Apply Functional Delta Method to
Variable Importance
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Let's write ψ_{x_p} as ψ for short. We know

$$\begin{aligned}\psi(f)-\psi(f_0) &= \frac{1}{\sqrt{n}}\psi'_{F_0}(f)+R_n \\ &= \frac{1}{n}\sum_i \psi'_{F_0}(\delta_{x_i}-F_0)+R_n. \end{aligned} \tag{1}$$

On the other hand,

$$\psi'_{F_0}(\delta_x-F_0)=\frac{d}{dt}\mid_{t=0}\psi[(1-t)F_0+t\delta_x]=IF_{\psi,F_0}(x).$$

In our case, $\psi(f)=||\frac{\partial}{\partial x_p}f||_n^2$, thus

$$\begin{aligned}\frac{d}{dt}\mid_{t=0}\psi[(1-t)F_0+t\delta_x] =& \frac{d}{dt}\mid_{t=0}||\frac{\partial}{\partial x_p}[(1-t)F_0+t\delta_x]||_n^2 \\ =& \frac{d}{dt}\mid_{t=0}(1-t)^2\psi(f_0) \\ =& \psi(f_0). \end{aligned} \tag{2}$$

Therefore, (1) turns into

$$\psi(f)-\psi(f_0)=\frac{1}{n}\sum_i \psi(f_0)+R_n.$$