Asymptotic Distribution
of Variable Importance
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Denote $D_p : f \to \frac{\partial}{\partial x_p} f$ the differentiation operator, $H_p = D_p^\top D_p$ the inner product of D_p, define

$$\psi_p(f) = \|D_p(f)\|_n^2 = \langle D_p f, D_p f \rangle = f^\top H_p f, \tag{1}$$

then we have

$$\frac{\partial}{\partial f} \psi_p(f) = 2H_p f.$$

Therefore, if given

$$\sqrt{n}(\hat{f} - f_0) \xrightarrow{d} N(0, \sigma^2),$$

by functional delta method, we have

$$\sqrt{n}(\psi_p(\hat{f}) - \psi_p(\hat{f}_0)) \xrightarrow{d} N(0, 4\sigma^2 \|H_p f_0\|_n^2).$$

If we generalize this result to multivariate: Denote $\boldsymbol{\psi} = [\psi_1, \dots, \psi_P]$ for ψ_p as defined in (1), then $\boldsymbol{\psi}(\hat{f})$ asymptotically converges toward a multivariate normal distribution surrounding $\boldsymbol{\psi}(f_0)$, i.e.,

$$\sqrt{n}(\boldsymbol{\psi}(\hat{f}) - \boldsymbol{\psi}(\hat{f}_0)) \xrightarrow{d} N(0, V_0),$$

where V_0 is a $P \times P$ matrix such that $(V_0)_{p1,p2} = 4\sigma^2 \langle H_{p1} f_0, H_{p2} f_0 \rangle_n$.