Roussanov, Nikolai, "Diversification and its discontents: Idiosyncratic and entrepreneurial risk in the quest for social status," *The Journal of Finance*, 2010, *65* (5), 1755–1788.

Seel, Christian and Philipp Strack, "Gambling in contests," *Journal of Economic Theory*, 2013, *148* (5), 2033–2048.

Sorensen, Morten, Neng Wang, and Jinqiang Yang, "Valuing private equity," *Review of Financial Studies*, 2014, *27* (7), 1977–2021.

Starks, Laura T, "Performance incentive fees: An agency theoretic approach," *Journal of Financial and Quantitative Analysis*, 1987, *22* (01), 17–32.

Strack, Philipp, "Risk-taking in contests: The impact of fund-manager compensation on investor welfare," *Available at SSRN*, 2016.

Taylor, Jonathan, "Risk-taking behavior in mutual fund tournaments," *Journal of Economic Behavior & Organization*, 2003, *50* (3), 373–383.

van Zwet, Willem Rutger, *Convex Transformations of Random Variables*, Vol. 7, Mathematisch Centrum, 1964.

Wolfstetter, Elmar, U Dulleck, R Inderst, P Kuhbier, and M Lands-Berger, *Stochastic Dominance: Theory and Applications*, Vol. 11, Humboldt-Univ., Wirtschaftswiss. Fak., 1993.

A. Tables and Graphs

A.1. Tables

Table 1. Optimal parameters and statistics of the bonus-contract model

Panel A presents the detailed optimal parameters under different choices of ι in the bonus-contract model. We can conclude that, as ι increases towards 1: (i) $\alpha^* \equiv 0$ within the computation error tolerance; (ii) $\mathbb{E}[X^*] = 1$, i.e., the capacity constraint binds; and (iii) the optimal x_2 decreases and converges to 1.5 gradually.

Note that when ι grows to be very close to 1, the system of optimization is highly sensitive to the choice of starting value. For example, when $\iota = 9999/10000$, the initial value needs to be adjusted to approximately $(\alpha, \beta) = [0.00001, 0.56451]$ to make the program able to find the converging point. This might be due to that the whole system, specifically all the functions, depends highly on α and β. Moreover, we do not include the results for relatively small ι here, since the pattern is obvious from the description in Proposition 1.

Panel B presents the results from the investors' perspective, including mean, variance and certainty equivalent of investors' wealth in the bonus-contract model. Both the mean and the certainty equivalent decrease as ι increases, which demonstrates that the risk premium required by the CARA investors for holding the portfolio becomes higher.

Panel A - optimal parameters, and capacities

ι	(α^*, β^*)	x_2	$\mathbb{E}[X^*]$
3/4	(0.00000226, 0.50637924)	1.9741	1.0000
5/6	(0.00000008, 0.52051716)	1.9122	1.0000
7/8	(0.00000007, 0.52940184)	1.8684	1.0000
9/10	(0.00000007, 0.53521787)	1.8365	1.0000
99/100	(0.00000040, 0.55933332)	1.6224	1.0000
999/1000	(0.00000005, 0.56216772)	1.5409	1.0000
9999/10000	(0.00000093, 0.56246593)	1.5132	1.0000

Panel B - statistics, and certainty equivalent of investor wealth

ι	$\mathbb{E}[W]$	$\mathrm{Var}(W)$	$C_u(W)$
0	0.8800	0.2516	0.0621
1/2	0.8800	0.3516	0.0445
3/4	0.8783	0.5083	0.0315
5/6	0.8751	0.5290	0.0257
7/8	0.8733	0.5268	0.0221
9/10	0.8722	0.5209	0.0197
99/100	0.8680	0.4379	0.0085
999/1000	0.8676	0.3969	0.0082

Table 2. Optimal parameters and statistics of the option-contract model

Panel A presents the detailed optimal parameters under different choices of ι in the option-contract model. We can conclude that, as ι increases towards 0.5: (i) $\alpha^{**} \equiv 0$ within the computation error tolerance; (ii) $\mathbb{E}[X^{**}] = 1$, i.e., the capacity constraint binds; (iii) the right end of the support, $\hat{x}_2$, explodes from 2 to infinity; and (iv) the equilibrium challenge distribution is more and more positively skewed, as shown by the last column. Note that the notation Skew$[X]$ in the table represents the Pearson's moment coefficient of skewness, i.e., $\mathbb{E}[(X-\mu)^3]/\big(\mathbb{E}[(X-\mu)^2]\big)^{3/2}$, where μ is the mean value.

Panel B presents the results from the investors' perspective, including mean, variance, skewness and certainty equivalent of investors' wealth in the option-contract model. Both the mean and the certainty equivalent decrease as ι increases, which demonstrates that the risk premium required by the CARA investors for holding the portfolio becomes higher when the managers become more focused on absolute performance. Moreover, the last column gives us the skewed utility of underdiversified M-V investors. As ι increases, the M-V utility increases first then decreases, which shows that the optimal compensation package for investors with skewed preference is at some intermediate level.

Panel A - optimal parameters, capacities, and skewness

ι	$(\alpha^{**}, \beta^{**})$	$\hat{x}_2$	$\mathbb{E}[X^{**}]$	Skew$[X^{**}]$
1/3	(0.00000140, 0.45238035)	2.8000	1.0000	0.7134
5/12	(0.00000016, 0.45833333)	4.0000	1.0000	1.4703
9/20	(0.00000062, 0.46785732)	5.5999	1.0000	2.2247
49/100	(0.00000007, 0.49092598)	21.5983	1.0000	6.1327
99/200	(0.00000030, 0.49524042)	41.5939	1.0000	8.9226

Panel B - statistics, certainty equivalent, and M-V utility of investor wealth

ι	$\mathbb{E}[\widehat{W}]$	Var$(\widehat{W})$	Skew$[\widehat{W}]$	$C_u(\widehat{W})$	M-V utility
1/3	0.9086	0.4241	0.5984	0.0383	0.9036
5/12	0.9000	0.6353	1.3227	0.0356	0.9493
9/20	0.8943	0.9040	2.0649	0.0342	0.9888
49/100	0.8837	0.4636	5.4720	0.0321	1.5206
197/400	0.8828	0.3313	5.3573	0.0321	1.5308
99/200	0.8819	0.2321	4.3453	0.0318	1.4149
199/400	0.8818	0.1696	2.1165	0.0321	1.1301

A.2. Graphs

FIGURE 11. Slope of the equilibrium performance distribution near origin as $\iota \to 1$

The picture gives the CDFs of $F^*(\cdot)$ near origin for $\iota = 9/10$, $99/100$, $999/1000$, and $9999/10000$ in the bonus-contract model. The slope of F^* grows steeper as ι increases, and finally converges to the extreme discrete case where there is a mass $1/3$ on zero level.

When $\iota = 9999/10000$, we can get from calculation that $\underline{y} \approx 0.3419$, which is very close to $1/3$ — the optimal weight on level 1.5 when $\iota = 1$. Therefore, our optimal F^* converges to the extreme case when there is only benchmark incentive, which shows that the overall result is robust and continuous with respect to ι.

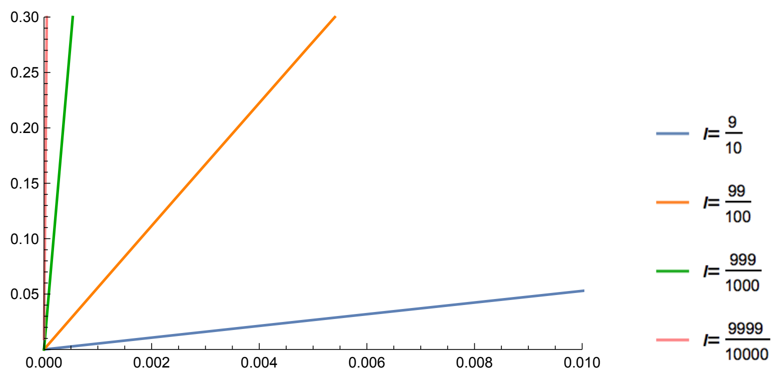
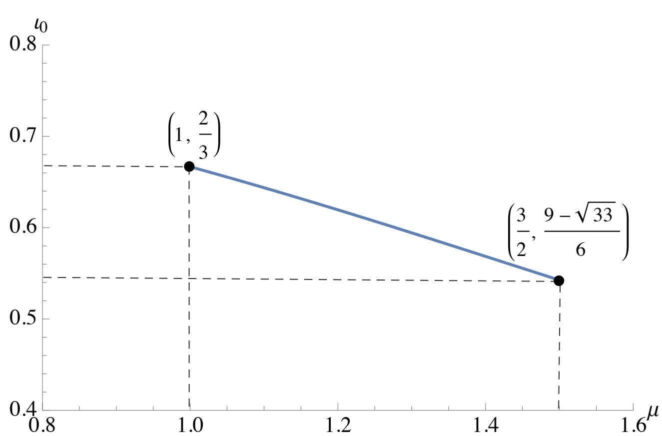


FIGURE 12. Value of the turning point ι_0 with respect to capacity μ

As shown in the following figure, the values of ι_0 and μ are negatively correlated, for any $\mu \in [1, 3/2]$. A larger capacity implies that the marginal cost of beating the benchmark is lower. Hence, ι_0 is smaller and the gap in the density function appears earlier, so that the managers could concentrate on using their resources to beat the benchmark. Especially, in our model the point $(\mu, \iota_0) = (1, 2/3)$ lies on the curve below. Thus there is no singular point any more.



B. Proofs

B.1. Proof of Lemma 1

Proof: The proof is outlined in the content. The key take away is that no player would place positive mass on any single performance level other than zero, or form a tie. We prove this by contradiction. Assume the optimal distribution chosen by manager X, denoted by F_X, has some positive weights on a point $\omega \in \operatorname{supp}\{F_X\}$, or both managers "tie" at ω with positive probability. Under this situation, Y can choose a performance level $\omega + \varepsilon$ with an infinitesimal positive ε. It costs Y little but generates a non-infinitesimal and positive gain in her payoff. This contradicts the optimality of the distribution F_X, and the tie would break as well.

In a symmetric equilibrium, this atomless result implies that the optimal solution in the symmetric equilibrium is absolutely continuous and intersects the origin. $\square$

B.2. Proof of Lemma 2

Proof: The expected payoff of manager X in each period can be defined as

$$\delta H(X, Y) \coloneqq \delta\big(\mathbb{P}(X > Y) - \mathbb{P}(X < Y)\big).$$

We have

$$1 - 2\mathbb{P}(Y \geq X) \leq H(X, Y) \leq 2\mathbb{P}(X \geq Y) - 1. \tag{B.1}$$

Take (X^*, Y^*) as the optimal strategy corresponding to the pair of distributions (F_X^*, F_Y^*) given in the lemma. First, we show that it is indeed an equilibrium solution. For any F_Y with $\mathbb{E}[Y] \leq \mu_Y$,

$$\mathbb{P}(Y \geq X^*) = \frac{1}{2\mu_X} \int_0^{2\mu_X} \mathbb{P}(Y \geq x)\, \mathrm{d}x \leq \frac{1}{2\mu_X} \int_0^{\infty} \mathbb{P}(Y \geq x)\, \mathrm{d}x \leq \frac{\mu_Y}{2\mu_X}. \tag{B.2}$$

Hence we have $H(X^*, Y) \geq 1 - \mu_Y/\mu_X$. Similarly, for any F_X with $\mathbb{E}[X] \leq \mu_X$,

$$\mathbb{P}(X \geq Y^*) = \Big(1 - \frac{\mu_Y}{\mu_X}\Big)\mathbb{P}(X \geq 0) + \frac{\mu_Y}{2\mu_X^2} \int_0^{2\mu_X} \mathbb{P}(X \geq y)\, \mathrm{d}y \leq 1 - \frac{\mu_Y}{\mu_X} + \frac{\mu_Y}{2\mu_X} = 1 - \frac{\mu_Y}{2\mu_X}. \tag{B.3}$$

Hence we have $H(X, Y^*) \leq 1 - \mu_Y/\mu_X$. From (B.1) we know that $H(X^*, Y^*) = 1 - \mu_Y/\mu_X$ and (F_X^*, F_Y^*) is the equilibrium solution. The result also implies that both capacity constraints bind.

Next, we prove the uniqueness.

Let $\widetilde{F}_X$ be an optimal distribution chosen by manager X. Then we have $H(\widetilde{X}, Y) \geq 1 - \mu_Y/\mu_X$ for any possible F_Y, i.e.,

$$\mathbb{P}(\widetilde{X} \geq Y) \geq 1 - \frac{\mu_Y}{2\mu_X}, \tag{B.4}$$

with $Y = Y^*$ the equality holds. From (B.3) we conclude that

$$\mathbb{P}(\widetilde{X} > 2\mu_X) \equiv 0. \tag{B.5}$$

We then show that, for every $u \in [0, 2\mu_X]$,

$$\mathbb{P}(\widetilde{X} \geq u) \geq 1 - \frac{u}{2\mu_X}. \tag{B.6}$$

For subcase (i) $u \in [\mu_Y, 2\mu_X]$, let $Y = (1 - \mu_Y/u)\mathbb{1}_0 + \mu_Y/u\mathbb{1}_u$, then

$$\mathbb{P}(\widetilde{X} \geq Y) = 1 - \frac{\mu_Y}{u} + \frac{\mu_Y}{u}\mathbb{P}(\widetilde{X} \geq u),$$

which by (B.4) yields (B.6); for subcase (ii) $u \in [0, \mu_Y)$, let $Y = (2\mu_X + \varepsilon - \mu_Y)/(2\mu_X + \varepsilon - u)\mathbb{1}_u + (\mu_Y - u)/(2\mu_X + \varepsilon - u)\mathbb{1}_{2\mu_X+\varepsilon}$ with infinitesimal $\varepsilon > 0$, then

$$\mathbb{P}(\widetilde{X} \geq Y) = \frac{2\mu_X + \varepsilon - \mu_Y}{2\mu_X + \varepsilon - u}\mathbb{P}(\widetilde{X} \geq u) + \frac{\mu_Y - u}{2\mu_X + \varepsilon - u}\mathbb{P}(\widetilde{X} \geq 2\mu_X + \varepsilon),$$

which by (B.4) and (B.5) yields (B.6) when $\varepsilon \to 0$. Now

$$\mu_X \geq \mathbb{E}[\widetilde{X}] \geq \int_0^{2\mu_X} \mathbb{P}(\widetilde{X} \geq u)\,\mathrm{d}u \geq \int_0^{2\mu_X} \Big(1 - \frac{u}{2\mu_X}\Big)\,\mathrm{d}u = \mu_X,$$

thus the equality holds in (B.6) for all $u \in [0, 2\mu_X]$. Therefore, $\widetilde{F}_X(\cdot) = F_X^*(\cdot)$ for every possible F_Y.

Let $\widetilde{F}_Y$ be an optimal distribution chosen by manager Y. Then we have $H(X, \widetilde{Y}) \leq 1 - \mu_Y/\mu_X$ for any possible F_X, i.e.,

$$\mathbb{P}(\widetilde{Y} \geq X) \geq \frac{\mu_Y}{2\mu_X}, \tag{B.7}$$

with $X = X^*$ the equality holds. Hence similarly, from (B.2) we have

$$\mathbb{P}(\widetilde{Y} > 2\mu_X) \equiv 0. \tag{B.8}$$

This actually implies that the weaker player "matches" the support of the stronger one. For the weaker manager Y, we need to figure out her behavior near zero. For every small $\varepsilon > 0$, let $F_X = U(\varepsilon, 2\mu_X - \varepsilon)$. Then $\mathbb{E}[X] = \mu_X$ and

$$\frac{1}{2\mu_X - 2\varepsilon}\Big(\mathbb{E}[\widetilde{Y}] - \varepsilon\mathbb{P}(\widetilde{Y} \geq \varepsilon)\Big) \geq \frac{1}{2\mu_X - 2\varepsilon}\int_\varepsilon^{2\mu_X - \varepsilon} \mathbb{P}(\widetilde{Y} \geq x)\,\mathrm{d}x = \mathbb{P}(\widetilde{Y} \geq X) \geq \frac{\mu_Y}{2\mu_X}.$$

Thus, $\mathbb{P}(\widetilde{Y} \geq \varepsilon) \leq \mu_Y/\mu_X$, which implies the following inequality when $\varepsilon \to 0$:

$$\mathbb{P}(\widetilde{Y} = 0) \geq 1 - \frac{\mu_Y}{\mu_X}.$$

It leaves to show that, for every $u \in (0, 2\mu_X]$,

$$\mathbb{P}(\widetilde{X} \geq u) \geq \frac{\mu_Y}{\mu_X}\Big(1 - \frac{u}{2\mu_X}\Big). \tag{B.9}$$

The rest part follows the similar construction method as proving the uniqueness of X. Finally, using the capacity constraint we have $\widetilde{F}_Y(\cdot) = F_Y^*(\cdot)$ for every possible F_X.

For the symmetric case, one can also refer to Fang and Noe (2015) for a detailed discussion. □

B.3. Proof of Lemma 3

PROOF: The proof is outlined in the content. More generally, the results in the lemma comes from an important technique in optimization theory, introduced as *concave envelope* or *concavification* in the literature.

First, we formally define the concave envelope,[21] denoted by $G^{\text{co}}(\cdot)$, of our distribution G:

Assume that G is a real-valued distribution with a positive, compact and convex support. A function $G^{\text{co}}(\cdot)$ is called the *concave envelope* of G when

(i) $G^{\text{co}}(\cdot)$ is concave on $\text{supp}\{G\}$,

(ii) $G^{\text{co}}(x) \geq G(x)$ for all $x \in \text{supp}\{G\}$,

(iii) there is no real-valued function H on $\text{supp}\{G\}$ satisfying both (i) and (ii), such that $H(\tilde{x}) < G^{\text{co}}(\tilde{x})$ for some $\tilde{x} \in \text{supp}\{G\}$.

[21] For the classical envelope theorem and its modified version, one can see, for example, Milgrom and Segal (2002) and Bonnans and Shapiro (2013).

To beat $G(\cdot)$, we have to maximize the probability of winning:

$$\max_{F_X} \mathbb{P}(X \geq \tilde{\varepsilon}_m) = \int_{\text{supp}\{G\}} G(x)\, \mathrm{d}F_X(x) = \mathbb{E}^{F_X}[G(X)] \leq \mathbb{E}^{F_X}[G^{\text{co}}(X)], \tag{B.10}$$

because $G^{\text{co}}(\cdot)$ is point-wisely greater than or equal to $G(\cdot)$. We can solve the optimization problem with respect to the concavification of the objective function, subject to the same constraint. Then the optimal payoff should be greater or equal to the original optimal payoff. Next we need to prove that this maximizer is indeed attainable, i.e., the inequality in (B.10) holds with equality. This is true because, by the Lagrangian in our optimization problem, the optimizer never takes on values where $G(\cdot)$ and $G^{\text{co}}(\cdot)$ disagree. Hence, the optimal solution with the concavified objective function is also optimal for the original target distribution. In other words, the optimal solution should at least be part of the concave envelope of G.

Note that the result holds for both continuous and discontinuous objective functions, as long as the concave envelope exists. It is a rather general and useful argument in optimization theory. Our results in the lemma are easily attained using this idea. □

B.4. Proof of Proposition 1

Proof: Define

$$T(x) = \frac{\alpha + \beta x - \iota G(x)}{1 - \iota}. \tag{B.11}$$

In order to make the optimal condition in (1.4) and the constraints hold, we have the following necessary conditions for $T(\cdot)$:

$$\begin{aligned} &T(x) \geq F(x), \qquad \mathrm{d}F\big\{x \in [0,2]:\ T(x) > F(x)\big\} = 0, \\ &\int_0^2 T(x)\,\mathrm{d}x \geq 1 \qquad\qquad \Rightarrow\ 2\alpha + 2\beta \geq 1, \\ &T(2) = \frac{\alpha + 2\beta - \iota}{1 - \iota} \geq 1 \quad \Rightarrow\ \alpha + 2\beta \geq 1. \end{aligned} \tag{B.12}$$

In the following we separate the proof into two parts according to the properties of $T(\cdot)$.

(i) $T(\cdot)$ *nondecreasing.* In this case $\beta \geq 3\iota/4$.

By the nondecreasing property of a distribution, according to the necessary conditions in (B.12), the optimal challenge distribution is $F^*(\cdot) \equiv T^*(\cdot)$ on $[0,2]$.

As stated in (B.12), we have transformed the constraints into functions of (α, β). Then equivalent to the original problem, we solve the following linear programming:

$$\begin{aligned} &\min_{\alpha,\beta \geq 0}\ \alpha + \beta; \\ &\text{s.t.}\ \ \beta \geq \frac{3\iota}{4},\ \alpha + 2\beta \geq 1,\ 2\alpha + 2\beta \geq 1. \end{aligned} \tag{B.13}$$

The optimal Lagrangian multipliers are

$$\begin{cases} \beta^* = \dfrac{1}{2},\ \alpha^* = 0 & \text{for } \iota \leq \dfrac{2}{3}, \\ \beta^* = \dfrac{3\iota}{4},\ \alpha^* = 0 & \text{for } \iota > \dfrac{2}{3}. \end{cases} \tag{B.14}$$

Plugging back the solutions of (α^*, β^*), our previous conjecture of $\alpha^* \equiv 0$ and $\mathbb{E}[X] = 1$ are indeed proved to be true.

Note that in this situation, the support of $F^*(\cdot)$ happens to be $[0,2]$ and $F^*(2) = T^*(2) = 1$. However, the solution under $\iota > 2/3$ contradicts with this property, which means that the nondecreasing property of

$F(\cdot)$ might be violated. Hence, we need to resort to the following case for the solution when ι is too big. Moreover, the result is consistent with Lemma 2 when $\iota = 0$.

(ii) $T(x)$ *is nonincreasing and has a local minimizer.* In this case $\beta < 3\iota/4$.

(ii-a) When $\iota < 1$, at least the continuity of the optimal distribution still holds. Through calculation we get the local minimizer as

$$\underline{x} = 1 + \sqrt{1 - \frac{4\beta}{3\iota}},$$

at which the value is denoted by $\underline{y} := T(\underline{x})$. There are two subcases: (i) $\underline{y}$ is greater or equal to the intercept of T, $\alpha/(1-\iota)$, which corresponds to $9\iota/16 \leq \beta < 3\iota/4$; or (ii) $\underline{y}$ is smaller than the intercept $\alpha/(1-\iota)$, which corresponds to $0 \leq \beta < 9\iota/16$. We will show later these this two subcases can be dealt with together.

By the idea of (B.12), we have that the optimal form of the challenge distribution $F(\cdot)$ should be a piecewise function:

$$F(x) = \begin{cases} T(x) & \text{for } x \in [0, x_1), \\ T(\underline{x}) & \text{for } x \in [x_1, \underline{x}), \\ T(x) & \text{for } x \in [\underline{x}, x_2), \\ 1 & \text{for } x \in [x_2, 2]; \end{cases} \tag{B.15}$$

where x_1 is another solution besides the minimizer of $T(x) = \underline{y}$, in subcase (ii) would be automatically set to be 0; and $\operatorname{supp}\{F\} = [0, x_2]$ with $F(x_2) = 1$.

Now the capacity constraint gives another relationship of (α, β). We use h to denote this transformed constraint:

$$\mathbb{E}[X] \leq 1 \quad \Rightarrow \quad \int_0^2 F(x)\,\mathrm{d}x \geq 1 \quad \Rightarrow \ h(\alpha, \beta) \geq 1.$$

Then solving (1.3) is equivalent to solving the following programming:

$$\begin{aligned} &\min_{\alpha,\beta \geq 0} \ \alpha + \beta; \\ &\text{s.t. } \ 0 \leq \beta < \frac{3\iota}{4}, \ \alpha + 2\beta \geq 1, \ h(\alpha, \beta) \geq 1. \end{aligned} \tag{B.16}$$

Therefore, solving the programming gives that $\alpha^* = 0$ and $h(\alpha^*, \beta^*) = 1$. Plugging in the optimal Lagrangian multiplier we get the solution F^*, as shown in the proposition. The detailed parameter values of the results are given in Panel A of Table 1. If $\iota \leq 2/3$, the result overlaps with the previous case; if $\iota > 2/3$, between x_1 and $\underline{x}$ there is a gap within the support of the PDFs. Note that when solving the problem, we need to have $0 < \underline{y} < 1$. Otherwise it is possible to show that the solution is not optimal. Usually, when $\underline{y}$ is too big, the whole system explodes without being able to converge to a fixed point.

(ii-b) When $\iota = 1$, there is only an incentive to best the benchmark, with a distribution G. Due to the property of G, it is convex below median and concave above median. Define x^* to be:

$$x^* = \arg\max_x \frac{G(x)}{x}.$$

For a unimodal G, x^* exists, exceeds the median, and is unique. For our specific setting of $G(\cdot)$ in (1.1), $x^* = 3/2$.

As proved in 3, the optimal solution should be taken at the two possible levels, 0 and x^*. Assume the weight on zero is ω_0. Then according to the capacity constraint, we have

$$\mathbb{E}[X] = \omega_0 \times 0 + (1 - \omega_0) \times \frac{3}{2} \leq 1.$$

Hence $\omega_0 \geq 1/3$. Since we are maximizing

$$\mathbb{P}(X \geq \tilde{\varepsilon}_m) = \mathbb{E}^{F_X}[G(X)] = G(3/2) \times (1 - \omega_0),$$

we get $\omega_0 = 1/3$ and $1 - \omega_0 = 2/3$. Thus the capacity constraint binds. $\square$

B.5. Proof of Proposition 2

Proof: We prove using the probability theory.

(i) The expectation of investors' wealth can be rewritten as

$$\mathbb{E}[W] = \mathbb{E}[aX - b\,\mathbb{1}_{X \geq \tilde{\varepsilon}_m}] = a - b\,\mathbb{P}(X \geq \tilde{\varepsilon}_m) = a - b\int_0^2 G(x)\,\mathrm{d}F^*(x),$$

where G is as defined in (1.1). It is not hard to show that the probability of beating the random benchmark increases as ι increases. Thus the mean of the investors' wealth decreases as the benchmark incentives grow.

(ii) From the definition of cumulative distribution, we have

$$\begin{aligned}
F_W(w) &= \mathbb{P}(W \leq w) = \mathbb{P}\big(aX - b\,\mathbb{1}_{X \geq \tilde{\varepsilon}_m} \leq w\big) \\
&= \int_0^2 \Big(\int_0^{\tilde{\varepsilon}_m \wedge \frac{w}{a}} \mathrm{d}F^*(x) + \mathbb{1}_{\{\frac{w+b}{a} \geq \tilde{\varepsilon}_m\}} \int_{\tilde{\varepsilon}_m}^{\frac{w+b}{a}} \mathrm{d}F^*(x)\Big)\,\mathrm{d}G(\tilde{\varepsilon}_m) \\
&= \int_0^{\frac{w}{a}} \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^{\frac{w+b}{a}} \mathrm{d}F^*(x) + \int_{\frac{w}{a}}^{\frac{w+b}{a}} \int_{\tilde{\varepsilon}_m}^{\frac{w+b}{a}} \mathrm{d}F^*(x)\,\mathrm{d}G(\tilde{\varepsilon}_m) + \int_{\frac{w}{a}}^2 \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^{\frac{w}{a}} \mathrm{d}F^*(x) \\
&= \int_0^{\frac{w}{a}} \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^{\frac{w+b}{a}} \mathrm{d}F^*(x) + \int_{\frac{w}{a}}^{\frac{w+b}{a}} \int_{\frac{w}{a}}^{x} \mathrm{d}G(\tilde{\varepsilon}_m)\,\mathrm{d}F^*(x) + \int_{\frac{w}{a}}^2 \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^{\frac{w}{a}} \mathrm{d}F^*(x) \\
&= \int_0^{\frac{w}{a}} \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^{\frac{w}{a}} \mathrm{d}F^*(x) + \int_{\frac{w}{a}}^{\frac{w+b}{a}} \int_0^{x} \mathrm{d}G(\tilde{\varepsilon}_m)\,\mathrm{d}F^*(x) + \int_{\frac{w}{a}}^2 \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^{\frac{w}{a}} \mathrm{d}F^*(x) \\
&= F^*\Big(\frac{w}{a}\Big) + \int_{\frac{w}{a}}^{\frac{w+b}{a}} G(x)\,\mathrm{d}F^*(x).
\end{aligned}$$

Because the definitions of G and F^* are both restricted in $[0, 2]$, we need to pay attention to the cases where $(w + b)/a \geq 2$ and $w/a \geq 2$. If $w \in (2a - b, 2a]$, then all the above calculation holds except that we truncate the results at the upper bound 2, i.e.,

$$F_W(w) = F^*\Big(\frac{w}{a}\Big) + \int_{\frac{w}{a}}^2 G(x)\,\mathrm{d}F^*(x).$$

If $w > 2a$, then the formula reduces to

$$F_W(w) = \int_0^2 \Big(\int_0^{\tilde{\varepsilon}_m} \mathrm{d}F^*(x) + \int_{\tilde{\varepsilon}_m}^2 \mathrm{d}F^*(x)\Big)\,\mathrm{d}G(\tilde{\varepsilon}_m) = \int_0^2 \mathrm{d}G(\tilde{\varepsilon}_m) \int_0^2 \mathrm{d}F^*(x) = 1.$$

□

B.6. Proof of Proposition 3

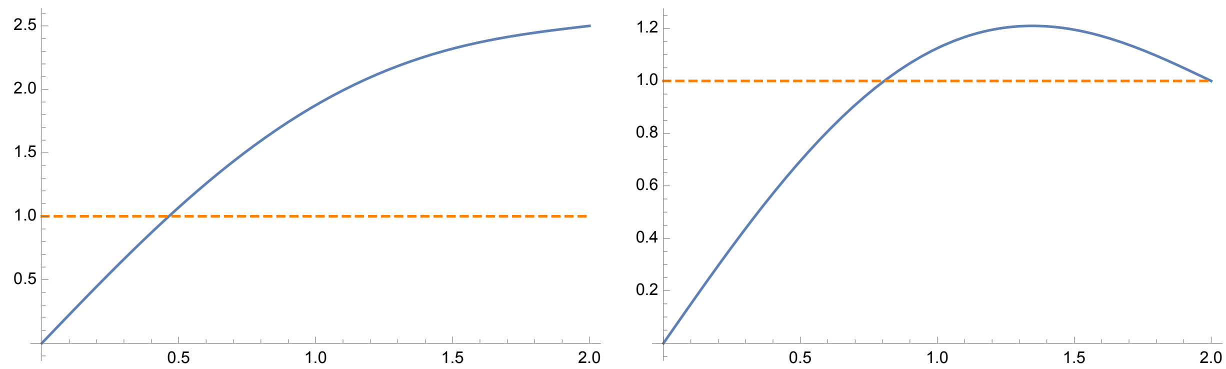
Proof: Define

$$\widehat{T}(x) = \frac{\alpha + \beta x - \iota\widehat{G}(x)}{1 - \iota}. \tag{B.17}$$

Following the same rationale, we have the following necessary conditions similar to (B.12):

$$\begin{aligned}
&\widehat{T}(x) \geq F(x), \qquad \mathrm{d}F\big\{x \in \mathrm{supp}\{F\} :\ \widehat{T}(x) > F(x)\big\} = 0, \\
&\widehat{T}'(x) = \frac{\beta - \iota\widehat{G}'(x)}{1 - \iota} \geq 0 \qquad \Rightarrow\ \beta \geq \iota, \\
&\widehat{T}(2) = \frac{\alpha + 2\beta - \iota}{1 - \iota} \leq 1 \qquad \Rightarrow\ \alpha + 2\beta \leq 1.
\end{aligned} \tag{B.18}$$

Different from the previous case, now $\widehat{T}$ does not have any local minimizer as compared to T in (B.11). As an example, we plot out $\widehat{T}(\cdot)$ when $\iota > 0.5$, $\beta > \iota$ (Panel A) and $\iota > 0.5$, $\beta < \iota$ (Panel B) in the following figure to have a general overview.



(A) $\iota = 2/3,\ \beta = 3/4$ (B) $\iota = 2/3,\ \beta = 1/2$

FIGURE 13. The figure plots the shape of $\widehat{T}(\cdot)$ when ι exceeds 0.5. Such features suggest that it is no longer possible for the optimal F to track $\widehat{T}$.

The same as in the indicator case, we can prove that $\alpha^{**} \equiv 0$ and capacity constraint binds for the optimal solution. Concerning (B.18), we have to guarantee two conditions for this optimal solution to exist: (i) $\widehat{T}$ is nondecreasing; (ii) $\widehat{T}(2) \leq 1$. If any of the above condition was violated, then as shown in Figure 13, we would get an F^{**} concentrating near zero level (before $\widehat{T}$ reaches 1), which contradicts the binding capacity constraint and cannot be the optimal distribution for sure. Hence, the managers start to gamble by ignoring the rank-order contest when any of the above two conditions is violated.

According to the property of $\widehat{G}$, we have that $\widehat{G}'(\cdot)$ has a maximizer at $x = 2$ and $\widehat{G}'(2) = \widehat{G}(2) = 1$. Hence the two conditions yield: (i) $\beta \geq \iota$ and (ii) $\beta \leq 0.5$. When $\iota = \beta = 0.5$, we have that $\operatorname{supp}\{F\} = [0, 2]$, whereas

$$\mathbb{E}[X] = \int_0^2 \Big(1 - F(x)\Big)\,\mathrm{d}x = \int_0^2 \Big(1 - \widehat{T}(x)\Big)\,\mathrm{d}x = \frac{3}{5} < 1,$$

which contradicts the requirement that each manager uses up all of her capacity. Hence, we conclude that for the optimal solution to exist, we must have $\iota < 0.5$ and $\beta \geq \iota$. □

B.7. Proof of Proposition 4

PROOF: Following the same rationale as in the bonus-contract case, in the equilibrium each manager uses up all of her capacity. Thus the capacity constraint binds:

$$\mathbb{E}[X] = \int_0^\infty \Big(1 - F(x)\Big)\,\mathrm{d}x = 1.$$

Moreover, $\alpha^{**} \equiv 0$, i.e., placing mass at zero level would not help the manager win. To solve for optimal β^{**}, we first figure out the support of $F(\cdot)$, denoted by $[0, \hat{x}_2]$. We already know that $F(\cdot) \equiv \widehat{T}(\cdot)$ over its support. To find $\hat{x}_2$, we have

$$F(\hat{x}_2) = \widehat{T}(\hat{x}_2) = \frac{\hat{x}_2\beta - \iota(\hat{x}_2 - 1)}{1 - \iota} = 1,$$

which implies $\hat{x}_2 = (1 - 2\iota)/(\beta - \iota)$.

Next, we solve for β^{**} using the binding capacity constraint, i.e.,

$$\begin{aligned}
&\int_0^{\frac{1-2\iota}{\beta-\iota}} \Big(1 - F(x)\Big)\,\mathrm{d}x \\
=&\int_0^{\frac{1-2\iota}{\beta-\iota}} \Big(1 - \widehat{T}(x)\Big)\,\mathrm{d}x \\
=&\int_0^{2} \Big(1 - \frac{\beta x - \iota\,(x^3/4 - x^4/16)}{1-\iota}\Big)\,\mathrm{d}x + \int_2^{\frac{1-2\iota}{\beta-\iota}} \Big(1 - \frac{\beta x - \iota\,(x-1)}{1-\iota}\Big)\,\mathrm{d}x \\
=&\frac{10\beta + 7\iota - 10}{5(\iota-1)} - \frac{(1-2\beta)^2}{2(\beta-\iota)(\iota-1)} = 1.
\end{aligned} \tag{B.19}$$

Since both the integral interval and integrand depend on β and ι, it is reasonable that the whole system is nonlinear. Solving the above equation (B.19) gives us a unique β^{**}:

$$\beta^{**} = \frac{4\iota^2 - 10\iota + 5}{2(5-8\iota)}.$$

Now we check that the necessary conditions in Proposition 3, (i) $\beta^{**} \geq \iota$ and (ii) $\beta^{**} \leq 0.5$, indeed hold. This is true because we have

$$\begin{aligned}
\frac{4\iota^2 - 10\iota + 5}{2(5-8\iota)} \geq \iota \quad &\Leftrightarrow \quad (\iota - \frac{1}{2})^2 \geq 0; \\
\frac{4\iota^2 - 10\iota + 5}{2(5-8\iota)} \leq \frac{1}{2} \quad &\Leftrightarrow \quad \iota(1-2\iota) \geq 0.
\end{aligned}$$

Finally, we can conclude that the optimal distribution F^{**} chosen by the managers is a nondecreasing function as stated in (2.3) and shown in Figure 5.

We are able to solve the optimal system and confirm that all of our previous analysis is true, as shown by the results in Panel A of Table 2. $\square$

B.8. Proof of Theorem 1

Proof: Mean preserving spread is a special case of second-order stochastic dominance. Hence we only show mean preserving spread property in the following proof.

Assume $0 \leq \iota_s < \iota_b < 0.5$ (b short for "big" and s short for "small"). Denote the corresponding challenge distributions by F_b^{**} and F_s^{**}. By the binding capacity constraint, we have that both distributions have the same mean, i.e.,

$$\mathbb{E}^{F_b^{**}}[X] = \mathbb{E}^{F_s^{**}}[X] = 1.$$

Thus we only need to show that F_s^{**} crosses F_b^{**} from below only once.

Define $\Delta F(x) = F_b^{**}(x) - F_s^{**}(x)$. This is a function intersecting the origin. It is equivalent to show that the difference function $\Delta F(x)$ is positive then negative, and has only one zero. According to the piecewise form of F^{**} in (2.3) and $\widehat{G}$ in (2.1), we have

$$\Delta F(x) = \begin{cases}
\Big(\frac{4\iota_b^2 - 10\iota_b^2 + 5}{16\iota_b^2 - 26\iota_b^2 + 10} - \frac{4\iota_s^2 - 10\iota_s^2 + 5}{16\iota_s^2 - 26\iota_s^2 + 10}\Big)x - \frac{\iota_b - \iota_s}{(1-\iota_b)(1-\iota_s)}\Big(\frac{x^3}{4} - \frac{x^4}{16}\Big) & \text{for } x \in [0,2), \\
\Big(\frac{4\iota_b^2 - 10\iota_b^2 + 5}{16\iota_b^2 - 26\iota_b^2 + 10} - \frac{4\iota_s^2 - 10\iota_s^2 + 5}{16\iota_s^2 - 26\iota_s^2 + 10}\Big)x - \frac{\iota_b - \iota_s}{(1-\iota_b)(1-\iota_s)}(x-1) & \text{for } x \in [2, x_{2,s}), \\
\Big(\frac{4\iota_b^2 - 10\iota_b^2 + 5}{16\iota_b^2 - 26\iota_b^2 + 10} - \frac{\iota_b}{1-\iota_b}\Big)x + \frac{2\iota_b - 1}{1-\iota_b} & \text{for } x \in [x_{2,s}, x_{2,b}), \\
0 & \text{for } x \in [x_{2,b}, \infty).
\end{cases}$$

On the interval $[0,2)$, we can show that the first derivative of ΔF is positive then negative, with only one zero point. This implies that ΔF has only one positive maximizer and no minimizer over this region. Over $[2,\infty)$, ΔF is linear in each sub-interval, and decreases over this whole region. Thus, there is only one zero of ΔF, either in $[0,2)$ if $\Delta F(2)<0$, or in $[2,\infty)$ if $\Delta F(2)\geq 0$. Moreover, before reaching the zero point, ΔF is always positive.

Therefore, it is sufficient for us to conclude that F_b^{**} is a mean preserving spread of F_s^{**}. Then we also have F_s^{**} second-order stochastically dominates F_b^{**}. □

B.9. Proof of Proposition 5

PROOF: We prove using the definitions and the properties of conditional expectation.

(i) The expectation of investors' wealth can be rewritten as

$$\mathbb{E}[\widehat{W}] = \mathbb{E}[aX - b(X-\tilde{\varepsilon}_m)^+] = \mathbb{E}\big[\mathbb{E}[aX - b(X-\tilde{\varepsilon}_m)^+ \mid \tilde{\varepsilon}_m]\big].$$

For an arbitrary challenge distribution $F(\cdot)$, we can derive

$$\begin{aligned}\mathbb{E}\big[\mathbb{E}[(X-\tilde{\varepsilon}_m)^+ \mid \tilde{\varepsilon}_m]\big] &= \int_0^2 \Big(\int_0^\infty (x-\tilde{\varepsilon}_m)^+\,\mathrm{d}F(x)\Big)\,\mathrm{d}G(\tilde{\varepsilon}_m)\\ &= \int_0^\infty \Big(\int_0^{x\wedge 2} (x-\tilde{\varepsilon}_m)\,\mathrm{d}G(\tilde{\varepsilon}_m)\Big)\,\mathrm{d}F(x) = \int_0^2 \Big(\frac{x^3}{4}-\frac{x^4}{16}\Big)\,\mathrm{d}F(x) + \int_2^\infty (x-1)\,\mathrm{d}F(x).\end{aligned}$$

Thus we have

$$\mathbb{E}[\widehat{W}] = a - b\int_0^\infty \widehat{G}(x)\,\mathrm{d}F^{**}(x),$$

where $\widehat{G}$ is as defined in (2.1).

(ii) From the definition of cumulative distribution, we have

$$\begin{aligned}F_{\widehat{W}}(w) &= \mathbb{P}(\widehat{W}\leq w) = \mathbb{P}\big(aX - b(X-\tilde{\varepsilon}_m)^+ \leq w\big)\\ &= \int_0^2 \Big(\int_0^{\tilde{\varepsilon}_m\wedge\frac{w}{a}} \mathrm{d}F^{**}(x) + \mathbb{1}_{\{\frac{w-b\tilde{\varepsilon}_m}{a-b}\geq\tilde{\varepsilon}_m\}}\int_{\tilde{\varepsilon}_m}^{\frac{w-b\tilde{\varepsilon}_m}{a-b}} \mathrm{d}F^{**}(x)\Big)\,\mathrm{d}G(\tilde{\varepsilon}_m)\\ &= \int_0^2 \Big(\int_0^{\tilde{\varepsilon}_m\wedge\frac{w}{a}} \mathrm{d}F^{**}(x) + \mathbb{1}_{\{\frac{w}{a}\geq\tilde{\varepsilon}_m\}}\int_{\tilde{\varepsilon}_m}^{\frac{w-b\tilde{\varepsilon}_m}{a-b}} \mathrm{d}F^{**}(x)\Big)\,\mathrm{d}G(\tilde{\varepsilon}_m)\\ &= \int_0^{\frac{w}{a}}\int_0^{\frac{w-b\tilde{\varepsilon}_m}{a-b}} \mathrm{d}F^{**}(x)\,\mathrm{d}G(\tilde{\varepsilon}_m) + \int_{\frac{w}{a}}^2\int_0^{\frac{w}{a}} \mathrm{d}F^{**}(x)\,\mathrm{d}G(\tilde{\varepsilon}_m)\\ &= \int_0^{\frac{w}{a}} \mathrm{d}G(\tilde{\varepsilon}_m)\int_0^{\frac{w}{a}} \mathrm{d}F^{**}(x) + \int_{\frac{w}{a}}^{\frac{w}{a-b}}\int_0^{\frac{b-a}{b}(x-\frac{w}{a-b})} \mathrm{d}G(\tilde{\varepsilon}_m)\,\mathrm{d}F^{**}(x) + \int_{\frac{w}{a}}^2 \mathrm{d}G(\tilde{\varepsilon}_m)\int_0^{\frac{w}{a}} \mathrm{d}F^{**}(x)\\ &= F^{**}\Big(\frac{w}{a}\Big) + \int_{\frac{w}{a}}^{\frac{w}{a-b}} \Big[\frac{3}{4}\Big(\frac{b-a}{b}\Big)^2\Big(x-\frac{w}{a-b}\Big)^2 - \frac{1}{4}\Big(\frac{b-a}{b}\Big)^3\Big(x-\frac{w}{a-b}\Big)^3\Big]\,\mathrm{d}F^{**}(x).\end{aligned}$$

Because $\operatorname{supp}\{G\}=[0,2]$, we need to pay attention to the upper bound of inner integral:

$$\frac{b-a}{b}\Big(x-\frac{w}{a-b}\Big)\leq 2 \quad\Leftrightarrow\quad x\geq\frac{w-2b}{a-b}.$$

Then we have:

$$F_{\widehat{W}}(w) = F^{**}\Big(\frac{w}{a}\Big) + \int_{\frac{w}{a}}^{\frac{w}{a-b}}\int_0^{\frac{b-a}{b}(x-\frac{w}{a-b})} \mathrm{d}G(\tilde{\varepsilon}_m)\,\mathrm{d}F^{**}(x) = F^{**}\Big(\frac{w}{a}\Big) + \int_{\frac{w}{a}}^{\frac{w-2b}{a-b}} \mathrm{d}F^{**}(x) + \int_{\frac{w-2b}{a-b}}^{\frac{w}{a-b}} G\Big(\frac{b-a}{b}\Big(x-\frac{w}{a-b}\Big)\Big)\,\mathrm{d}F^{**}(x).$$

If $w \geq 2a$, then $(w-2b)/(a-b) \geq w/a$, thus there is no problem with the above derivation; otherwise when $w < 2a$, we have that $x > w/a > (w-2b)/(a-b)$, which means the distribution reduces to that in part (ii) of Proposition 5. □

B.10. Proof of Theorem 2

Proof: Assume $0 \leq \iota_s < \iota_b < 0.5$. Denote the corresponding distributions of wealth by $F_{\widehat{W},b}$ and $F_{\widehat{W},s}$, and the corresponding wealth by $\widehat{W}_b$ and $\widehat{W}_s$.

First we show that the mean of investors' wealth decreases as ι increases. According to the result in part (i) of Proposition 5, we can calculate and get

$$\int_0^\infty \widehat{G}(x)\,\mathrm{d}F^{**}(x) = \frac{2\iota}{5(1-\iota)} + \frac{52\iota^2 - 55\iota + 15}{80\iota^2 - 130\iota + 50},$$

with a positive differentiation

$$\frac{\partial}{\partial \iota}\Big(\int_0^\infty \widehat{G}(x)\,\mathrm{d}F^{**}(x)\Big) = \frac{2}{(5-8\iota)^2}.$$

Hence, $\mathbb{E}[\widehat{W}_s] \geq \mathbb{E}[\widehat{W}_b]$, for any $0 \leq \iota_s < \iota_b < 0.5$.

Then we only need to show the single crossing property, i.e., $F_{\widehat{W},s}$ crosses $F_{\widehat{W},b}$ from below only once.[22] The proof follows exactly the same rationale as that in Theorem 1. Similarly, we define $\Delta F_{\widehat{W}} = F_{\widehat{W},b} - F_{\widehat{W},s}$. On the interval $[0, 2a)$, $\Delta F_{\widehat{W}}$ has only one positive maximizer and no minimizer. Over $[2a, \infty)$, $\Delta F_{\widehat{W}}$ decreases. Thus, there is only one zero of $\Delta F_{\widehat{W}}$, either in $[0, 2a)$ if $\Delta F(2a) < 0$, or in $[2a, \infty)$ if $\Delta F(2a) \geq 0$. Moreover, before reaching the zero point, $\Delta F_{\widehat{W}}$ is always positive.

Therefore, $F_{\widehat{W},b}$ is riskier than $F_{\widehat{W},s}$ in the sense of second-order stochastic dominance. □

B.11. Proof of Example 1

Proof: We consider the scaled performance in the following proof, X/μ_X and Y/μ_Y. Then the objective functions reduce to a pair where only the ι (in this example represented by some function of b, c, and $\tilde{\rho}$) is endogenous and different for the two funds. The idea of the programming is as follows: Given a six-dimensional vector as for the values at six evenly distributed connecting points within $[0, 4]$, we apply cubic spline interpolation to get a smooth function and truncate it to be nonnegative. Admitting the smooth property, we optimize the objective over this well-defined density function, subject to the expectation and integration constraints. By contraction mapping, for a given pair of densities $(f_X^{(0)}, f_Y^{(0)})$, we iterate the optimization system to find the converging point. Equivalently, we first optimize the objective function for X with a given initial distribution $f_Y^{(0)}$, and then plug in the solution $f_X^{(0)}$ to Y's objective and optimize for Y, so on so forth. Provided that we could reach a converging pair (f_X^*, f_Y^*) by doing the iteration sequentially, this fixed point should be one of the equilibria.

(i) *For $b = 0.2$, $c = 0.7$ and $1 - c = 0.3$ (Panel A)*:

By doing the optimization alternatively, we finally end up in a circulation of optimizers: $f_X^1 \to f_Y^1 \to f_X^2 \to f_Y^2 \to f_X^1 \to \cdots$. From the idea of "Rock-Paper-Scissors", we conjecture that the fixed point for each manager should be the weighted average of the two optimizers above, for example, $f_i^* = (f_i^1 + f_i^2)/2$ for $i = X, Y$. We check by plotting enough points of the objective values for both managers over the supports of f_i^*. Within the error tolerance, all the points in the domain of the density line up linearly, as shown in Figure

[22] For the relationship between single crossing property and second-order stochastic dominance provided the mean condition holds, one can refer to Proposition 5 in Wolfstetter et al. (1993).

14. Recall that checking the colinearity comes directly from the necessary conditions as in (1.4). Hence, we could regard our results as one of the equilibria. Furthermore, this solution is robust to different choices of the starting value $f_Y^{(0)}$.

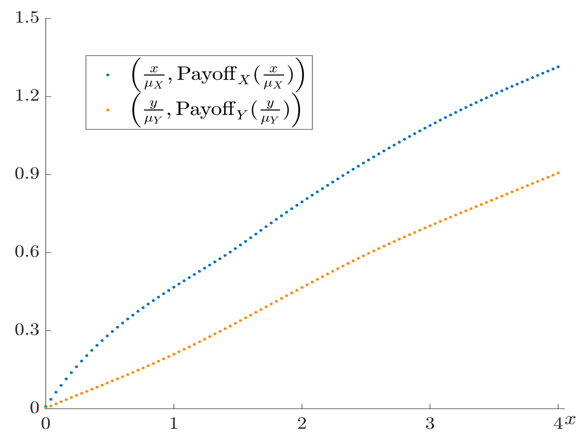


FIGURE 14. This figure tests the colinearity of the payoffs for player X and Y. According to (1.4), the points where $\mathrm{d}F^* > 0$ should line up linearly, which is roughly the case here within the error tolerance.

(ii) *For* $b = 0.2$*,* $c = 0.99$ *and* $1 - c = 0.01$ *(Panel B)*:

Now we examine an extreme case. In this case it is more straight-forward, since by doing the optimization alternatively, the system directly converges to a fixed pair: $f_X \to f_Y \to f_X \to \cdots$. It is not hard to show that $f_i^* = f_i$ $(i = X, Y)$ is acceptable as one of the equilibria. Furthermore, this solution is robust to different choices of the starting value $f_Y^{(0)}$. $\square$

B.12. Proof of Proposition 6

PROOF: We assume the upper capacity $\mu \in [1, 3/2]$. Following Lemma 1, we should have that the optimal F^* is continuous, and $\alpha^* \equiv 0$. Assume the maximal possible support of the equilibrium distribution is $[0, \overline{x}]$, where $\overline{x} \geq 2$. Following the same rationale as we used to determine the necessary conditions in (B.12), we have

$$\begin{aligned}
&T(x) \geq F(x), \qquad \mathrm{d}F\big\{x \in [0, \overline{x}] : \ T(x) > F(x)\big\} = 0, \\
&\int_0^{\overline{x}} T(x)\,\mathrm{d}x \geq \int_0^{\overline{x}} F(x)\,\mathrm{d}x \geq \overline{x} - \mu, \\
&T(\overline{x}) = \int_0^{\overline{x}} \mathrm{d}T(x) = \frac{\beta\overline{x} - \iota}{1 - \iota} \geq 1;
\end{aligned} \tag{B.20}$$

where T is as defined in (B.11). Thus the optimal level of β is given by

$$\beta = \frac{1}{\overline{x}} \leq \frac{1}{2}.$$

Solving the inequality of capacity constraint, we get

$$\int_0^{1/\beta} \Big(1 - T(x)\Big)\,\mathrm{d}x \leq \int_0^{1/\beta} \Big(1 - F(x)\Big)\,\mathrm{d}x \leq \mu \qquad \Rightarrow \qquad \beta \geq \frac{1}{2(\iota + \mu - \mu\iota)}.$$

We can check that, in order for the solution to exist, we must have

$$\begin{aligned} \frac{1}{2(\iota+\mu-\mu\iota)} &\leq \frac{1}{2}, \\ \Leftrightarrow \quad (\mu-1)(1-\iota) &\geq 0, \end{aligned} \tag{B.21}$$

which holds since $\mu \geq 1$. Hence we require

$$\beta \in \left[\frac{1}{2(\iota+\mu-\mu\iota)}, \frac{1}{2}\right].$$

In the optimization system, we want β to be as small as possible, so the optimal solution should be taken at

$$\beta^* = \frac{1}{2(\iota+\mu-\mu\iota)};$$

and optimal solution for α^* is zero. Hence, function T^* becomes

$$T^*(x) = \frac{x/(2\iota+2\mu-2\mu\iota) - \iota G(x)}{1-\iota}.$$

If T^* is nondecreasing (which is the part (i) in the proof of Proposition 1), we must have

$$T^{*\prime}(x) = \frac{1}{2(\iota+\mu-\mu\iota)(1-\iota)} - \frac{\iota G'(x)}{1-\iota} \geq 0,$$

for all $x \in [0,2]$. Due to the property of $G'(\cdot)$, it has only one maximizer at the point $(1, 3/4)$. Thus, for the above inequality to hold for all x, we must have

$$3(\iota+\mu-\mu\iota) \leq 2.$$

For $\mu = 1$, the requirement reduces to

$$\iota \leq \iota_0 = \frac{2}{3};$$

otherwise, if $\mu \in (1, 3/2]$,

$$\iota \leq \iota_0 = \frac{3\mu - \sqrt{9\mu^2 - 24\mu + 24}}{6(\mu-1)}.$$

Up till now we have found the correct marginal threshold, ι_0, in the case with general capacity constraint. All the other proofs follow exactly the same rationale as in Proposition 1.

As shown in Figure 12 in Appendix A, our piecewise definition of ι_0 is in fact a continuous function of μ for $\mu \in [1, 3/2]$. □

B.13. Proof of Proposition 7

Proof: The necessary conditions implied by Lagrangian method, previously Equation (1.4), becomes

$$\iota G(x) + (1-\iota)\big(F(x)\big)^{n-1} - \alpha - \beta x \leq 0, \qquad \forall x \in \operatorname{supp}\{F\};$$

$$\mathrm{d}F\big\{x \in \operatorname{supp}\{F\} :\ \iota G(x) + (1-\iota)\big(F(x)\big)^{n-1} - \alpha - \beta x < 0\big\} = 0.$$

The rest of the proof follows the same rationale as in Proposition 1.

In part (i), the extreme case with $\iota = 0$ is in fact the result proved in Section 3 of Fang and Noe (2015), setting the number of prizes to be one. Our results coincide with the reduced Complementary Beta Distribution.

In part (ii), because the power function will not change the monotonicity and local minimizer/maximizer of function $T(\cdot)$, $\underline{x}$ remains unchanged. Naturally x_1 is now updated to be the second solution of $\big(T(x)\big)^{\frac{1}{n-1}} = T(\underline{x})$ besides $\underline{x}$. □

B.14. Proof of Theorem 3

PROOF: It is not hard to prove by definition. The mean remains unchanged if we fix an exogenous capacity limit. The idea follows exactly from the proof of Theorem 1. Intuitively, the equilibrium distribution when $n = 2$ crosses the distribution when $n = 3$ from below only once, as can be seen from Figure 10. $\square$

B.15. Proof of Lemma 4

PROOF: Since X and $Q(\cdot)$ are identical in probability distribution, all the results follow naturally from integration by substitution. For the detailed proof, see Section 2 of He and Zhou (2011). $\square$

B.16. Proof of Proposition 8

PROOF: Assume we have properly defined Q and its derivatives. Using the calculus of variations, assume $Q = Q(\cdot)$ is an extremal. Take any arbitrary function $\widetilde{Q}$ that has at least one derivative, and $\widetilde{Q}(0) = \widetilde{Q}(1) = 0$. Then for any infinitesimal ε, we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^1 \Big(\iota\, G(Q + \varepsilon \widetilde{Q}) + (1-\iota)\, F_Y(Q + \varepsilon \widetilde{Q}) - \beta(Q + \varepsilon \widetilde{Q}) \Big)\, \mathrm{d}t \bigg|_{\varepsilon = 0} &= 0 \\ \Leftrightarrow \int_0^1 \widetilde{Q} \Big(\iota\, G'(Q + \varepsilon \widetilde{Q}) + (1-\iota)\, F_Y'(Q + \varepsilon \widetilde{Q}) - \beta \Big)\, \mathrm{d}t \bigg|_{\varepsilon = 0} &= 0 \\ \Leftrightarrow \int_0^1 \widetilde{Q} \Big(\iota\, G'(Q) + (1-\iota)\, F_Y'(Q) - \beta \Big)\, \mathrm{d}t &= 0. \end{aligned} \tag{B.22}$$

Since $\widetilde{Q}$ is arbitrary, we must have

$$\iota\, G'\big(Q(t)\big) + (1-\iota)\, F_Y'\big(Q(t)\big) - \beta = 0,$$

i.e., to maximize the Lagrangian problem (4.2) is equivalent to consider the point-wise maximization and first order condition of the integrand. As we know *a posteriori*, $F_Y(\cdot) = F_X(\cdot)$ in a symmetric equilibrium. Thus, the first order condition can be reformulated as follows:

$$\begin{aligned} \iota\, G'\big(Q(t)\big) + (1-\iota)\, (Q^{-1})'\big(Q(t)\big) - \beta = 0, \\ \iota\, G'\big(Q(t)\big) + (1-\iota)/Q'(t) - \beta = 0. \end{aligned} \tag{B.23}$$

Simplifying the equation gives us the ODE in the proposition. $\square$

B.17. Proof of Proposition 9

PROOF: Denote by v and $v(\beta)$ the optimal values of (4.1) and (4.2), respectively. Then by weak duality we have

$$v \le \inf_{\beta \ge 0} v(\beta).$$

The binding capacity condition further implies

$$v \le \inf_{\beta \ge 0} v(\beta) \le v(\beta^*) = \mathcal{L}\big(Q^*(\cdot), \beta^*\big) = \int_0^1 \Big(\iota\, G\big(Q^*(t)\big) + (1-\iota)\, F_Y\big(Q^*(t)\big) \Big)\, \mathrm{d}t \le v.$$

This implies that the strong duality holds. Thus $Q^*(\cdot)$ is the equilibrium solution to (4.1). According to Lemma 4, the corresponding F_X^* determined by its quantile function $Q^*(\cdot)$ gives the equilibrium solution to the original problem (1.3).

Now it remains to show the uniqueness of $Q^*(\cdot)$, provided some certain condition of G' holds. Recall the ODE as follows:

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \frac{1-\iota}{\beta - \iota G'\big(Q(t)\big)} := H\big(Q(t)\big).$$

Assume that $H(\cdot)$ is Lipschitz continuous, i.e.,

$$\left|\frac{H(x_1) - H(x_2)}{x_1 - x_2}\right| \le K, \quad \forall x_1, x_2 \ge 0,$$

where K is some positive constant.

If there are two optimal solutions, $Q_1(\cdot)$ and $Q_2(\cdot)$, consider $\widetilde{Q} = Q_1 - Q_2$. Then we have

$$\begin{aligned}
&\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}t} = \big(H(Q_1) - H(Q_2)\big) \\
\Rightarrow\ &\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}t} \cdot \widetilde{Q} = \big(H(Q_1) - H(Q_2)\big)\widetilde{Q} \\
\Rightarrow\ &\frac{\mathrm{d}\widetilde{Q}^2}{\mathrm{d}t} \le 2K\widetilde{Q}^2, \qquad\qquad t \in [0,1].
\end{aligned}$$

Hence, if $\widetilde{Q}\big|_{t=0} = 0$, then we must have $\widetilde{Q}(\cdot) \equiv 0$. In other words, $Q_1(\cdot) \equiv Q_2(\cdot)$. Thus the uniqueness result follows. $\square$