Basic Concepts of Cooperative Game Theory

He Wang

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Cooperative Games

- Cooperative games model scenarios, where
  - each agent is still selfish but
  - agents can benefit by cooperating
  - binding agreements are possible

- In cooperative games, actions are taken by groups of agents
  - **Transferable utility games**: payoffs are given to the group and then divided among its members
    - Partition Function Games: the payoff obtained by a coalition depends on the actions chosen by other coalition.
    - **Charateristic Function Games**: the payoff of each coalition only depends on the action of that coalition.
  - Non-transferable utility games: group actions result in payoffs to individual group members

Characteristic Function Games

Definition: Characteristic Function Games

A *characteristic function game* G is given by a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite, non-empty set of agents and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, which maps each coalition $C \subseteq N$ to a real number $v(C)$. The number of $v(C)$ is usually referred to as the value of the coalition C.

- Coalition: a subset of the players N.
- Grand coalition: N.
- Value of the coalition: charateristic function $v(C)$.
  - NOT indicate how the coalition value $v(C)$ should be divided amongst the members of C.
  - $\to$ solution concepts. [later]

Standard Assumptions

1. $v(\emptyset) = 0$
2. Two cases:
   - Case 1 (make a profit): $v(C) \geq 0$ for all $C \subseteq N$. **[mostly]**
   - Case 2 (share costs): $v(C) \leq 0$ for all $C \subseteq N$.
3. Transferable utility games (TU games):
   - $v(C)$ can be divided among the members of C in any way that the members of C choose.
   - Most of the games, except Chapter 5.

Example: Ice-Cream Game

Charlie (C), Marcie (M), and Pattie (P) want to pool their savings to buy ice cream. Charlie has 6 dollars, Marcie has 4 dollars, Pattie has 3 dollars, and the ice cream tubs come in three different sizes: 500g, which costs \$7; 750g, which costs \$9; and 1000g, which costs \$11.

 C: \$6,  M: \$4,  P: \$3

 w = 500 p = \$7  w = 750 p = \$9  w = 1000 p = \$11

Then, $N = \{C, M, P\}$ and the characteristic function v:

S	$\emptyset$	$\{C\}$	$\{M\}$	$\{P\}$	$\{C, M\}$	$\{C, P\}$	$\{M, P\}$	$\{C, M, P\}$
$v(S)$	0	0	0	0	750	750	500	1000

Outcomes

Definition: Coalition Structure

Given a characteristic function game $G = (N, v)$, a *coalition structure* over N is a collection of non-empty subsets $CS = \{C^1, \ldots, C^k\}$ such that

- $\cup_{j=1}^{k} C^j = N$,
- $C^i \cap C^j = \emptyset$ for any $i, j \in \{1, \ldots, k\}$ such that $i \neq j$.

- Every player must appear in some coalition.
- A player cannot appear in more than one coalition.

Outcomes (cont'd)

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Definition: Payoff

A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is a *payoff* vector for a coalition structure $CS = \{C^1, \ldots, C^k\}$ over $N = \{1, \ldots, n\}$ if

- $x_i \geq 0$ for all $i \in N$
- $\sum_{i \in C^j} x_i \leq v(C^j)$ for any $j \in \{1, \ldots, k\}$.

- Every player receives a non-negative payoff.
- Feasibility: The total amount paid out to a coalition cannot exceed the value of that coalition.

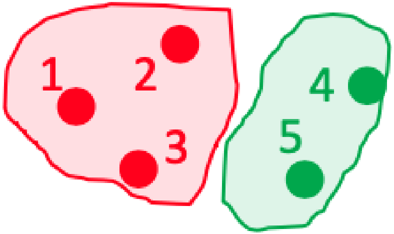
Outcomes (cont'd)

Definition: Outcomes

An outcome of G is a pair $(CS, \mathbf{x})$, where CS is a coalition structure over G and $\mathbf{x}$ is a payoff vector CS.

- The total payoff of any coalition $C \subseteq N$ under $\mathbf{x}$: $x(C) = \sum_{i \in C} x_i$.
- Efficient payoff: $\sum_{i \in C^j} x_i = v(C^j)$ for every $j \in \{1, \ldots, k\}$.
- If a payoff vector is imputation:
  - Efficient,
  - Individual rationality: $x_i \geq v(\{i\}$.
- How to evaluate the outcome? [later]

Outcomes: Example



- Suppose $v(\{1,2,3\}) = 9$, $v(\{4,5\}) = 4$.
- Then $((\{1,2,3\},\{4,5\}),(3,3,3,3,1))$ is an outcome.
- But, $((\{1,2,3\},\{4,5\}),(2,3,2,3,3))$ is NOT an outcome: transfers between coalitions are not allowed.

Subclasses of Characteristic Function Games

1. Monotone games.
2. Superadditive games.
3. Convex games.
4. Simple games.

Monotone Games

Definition: Monotone Games

A characteristic function game $G = (N, v)$ is said to be *monotone* if it satisfies $v(C) \leq v(D)$ for every pair of coalitions $C, D \subseteq N$ such that $C \subseteq D$.

- In other words: adding an agent to an existing coalition can only increase the overall productivity of this coalition; games with this property are said to be *monotone*.

Superadditive Games

Definition: Superadditive Games

A characteristic function game $G = (N, v)$ is said to be *superadditive* if it satisfies $v(C \cup D) \geq v(C) + v(D)$ for every pair of **disjoint** coalitions $C, D \subseteq N$.

- In other words: it is always profitable for two groups of players to join forces.
  $\Longrightarrow$ We can assume that players form the grand coalition, the outcome is $(N, \mathbf{x})$ where $\sum_{i \in N} x_i = v(N)$.
- Assume $v(C) \geq 0$: Superadditive games implies monotone games.
- Any non-superadditive game $G = (N, v)$ can be transformed into a superadditive game $G^* = (N, v^*)$.
  - G^*: the *supperadditive cover* of G
  - $v^*(C) = \max_{CS \in \mathcal{CS}_C} v(CS)$.

Convex Games

Definition: Convex Games (1)

A characteristic function game $G = (N, v)$ is said to be *convex* if it satisfies $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$ for every pair of coalitions $C, D \subseteq N$.

- In other words: a player is more useful when she joins a bigger coalition.

Definition: Convex Games (2)

A characteristic function game $G = (N, v)$ is *convex* if and only if for every pair of coalitions T, S such that $T \subseteq S$, and every player $N \setminus S$ it holds that

$$v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$$

- Assume $v(C) \geq 0$: Convex games implies superadditive games.

Simple Games

Definition: Simple Games

A characteristic function game $G = (N, v)$ is said to be *simple* if it monotone and its characteristic function only takes values 0 and 1, i.e., $v(C) \in \{0, 1\}$ for any $C \subseteq N$.

- Such games can model situations where there is a task to be completed.
- Note that simple games are superadditive only if the complement of each winning coalition (some coalition C with $v(C) = 1$) is losing (some coalition C with $v(C) = 0$).

How to Evaluate the Outcomes?

- Two sets of criteria:
  - *Fairness*: How well each agent's payoff reflects his **contribution**.
  - *Stability*: What are the **incentives** for the agent to **stay** in the coalition structure.
- Two families of solution concepts:
  - *Fairness*: (1) Shapley value, (2) Banzhaf index
  - *Stability*: (3) Core and core-related concepts

Shapley Value

- Formulated w.r.t. the grand coalition:
  - defines a way of distributing the value $v(N)$
  - implicitly assumed that the games to which it is applied are superadditive
  - well-defined for non-superadditive games

- Key point: The payment that each agent receives should be *proportional to his contribution*. [Fairness]

- Previous attempt:
  - Naive implementation: $x_i = v(N) - v(N \setminus \{i\})$. ✗ efficient
  - Consider predecessors in the ordering. ✓ efficient
    - the agents' payoffs in this scheme strongly depend on the selected ordering of the agents.
    - Example: $N = \{1, 2\}$, $v(\emptyset) = 0, v(\{1\}) = v(\{2\}) = 5, v(\{1, 2\}) = 20$. If the ordering is $\pi = (1, 2)$: $x_1 = v(1) - v(\emptyset) = 5$, and $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$. ✗ symmetric
  - Insight: To remove the dependence on ordering, **averaging over all possible orderings, or permutations, of the players.**

Shapley Value: Notations

- Π_N: the set of all permutations of N.
- $\pi \in \Pi_N$: a permutation.
- $S_\pi(i)$: the set of all predecessors of i in π, i.e., $S_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}$.

Figure: The illustration of $S_\pi(i)$.

Example: $G = (N, v)$

- $N = \{1, 2\}$, $v(\emptyset) = 0, v(\{1\}) = v(\{2\}) = 5, v(\{1, 2\}) = 20$.
- $\Pi_N = \{(1, 2), (2, 1)\}$.
- if $\pi_1 = (1, 2)$: $S_\pi(1) = \emptyset$, $S_\pi(2) = 1$.
- if $\pi_2 = (2, 1)$: $S_\pi(2) = \emptyset$, $S_\pi(1) = 2$.

Shapley Value (cont'd)

Definition: Marginal Contribution

The marginal contribution of an agent i with respect to a permutation π in a game $G = (N, v)$ is denoted by $\Delta_\pi^G(i)$ and is given by

$$\Delta_\pi^G(i) = v\left(S_\pi(i) \cup \{i\}\right) - v\left(S_\pi(i)\right).$$

- How much i increases the value of the coalition consisting of its predecessors in π when he joins them.
- Example: $G = (N, v)$
  - $N = \{1, 2\}$, $v(\emptyset) = 0, v(\{1\}) = v(\{2\}) = 5, v(\{1, 2\}) = 20.$
  - $\Delta_{\pi_1}^G(1) = 5$ and $\Delta_{\pi_1}^G(2) = 15.$
  - $\Delta_{\pi_2}^G(2) = 5$ and $\Delta_{\pi_1}^G(1) = 15.$
  - By averaging all possible orderings, let $x_1 = \frac{\Delta_{\pi_1}^G(1) + \Delta_{\pi_2}^G(1)}{2} = \frac{5+15}{2} = 10 = x_2.$
  - The resulting outcome is fair!

Shapley Value

Definition: Shapley Value

Given a characteristic function game $G = (N, v)$ with $|N| = n$, the Shapley value of a player $i \in N$ is denoted by $\phi_i(G)$ and is given by

$$\phi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \Delta_\pi^G(i).$$

- **Probabilistic Interpretation**: If we choose a permutation of players uniformly at random, among all possible permutations of N,
  - The ϕ_i is i's **expected marginal contributions** to the coalition of its predecessors, over all permutations of players.
- Previous example: $\phi_1 = \phi_2 = 10$.

Shapley Value: Attractive Properties

Axiomatic Characterization of Shapley Value

1. *Efficiency*: $\sum_{i \in N} \phi_i(G) = v(N)$.
2. *Dummy player*: if i is a **dummy** (if $v(C) = v(C \cup \{i\})$ for any $C \subseteq N$), then $\phi_i(G) = 0$.
3. *symmetricity*: if i and j are **symmetric** in G (if $v(C \cup \{i\}) = v(C \cup \{j\})$ for any coalition $C \subseteq N \setminus \{i, j\}$.), then $\phi_i(G) = \phi_j(G)$;
4. *Additivity*: $\varphi_i\left(G^1 + G^2\right) = \varphi_i\left(G^1\right) + \varphi_i\left(G^2\right)$ for all $i \in N$.
   - $G^1 = (N, v^1)$ and $G^2 = (N, v^2)$.
   - $G^+ = G^1 + G^2$ is the game with the set of players N and characteristic function v^+ given by $v^+(C) = v^1(C) + v^2(C)$ for all $C \subseteq N$.

A payoff division scheme satisfies the above properties 1-4 $\Longleftrightarrow$ Shapley value

Banzhaf Index

Definition: Banzhaf Index

Given a characteristic function game $G = (N, v)$ with $|N| = n$, the Banzhaf Index of a player $i \in N$ is denoted by $\beta_i(G)$ and is given by

$$\beta_i(G) = \frac{1}{2^{n-1}} \sum_{C \subseteq N \setminus \{i\}} [v(C \cup \{i\}) - v(C)].$$

- Expected marginal contributions in terms of all coalitions in the game.
- Satisfy 2-4, but efficiency is important ....

Banzhaf Index (cont'd)

Definition: Normalized Banzhaf Index

The normalized Banzhaf index of a player $i \in N$ is defined as

$$\eta_i(G) = \frac{\beta_i(G)}{\sum_{i \in N} \beta_i(G)}$$

- Expected marginal contributions in terms of all permutations of players.
- Satisfy 1-3 with the loss of the additivity property.

The Relationship between SV and BI

Consider the simple games,

- Similarity: They measure the **power** of a player, i.e., the probability that she can influence the outcome of the game.
- Difference:
  - Shapley Value (SV): If agents join the coalition in a random order, $\phi_i(G)$ is exactly the probability that player i turns a losing coalition into a winning one [The player i is *pivotal*].
  - Banzhaf Index (BI): A given agent i turns a losing coalition into a winning one if each of the other agents decides whether to join the coalition by independently tossing a fair coin.
- Like permutation (SV) and combination (BI).

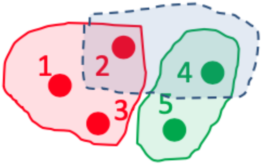
Core

Definition: Core

The set of stable outcomes, i.e., outcomes where no subset of players has an incentive to deviate, is called the *core* of G, i.e.,.

$$\mathcal{C}(G) = \{(CS, \mathbf{x}) \mid \Sigma_{i \in C} x_i \geq v(C) \text{ for any } C \subseteq N\}$$

- Under the payoff distribution scheme $\mathbf{x}$, each coalition earns at least as much as it can make on its own. Thus, the outcomes in the core are stable in the sense that no coalition has any incentive to "defect"— **no coalition can do better**.
- Example
  - Suppose $v(\{1,2,3\}) = 9$, $v(\{4,5\}) = 4$ and $v(\{2,4\}) = 7$.
  - Then $((\{1,2,3\}, \{4,5\}), (3,3,3,3,1))$ is NOT in the core.



Core: Ice-Cream Game

S	$\emptyset$	$\{C\}$	$\{M\}$	$\{P\}$	$\{C, M\}$	$\{C, P\}$	$\{M, P\}$	$\{C, M, P\}$
$v(S)$	0	0	0	0	500	500	0	750

- This is a superadditive game.
- $\mathbf{x} = (200, 200, 350)$ is NOT in the core:
  - $v(\{C, M\}) > x_c + x_M$
- $\mathbf{x} = (250, 250, 250)$ is in the core:
  - no subgroup of players can deviate so that each member of the subgroup gets more
- $\mathbf{x} = (750, 0, 0)$ is also in the core:
  - Marcie and Pattie cannot get more on their own!

Core: Efficiency

- From the definition of outcomes $(CS, \mathbf{x})$, we have that

$$x(C) \leq v(C) \quad \forall C \in CS$$

- If $(CS, \mathbf{x}) \in \mathcal{C}(G)$, we have that

$$x(C) \geq v(C) \quad \forall C \in CS$$

- Thus, **stability implies efficiency**: $x(C) = v(C), \forall C \in CS$.
  - Any outcome in the core maximizes the **social welfare** (the total payoff of all players).
- If an outcome $(CS, \mathbf{x})$ is in the core of a characteristic function game $G = (N, v)$ then $v(CS) \geq v(CS')$ for every coalition structure $CS' \in CS_N$.
  - CS_N: the space of all coalition structures over N. For example:

$$\mathcal{CS}_{\{1,2,3\}} = \{\{\{1\},\{2\},\{3\}\}, \quad \{\{1\},\{2,3\}\}, \quad \{\{2\},\{1,3\}\}, \quad \{\{3\},\{1,2\}\}, \quad \{\{1,2,3\}\}\}$$

  - Social welfare of the coalition structure CS: $v(CS) = \sum_{C \in CS} v(C)$.

Games with Empty Core

- Some games have empty cores.
- Example (3-player majority game): Consider a game $G = (N, v)$, where $N = \{1, 2, 3\}$ and $v(C) = 1$ if $|C| \geq 2$ and $v(C) = 0$ otherwise.
  - $v(N) = 1$, any outcome $(CS, (x_1, x_2, x_3)) \in \mathcal{C}(G)$ must satisfy:
    1. $x_1 \geq 0$,
    2. $x_2 \geq 0$,
    3. $x_3 \geq 0$,
    4. $x_1 + x_2 + x_3 \geq 1$.
  - Thus, for some player $i \in \{1, 2, 3\}$, $x_i \geq \frac{1}{3}$. (*)
  - This characteristic function makes that, for any $CS \in CS_N$ we have $v(CS) = x_1 + x_2 + x_3 \leq 1$.
  - For $C = N \setminus \{i\}$, we have $v(C) = 1$, $x(C) \leq \frac{2}{3}$ due to (*).
  - which means that $(CS, (x_1, x_2, x_3))$ is not in the core.

The Core of Superadditive Games

First, w.l.o.g, $CS = \{N\}$. Then, simplify the notation and define the core as the set of all vectors $\mathbf{x}$ that satisfy:

- $x_i \geq 0$ for all $i \in N$,
- $x(N) = v(N)$,
- $x(C) \geq v(C)$ for all $C \subseteq N$.

The Core of Non-supperadditive Games

We cannot assume the grand coalition, since it may cause the loss of stability. However, we can view **the elements of the core** as **payoff vectors for the grand coalition**, based on the following proposition.

Proposition

A characteristic function game $G = (N, v)$ has a non-empty core if and only if its superadditive cover $G^* = (N, v^*)$ has a non-empty core.

But,...

- If extend to other solution concepts, like SV, may require cross-coalitional transfers.
- Substantial computational efforts.

The Core of Simple Games

Theorem

A simple game $G = (N, v)$ has a non-empty core if and only if it has a veto player. Moreover, an outcome $(x_1, \ldots, x_n)$ is in the core of G if and only if $x_i = 0$ for every player i who is not a veto player in G.

- A player i is a *veto* player if $v(C) = 0$ for any $C \subseteq N \setminus \{i\}$. $\iff v(N \setminus \{i\}) = 0$.
- Check non-emptiness of the core: whether $v(N \setminus \{i\}) = 0$ for $i \in N$.

The Core of Convex Games

Theorem

If $G = (N, v)$ is a convex game

- Non-empty core.
- Shapley value is in the core.

- Proof:
  1. Construct one payoff scheme (marginal contribution with respect to an arbitrary permutation).
  2. Show this payoff scheme in the core.
  3. Shapley value can be viewed as a convex combination of such constructed payoff, and core is convex set.

ϵ-Core

- Insight:If the core is empty, we may want to find **approximately stable** outcomes.
- Need to relax the notation of core:
  - Core: $(CS, \mathbf{x})$: $\quad x(C) \geq v(C), \quad$ for all $C \subseteq N$.
  - ϵ-Core: $(CS, \mathbf{x}) : x(C) \geq v(C) - \epsilon$, for all $C \subseteq N$.
- No coalition can benefit significantly by deviating.
- In the rest of this section, limit to superadditive games only. (but can be extended to general case.)
- Core $=$ 0-Core $\subseteq$ ϵ-Core.

- Example (3-player majority game): Consider a game $G = (N, v)$, where $N = \{1, 2, 3\}$ and $v(C) = 1$ if $|C| \geq 2$ and $v(C) = 0$ otherwise.
  - $\frac{1}{3}$-core is non-empty: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \frac{1}{3}$-core.
  - ϵ-core is empty for any $\epsilon < \frac{1}{3}$:
  - For some player $i \in \{1, 2, 3\}$, $x_i \geq \frac{1}{3}$, so we have $v(N \setminus \{i\}) = 1$, $x(N \setminus \{i\}) \leq \frac{2}{3}$.

Least-Core

- If an outcome (CS, x) is in ϵ-core, the **deficit** $v(C) - x(C)$ of any coalition is at most ϵ.
- We are interested in outcomes that **minimize** the **worst-case deficit**

Definition: Least-Core

Given a superadditive game G, let

$$\varepsilon^*(G) = \inf\{\varepsilon \mid \varepsilon \text{ -core of } G \text{ is non-empty } \}.$$

The *least core* of G is its $\epsilon^*(G)$-core. The quantity $\epsilon^*(G)$ is called the *value of the least core*.

- Non-empty.
- Example(3-player majority game): the least-core $= \frac{1}{3}$-core.

Cost of Stability

- The least core: relax the stability constraints in the definition of core.
- The *cost of stability*: relax the feasibility constraint.
  - There exists a benevolent external party which wishes to **stabilize** the game, by offering subsidies to players if they stay in the grand coalition.

Definition: Cost of Stability

Given a superadditive game $G = (N, v)$ and a $\Delta \geq 0$, let $G^\Delta = (N, v^\Delta)$ be a characteristic function game over the set of players N given by $v^\Delta(N) = v(N) + \Delta$, $v^\Delta(C) = v(C)$ for all $C \subset N$. The *cost of stability (CoS)* of G is the smallest (nonnegative) value of Δ such that G^Δ has a non-empty core; we will denote it by CoS(G).

- It is not clear what the goal of the benevolent external party should be.
- $\epsilon^*(G) > 0 \iff$ CoS$(G) > 0$.

Recall: The Core of Superadditive Games

First, w.l.o.g, $CS = \{N\}$. Then, simplify the notation and define the core as the set of all vectors $\mathbf{x}$ that satisfy:

- $x_i \geq 0$ for all $i \in N$,
- $x(N) = v(N)$,
- $x(C) \geq v(C)$ for all $C \subseteq N$.

Check Non-Emptiness of the Core: Least-Core

$$
\begin{aligned}
\min \quad & \epsilon \\
\text{s. t.} \quad & x_i \geq, \qquad \forall i \in N \\
& \sum_{i \in N} x_i = v(N) \\
& \sum_{i \in C} x_i \geq v(C) - \varepsilon, \quad \forall C \subseteq N
\end{aligned}
$$

- LP with $2^n + n + 1$ constraints.
- How to find the solution in poly(n) time?

Check Non-Emptiness of the Core: CoS

$$
\begin{aligned}
\min \quad & \Delta \\
\text{s. t.} \quad & x_i \geq 0 \text{ for each } i \in N \\
& \sum_{i \in N} x_i = v(N) + \Delta \\
& \sum_{i \in C} x_i \geq v(C) \text{ for each } C \subseteq N
\end{aligned}
$$

Check Non-Emptiness of the Core: Bondareva-Shapley Theorem

Bondareva-Shapley Theorem

A superadditive game $G = (N, v)$ has a non-empty core if and only if for every balanced collection of sets $\mathcal{S} \subseteq 2^N \setminus \{\emptyset\}$ and any balancing weight system $(\delta_S)_{S \in \mathcal{S}}$ for $\mathcal{S}$ it holds that

$$\sum_{S \in \mathcal{S}} \delta_S v(S) \leq v(N)$$

- Balancing weight system: See the page 31 of the book.
- Complete characterization.
- NOT efficient method of checking whether a given game has a non-empty core, as the number of balanced set systems is superexponential in the number of agents n.

Nucleolus

- Deficit of C w.r.t. $\mathbf{x}$:

$$d(\mathbf{x}, C) = v(C) - x(C)$$

  - the quantity measures C's incentive to deviate under $\mathbf{x}$.

- 2^n-dimensional *deficit vector*

$$\mathbf{d}(\mathbf{x}) = (d(\mathbf{x}, C_1), \ldots, d(\mathbf{x}, C_{2^n}))$$

  where $C_1, \ldots, C_{2^n}$ is the list of all subsets of N ordered by their deficit under $\mathbf{x}$, from the largest to the smallest.

- $\mathbf{d}(\mathbf{x})$ is *lexicographically smaller* than $\mathbf{d}(\mathbf{y})$: $\mathbf{d}(\mathbf{x}) <_{\text{lex}} \mathbf{d}(\mathbf{y})$.

Nucleolus (cont'd)

Definition: Nucleolus

The nucleolus $\mathcal{N}(G)$ of a superadditive game is the set

$$\mathcal{N}(G) = \{\mathbf{x} \in \mathcal{I}(N) \mid \mathbf{d}(\mathbf{x}) \leq_{\text{lex}} \mathbf{d}(\mathbf{y}) \text{ for all } \mathbf{y} \in \mathcal{I}(N)\}$$

- $\mathcal{I}(N)$: the set of all imputations for a grand coalition.
- For any superadditive game, the set $\mathcal{N}(G)$ is non-empty, and contains exactly one elements.
- A unique way to divide the payoff of the grand coalition.

Definition: Pre-Nucleolus

The nucleolus $\mathcal{N}(G)$ of a superadditive game is the set

$$\mathcal{N}(G) = \{\mathbf{x} \mid \mathbf{d}(\mathbf{x}) \leq_{\text{lex}} \mathbf{d}(\mathbf{y}) \text{ for all } \mathbf{y}\}$$

- When $v(C) = 0$ if $|C| \leq 1$: Nucleolus = Pre-nucleolus.

Nucleolus (cont'd)

- The least core $= \{\mathbf{x} | \min_{\mathbf{x}} d_1(\mathbf{x}),$ where $d_1(\mathbf{x}) = \max\{d(\mathbf{x}, C) | C \subseteq N\}\}$
- Procedure for computing the (pre)-nucleolus:
  - Start by computing the set of payoff vectors that minimize the largest deficit d_1, i.e., the least core. (use the LP)
  - Among all payoff vectors in the least core, we pick the ones that minimize the second largest deficit $d_2(\mathbf{x}) = \max\{d(\mathbf{x}, C) | C \subseteq N\}, d(\mathbf{x}, C) < d_1\}$,and remove all other payoff vectors.
  - Repeat until the surviving set stabilizes.
- A variant of this procedure converges in at most n steps.

Kernel

Definition: Surplus

For any player i, its *surplus* over the player j w.r.t a payoff vector $\mathbf{x}$ as the quantity

$$S_{i,j}(\mathbf{x}) = \max\{v(C) - x(C) \mid C \subseteq N, i \in C, j \notin C\}.$$

- $S_{i,j}(\mathbf{x})$: the amount that player i can earn without the cooperation of player j, by asking a set $C \setminus \{i\}$ to join him in a deviation, and paying each player in $C \setminus \{i\}$ what it used to be paid under $\mathbf{x}$.
- if $S_{i,j}(\mathbf{x}) > S_{j,i}(\mathbf{x})$, player i should be able to demand a fraction of player j's payoff — unless $x_j = v(\{j\})$.

Kernel (cont'd)

Definition: Kernel

The kernel $\mathcal{K}(G)$ of a superadditive game G is the set of all imputations $\mathbf{x}$ such that for any pair of players (i, j) we have either: (1) $S_{i,j}(\mathbf{x}) = S_{j,i}(\mathbf{x})$; or (2) $S_{i,j}(\mathbf{x}) > S_{j,i}(\mathbf{x})$ and $x_j = v(\{j\})$; or (3) $S_{i,j}(\mathbf{x}) < S_{j,i}(\mathbf{x})$ and $x_i = v(\{i\})$.

- No player can credibly demand a fraction of another player's payoff.
- Contain the nucleolus.
- Non-empty.

Objection and Counterobjection

Definition: Objection

A pair $(\mathbf{y}, C)$ is said to be an *objection* to $\mathbf{x}$ if C is a subset of N and $\mathbf{y}$ is a vector in $\mathbb{R}^n$ that satisfies

1. $y(C) \leq v(C)$,
2. $y_i \geq x_i$ for each $i \in C$, and
3. $y_i > x_i$ for at least one $i \in C$

Definition: Objection

A pair $(\mathbf{z}, D)$ is said to be an *counterobjection* to an obejction $(\mathbf{y}, C)$ if D is a subset of N and $\mathbf{z}$ is a vector in $\mathbb{R}^n$ that satisfies

1. $z(D) \leq z(D)$,
2. $z_i \geq y_i$ for all $i \in D \cap C$, and
3. $z_i \geq x_i$ for all $i \in D \setminus C$,

with at least one of the inequalities in 2 and 3 being strict.

Bargaining Set

- An objection is said to be *justified* if it does not admit a counterobjection.

Definition: Bargaining Set

The bargaining set $\mathcal{B}(G)$ of a superadditive game $G = (N, v)$ is the set of all payoff vectors that do not admit a justified objection.

- Core $\subset$ Bargaining set
- Least-core $\subset$ Bargaining set
- Kernel $\subset$ Bargaining set

Stable Set

- Very first solution concept.
- Idea of Dominance:
  - $\mathbf{y}$ dom $\mathbf{z}$: $\mathbf{y}$ *dominates* $\mathbf{z}$ via some non-empty coalition $C \subseteq N$ if $y(C) \leq v(C)$ and $y_i > z_i$ for all $i \in C$.
- Given a set of imputations $J \subseteq \mathcal{I}(N)$, let

$$\text{Dom}(J) = \{\mathbf{z} \in \mathcal{I}(N) | \text{there exists a } \mathbf{y} \in J \text{ such that } \mathbf{y} \text{ dom } \mathbf{z}\}.$$

- If $\mathbf{z} \in \text{Dom}(J)$, there exists a non-empty C and $\mathbf{y} \in J$ s.t. each player in C prefers $\mathbf{y}$ to $\mathbf{z}$.

Stable Set (cont'd)

Definition: Stable Set

Given a superadditive game $G = (N, v)$, a set of imputations J is called a stable set of G if $J, \mathrm{Dom}(J)$ is a partition of $\mathcal{I}(N)$, i.e.,

1. Internal stability: $J \cap \mathrm{Dom}(J) = \emptyset$,
2. External stability: $\mathcal{I}(N) \setminus J \subseteq \mathrm{Dom}(J)$.

- From no stable sets to many stable sets.
- Supperadditive game: all stable sets $\subseteq$ core.
- Convex game: a unique stable set $=$ core.