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第一章 绪论

1. 设 $x > 0$, x 的相对误差为 δ , 求 $\ln x$ 的误差。

解: 近似值 x^* 的相对误差为 $\delta = e_r^* = \frac{e^*}{x^*} = \frac{x^* - x}{x^*}$

而 $\ln x$ 的误差为 $e(\ln x^*) = \ln x^* - \ln x \approx \frac{1}{x^*} e^*$

进而有 $\varepsilon(\ln x^*) \approx \delta$

2. 设 x 的相对误差为 2%, 求 x^n 的相对误差。

解: 设 $f(x) = x^n$, 则函数的条件数为 $C_p = \left| \frac{xf'(x)}{f(x)} \right|$

又 $\because f'(x) = nx^{n-1}$, $\therefore C_p = \left| \frac{x \cdot nx^{n-1}}{x^n} \right| = n$

又 $\because \varepsilon_r((x^*)^n) \approx C_p \cdot \varepsilon_r(x^*)$

且 $e_r(x^*)$ 为 2

$\therefore \varepsilon_r((x^*)^n) \approx 0.02n$

3. 下列各数都是经过四舍五入得到的近似数, 即误差限不超过最后一位的半个单位, 试指出它们是几位有效数字: $x_1^* = 1.1021$, $x_2^* = 0.031$, $x_3^* = 385.6$, $x_4^* = 56.430$, $x_5^* = 7 \times 1.0$.

解: $x_1^* = 1.1021$ 是五位有效数字;

$x_2^* = 0.031$ 是二位有效数字;

$x_3^* = 385.6$ 是四位有效数字;

$x_4^* = 56.430$ 是五位有效数字;

$x_5^* = 7 \times 1.0$ 是二位有效数字。

4. 利用公式(2.3)求下列各近似值的误差限: (1) $x_1^* + x_2^* + x_4^*$, (2) $x_1^* x_2^* x_3^*$, (3) x_2^* / x_4^* .

其中 $x_1^*, x_2^*, x_3^*, x_4^*$ 均为第 3 题所给的数。

解:

$$\varepsilon(x_1^*) = \frac{1}{2} \times 10^{-4}$$

$$\varepsilon(x_2^*) = \frac{1}{2} \times 10^{-3}$$

$$\varepsilon(x_3^*) = \frac{1}{2} \times 10^{-1}$$

$$\varepsilon(x_4^*) = \frac{1}{2} \times 10^{-3}$$

$$\varepsilon(x_5^*) = \frac{1}{2} \times 10^{-1}$$

$$\begin{aligned} (1) & \varepsilon(x_1^* + x_2^* + x_4^*) \\ &= \varepsilon(x_1^*) + \varepsilon(x_2^*) + \varepsilon(x_4^*) \\ &= \frac{1}{2} \times 10^{-4} + \frac{1}{2} \times 10^{-3} + \frac{1}{2} \times 10^{-3} \\ &= 1.05 \times 10^{-3} \end{aligned}$$

$$\begin{aligned} (2) & \varepsilon(x_1^* x_2^* x_3^*) \\ &= |x_1^* x_2^*| \varepsilon(x_3^*) + |x_2^* x_3^*| \varepsilon(x_1^*) + |x_1^* x_3^*| \varepsilon(x_2^*) \\ &= |1.1021 \times 0.031| \times \frac{1}{2} \times 10^{-1} + |0.031 \times 385.6| \times \frac{1}{2} \times 10^{-4} + |1.1021 \times 385.6| \times \frac{1}{2} \times 10^{-3} \\ &\approx 0.215 \end{aligned}$$

$$\begin{aligned} (3) & \varepsilon(x_2^* / x_4^*) \\ &\approx \frac{|x_2^*| \varepsilon(x_4^*) + |x_4^*| \varepsilon(x_2^*)}{|x_4^*|^2} \\ &= \frac{0.031 \times \frac{1}{2} \times 10^{-3} + 56.430 \times \frac{1}{2} \times 10^{-3}}{56.430 \times 56.430} \\ &= 10^{-5} \end{aligned}$$

5 计算球体积要使相对误差限为 1，问度量半径 R 时允许的相对误差限是多少？

解：球体体积为 $V = \frac{4}{3} \pi R^3$

则何种函数的条件数为

$$C_p = \left| \frac{R[V']}{V} \right| = \left| \frac{R[4\pi R^2]}{\frac{4}{3}\pi R^3} \right| = 3$$

$$\therefore \varepsilon_r(V^*) \approx C_p \varepsilon_r(R^*) = 3\varepsilon_r(R^*)$$

$$\text{又} \because \varepsilon_r(V^*) = 1\%1$$

故度量半径 R 时允许的相对误差限为 $\varepsilon_r(V^*) = \frac{1}{3} * 1\% = \frac{1}{300}$

6. 设 $Y_0 = 28$, 按递推公式 $Y_n = Y_{n-1} - \frac{1}{100}\sqrt{783}$ ($n=1,2,\dots$)

计算到 Y_{100} 。若取 $\sqrt{783} \approx 27.982$ (5 位有效数字), 试问计算 Y_{100} 将有多大误差?

$$\text{解: } \because Y_n = Y_{n-1} - \frac{1}{100}\sqrt{783}$$

$$\therefore Y_{100} = Y_{99} - \frac{1}{100}\sqrt{783}$$

$$Y_{99} = Y_{98} - \frac{1}{100}\sqrt{783}$$

$$Y_{98} = Y_{97} - \frac{1}{100}\sqrt{783}$$

.....

$$Y_1 = Y_0 - \frac{1}{100}\sqrt{783}$$

$$\text{依次代入后, 有 } Y_{100} = Y_0 - 100 \times \frac{1}{100}\sqrt{783}$$

$$\text{即 } Y_{100} = Y_0 - \sqrt{783},$$

$$\text{若取 } \sqrt{783} \approx 27.982, \therefore Y_{100} = Y_0 - 27.982$$

$$\therefore \varepsilon(Y_{100}^*) = \varepsilon(Y_0) + \varepsilon(27.982) = \frac{1}{2} \times 10^{-3}$$

$$\therefore Y_{100} \text{ 的误差限为 } \frac{1}{2} \times 10^{-3}.$$

7. 求方程 $x^2 - 56x + 1 = 0$ 的两个根, 使它至少具有 4 位有效数字 ($\sqrt{783} = 27.982$)。

$$\text{解: } x^2 - 56x + 1 = 0,$$

$$\text{故方程的根应为 } x_{1,2} = 28 \pm \sqrt{783}$$

$$\text{故 } x_1 = 28 + \sqrt{783} \approx 28 + 27.982 = 55.982$$

$\therefore x_1$ 具有 5 位有效数字

$$x_2 = 28 - \sqrt{783} = \frac{1}{28 + \sqrt{783}} \approx \frac{1}{28 + 27.982} = \frac{1}{55.982} \approx 0.017863$$

x_2 具有 5 位有效数字

8. 当 N 充分大时, 怎样求 $\int_N^{N+1} \frac{1}{1+x^2} dx$?

$$\text{解 } \int_N^{N+1} \frac{1}{1+x^2} dx = \arctan(N+1) - \arctan N$$

设 $\alpha = \arctan(N+1), \beta = \arctan N$ 。

则 $\tan \alpha = N+1, \tan \beta = N$.

$$\begin{aligned} & \int_N^{N+1} \frac{1}{1+x^2} dx \\ &= \alpha - \beta \\ &= \arctan(\tan(\alpha - \beta)) \\ &= \arctan \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \\ &= \arctan \frac{N+1 - N}{1 + (N+1)N} \\ &= \arctan \frac{1}{N^2 + N + 1} \end{aligned}$$

9. 正方形的边长大约为了 100cm , 应怎样测量才能使其面积误差不超过 1cm^2 ?

解: 正方形的面积函数为 $A(x) = x^2$

$$\therefore \varepsilon(A^*) = 2A^* \varepsilon(x^*).$$

当 $x^* = 100$ 时, 若 $\varepsilon(A^*) \leq 1$,

$$\text{则 } \varepsilon(x^*) \leq \frac{1}{2} \times 10^{-2}$$

故测量中边长误差限不超过 0.005cm 时, 才能使其面积误差不超过 1cm^2

10. 设 $S = \frac{1}{2}gt^2$, 假定 g 是准确的, 而对 t 的测量有 ± 0.1 秒的误差, 证明当 t 增加时 S 的绝对误差增加, 而相对误差却减少。

$$\text{解: } \because S = \frac{1}{2}gt^2, t > 0$$

$$\therefore \varepsilon(S^*) = g t \varepsilon(t)$$

当 t^* 增加时, S^* 的绝对误差增加

$$\begin{aligned}\varepsilon_r(S^*) &= \frac{\varepsilon(S^*)}{|S^*|} \\ &= \frac{gt^2 \varepsilon(t^*)}{\frac{1}{2}g(t^*)^2} \\ &= 2 \frac{\varepsilon(t^*)}{t^*}\end{aligned}$$

当 t^* 增加时, $\varepsilon(t^*)$ 保持不变, 则 S^* 的相对误差减少。

11. 序列 $\{y_n\}$ 满足递推关系 $y_n = 10y_{n-1} - 1$ ($n=1,2,\dots$),

若 $y_0 = \sqrt{2} \approx 1.41$ (三位有效数字), 计算到 y_{10} 时误差有多大? 这个计算过程稳定吗?

解: $\because y_0 = \sqrt{2} \approx 1.41$

$$\therefore \varepsilon(y_0^*) = \frac{1}{2} \times 10^{-2}$$

$$\text{又} \because y_n = 10y_{n-1} - 1$$

$$\therefore y_1 = 10y_0 - 1$$

$$\therefore \varepsilon(y_1^*) = 10\varepsilon(y_0^*)$$

$$\text{又} \because y_2 = 10y_1 - 1$$

$$\therefore \varepsilon(y_2^*) = 10\varepsilon(y_1^*)$$

$$\therefore \varepsilon(y_2^*) = 10^2 \varepsilon(y_0^*)$$

.....

$$\therefore \varepsilon(y_{10}^*) = 10^{10} \varepsilon(y_0^*)$$

$$= 10^{10} \times \frac{1}{2} \times 10^{-2}$$

$$= \frac{1}{2} \times 10^8$$

计算到 y_{10} 时误差为 $\frac{1}{2} \times 10^8$, 这个计算过程不稳定。

12. 计算 $f = (\sqrt{2} - 1)^6$, 取 $\sqrt{2} \approx 1.4$, 利用下列等式计算, 哪一个得到的结果最好?

$$\frac{1}{(\sqrt{2}+1)^6}, \quad (3-2\sqrt{2})^3, \quad \frac{1}{(3+2\sqrt{2})^3}, \quad 99-70\sqrt{2}.$$

解：设 $y = (x-1)^6$,

若 $x = \sqrt{2}$, $x^* = 1.4$, 则 $\varepsilon(x^*) = \frac{1}{2} \times 10^{-1}$ 。

若通过 $\frac{1}{(\sqrt{2}+1)^6}$ 计算 y 值, 则

$$\begin{aligned}\varepsilon(y^*) &= - \left| -6 \times \frac{1}{(x^*+1)^7} \right| \varepsilon(x^*) \\ &= \frac{6}{(x^*+1)^7} y^* \varepsilon(x^*) \\ &= 2.53 y^* \varepsilon(x^*)\end{aligned}$$

若通过 $(3-2\sqrt{2})^3$ 计算 y 值, 则

$$\begin{aligned}\varepsilon(y^*) &= \left| -3 \times 2 \times (3-2x^*)^2 \right| \varepsilon(x^*) \\ &= \frac{6}{3-2x^*} y^* \varepsilon(x^*) \\ &= 30 y^* \varepsilon(x^*)\end{aligned}$$

若通过 $\frac{1}{(3+2\sqrt{2})^3}$ 计算 y 值, 则

$$\begin{aligned}\varepsilon(y^*) &= - \left| -3 \times \frac{1}{(3+2x^*)^4} \right| \varepsilon(x^*) \\ &= 6 \times \frac{1}{(3+2x^*)^7} y^* \varepsilon(x^*) \\ &= 1.0345 y^* \varepsilon(x^*)\end{aligned}$$

通过 $\frac{1}{(3+2\sqrt{2})^3}$ 计算后得到的结果最好。

13. $f(x) = \ln(x - \sqrt{x^2 - 1})$, 求 $f(30)$ 的值。若开平方用 6 位函数表, 问求对数时误差有多大?

大? 若改用另一等价公式。 $\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 - 1})$

计算, 求对数时误差有多大?

解

$$\because f(x) = \ln(x - \sqrt{x^2 - 1}), \therefore f(30) = \ln(30 - \sqrt{899})$$

设 $u = \sqrt{899}$, $y = f(30)$

则 $u^* = 29.9833$

$$\therefore \varepsilon(u^*) = \frac{1}{2} \times 10^{-4}$$

故

$$\begin{aligned}\varepsilon(y^*) &\approx -\frac{1}{|30-u^*|} \varepsilon(u^*) \\ &= \frac{1}{0.0167} \varepsilon(u^*) \\ &\approx 3 \times 10^{-3}\end{aligned}$$

若改用等价公式

$$\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 - 1})$$

$$\text{则 } f(30) = -\ln(30 + \sqrt{899})$$

此时，

$$\begin{aligned}\varepsilon(y^*) &= \left| -\frac{1}{30+u^*} \right| \varepsilon(u^*) \\ &= \frac{1}{59.9833} \cdot \varepsilon(u^*) \\ &\approx 8 \times 10^{-7}\end{aligned}$$

第二章 插值法

1. 当 $x = 1, -1, 2$ 时, $f(x) = 0, -3, 4$, 求 $f(x)$ 的二次插值多项式。

解:

$$x_0 = 1, x_1 = -1, x_2 = 2,$$

$$f(x_0) = 0, f(x_1) = -3, f(x_2) = 4;$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = -\frac{1}{2}(x+1)(x-2)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{1}{6}(x-1)(x-2)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{3}(x-1)(x+1)$$

则二次拉格朗日插值多项式为

$$\begin{aligned} L_2(x) &= \sum_{k=0}^2 y_k l_k(x) \\ &= -3l_0(x) + 4l_2(x) \\ &= -\frac{1}{2}(x-1)(x-2) + \frac{4}{3}(x-1)(x+1) \\ &= \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3} \end{aligned}$$

2. 给出 $f(x) = \ln x$ 的数值表

X	0.4	0.5	0.6	0.7	0.8
lnx	-0.916291	-0.693147	-0.510826	-0.356675	-0.223144

用线性插值及二次插值计算 $\ln 0.54$ 的近似值。

解: 由表格知,

$$x_0 = 0.4, x_1 = 0.5, x_2 = 0.6, x_3 = 0.7, x_4 = 0.8;$$

$$f(x_0) = -0.916291, f(x_1) = -0.693147$$

$$f(x_2) = -0.510826, f(x_3) = -0.356675$$

$$f(x_4) = -0.223144$$

若采用线性插值法计算 $\ln 0.54$ 即 $f(0.54)$,

则 $0.5 < 0.54 < 0.6$

$$l_1(x) = \frac{x - x_2}{x_1 - x_2} = -10(x - 0.6)$$

$$l_2(x) = \frac{x - x_1}{x_2 - x_1} = -10(x - 0.5)$$

$$L_1(x) = f(x_1)l_1(x) + f(x_2)l_2(x)$$

$$= 6.93147(-0.6) - 5.1082$$

$$\therefore L_1(0.54) = -0.6202186 \approx -0.620219$$

若采用二次插值法计算 $\ln 0.54$ 时,

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = 50(x - 0.5)(x - 0.6)$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = -100(x - 0.4)(x - 0.6)$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = 50(x - 0.4)(x - 0.5)$$

$$L_2(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x)$$

$$= 50 \times 0.916291(-0.5)(+0.6) - 0.916291 \times 100 \times 0.40826050(0.54)($$

$$\therefore L_2(0.54) = -0.61531984 \approx -0.61532$$

3. 给全 $\cos x, 0^\circ \leq x \leq 90^\circ$ 的函数表, 步长 $h = 1' = (1/60)^\circ$, 若函数表具有 5 位有效数字, 研究用线性插值求 $\cos x$ 近似值时的总误差界。

解: 求解 $\cos x$ 近似值时, 误差可以分为两个部分, 一方面, x 是近似值, 具有 5 位有效数字, 在此后的计算过程中产生一定的误差传播; 另一方面, 利用插值法求函数 $\cos x$ 的近似值时, 采用的线性插值法插值余项不为 0, 也会有一定的误差。因此, 总误差界的计算应综合以上两方面的因素。

当 $0^\circ \leq x \leq 90^\circ$ 时,

$$\text{令 } f(x) = \cos x$$

$$\text{取 } x_0 = 0, h = \left(\frac{1}{60}\right)^\circ = \frac{1}{60} \times \frac{\pi}{180} = \frac{\pi}{10800}$$

$$\text{令 } x_i = x_0 + ih, i = 0, 1, \dots, 5400$$

$$\text{则 } x_{5400} = \frac{\pi}{2} = 90^\circ$$

当 $x \in [x_k, x_{k+1}]$ 时, 线性插值多项式为

$$L_1(x) = f(x_k) \frac{x - x_{k+1}}{x_k - x_{k+1}} + f(x_{k+1}) \frac{x - x_k}{x_{k+1} - x_k}$$

插值余项为

$$R(x) = |\cos x - L_1(x)| = \left| \frac{1}{2} f''(\xi)(x - x_k)(x - x_{k+1}) \right|$$

又 \because 在建立函数表时,表中数据具有5位有效数字,且 $\cos x \in [0,1]$,故计算中有误差传播过程。

$$\therefore \varepsilon(f^*(x_k)) = \frac{1}{2} \times 10^{-5}$$

$$R_2(x) = \left| \varepsilon(f^*(x_k)) \frac{x - x_{k+1}}{x_k - x_{k+1}} \right| + \left| \varepsilon(f^*(x_{k+1})) \frac{x - x_k}{x_{k+1} - x_k} \right|$$

$$\leq \varepsilon(f^*(x_k)) \left(\left| \frac{x - x_{k+1}}{x_k - x_{k+1}} \right| + \left| \frac{x - x_k}{x_{k+1} - x_k} \right| \right)$$

$$= \varepsilon(f^*(x_k)) \frac{1}{h} (x_{k+1} - x + x - x_k)$$

$$= \varepsilon(f^*(x_k))$$

\therefore 总误差界为

$$R = R_1(x) + R_2(x)$$

$$= \left| \frac{1}{2} (-\cos \xi)(x - x_k)(x - x_{k+1}) \right| + \varepsilon(f^*(x_k))$$

$$\leq \frac{1}{2} \times (x - x_k)(x_{k+1} - x) + \varepsilon(f^*(x_k))$$

$$\leq \frac{1}{2} \times \left(\frac{1}{2}h\right)^2 + \varepsilon(f^*(x_k))$$

$$= 1.06 \times 10^{-8} + \frac{1}{2} \times 10^{-5}$$

$$= 0.50106 \times 10^{-5}$$

4. 设为互异节点, 求证:

$$(1) \sum_{j=0}^n x_j^k l_j(x) \equiv x^k \quad (k = 0, 1, \dots, n,$$

$$(2) \sum_{j=0}^n (x_j - x)^k l_j(x) \equiv 0 \quad (k = 0, 1, \dots, n,$$

证明

$$(1) \quad \text{令 } f(x) = x^k$$

若插值节点为 $x_j, j=0,1,\cdots,n$, 则函数 $f(x)$ 的 n 次插值多项式为 $L_n(x) = \sum_{j=0}^n x_j^k l_j(x)$ 。

$$\text{插值余项为 } R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_{n+1}(x)$$

又 $\because k \leq n$,

$$\therefore f^{(n+1)}(\xi) = 0$$

$$\therefore R_n(x) = 0$$

$$\therefore \sum_{j=0}^n x_j^k l_j(x) = x^k \quad (k = 0, 1, \dots, n),$$

$$\begin{aligned} (2) & \sum_{j=0}^n (x_j - x)^k l_j(x) \\ &= \sum_{j=0}^n \left(\sum_{i=0}^n C_k^j x_j^i (-x)^{k-i} \right) l_j(x) \\ &= \sum_{i=0}^n C_k^i (-x)^{k-i} \left(\sum_{j=0}^n x_j^i l_j(x) \right) \end{aligned}$$

又 $\because 0 \leq i \leq n$ 由上题结论可知

$$\sum_{j=0}^n x_j^i l_j(x) = x^i$$

$$\begin{aligned} \therefore \text{原式} &= \sum_{i=0}^n C_k^i (-x)^{k-i} x^i \\ &= (x - x)^k \\ &= 0 \end{aligned}$$

\therefore 得证。

5 设 $f(x) \in C^2[a, b]$ 且 $f(a) = f(b) = 0$, 求证:

$$\max_{a \leq x \leq b} |f(x)| \leq \frac{1}{8} (b-a)^2 \max_{a \leq x \leq b} |f''(x)|.$$

解: 令 $x_0 = a, x_1 = b$, 以此为插值节点, 则线性插值多项式为

$$\begin{aligned} L_1(x) &= f(x_0) \frac{x-x_1}{x_0-x_1} + f(x_1) \frac{x-x_0}{x-x_0} \\ &= f(a) \frac{x-b}{a-b} + f(b) \frac{x-a}{x-a} \end{aligned}$$

$$\text{又} \because f(a) = f(b) = 0$$

$$\therefore L_1(x) = 0$$

$$\text{插值余项为 } R(x) = f(x) - L_1(x) = \frac{1}{2} f''(x)(x-x_0)(x-x_1)$$

$$\therefore f(x) = \frac{1}{2} f''(x)(x-x_0)(x-x_1)$$

$$\text{又} \because |(x-x_0)(x-x_1)|$$

$$\leq \left\{ \frac{1}{2} [(x-x_0) + (x_1-x)] \right\}^2$$

$$= \frac{1}{4} (x_1 - x_0)^2$$

$$= \frac{1}{4} (b-a)^2$$

$$\therefore \max_{a \leq x \leq b} |f(x)| \leq \frac{1}{8} (b-a)^2 \max_{a \leq x \leq b} |f''(x)|.$$

6. 在 $-4 \leq x \leq 4$ 上给出 $f(x) = e^x$ 的等距节点函数表, 若用二次插值求 e^x 的近似值, 要使截

断误差不超过 10^{-6} , 问使用函数表的步长 h 应取多少?

解: 若插值节点为 x_{i-1}, x_i 和 x_{i+1} , 则分段二次插值多项式的插值余项为

$$R_2(x) = \frac{1}{3!} f'''(\xi)(x-x_{i-1})(x-x_i)(x-x_{i+1})$$

$$\therefore |R_2(x)| \leq \frac{1}{6} (x-x_{i-1})(x-x_i)(x-x_{i+1}) \max_{-4 \leq x \leq 4} |f'''(x)|$$

设步长为 h , 即 $x_{i-1} = x_i - h, x_{i+1} = x_i + h$

$$\therefore |R_2(x)| \leq \frac{1}{6} e^4 \cdot \frac{2}{3\sqrt{3}} h^3 = \frac{\sqrt{3}}{27} e^4 h^3.$$

若截断误差不超过 10^{-6} , 则

$$|R_2(x)| \leq 10^{-6}$$

$$\therefore \frac{\sqrt{3}}{27} e^4 h^3 \leq 10^{-6}$$

$$\therefore h \leq 0.0065.$$

7. 若 $y_n = 2^n$, 求 $\Delta^4 y_n$ 及 $\delta^4 y_n$.

解: 根据向前差分算子和中心差分算子的定义进行求解。

$$y_n = 2^n$$

$$\begin{aligned}
\Delta^4 y_n &= (E-1)^4 y_n \\
&= \sum_{j=0}^4 (-1)^j \binom{4}{j} E^{4-j} y_n \\
&= \sum_{j=0}^4 (-1)^j \binom{4}{j} y_{4+n-j} \\
&= \sum_{j=0}^4 (-1)^j \binom{4}{j} 2^j \cdot y_n \\
&= (2-1)^4 y_n \\
&= y_n \\
&= 2^n
\end{aligned}$$

$$\begin{aligned}
\delta^4 y_n &= (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^4 y_n \\
&= (E^{-\frac{1}{2}})^4 (E-1)^4 y_n \\
&= E^{-2} \Delta^4 y_n \\
&= y_{n-2} \\
&= 2^{n-2}
\end{aligned}$$

8. 如果 $f(x)$ 是 m 次多项式, 记 $\Delta f(x) = f(x+h) - f(x)$, 证明 $f(x)$ 的 k 阶差分

$\Delta^k f(x) (0 \leq k \leq m)$ 是 $m-k$ 次多项式, 并且 $\Delta^{m+1} f(x) = 0$ (l 为正整数)。

解: 函数 $f(x)$ 的 *Taylor* 展式为

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \cdots + \frac{1}{m!} f^{(m)}(x)h^m + \frac{1}{(m+1)!} f^{(m+1)}(\xi)h^{m+1}$$

其中 $\xi \in (x, x+h)$

又 $\because f(x)$ 是次数为 m 的多项式

$$\therefore f^{(m+1)}(\xi) = 0$$

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= f'(x)h + \frac{1}{2} f''(x)h^2 + \cdots + \frac{1}{m!} f^{(m)}(x)h^m$$

$\therefore \Delta f(x)$ 为 $m-1$ 阶多项式

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$\therefore \Delta^2 f(x)$ 为 $m-2$ 阶多项式

依此过程递推，得 $\Delta^k f(x)$ 是 $m-k$ 次多项式

$\therefore \Delta^m f(x)$ 是常数

\therefore 当 l 为正整数时，

$$\Delta^{m+1} f(x) = 0$$

9. 证明 $\Delta(f_k g_k) = f_k \Delta g_k + g_{k+1} \Delta f_k$

证明

$$\begin{aligned} \Delta(f_k g_k) &= f_{k+1} g_{k+1} - f_k g_k \\ &= f_{k+1} g_{k+1} - f_k g_{k+1} + f_k g_{k+1} - f_k g_k \\ &= g_{k+1} (f_{k+1} - f_k) + f_k (g_{k+1} - g_k) \\ &= g_{k+1} \Delta f_k + f_k \Delta g_k \\ &= f_k \Delta g_k + g_{k+1} \Delta f_k \end{aligned}$$

\therefore 得证

10. 证明 $\sum_{k=0}^{n-1} f_k \Delta g_k = f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k$

证明：由上题结论可知

$$\begin{aligned} f_k \Delta g_k &= \Delta(f_k g_k) - g_{k+1} \Delta f_k \\ \therefore \sum_{k=0}^{n-1} f_k \Delta g_k &= \sum_{k=0}^{n-1} (\Delta(f_k g_k) - g_{k+1} \Delta f_k) \\ &= \sum_{k=0}^{n-1} \Delta(f_k g_k) - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k \\ &\because \Delta(f_k g_k) = f_{k+1} g_{k+1} - f_k g_k \\ \therefore \sum_{k=0}^{n-1} \Delta(f_k g_k) &= (f_1 g_1 - f_0 g_0) + (f_2 g_2 - f_1 g_1) + \cdots + (f_n g_n - f_{n-1} g_{n-1}) \\ &= f_n g_n - f_0 g_0 \\ \therefore \sum_{k=0}^{n-1} f_k \Delta g_k &= f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k \end{aligned}$$

得证。

11. 证明 $\sum_{j=0}^{n-1} \Delta^2 y_j = \Delta y_n - \Delta y_0$

$$\begin{aligned}
\text{证明 } \sum_{j=0}^{n-1} \Delta^2 y_j &= \sum_{j=0}^{n-1} (\Delta y_{j+1} - \Delta y_j) \\
&= (\Delta y_1 - \Delta y_0) + (\Delta y_2 - \Delta y_1) + \cdots + (\Delta y_n - \Delta y_{n-1}) \\
&= \Delta y_n - \Delta y_0
\end{aligned}$$

得证。

12. 若 $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ 有 n 个不同实根 x_1, x_2, \cdots, x_n ,

$$\text{证明: } \sum_{j=1}^n \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, 0 \leq k \leq n-2; \\ n_0^{-1}, k = n-1 \end{cases}$$

证明: $\because f(x)$ 有个不同实根 x_1, x_2, \cdots, x_n

$$\text{且 } f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$$

$$\therefore f(x) = a_n(x-x_1)(x-x_2)\cdots(x-x_n)$$

$$\text{令 } \omega_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$$

$$\text{则 } \sum_{j=1}^n \frac{x_j^k}{f'(x_j)} = \sum_{j=1}^n \frac{x_j^k}{a_n \omega'_n(x_j)}$$

$$\text{而 } \omega'_n(x) = (x-x_2)(x-x_3)\cdots(x-x_n) + (x-x_1)(x-x_3)\cdots(x-x_n)$$

$$+ \cdots + (x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

$$\therefore \omega'_n(x_j) = (x_j-x_1)(x_j-x_2)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)$$

$$\text{令 } g(x) = x^k,$$

$$g[x_1, x_2, \cdots, x_n] = \sum_{j=1}^n \frac{x_j^k}{\omega'_n(x_j)}$$

$$\text{则 } g[x_1, x_2, \cdots, x_n] = \sum_{j=1}^n \frac{x_j^k}{\omega'_n(x_j)}$$

$$\text{又 } \therefore \sum_{j=1}^n \frac{x_j^k}{f'(x_j)} = \frac{1}{a_n} g[x_1, x_2, \cdots, x_n]$$

$$\therefore \sum_{j=1}^n \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, 0 \leq k \leq n-2; \\ n_0^{-1}, k = n-1 \end{cases}$$

\therefore 得证。

13. 证明 n 阶均差有下列性质:

(1) 若 $F(x) = cf(x)$, 则 $F[x_0, x_1, \dots, x_n] = cf[x_0, x_1, \dots, x_n]$;

(2) 若 $F(x) = f(x) + g(x)$, 则 $F[x_0, x_1, \dots, x_n] = f[x_0, x_1, \dots, x_n] + g[x_0, x_1, \dots, x_n]$.

证明:

$$\begin{aligned}
 (1) \quad \because f[x_1, x_2, \dots, x_n] &= \sum_{j=0}^n \frac{f(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 F[x_1, x_2, \dots, x_n] &= \sum_{j=0}^n \frac{F(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 &= \sum_{j=0}^n \frac{cf(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 &= c \left(\sum_{j=0}^n \frac{f(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \right) \\
 &= cf[x_0, x_1, \dots, x_n]
 \end{aligned}$$

\therefore 得证。

(2) $\because F(x) = f(x) + g(x)$

$$\begin{aligned}
 \therefore F[x_0, \dots, x_n] &= \sum_{j=0}^n \frac{F(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 &= \sum_{j=0}^n \frac{f(x^j) + g(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 &= \sum_{j=0}^n \frac{f(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 &\quad + \sum_{j=0}^n \frac{g(x^j)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)} \\
 &= f[x_0, \dots, x_n] + g[x_0, \dots, x_n]
 \end{aligned}$$

\therefore 得证。

14. $f(x) = x^7 + x^4 + 3x + 1$, 求 $F[2^0, 2^1, \dots, 2^7]$ 及 $F[2^0, 2^1, \dots, 2^8]$ 。

解: $\because f(x) = x^7 + x^4 + 3x + 1$

若 $x_i = 2^i, i = 0, 1, \dots, 8$

$$\text{则 } f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

$$\therefore f[x_0, x_1, \dots, x_7] = \frac{f^{(7)}(\xi)}{7!} = \frac{7!}{7!} = 1$$

$$f[x_0, x_1, \dots, x_8] = \frac{f^{(8)}(\xi)}{8!} = 0$$

15. 证明两点三次埃尔米特插值余项是

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!} (x - x_k)^2 (x - x_{k+1})^2, \quad \xi \in [x_k, x_{k+1}]$$

解:

若 $x \in [x_k, x_{k+1}]$, 且插值多项式满足条件

$$H_3(x_k) = f(x_k), H'_3(x_k) = f'(x_k)$$

$$H_3(x_{k+1}) = f(x_{k+1}), H'_3(x_{k+1}) = f'(x_{k+1})$$

插值余项为 $R(x) = f(x) - H_3(x)$

由插值条件可知 $R(x_k) = R(x_{k+1}) = 0$

且 $R'(x_k) = R'(x_{k+1}) = 0$

$\therefore R(x)$ 可写成 $R(x) = g(x)(x - x_k)^2(x - x_{k+1})^2$

其中 $g(x)$ 是关于 x 的待定函数,

现把 x 看成 $[x_k, x_{k+1}]$ 上的一个固定点, 作函数

$$\varphi(t) = f(t) - H_3(t) - g(x)(t - x_k)^2(t - x_{k+1})^2$$

根据余项性质, 有

$$\varphi(x_k) = 0, \varphi(x_{k+1}) = 0$$

$$\begin{aligned} \varphi(x) &= f(x) - H_3(x) - g(x)(x - x_k)^2(x - x_{k+1})^2 \\ &= f(x) - H_3(x) - R(x) \\ &= 0 \end{aligned}$$

$$\varphi'(t) = f'(t) - H'_3(t) - g(x)[2(t - x_k)(t - x_{k+1})^2 + 2(t - x_{k+1})(t - x_k)^2]$$

$$\therefore \varphi'(x_k) = 0$$

$$\varphi'(x_{k+1})=0$$

由罗尔定理可知，存在 $\xi \in (x_k, x)$ 和 $\xi \in (x, x_{k+1})$ ，使

$$\varphi'(\xi_1)=0, \varphi'(\xi_2)=0$$

即 $\varphi'(x)$ 在 $[x_k, x_{k+1}]$ 上有四个互异零点。

根据罗尔定理， $\varphi''(t)$ 在 $\varphi'(t)$ 的两个零点间至少有一个零点，

故 $\varphi''(t)$ 在 (x_k, x_{k+1}) 内至少有三个互异零点，

依此类推， $\varphi^{(4)}(t)$ 在 (x_k, x_{k+1}) 内至少有一个零点。

记为 $\xi \in (x_k, x_{k+1})$ 使

$$\varphi^{(4)}(\xi) = f^{(4)}(\xi) - H_3^{(4)}(\xi) - 4!g(x) = 0$$

$$\text{又} \because H_3^{(4)}(t) = 0$$

$$\therefore g(x) = \frac{f^{(4)}(\xi)}{4!}, \xi \in (x_k, x_{k+1})$$

其中 ξ 依赖于 x

$$\therefore R(x) = \frac{f^{(4)}(\xi)}{4!} (x - x_k)^2 (x - x_{k+1})^2$$

分段三次埃尔米特插值时，若节点为 $x_k (k=0, 1, \dots, n)$ ，设步长为 h ，即

$x_k = x_0 + kh, k=0, 1, \dots, n$ 在小区间 $[x_k, x_{k+1}]$ 上

$$R(x) = \frac{f^{(4)}(\xi)}{4!} (x - x_k)^2 (x - x_{k+1})^2$$

$$\therefore |R(x)| = \frac{1}{4!} |f^{(4)}(\xi)| (x - x_k)^2 (x - x_{k+1})^2$$

$$\begin{aligned}
&\leq \frac{1}{4!} (x-x_k)^2 (x_{k+1}-x)^2 \max_{a \leq x \leq b} |f^{(4)}(x)| \\
&\leq \frac{1}{4!} \left[\left(\frac{x-x_k + x_{k+1}-x}{2} \right)^2 \right]^2 \max_{a \leq x \leq b} |f^{(4)}(x)| \\
&= \frac{1}{4!} \times \frac{1}{2^4} h^4 \max_{a \leq x \leq b} |f^{(4)}(x)| \\
&= \frac{h^4}{384} \max_{a \leq x \leq b} |f^{(4)}(x)|
\end{aligned}$$

16. 求一个次数不高于 4 次的多项式 $P(x)$, 使它满足 $P(0)=P'(0)=0, P(1)=P'(1)=0, P(2)=0$

解: 利用埃米尔特插值可得到次数不高于 4 的多项式

$$x_0 = 0, x_1 = 1$$

$$y_0 = 0, y_1 = 1$$

$$m_0 = 0, m_1 = 1$$

$$H_3(x) = \sum_{j=0}^1 y_j \alpha_j(x) + \sum_{j=0}^1 m_j \beta_j(x)$$

$$\begin{aligned}
\alpha_0(x) &= \left(1 - 2 \frac{x-x_0}{x_0-x_1}\right) \left(\frac{x-x_1}{x_0-x_1}\right)^2 \\
&= (1+2x)(x-1)^2
\end{aligned}$$

$$\begin{aligned}
\alpha_1(x) &= \left(1 - 2 \frac{x-x_1}{x_1-x_0}\right) \left(\frac{x-x_0}{x_1-x_0}\right)^2 \\
&= (3-2x)x^2
\end{aligned}$$

$$\beta_0(x) = x(x-1)^2$$

$$\beta_1(x) = (x-1)x^2$$

$$\therefore H_3(x) = (3-2x)x^2 + (x-1)x^2 = -x^3 + 2x^2$$

$$\text{设 } P(x) = H_3(x) + A(x-x_0)^2(x-x_1)^2$$

其中, A 为待定常数

$$\because P(2) = 1$$

$$\therefore P(x) = -x^3 + 2x^2 + Ax^2(x-1)^2$$

$$\therefore A = \frac{1}{4}$$

$$\text{从而 } P(x) = \frac{1}{4}x^2(x-3)^2$$

17. 设 $f(x) = \frac{1}{x^2}$, 在 $-5 \leq x \leq 5$ 上取 $n=10$, 按等距节点求分段线性插值函数 $I_h(x)$,

计算各节点间中点处的 $I_h(x)$ 与 $f(x)$ 值，并估计误差。

解：

若 $x_0 = -5, x_{10} = 5$

则步长 $h = 1$,

$x_i = x_0 + ih, i = 0, 1, \dots, 10$

$$f(x) = \frac{1}{1+x^2}$$

在小区间 $[x_i, x_{i+1}]$ 上，分段线性插值函数为

$$\begin{aligned} I_h(x) &= \frac{x-x_{i+1}}{x_i-x_{i+1}} f(x_i) + \frac{x-x_i}{x_{i+1}-x_i} f(x_{i+1}) \\ &= (x_{i+1}-x) \frac{1}{1+x_i^2} + (x-x_i) \frac{1}{1+x_{i+1}^2} \end{aligned}$$

各节点间中点处的 $I_h(x)$ 与 $f(x)$ 的值为

当 $x = \pm 4.5$ 时， $f(x) = 0.0471, I_h(x) = 0.0486$

当 $x = \pm 3.5$ 时， $f(x) = 0.0755, I_h(x) = 0.0794$

当 $x = \pm 2.5$ 时， $f(x) = 0.1379, I_h(x) = 0.1500$

当 $x = \pm 1.5$ 时， $f(x) = 0.3077, I_h(x) = 0.3500$

当 $x = \pm 0.5$ 时， $f(x) = 0.8000, I_h(x) = 0.7500$

误差

$$\max_{x_i \leq x \leq x_{i+1}} |f(x) - I_h(x)| \leq \frac{h^2}{8} \max_{-5 \leq x \leq 5} |f''(\xi)|$$

$$\text{又} \because f(x) = \frac{1}{1+x^2}$$

$$\therefore f'(x) = \frac{-2x}{(1+x^2)^2},$$

$$f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(1+x^2)^4}$$

$$\text{令 } f'''(x) = 0$$

得 $f''(x)$ 的驻点为 $x_{1,2} = \pm 1$ 和 $x_3 = 0$

$$f''(x_{1,2}) = \frac{1}{2}, f''(x_3) = -2$$

$$\therefore \max_{-5 \leq x \leq 5} |f(x) - I_h(x)| \leq \frac{1}{4}$$

18. 求 $f(x) = x^2$ 在 $[a, b]$ 上分段线性插值函数 $I_h(x)$ ，并估计误差。

解：

在区间 $[a, b]$ 上， $x_0 = a, x_n = b, h_i = x_{i+1} - x_i, i = 0, 1, \dots, n-1$,

$$h = \max_{0 \leq i \leq n-1} h_i$$

$$\because f(x) = x^2$$

\therefore 函数 $f(x)$ 在小区间 $[x_i, x_{i+1}]$ 上分段线性插值函数为

$$\begin{aligned} I_h(x) &= \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) \\ &= \frac{1}{h_i} [x_i^2 (x_{i+1} - x) + x_{i+1}^2 (x - x_i)] \end{aligned}$$

误差为

$$\begin{aligned} \max_{x_i \leq x \leq x_{i+1}} |f(x) - I_h(x)| &\leq \frac{1}{8} \max_{a \leq \xi \leq b} |f''(\xi)| h_i^2 \\ \because f(x) &= x^2 \\ \therefore f'(x) &= 2x, f''(x) = 2 \\ \therefore \max_{a \leq x \leq b} |f(x) - I_h(x)| &\leq \frac{h^2}{4} \end{aligned}$$

19. 求 $f(x) = x^4$ 在 $[a, b]$ 上分段埃尔米特插值，并估计误差。

解：

在 $[a, b]$ 区间上， $x_0 = a, x_n = b, h_i = x_{i+1} - x_i, i = 0, 1, \dots, n-1$,

$$\text{令 } h = \max_{0 \leq i \leq n-1} h_i$$

$$\because f(x) = x^4, f'(x) = 4x^3$$

\therefore 函数 $f(x)$ 在区间 $[x_i, x_{i+1}]$ 上的分段埃尔米特插值函数为

$$\begin{aligned}
I_h(x) &= \left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)^2 \left(1+2\frac{x-x_i}{x_{i+1}-x_i}\right) f(x_i) \\
&+ \left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2 \left(1+2\frac{x-x_{i+1}}{x_i-x_{i+1}}\right) f(x_{i+1}) \\
&+ \left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)^2 (x-x_i) f'(x_i) \\
&+ \left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2 (x-x_{i+1}) f'(x_{i+1}) \\
&= \frac{x_i^4}{h_i^3} (x-x_{i+1})^2 (h_i+2x-2x_i) \\
&+ \frac{x_{i+1}^4}{h_i^3} (x-x_i)^2 (h_i-2x+2x_{i+1}) \\
&+ \frac{4x_i^3}{h_i^2} (x-x_{i+1})^2 (x-x_i) \\
&+ \frac{4x_{i+1}^3}{h_i^2} (x-x_i)^2 (x-x_{i+1})
\end{aligned}$$

误差为

$$\begin{aligned}
&|f(x) - I_h(x)| \\
&= \frac{1}{4!} |f^{(4)}(\xi)| (x-x_i)^2 (x-x_{i+1})^2 \\
&\leq \frac{1}{24} \max_{a \leq x \leq b} |f^{(4)}(\xi)| \left(\frac{h_i}{2}\right)^4
\end{aligned}$$

$$\text{又} \because f(x) = x^4$$

$$\therefore f^{(4)}(x) = 4! = 24$$

$$\therefore \max_{a \leq x \leq b} |f(x) - I_h(x)| \leq \max_{0 \leq i \leq n-1} \frac{h_i^4}{16} \leq \frac{h^4}{16}$$

20. 给定数据表如下:

X_j	0.25	0.30	0.39	0.45	0.53
Y_j	0.5000	0.5477	0.6245	0.6708	0.7280

试求三次样条插值, 并满足条件:

$$(1) S'(0.25) = 1.0000, S'(0.53) = 0.6868;$$

$$(2) S''(0.25) = S''(0.53) = 0.$$

解:

$$h_0 = x_1 - x_0 = 0.05$$

$$h_1 = x_2 - x_1 = 0.09$$

$$h_2 = x_3 - x_2 = 0.06$$

$$h_3 = x_4 - x_3 = 0.08$$

$$\because \mu_j = \frac{h_{j-1}}{h_{j-1} - h_j}, \lambda_j = \frac{h_j}{h_{j-1} - h_j}$$

$$\therefore \mu_1 = \frac{5}{14}, \mu_2 = \frac{3}{5}, \mu_3 = \frac{3}{7}, \mu_4 = 1$$

$$\lambda_1 = \frac{9}{14}, \lambda_2 = \frac{2}{5}, \lambda_3 = \frac{4}{7}, \lambda_0 = 1$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 0.9540$$

$$f[x_1, x_2] = 0.8533$$

$$f[x_2, x_3] = 0.7717$$

$$f[x_3, x_4] = 0.7150$$

$$(1) S'(x_0) = 1.0000, S'(x_4) = 0.6868$$

$$d_0 = \frac{6}{h_0} (f[x_1, x_2] - f'_0) = -5.5200$$

$$d_1 = 6 \frac{f[x_1, x_2] - f[x_0, x_1]}{h_0 + h_1} = -4.3157$$

$$d_2 = 6 \frac{f[x_2, x_3] - f[x_1, x_2]}{h_1 + h_2} = -3.2640$$

$$d_3 = 6 \frac{f[x_3, x_4] - f[x_2, x_3]}{h_2 + h_3} = -2.4300$$

$$d_4 = \frac{6}{h_3} (f'_4 - f[x_3, x_4]) = -2.1150$$

由此得矩阵形式的方程组为

$$\begin{pmatrix} 2 & 1 & & & \\ \frac{5}{14} & 2 & \frac{9}{14} & & \\ & \frac{3}{5} & 2 & \frac{2}{5} & \\ & & \frac{3}{7} & 2 & \frac{4}{7} \\ & & & 1 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} -5.5200 \\ -4.3157 \\ -3.2640 \\ -2.4300 \\ -2.1150 \end{pmatrix}$$

求解此方程组得

$$M_0 = -2.0278, M_1 = -1.4643$$

$$M_2 = -1.0313, M_3 = -0.8070, M_4 = -0.6539$$

∴ 三次样条表达式为

$$S(x) = M_j \frac{(x_{j+1} - x)^3}{6h_j} + M_{j+1} \frac{(x - x_j)^3}{6h_j} + (y_j - \frac{M_j h_j^2}{6}) \frac{x_{j+1} - x}{h_j} + (y_{j+1} - \frac{M_{j+1} h_j^2}{6}) \frac{x - x_j}{h_j} (j = 0, 1, \dots, n-1)$$

∴ 将 M_0, M_1, M_2, M_3, M_4 代入得

$$S(x) = \begin{cases} -6.7593(0.30 - x)^3 - 4.8810(x - 0.25)^3 + 10.0169(0.30 - x) + 10.9662(x - 0.25) \\ x \in [0.25, 0.30] \\ -2.7117(0.39 - x)^3 - 1.9098(x - 0.30)^3 + 6.1075(0.39 - x) + 6.9544(x - 0.30) \\ x \in [0.30, 0.39] \\ -2.8647(0.45 - x)^3 - 2.2422(x - 0.39)^3 + 10.4186(0.45 - x) + 10.9662(x - 0.39) \\ x \in [0.39, 0.45] \\ -1.6817(0.53 - x)^3 - 1.3623(x - 0.45)^3 + 8.3958(0.53 - x) + 9.1087(x - 0.45) \\ x \in [0.45, 0.53] \end{cases}$$

$$(2) S''(x_0) = 0, S''(x_4) = 0$$

$$d_0 = 2f_0'' = 0, d_1 = -4.3157, d_2 = -3.2640$$

$$d_3 = -2.4300, d_4 = 2f_4'' = 0$$

$$\lambda_0 = \mu_4 = 0$$

由此得矩阵开工的方程组为

$$M_0 = M_4 = 0$$

$$\begin{pmatrix} 2 & \frac{9}{14} & 0 \\ \frac{3}{5} & 2 & \frac{2}{5} \\ 0 & \frac{3}{7} & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} -4.3157 \\ -3.2640 \\ -2.4300 \end{pmatrix}$$

求解此方程组，得

$$M_0 = 0, M_1 = -1.8809$$

$$M_2 = -0.8616, M_3 = -1.0304, M_4 = 0$$

又∴ 三次样条表达式为

$$S(x) = M_j \frac{(x_{j+1} - x)^3}{6h_j} + M_{j+1} \frac{(x - x_j)^3}{6h_j} \\ + (y_j - \frac{M_j h_j^2}{6}) \frac{x_{j+1} - x}{h_j} + (y_{j+1} - \frac{M_{j+1} h_j^2}{6}) \frac{x - x_j}{h_j}$$

将 M_0, M_1, M_2, M_3, M_4 代入得

$$\therefore S(x) = \begin{cases} -6.2697(x-0.25)^3 + 10(0.3-x) + 10.9697(x-0.25) \\ x \in [0.25, 0.30] \\ -3.4831(0.39-x)^3 - 1.5956(x-0.3)^3 + 6.1138(0.39-x) + 6.9518(x-0.30) \\ x \in [0.30, 0.39] \\ -2.3933(0.45-x)^3 - 2.8622(x-0.39)^3 + 10.4186(0.45-x) + 11.1903(x-0.39) \\ x \in [0.39, 0.45] \\ -2.1467(0.53-x)^3 + 8.3987(0.53-x) + 9.1(x-0.45) \\ x \in [0.45, 0.53] \end{cases}$$

21. 若 $f(x) \in C^2[a, b]$, $S(x)$ 是三次样条函数, 证明:

$$(1) \int_a^b [f''(x)]^2 dx - \int_a^b [S''(x)]^2 dx \\ = \int_a^b [f''(x) - S''(x)]^2 dx + 2 \int_a^b S''(x) [f''(x) - S''(x)]^2 dx$$

(2) 若 $f(x_i) = S(x_i) (i=0, 1, \dots, n)$, 式中 x_i 为插值节点, 且 $a = x_0 < x_1 < \dots < x_n = b$, 则

$$\int_a^b S''(x) [f''(x) - S''(x)] dx \\ = S''(b) [f'(b) - S'(b)] - S''(a) [f'(a) - S'(a)]$$

证明:

$$(1) \int_a^b [f''(x) - S''(x)]^2 dx \\ = \int_a^b [f''(x)]^2 dx + \int_a^b [S''(x)]^2 dx - 2 \int_a^b f''(x) S''(x) dx \\ = \int_a^b [f''(x)]^2 dx - \int_a^b [S''(x)]^2 dx - 2 \int_a^b S''(x) [f''(x) - S''(x)] dx$$

从而有

$$\int_a^b [f''(x)]^2 dx - \int_a^b [S''(x)]^2 dx \\ = \int_a^b [f''(x) - S''(x)]^2 dx + 2 \int_a^b S''(x) [f''(x) - S''(x)] dx$$

$$\begin{aligned}
& (2) \int_a^b S''(x) [f''(x) - S''(x)] dx \\
&= \int_a^b S''(x) d[f'(x) - S'(x)] \\
&= S''(x) [f'(x) - S'(x)] \Big|_a^b - \int_a^b [f'(x) - S'(x)] d[S''(x)] \\
&= S''(b) [f'(b) - S'(b)] - S''(a) [f'(a) - S'(a)] - \int_a^b S'''(x) [f'(x) - S'(x)] dx \\
&= S''(b) [f'(b) - S'(b)] - S''(a) [f'(a) - S'(a)] - \sum_{k=0}^{n-1} S'''(\frac{x_k + x_{k+1}}{2}) \int_{x_k}^{x_{k+1}} [f'(x) - S'(x)] dx \\
&= S''(b) [f'(b) - S'(b)] - S''(a) [f'(a) - S'(a)] - \sum_{k=0}^{n-1} S'''(\frac{x_k + x_{k+1}}{2}) [f'(x) - S'(x)] \Big|_{x_k}^{x_{k+1}} \\
&= S''(b) [f'(b) - S'(b)] - S''(a) [f'(a) - S'(a)]
\end{aligned}$$

第三章 函数逼近与曲线拟合

1. $f(x) = \sin \frac{\pi}{2} x$, 给出 $[0,1]$ 上的伯恩斯坦多项式 $B_1(f, x)$ 及 $B_3(f, x)$ 。

解:

$$\because f(x) = \sin \frac{\pi}{2}, x \in [0,1]$$

伯恩斯坦多项式为

$$B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) P_k(x)$$

$$\text{其中 } P_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

当 $n=1$ 时,

$$P_0(x) = \binom{1}{0} (1-x)$$

$$P_1(x) = x$$

$$\therefore B_1(f, x) = f(0)P_0(x) + f(1)P_1(x)$$

$$= \binom{1}{0} (1-x) \sin\left(\frac{\pi}{2} \times 0\right) + x \sin \frac{\pi}{2}$$

$$= x$$

当 $n=3$ 时,

$$P_0(x) = \binom{3}{0} (1-x)^3$$

$$P_1(x) = \binom{3}{1} x(1-x)^2 = 3x(1-x)^2$$

$$P_2(x) = \binom{3}{1} x^2(1-x) = 3x^2(1-x)$$

$$P_3(x) = \binom{3}{3} x^3 = x^3$$

$$\begin{aligned}
\therefore B_3(f, x) &= \sum_{k=0}^3 f\left(\frac{k}{n}\right) P_k(x) \\
&= 0 + 3x(1-x)^2 \left[\sin \frac{\pi}{6} + 3x^2(1-x) \left[\sin \frac{\pi}{3} + x^3 \sin \frac{\pi}{2} \right] \right] \\
&= \frac{3}{2} x(1-x)^2 + \frac{3\sqrt{3}}{2} x^2(1-x) + x^3 \\
&= \frac{5-3\sqrt{3}}{2} x^3 + \frac{3\sqrt{3}-6}{2} x^2 + \frac{3}{2} x \\
&\approx 1.5x - 0.402x^2 - 0.098x^3
\end{aligned}$$

2. 当 $f(x) = x$ 时, 求证 $B_n(f, x) = x$

证明:

若 $f(x) = x$, 则

$$\begin{aligned}
B_n(f, x) &= \sum_{k=0}^n f\left(\frac{k}{n}\right) P_k(x) \\
&= \sum_{k=0}^n \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^n \frac{k}{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^n \frac{(n-1)\cdots[(n-1)-(k-1)+1]}{(k-1)!} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^n \binom{n-1}{k-1} x^k (1-x)^{n-k} \\
&= x \sum_{k=1}^n \binom{n-1}{k-1} x^{k-1} (1-x)^{(n-1)-(k-1)} \\
&= x [x + (1-x)]^{n-1} \\
&= x
\end{aligned}$$

3. 证明函数 $1, x, \dots, x^n$ 线性无关

证明:

若 $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0, \forall x \in R$

分别取 $x^k (k=0, 1, 2, \dots, n)$, 对上式两端在 $[0, 1]$ 上作带权 $\rho(x) \equiv 1$ 的内积, 得

$$\begin{pmatrix} 1 & \cdots & \frac{1}{n+1} \\ \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \cdots & \frac{1}{2n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

∴ 此方程组的系数矩阵为希尔伯特矩阵，对称正定非奇异，

∴ 只有零解 $a=0$ 。

∴ 函数 $1, x, \cdots, x^n$ 线性无关。

4. 计算下列函数 $f(x)$ 关于 $C[0,1]$ 的 $\|f\|_\infty, \|f\|_1$ 与 $\|f\|_2$:

$$(1) f(x) = (x-1)^3, x \in [0,1]$$

$$(2) f(x) = \left| x - \frac{1}{2} \right|,$$

$$(3) f(x) = x^m(1-x)^n, m \text{ 与 } n \text{ 为正整数},$$

$$(4) f(x) = (x+1)^{10} e^{-x}$$

解:

$$(1) \text{ 若 } f(x) = (x-1)^3, x \in [0,1], \text{ 则}$$

$$f'(x) = 3(x-1)^2 \geq 0$$

∴ $f(x) = (x-1)^3$ 在 $(0,1)$ 内单调递增

$$\begin{aligned} \|f\|_\infty &= \max_{0 \leq x \leq 1} |f(x)| \\ &= \max \{|f(0)|, |f(1)|\} \\ &= \max \{0, 1\} = 1 \end{aligned}$$

$$\begin{aligned} \|f\|_\infty &= \max_{0 \leq x \leq 1} |f(x)| \\ &= \max \{|f(0)|, |f(1)|\} \\ &= \max \{0, 1\} = 1 \end{aligned}$$

$$\begin{aligned} \|f\|_2 &= \left(\int_0^1 (1-x)^6 dx \right)^{\frac{1}{2}} \\ &= \left[\frac{1}{7} (1-x)^7 \right]_0^1^{\frac{1}{2}} \\ &= \frac{\sqrt{7}}{7} \end{aligned}$$

(2) 若 $f(x) = \left|x - \frac{1}{2}\right|, x \in [0, 1]$, 则

$$\|f\|_{\infty} = \max_{0 \leq x \leq 1} |f(x)| = \frac{1}{2}$$

$$\|f\|_1 = \int_0^1 |f(x)| dx$$

$$= 2 \int_{\frac{1}{2}}^1 (x - \frac{1}{2}) dx$$

$$= \frac{1}{4}$$

$$\|f\|_2 = \left(\int_0^1 f^2(x) dx\right)^{\frac{1}{2}}$$

$$= \left[\int_0^1 (x - \frac{1}{2})^2 dx\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{6}$$

(3) 若 $f(x) = x^m(1-x)^n$, m 与 n 为正整数

当 $x \in [0, 1]$ 时, $f(x) \geq 0$

$$f'(x) = mx^{m-1}(1-x)^n + x^m n(1-x)^{n-1}(-1)$$

$$= x^{m-1}(1-x)^{n-1} m(1 - \frac{n+m}{m}x)$$

当 $x \in (0, \frac{m}{n+m})$ 时, $f'(x) > 0$

$\therefore f(x)$ 在 $(0, \frac{m}{n+m})$ 内单调递增

当 $x \in (\frac{m}{n+m}, 1)$ 时, $f'(x) < 0$

$\therefore f(x)$ 在 $(\frac{m}{n+m}, 1)$ 内单调递减。

$$x \in (\frac{m}{n+m}, 1) f'(x) < 0$$

$$\|f\|_{\infty} = \max_{0 \leq x \leq 1} |f(x)| =$$

$$= \max \left\{ \left|f(0)\right|, \left|f\left(\frac{m}{n+m}\right)\right| \right\}$$

$$= \frac{m^m n^n}{(m+n)^{m+n}}$$

$$\begin{aligned}
\|f\|_1 &= \int_0^1 |f(x)| dx \\
&= \int_0^1 x^m (1-x)^n dx \\
&= \int_0^{\frac{\pi}{2}} (\sin^2 t)^m (1 - \sin^2 t)^n d \sin^2 t \\
&= \int_0^{\frac{\pi}{2}} \sin^{2m} t \cos^{2n} t \cos t \cdot 2 \sin t dt \\
&= \frac{n!m!}{(n+m+1)!}
\end{aligned}$$

$$\begin{aligned}
\|f\|_2 &= \left[\int_0^1 x^{2m} (1-x)^{2n} dx \right]^{\frac{1}{2}} \\
&= \left[\int_0^{\frac{\pi}{2}} \sin^{4m} t \cos^{4n} t d(\sin^2 t) \right]^{\frac{1}{2}} \\
&= \left[\int_0^{\frac{\pi}{2}} 2 \sin^{4m+1} t \cos^{4n+1} t dt \right]^{\frac{1}{2}} \\
&= \sqrt{\frac{(2n)!(2m)!}{[2(n+m)+1]!}}
\end{aligned}$$

(4) 若 $f(x) = (x+1)^{10} e^{-x}$

当 $x \in [0, 1]$ 时, $f(x) > 0$

$$\begin{aligned}
f'(x) &= 10(x+1)^9 e^{-x} + (x+1)^{10} (-e^{-x}) \\
&= (x+1)^9 e^{-x} (9-x) \\
&> 0
\end{aligned}$$

$\therefore f(x)$ 在 $[0, 1]$ 内单调递增。

$$\begin{aligned}
\|f\|_{\infty} &= \max_{0 \leq x \leq 1} |f(x)| = \\
&= \max\{|f(0)|, |f(1)|\} \\
&= \frac{2^{10}}{e} \\
\|f\|_1 &= \int_0^1 |f(x)| dx \\
&= \int_0^1 (x+1)^{10} e^{-x} dx \\
&= -(x+1)^{10} e^{-x} \Big|_0^1 + \int_0^1 10(x+1)^9 e^{-x} dx \\
&= 5 - \frac{10}{e} \\
\|f\|_2 &= \left[\int_0^1 (x+1)^{20} e^{-2x} dx \right]^{\frac{1}{2}} \\
&= 7 \left(\frac{3}{4} - \frac{4}{e^2} \right)
\end{aligned}$$

5. 证明 $\|f - g\| \geq \|f\| - \|g\|$

证明:

$$\begin{aligned}
&\|f\| \\
&= \|(f - g) + g\| \\
&\leq \|f - g\| + \|g\| \\
&\therefore \|f - g\| \geq \|f\| - \|g\|
\end{aligned}$$

6. 对 $f(x), g(x) \in C^1[a, b]$, 定义

$$\begin{aligned}
(1) (f, g) &= \int_a^b f'(x) g'(x) dx \\
(2) (f, g) &= \int_a^b f'(x) g'(x) dx + f(a) g(a)
\end{aligned}$$

问它们是否构成内积。

解:

(1) 令 $f(x) \equiv C$ (C 为常数, 且 $C \neq 0$)

则 $f'(x) = 0$

$$\text{而 } (f, f) = \int_a^b f'(x) f'(x) dx$$

这与当且仅当 $f \equiv 0$ 时, $(f, f) = 0$ 矛盾

\therefore 不能构成 $C^1[a, b]$ 上的内积。

(2) 若 $(f, g) = \int_a^b f'(x)g'(x)dx + f(a)g(a)$, 则

$$(g, f) = \int_a^b g'(x)f'(x)dx + g(a)f(a) = (f, g), \forall \alpha \in K$$

$$(\alpha f, g) = \int_a^b [\alpha f'(x)]'g'(x)dx + \alpha f(a)g(a)$$

$$= \alpha \left[\int_a^b f'(x)g'(x)dx + f(a)g(a) \right]$$

$$= \alpha(f, g)$$

$\forall h \in C^1[a, b]$, 则

$$(f + g, h) = \int_a^b [f'(x) + g'(x)]'h'(x)dx + [f(a)g(a)]h(a)$$

$$= \int_a^b f'(x)h'(x)dx + f(a)h(a) + \int_a^b g'(x)h'(x)dx + g(a)h(a)$$

$$= (f, h) + (h, g)$$

$$(f, f) = \int_a^b [f'(x)]^2 dx + f^2(a) \geq 0$$

若 $(f, f) = 0$, 则

$$\int_a^b [f'(x)]^2 dx = 0, \text{ 且 } f^2(a) = 0$$

$$\therefore f'(x) \equiv 0, f(a) = 0$$

$$\therefore f(x) \equiv 0$$

即当且仅当 $f = 0$ 时, $(f, f) = 0$.

故可以构成 $C^1[a, b]$ 上的内积。

7. 令 $T_n^*(x) = T_n(2x-1), x \in [0, 1]$, 试证 $\{T_n^*(x)\}$ 是在 $[0, 1]$ 上带权 $\rho(x) = \frac{1}{\sqrt{x-x^2}}$ 的正交多

项式, 并求 $T_0^*(x), T_1^*(x), T_2^*(x), T_3^*(x)$ 。

解:

若 $T_n^*(x) = T_n(2x-1), x \in [0, 1]$, 则

$$\int_0^1 T_n^*(x)T_m^*(x)\rho(x)dx$$

$$= \int_0^1 T_n(2x-1)T_m(2x-1)\frac{1}{\sqrt{x-x^2}}dx$$

令 $t = (2x-1)$, 则 $t \in [-1, 1]$, 且 $x = \frac{t+1}{2}$, 故

$$\begin{aligned}
& \int_0^1 T_n^*(x) T_m^*(x) \rho(x) dx \\
&= \int_{-1}^1 T_n(t) T_m(t) \frac{1}{\sqrt{\frac{t+1}{2} - (\frac{t+1}{2})^2}} d(\frac{t+1}{2}) \\
&= \int_{-1}^1 T_n(t) T_m(t) \frac{1}{\sqrt{1-t^2}} dt
\end{aligned}$$

又 \because 切比雪夫多项式 $\{T_k^*(x)\}$ 在区间 $[0,1]$ 上带权 $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ 正交, 且

$$\int_{-1}^1 T_n(x) T_m(x) d \frac{x}{\sqrt{1-t^2}} = \begin{cases} 0, n \neq m \\ \frac{\pi}{2}, n = m \neq 0 \\ \pi, n = m = 0 \end{cases}$$

$\therefore \{T_n^*(x)\}$ 是在 $[0,1]$ 上带权 $\rho(x) = \frac{1}{\sqrt{x-x^2}}$ 的正交多项式。

又 $\because T_0(x) = 1, x \in [-1,1]$

$$\therefore T_0^*(x) = T_0(2x-1) = 1, x \in [0,1]$$

$$\because T_1(x) = x, x \in [-1,1]$$

$$\therefore T_1^*(x) = T_1(2x-1) = 2x-1, x \in [0,1]$$

$$\because T_2(x) = 2x^2 - 1, x \in [-1,1]$$

$$\therefore T_2^*(x) = T_2(2x-1)$$

$$= 2(2x-1)^2 - 1$$

$$= 8x^2 - 8x - 1, x \in [0,1]$$

$$\because T_3(x) = 4x^3 - 3x, x \in [-1,1]$$

$$\therefore T_3^*(x) = T_3(2x-1)$$

$$= 4(2x-1)^3 - 3(2x-1)$$

$$= 32x^3 - 48x^2 + 18x - 1, x \in [0,1]$$

8. 对权函数 $\rho(x) = 1-x^2$, 区间 $[-1,1]$, 试求首项系数为 1 的正交多项式 $\varphi_n(x), n=0,1,2,3$.

解:

若 $\rho(x) = 1-x^2$, 则区间 $[-1,1]$ 上内积为

$$(f, g) = \int_{-1}^1 f(x) g(x) \rho(x) dx$$

定义 $\varphi_0(x)=1$ ， 则

$$\varphi_{n+1}(x)=(x-\alpha_n)\varphi_n(x)-\beta_n\varphi_{n-1}(x)$$

其中

$$\alpha_n=(x\varphi_n(x),\varphi_n(x))/(\varphi_n(x),\varphi_n(x))$$

$$\beta_n=(\varphi_n(x),\varphi_n(x))/(\varphi_{n-1}(x),\varphi_{n-1}(x))$$

$$\therefore \alpha_0=(x,1)/(1,1)$$

$$=\frac{\int_{-1}^1 x(1+x^2)dx}{\int_{-1}^1 (1+x^2)dx}$$

$$=0$$

$$\therefore \varphi_1(x)=x$$

$$\alpha_1=(x^2,x)/(x,x)$$

$$=\frac{\int_{-1}^1 x^3(1+x^2)dx}{\int_{-1}^1 x^2(1+x^2)dx}$$

$$=0$$

$$\beta_1=(x,x)/(1,1)$$

$$=\frac{\int_{-1}^1 x^2(1+x^2)dx}{\int_{-1}^1 (1+x^2)dx}$$

$$=\frac{\frac{16}{15}}{\frac{8}{3}}=\frac{2}{5}$$

$$\therefore \varphi_2(x)=x^2-\frac{2}{5}$$

$$\alpha_2 = (x^3 - \frac{2}{5}x, x^2 - \frac{2}{5}) / (x^2 - \frac{2}{5}, x^2 - \frac{2}{5})$$

$$= \frac{\int_{-1}^1 (x^3 - \frac{2}{5}x)(x^2 - \frac{2}{5})(1+x^2)dx}{\int_{-1}^1 (x^2 - \frac{2}{5})(x^2 - \frac{2}{5})(1+x^2)dx}$$

$$= 0$$

$$\beta_2 = (x^2 - \frac{2}{5}, x^2 - \frac{2}{5}) / (x, x)$$

$$= \frac{\int_{-1}^1 (x^2 - \frac{2}{5})(x^2 - \frac{2}{5})(1+x^2)dx}{\int_{-1}^1 x^2(1+x^2)dx}$$

$$= \frac{\frac{136}{525}}{\frac{16}{15}} = \frac{17}{70}$$

$$\therefore \varphi_3(x) = x^3 - \frac{2}{5}x^2 - \frac{17}{70}x = x^3 - \frac{9}{14}x$$

9. 试证明由教材式 (2.14) 给出的第二类切比雪夫多项式族 $\{u_n(x)\}$ 是 $[0,1]$ 上带权

$\rho(x) = \sqrt{1-x^2}$ 的正交多项式。

证明:

$$\text{若 } U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$$

令 $x = \cos \theta$, 可得

$$\int_{-1}^1 U_m(x)U_n(x)\sqrt{1-x^2}dx = \int_{-1}^1 \frac{\sin[(m+1)\arccos x]\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}dx$$

$$= \int_{\pi}^0 \frac{\sin[(m+1)\theta]\sin[(n+1)\theta]}{\sqrt{1-\cos^2 \theta}}d\theta$$

$$= \int_0^{\pi} \sin[(m+1)\theta]\sin[(n+1)\theta]d\theta$$

当 $m = n$ 时,

$$\int_0^{\pi} \sin^2[(m+1)\theta]d\theta = \int_0^{\pi} \frac{1-\cos[2(m+1)\theta]}{2}d\theta$$

$$= \frac{\pi}{2}$$

当 $m \neq n$ 时,

$$\begin{aligned}
& \int_0^\pi \sin[(m+1)\theta] \sin[(n+1)\theta] d\theta \\
&= \int_0^\pi \sin[(m+1)\theta] d\left\{ \frac{1}{n+1} \cos(n+1)\theta \right\} \\
&= \int_0^\pi \frac{1}{n+1} \cos(n+1)\theta d\{\sin[(m+1)\theta]\} \\
&= \int_0^\pi -\frac{m+1}{n+1} \cos(n+1)\theta \cos(m+1)\theta d\theta \\
&= -\int_0^\pi \frac{m+1}{n+1} \cos[(m+1)\theta] d\left\{ \frac{1}{n+1} \sin[(n+1)\theta] \right\} \\
&= -\int_0^\pi \frac{m+1}{(n+1)^2} \sin[(n+1)\theta] d\{\cos[(m+1)\theta]\} \\
&= \int_0^\pi \left(\frac{m+1}{n+1}\right)^2 \sin[(n+1)\theta] \sin[(m+1)\theta] d\theta \\
&= 0 \\
&\therefore [1 - \left(\frac{m+1}{n+1}\right)^2] \int_0^\pi \sin[(n+1)\theta] \sin[(m+1)\theta] d\theta = 0
\end{aligned}$$

又 $\because m \neq n$, 故 $\left(\frac{m+1}{n+1}\right)^2 \neq 1$

$$\therefore \int_0^\pi \sin[(n+1)\theta] \sin[(m+1)\theta] d\theta = 0$$

得证。

10. 证明切比雪夫多项式 $T_n(x)$ 满足微分方程

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$$

证明:

切比雪夫多项式为

$$T_n(x) = \cos(n \arccos x), |x| \leq 1$$

从而有

$$\begin{aligned}
T'_n(x) &= -\sin(n \arccos x) \cdot n \cdot \left(\frac{-1}{\sqrt{1-x^2}} \right) \\
&= \frac{n}{\sqrt{1-x^2}} \sin(n \arccos x) \\
T''_n(x) &= \frac{n}{(1-x^2)^{\frac{3}{2}}} \sin(n \arccos x) - \frac{n^2}{1-x^2} \cos(n \arccos x) \\
\therefore (1-x^2)T''_n(x) - xT'_n(x) + n^2T_n(x) \\
&= \frac{nx}{\sqrt{1-x^2}} \sin(n \arccos x) - n^2 \cos(n \arccos x) \\
&\quad - \frac{nx}{\sqrt{1-x^2}} \sin(n \arccos x) + n^2 \cos(n \arccos x) \\
&= 0
\end{aligned}$$

得证。

11. 假设 $f(x)$ 在 $[a, b]$ 上连续, 求 $f(x)$ 的零次最佳一致逼近多项式?

解:

$\because f(x)$ 在闭区间 $[a, b]$ 上连续

\therefore 存在 $x_1, x_2 \in [a, b]$, 使

$$f(x_1) = \min_{a \leq x \leq b} f(x),$$

$$f(x_2) = \max_{a \leq x \leq b} f(x),$$

$$\text{取 } P = \frac{1}{2}[f(x_1) + f(x_2)]$$

则 x_1 和 x_2 是 $[a, b]$ 上的 2 个轮流为“正”、“负”的偏差点。

由切比雪夫定理知

P 为 $f(x)$ 的零次最佳一致逼近多项式。

12. 选取常数 a , 使 $\max_{0 \leq x \leq 1} |x^3 - ax|$ 达到极小, 又问这个解是否唯一?

解:

$$\text{令 } f(x) = x^3 - ax$$

则 $f(x)$ 在 $[-1, 1]$ 上为奇函数

$$\therefore \max_{0 \leq x \leq 1} |x^3 - ax|$$

$$= \max_{-1 \leq x \leq 1} |x^3 - ax|$$

$$= \|f\|_{\infty}$$

又 $\because f(x)$ 的最高次项系数为1, 且为3次多项式。

$\therefore \omega_3(x) = \frac{1}{2^3}T_3(x)$ 与0的偏差最小。

$$\omega_3(x) = \frac{1}{4}T_3(x) = x^3 - \frac{3}{4}x$$

$$\text{从而有 } a = \frac{3}{4}$$

13. 求 $f(x) = \sin x$ 在 $[0, \frac{\pi}{2}]$ 上的最佳一次逼近多项式, 并估计误差。

解:

$$\because f(x) = \sin x, x \in [0, \frac{\pi}{2}]$$

$$f'(x) = \cos x, f''(x) = -\sin x \leq 0$$

$$a_1 = \frac{f(b) - f(a)}{b - a} = \frac{2}{\pi},$$

$$\cos x_2 = \frac{2}{\pi},$$

$$\therefore x_2 = \arccos \frac{2}{\pi} \approx 0.88069$$

$$f(x_2) = 0.77118$$

$$a_0 = \frac{f(a) + f(x_2)}{2} - \frac{f(b) - f(a)}{b - a} \cdot \frac{a + x_2}{2}$$

$$= 0.10526$$

于是得 $f(x)$ 的最佳一次逼近多项式为

$$P_1(x) = 0.10526 + \frac{2}{\pi}x$$

即

$$\sin x \approx 0.10526 + \frac{2}{\pi}x, 0 \leq x \leq \frac{\pi}{2}$$

误差限为

$$\|\sin x - P_1(x)\|_{\infty}$$

$$= |\sin 0 - P_1(0)|$$

$$= 0.10526$$

14. 求 $f(x) = e^x$ 在 $[0, 1]$ 上的最佳一次逼近多项式。

解:

$$\because f(x) = e^x, x \in [0, 1]$$

$$\therefore f'(x) = e^x,$$

$$f''(x) = e^x > 0$$

$$a_1 = \frac{f(b) - f(a)}{b - a} = e - 1$$

$$e^{x_2} = e - 1$$

$$x_2 = \ln(e - 1)$$

$$f(x_2) = e^{x_2} = e - 1$$

$$a_0 = \frac{f(a) + f(x_2)}{2} - \frac{f(b) - f(a)}{b - a} \cdot \frac{a + x_2}{2}$$

$$= \frac{1 + (e - 1)}{2} - (e - 1) \frac{\ln(e - 1)}{2}$$

$$= \frac{1}{2} \ln(e - 1)$$

于是得 $f(x)$ 的最佳一次逼近多项式为

$$P_1(x) = \frac{e}{2} + (e - 1) \left[x - \frac{1}{2} \ln(e - 1) \right]$$

$$= (e - 1)x + \frac{1}{2} [e - (e - 1) \ln(e - 1)]$$

15. 求 $f(x) = x^4 + 3x^3 - 1$ 在区间 $[0, 1]$ 上的三次最佳一致逼近多项式。

解:

$$\because f(x) = x^4 + 3x^3 - 1, x \in [0, 1]$$

$$\text{令 } t = 2\left(x - \frac{1}{2}\right), \text{ 则 } t \in [-1, 1]$$

$$\text{且 } x = \frac{1}{2}t + \frac{1}{2}$$

$$\therefore f(t) = \left(\frac{1}{2}t + \frac{1}{2}\right)^4 + 3\left(\frac{1}{2}t + \frac{1}{2}\right)^3 - 1$$

$$= \frac{1}{16}(t^4 + 10t^3 + 24t^2 + 22t - 9)$$

$$\text{令 } g(t) = 16f(t), \text{ 则 } g(t) = t^4 + 10t^3 + 24t^2 + 22t - 9$$

若 $g(t)$ 为区间 $[-1, 1]$ 上的最佳三次逼近多项式 $P_3^*(t)$ 应满足

$$\max_{-1 \leq t \leq 1} |g(t) - P_3^*(t)| = \min$$

$$\text{当 } g(t) - P_3^*(t) = \frac{1}{2^3} T_4(t) = \frac{1}{8}(8t^4 - 8t^2 + 1)$$

时, 多项式 $g(t) - P_3^*(t)$ 与零偏差最小, 故

$$\begin{aligned} {}_3^*(t) &= g(t) - \frac{1}{2^3} T_4(t) \\ &= 10t^3 + 25t^2 + 22t - \frac{73}{8} \end{aligned}$$

进而, $f(x)$ 的三次最佳一致逼近多项式为 $\frac{1}{16} P_3^*(t)$, 则 $f(x)$ 的三次最佳一致逼近多项式为

$$\begin{aligned} P_3^*(t) &= \frac{1}{16} [10(2x-1)^3 + 25(2x-1)^2 + 22(2x-1) - \frac{73}{8}] \\ &= 5x^3 - \frac{5}{4}x^2 + \frac{1}{4}x - \frac{129}{128} \end{aligned}$$

16. $f(x) = |x|$, 在 $[-1, 1]$ 上求关于 $\Phi = \text{span}\{1, x^2, x^4\}$ 的最佳平方逼近多项式。

解:

$$\because f(x) = |x|, x \in [-1, 1]$$

$$\text{若 } (f, g) = \int_{-1}^1 f(x)g(x)dx$$

且 $\varphi_0 = 1, \varphi_1 = x^2, \varphi_2 = x^4$, 则

$$\|\varphi_0\|_2^2 = 2, \|\varphi_1\|_2^2 = \frac{2}{5}, \|\varphi_2\|_2^2 = \frac{2}{9},$$

$$(f, \varphi_0) = 1, (f, \varphi_1) = \frac{1}{2}, (f, \varphi_2) = \frac{1}{3},$$

$$(\varphi_0, \varphi_1) = 1, (\varphi_0, \varphi_2) = \frac{2}{5}, (\varphi_1, \varphi_2) = \frac{2}{7},$$

则法方程组为

$$\begin{pmatrix} 2 & \frac{2}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{2}{5} & \frac{2}{7} \\ \frac{2}{5} & \frac{2}{7} & \frac{2}{9} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$$

解得

$$\begin{cases} a_0 = 0.1171875 \\ a_1 = 1.640625 \\ a_2 = -0.8203125 \end{cases}$$

故 $f(x)$ 关于 $\Phi = \text{span}\{1, x^2, x^4\}$ 的最佳平方逼近多项式为

$$\begin{aligned} S^*(x) &= a_0 + a_1 x^2 + a_2 x^4 \\ &= 0.1171875 + 1.640625x^2 - 0.8203125x^4 \end{aligned}$$

17. 求函数 $f(x)$ 在指定区间上对于 $\Phi = \text{span}\{1, x\}$ 的最佳逼近多项式:

$$(1) f(x) = \frac{1}{x}, [1, 3]; (2) f(x) = e^x, [0, 1];$$

$$(3) f(x) = \cos \pi x, [0, 1]; (4) f(x) = \ln x, [1, 2];$$

解:

$$(1) \because f(x) = \frac{1}{x}, [1, 3];$$

$$\text{若 } (f, g) = \int_1^3 f(x)g(x)dx$$

且 $\varphi_0 = 1, \varphi_1 = x$, , 则有

$$\|\varphi_0\|_2^2 = 2, \|\varphi_1\|_2^2 = \frac{26}{3},$$

$$(\varphi_0, \varphi_1) = 4,$$

$$(f, \varphi_0) = \ln 3, (f, \varphi_1) = 2,$$

则法方程组为

$$\begin{pmatrix} 2 & 4 \\ 4 & \frac{26}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \ln 3 \\ 2 \end{pmatrix}$$

从而解得

$$\begin{cases} a_0 = 1.1410 \\ a_1 = -0.2958 \end{cases}$$

故 $f(x)$ 关于 $\Phi = \text{span}\{1, x\}$ 的最佳平方逼近多项式为

$$\begin{aligned} S^*(x) &= a_0 + a_1 x \\ &= 1.1410 - 0.2958x \end{aligned}$$

$$(2) \because f(x) = e^x, [0, 1]$$

$$\text{若 } (f, g) = \int_0^1 f(x)g(x)dx$$

且 $\varphi_0 = 1, \varphi_1 = x$, , 则有

$$\|\varphi_0\|_2^2 = 1, \|\varphi_1\|_2^2 = \frac{1}{3},$$

$$(\varphi_0, \varphi_1) = \frac{1}{2},$$

$$(f, \varphi_0) = e - 1, (f, \varphi_1) = 1,$$

则法方程组为

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} e-1 \\ 1 \end{pmatrix}$$

从而解得

$$\begin{cases} a_0 = 0.1878 \\ a_1 = 1.6244 \end{cases}$$

故 $f(x)$ 关于 $\Phi = \text{span}\{1, x\}$ 的最佳平方逼近多项式为

$$\begin{aligned} S^*(x) &= a_0 + a_1 x \\ &= 0.1878 + 1.6244x \end{aligned}$$

$$(3) \because f(x) = \cos \pi x, x \in [0, 1]$$

$$\text{若 } (f, g) = \int_0^1 f(x)g(x)dx$$

且 $\varphi_0 = 1, \varphi_1 = x$, , 则有

$$\|\varphi_0\|_2^2 = 1, \|\varphi_1\|_2^2 = \frac{1}{3},$$

$$(\varphi_0, \varphi_1) = \frac{1}{2},$$

$$(f, \varphi_0) = 0, (f, \varphi_1) = -\frac{2}{\pi^2},$$

则法方程组为

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{\pi^2} \end{pmatrix}$$

从而解得

$$\begin{cases} a_0 = 1.2159 \\ a_1 = -0.24317 \end{cases}$$

故 $f(x)$ 关于 $\Phi = \text{span}\{1, x\}$ 的最佳平方逼近多项式为

$$\begin{aligned} S^*(x) &= a_0 + a_1 x \\ &= 1.2159 - 0.24317x \end{aligned}$$

$$(4) \because f(x) = \ln x, x \in [1, 2]$$

$$\text{若 } (f, g) = \int_1^2 f(x)g(x)dx$$

且 $\varphi_0 = 1, \varphi_1 = x$, 则有

$$\|\varphi_0\|_2^2 = 1, \|\varphi_1\|_2^2 = \frac{7}{3},$$

$$(\varphi_0, \varphi_1) = \frac{3}{2},$$

$$(f, \varphi_0) = 2\ln 2 - 1, (f, \varphi_1) = 2\ln 2 - \frac{3}{4},$$

则法方程组为

$$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{7}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 2\ln 2 - 1 \\ 2\ln 2 - \frac{3}{4} \end{pmatrix}$$

从而解得

$$\begin{cases} a_0 = -0.6371 \\ a_1 = 0.6822 \end{cases}$$

故 $f(x)$ 关于 $\Phi = \text{span}\{1, x\}$ 最佳平方逼近多项式为

$$\begin{aligned} S^*(x) &= a_0 + a_1 x \\ &= -0.6371 + 0.6822x \end{aligned}$$

18. $f(x) = \sin \frac{\pi}{2} x$, 在 $[-1, 1]$ 上按勒让德多项式展开求三次最佳平方逼近多项式。

解:

$$\because f(x) = \sin \frac{\pi}{2} x, x \in [-1, 1]$$

按勒让德多项式 $\{P_0(x), P_1(x), P_2(x), P_3(x)\}$ 展开

$$(f(x), P_0(x)) = \int_{-1}^1 \sin \frac{\pi}{2} x dx = \frac{2}{\pi} \cos \frac{\pi}{2} x \Big|_1^{-1} = 0$$

$$(f(x), P_1(x)) = \int_{-1}^1 x \sin \frac{\pi}{2} x dx = \frac{8}{\pi^2}$$

$$(f(x), P_2(x)) = \int_{-1}^1 \left(\frac{3}{2}x^2 - \frac{1}{2}\right) \sin \frac{\pi}{2} x dx = 0$$

$$(f(x), P_3(x)) = \int_{-1}^1 \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \sin \frac{\pi}{2} x dx = \frac{48(\pi^2 - 10)}{\pi^4}$$

则

$$a_0^* = (f(x), P_0(x)) / 2 = 0$$

$$a_1^* = 3(f(x), P_1(x)) / 2 = \frac{12}{\pi^2}$$

$$a_2^* = 5(f(x), P_2(x)) / 2 = 0$$

$$a_3^* = 7(f(x), P_3(x)) / 2 = \frac{168(\pi^2 - 10)}{\pi^4}$$

从而 $f(x)$ 的三次最佳平方逼近多项式为

$$S_3^*(x) = a_0^* P_0(x) + a_1^* P_1(x) + a_2^* P_2(x) + a_3^* P_3(x)$$

$$= \frac{12}{\pi^2} x + \frac{168(\pi^2 - 10)}{\pi^4} \left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$$

$$= \frac{420(\pi^2 - 10)}{\pi^4} x^3 + \frac{120(21 - 2\pi^2)}{\pi^4}$$

$$\approx 1.5531913x - 0.5622285x^3$$

19. 观测物体的直线运动，得出以下数据：

时间 t(s)	0	0.9	1.9	3.0	3.9	5.0
距离 s(m)	0	10	30	50	80	110

求运动方程。

解：

被观测物体的运动距离与运动时间大体为线性函数关系，从而选择线性方程

$$s = a + bt$$

$$\text{令 } \Phi = \text{span}\{1, t\}$$

则

$$\|\varphi_0\|_2^2 = 6, \|\varphi_1\|_2^2 = 53.63,$$

$$(\varphi_0, \varphi_1) = 14.7,$$

$$(\varphi_0, s) = 280, (\varphi_1, s) = 1078,$$

则法方程组为

$$\begin{pmatrix} 6 & 14.7 \\ 14.7 & 53.63 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 280 \\ 1078 \end{pmatrix}$$

从而解得

$$\begin{cases} a = -7.855048 \\ b = 22.25376 \end{cases}$$

故物体运动方程为

$$S = 22.25376t - 7.855048$$

20. 已知实验数据如下:

x_i	19	25	31	38	44
y_j	19.0	32.3	49.0	73.3	97.8

用最小二乘法求形如 $s = a + bx^2$ 的经验公式, 并计算均方误差。

解:

若 $s = a + bx^2$, 则

$$\Phi = \text{span}\{1, x^2\}$$

则

$$\|\varphi_0\|_2^2 = 5, \|\varphi_1\|_2^2 = 7277699,$$

$$(\varphi_0, \varphi_1) = 5327,$$

$$(f, \varphi_0) = 271.4, (f, \varphi_1) = 369321.5,$$

则法方程组为

$$\begin{pmatrix} 5 & 5327 \\ 5327 & 7277699 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 271.4 \\ 369321.5 \end{pmatrix}$$

从而解得

$$\begin{cases} a = 0.9726046 \\ b = 0.0500351 \end{cases}$$

$$\text{故 } y = 0.9726046 + 0.0500351x^2$$

$$\text{均方误差为 } \delta = \left[\sum_{j=0}^4 (y(x_j) - y_j)^2 \right]^{\frac{1}{2}} = 0.1226$$

21. 在某佛堂反应中, 由实验得分解物浓度与时间关系如下:

时间 t	0	5	10	15	20	25	30	35	40	45	50
	55										
浓 度	0	1.27	2.16	2.86	3.44	3.87	4.15	4.37	4.51	4.58	4.62
$y(\times 10^{-4})$	4.64										

用最小二乘法求 $y = f(t)$ 。

解:

观察所给数据的特点, 采用方程

$$y = ae^{\frac{-b}{t}}, (a, b > 0)$$

两边同时取对数, 则

$$\ln y = \ln a - \frac{b}{t}$$

$$\text{取 } \Phi = \text{span} \left\{ 1, -\frac{1}{t} \right\}, S = \ln y, x = -\frac{1}{t}$$

$$\text{则 } S = a^* + b^* x$$

$$\|\varphi_0\|_2^2 = 11, \|\varphi_1\|_2^2 = 0.062321,$$

$$(\varphi_0, \varphi_1) = -0.603975,$$

$$(\varphi_0, f) = -87.674095, (\varphi_1, f) = 5.032489,$$

则法方程组为

$$\begin{pmatrix} 11 & -0.603975 \\ -0.603975 & 0.062321 \end{pmatrix} \begin{pmatrix} a^* \\ b^* \end{pmatrix} = \begin{pmatrix} -87.674095 \\ 5.032489 \end{pmatrix}$$

从而解得

$$\begin{cases} a^* = -7.5587812 \\ b^* = 7.4961692 \end{cases}$$

因此

$$a = e^{a^*} = 5.2151048$$

$$b = b^* = 7.4961692$$

$$\therefore y = 5.2151048 e^{\frac{7.4961692}{t}}$$

22. 给出一张记录 $\{f_k\} = (4, 3, 2, 1, 0, 1, 2, 3)$, 用 FFT 算法求 $\{c_k\}$ 的离散谱。

解:

$$\{f_k\} = (4, 3, 2, 1, 0, 1, 2, 3),$$

$$\text{则 } k = 0, 1, \dots, 7, N = 8$$

$$\omega^0 = \omega^4 = 1,$$

$$\omega^1 = \omega^5 = e^{\frac{-\pi i}{4}},$$

$$\omega^2 = \omega^6 = e^{\frac{-\pi i}{2}} = -i,$$

$$\omega^3 = \omega^7 = e^{\frac{-3\pi i}{4}},$$

k	0	1	2	3	4	5	6	7
x_k	4	3	2	1	0	1	2	3
A_1	4	4	4	2ω	4	0	4	$-2\omega^3$
A_2	8	4	0	4	8	$2\sqrt{2}$	0	$-2\sqrt{2}$
C_j	16	$4+2\sqrt{2}$	0	$4-2\sqrt{2}$	0	$4-2\sqrt{2}$	0	$4+2\sqrt{2}$

23. 用辗转相除法将 $R_{22}(x) = \frac{3x^2 + 6x}{x^2 + 6x + 6}$ 化为连分式。

解

$$\begin{aligned}
 R_{22}(x) &= \frac{3x^2 + 6x}{x^2 + 6x + 6} \\
 &= 3 - \frac{12x + 18}{x^2 + 6x + 6} \\
 &= 3 - \frac{12}{x + \frac{9}{2} - \frac{\frac{4}{3}}{x + \frac{3}{2}}} \\
 &= 3 - \frac{12}{x + 4.5} - \frac{0.75}{x + 1.5}
 \end{aligned}$$

24. 求 $f(x) = \sin x$ 在 $x = 0$ 处的 $(3, 3)$ 阶帕德逼近 $R_{33}(x)$ 。

解:

由 $f(x) = \sin x$ 在 $x = 0$ 处的泰勒展开为

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

得 $C_0 = 0$,

$$C_1 = 1,$$

$$C_2 = 0,$$

$$C_3 = -\frac{1}{3!} = -\frac{1}{6},$$

$$C_4 = 0,$$

$$C_5 = \frac{1}{5!} = \frac{1}{120},$$

$$C_6 = 0,$$

从而

$$-C_1 b_3 - C_2 b_2 - C_3 b_1 = C_4$$

$$-C_2 b_3 - C_3 b_2 - C_4 b_1 = C_5$$

$$-C_3 b_3 - C_4 b_2 - C_5 b_1 = C_6$$

即

$$-\begin{pmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & -\frac{1}{6} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{120} \end{pmatrix} \begin{pmatrix} b_3 \\ b_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{120} \\ 0 \end{pmatrix}$$

从而解得

$$\begin{cases} b_3 = 0 \\ b_2 = \frac{1}{20} \\ b_1 = 0 \end{cases}$$

$$\text{又} \because a_k = \sum_{j=0}^{k-1} C_j b_{k-j} + C_k \quad (k=0,1,2,3)$$

则

$$a_0 = C_0 = 0$$

$$a_1 = C_0 b_1 + C_1 = 0$$

$$a_2 = C_0 b_2 + C_1 b_1 = 0$$

$$a_3 = C_0 b_3 + C_1 b_2 + C_2 b_1 + C_3 = -\frac{7}{60}$$

故

$$\begin{aligned}
 R_{33}(x) &= \frac{a_0 + a_1x + a_2x^2 + a_3x^3}{1 + b_1x + b_2x^2 + b_3x^3} \\
 &= \frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2} \\
 &= \frac{60x - 7x^3}{60 + 3x^3}
 \end{aligned}$$

25. 求 $f(x) = e^x$ 在 $x=0$ 处的 $(2,1)$ 阶帕德逼近 $R_{21}(x)$ 。

解:

由 $f(x) = e^x$ 在 $x=0$ 处的泰勒展开为

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

得

$$C_0 = 1,$$

$$C_1 = 1,$$

$$C_2 = \frac{1}{2!} = \frac{1}{2},$$

$$C_3 = \frac{1}{3!} = \frac{1}{6},$$

从而

$$-C_2b_1 = C_3$$

即

$$-\frac{1}{2}b_1 = \frac{1}{6}$$

解得

$$b_1 = -\frac{1}{3}$$

$$\text{又} \because a_k = \sum_{j=0}^{k-1} C_j b_{k-j} + C_k \quad (k=0,1,2)$$

则

$$a_0 = C_0 = 1$$

$$a_1 = C_0b_1 + C_1 = \frac{2}{3}$$

$$a_2 = C_1b_1 + C_2 = \frac{1}{6}$$

故

$$R_{21}(x) = \frac{a_0 + a_1x + a_2x^2}{1 + b_1x}$$

$$= \frac{1 + \frac{2}{3}x + \frac{1}{6}x^2}{1 - \frac{1}{3}x}$$

$$= \frac{6 + 4x + x^2}{6 - 2x}$$

第四章 数值积分与数值微

分

1. 确定下列求积公式中的特定参数, 使其代数精度尽量高, 并指明所构造出的求积公式所具有的代数精度:

$$(1) \int_{-h}^h f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h);$$

$$(2) \int_{-2h}^{2h} f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h);$$

$$(3) \int_{-1}^1 f(x) dx \approx [f(-1) + 2f(x_1) + 3f(x_2)]/3;$$

$$(4) \int_0^h f(x) dx \approx h[f(0) + f(h)]/2 + ah^2[f'(0) - f'(h)];$$

解:

求解求积公式的代数精度时, 应根据代数精度的定义, 即求积公式对于次数不超过 m 的多项式均能准确地成立, 但对于 $m+1$ 次多项式就不准确成立, 进行验证性求解。

$$(1) \text{ 若 } (1) \int_{-h}^h f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

令 $f(x) = 1$, 则

$$2h = A_{-1} + A_0 + A_1$$

令 $f(x) = x$, 则

$$0 = -A_{-1}h + A_1h$$

令 $f(x) = x^2$, 则

$$\frac{2}{3}h^3 = h^2A_{-1} + h^2A_1$$

从而解得

$$\begin{cases} A_0 = \frac{4}{3}h \\ A_1 = \frac{1}{3}h \\ A_{-1} = \frac{1}{3}h \end{cases}$$

令 $f(x) = x^3$, 则

$$\int_{-h}^h f(x) dx = \int_{-h}^h x^3 dx = 0$$

$$A_{-1}f(-h) + A_0f(0) + A_1f(h) = 0$$

故 $\int_{-h}^h f(x)dx = A_{-1}f(-h) + A_0f(0) + A_1f(h)$ 成立。

令 $f(x) = x^4$ ，则

$$\int_{-h}^h f(x)dx = \int_{-h}^h x^4 dx = \frac{2}{5}h^5$$

$$A_{-1}f(-h) + A_0f(0) + A_1f(h) = \frac{2}{3}h^5$$

故此时，

$$\int_{-h}^h f(x)dx \neq A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

$$\text{故 } \int_{-h}^h f(x)dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

具有 3 次代数精度。

$$(2) \text{ 若 } \int_{-2h}^{2h} f(x)dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

令 $f(x) = 1$ ，则

$$4h = A_{-1} + A_0 + A_1$$

令 $f(x) = x$ ，则

$$0 = -A_{-1}h + A_1h$$

令 $f(x) = x^2$ ，则

$$\frac{16}{3}h^3 = h^2 A_{-1} + h^2 A_1$$

从而解得

$$\begin{cases} A_0 = -\frac{4}{3}h \\ A_1 = \frac{8}{3}h \\ A_{-1} = \frac{8}{3}h \end{cases}$$

令 $f(x) = x^3$ ，则

$$\int_{-2h}^{2h} f(x)dx = \int_{-2h}^{2h} x^3 dx = 0$$

$$A_{-1}f(-h) + A_0f(0) + A_1f(h) = 0$$

故 $\int_{-2h}^{2h} f(x)dx = A_{-1}f(-h) + A_0f(0) + A_1f(h)$ 成立。

令 $f(x) = x^4$ ，则

$$\int_{-2h}^{2h} f(x)dx = \int_{-2h}^{2h} x^4 dx = \frac{64}{5} h^5$$

$$A_{-1}f(-h) + A_0f(0) + A_1f(h) = \frac{16}{3} h^5$$

故此时，

$$\int_{-2h}^{2h} f(x)dx \neq A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

因此，

$$\int_{-2h}^{2h} f(x)dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

具有 3 次代数精度。

$$(3) \text{ 若 } \int_{-1}^1 f(x)dx \approx [f(-1) + 2f(x_1) + 3f(x_2)]/3$$

令 $f(x) = 1$ ，则

$$\int_{-1}^1 f(x)dx = 2 = [f(-1) + 2f(x_1) + 3f(x_2)]/3$$

令 $f(x) = x$ ，则

$$0 = -1 + 2x_1 + 3x_2$$

令 $f(x) = x^2$ ，则

$$2 = 1 + 2x_1^2 + 3x_2^2$$

从而解得

$$\begin{cases} x_1 = -0.2899 \\ x_2 = 0.5266 \end{cases} \text{ 或 } \begin{cases} x_1 = 0.6899 \\ x_2 = 0.1266 \end{cases}$$

令 $f(x) = x^3$ ，则

$$\int_{-1}^1 f(x)dx = \int_{-1}^1 x^3 dx = 0$$

$$[f(-1) + 2f(x_1) + 3f(x_2)]/3 \neq 0$$

故 $\int_{-1}^1 f(x)dx = [f(-1) + 2f(x_1) + 3f(x_2)]/3$ 不成立。

因此，原求积公式具有 2 次代数精度。

$$(4) \text{ 若 } \int_0^h f(x)dx \approx h[f(0) + f(h)]/2 + ah^2[f'(0) - f'(h)]$$

令 $f(x)=1$ ，则

$$\int_0^h f(x)dx = h,$$

$$h[f(0)+f(h)]/2+ah^2[f'(0)-f'(h)]=h$$

令 $f(x)=x$ ，则

$$\int_0^h f(x)dx = \int_0^h xdx = \frac{1}{2}h^2$$

$$h[f(0)+f(h)]/2+ah^2[f'(0)-f'(h)] = \frac{1}{2}h^2$$

令 $f(x)=x^2$ ，则

$$\int_0^h f(x)dx = \int_0^h x^2dx = \frac{1}{3}h^3$$

$$h[f(0)+f(h)]/2+ah^2[f'(0)-f'(h)] = \frac{1}{2}h^3 - 2ah^2$$

故有

$$\frac{1}{3}h^3 = \frac{1}{2}h^3 - 2ah^2$$

$$a = \frac{1}{12}$$

令 $f(x)=x^3$ ，则

$$\int_0^h f(x)dx = \int_0^h x^3dx = \frac{1}{4}h^4$$

$$h[f(0)+f(h)]/2+\frac{1}{12}h^2[f'(0)-f'(h)] = \frac{1}{2}h^4 - \frac{1}{4}h^4 = \frac{1}{4}h^4$$

令 $f(x)=x^4$ ，则

$$\int_0^h f(x)dx = \int_0^h x^4dx = \frac{1}{5}h^5$$

$$h[f(0)+f(h)]/2+\frac{1}{12}h^2[f'(0)-f'(h)] = \frac{1}{2}h^5 - \frac{1}{3}h^5 = \frac{1}{6}h^5$$

故此时，

$$\int_0^h f(x)dx \neq h[f(0)+f(h)]/2+\frac{1}{12}h^2[f'(0)-f'(h)],$$

因此, $\int_0^h f(x)dx \approx h[f(0) + f(h)]/2 + \frac{1}{12}h^2[f'(0) - f'(h)]$

具有 3 次代数精度。

2. 分别用梯形公式和辛普森公式计算下列积分:

$$(1) \int_0^1 \frac{x}{4+x^2} dx, n=8;$$

$$(2) \int_0^1 \frac{(1-e^{-x})^{\frac{1}{2}}}{x} dx, n=10;$$

$$(3) \int_1^9 \sqrt{x} dx, n=4;$$

$$(4) \int_0^{\frac{\pi}{6}} \sqrt{4-\sin^2 \varphi} d\varphi, n=6;$$

解:

$$(1) n=8, a=0, b=1, h=\frac{1}{8}, f(x)=\frac{x}{4+x^2}$$

复化梯形公式为

$$T_8 = \frac{h}{2} [f(a) + 2 \sum_{k=1}^7 f(x_k) + f(b)] = 0.11140$$

复化辛普森公式为

$$S_8 = \frac{h}{6} [f(a) + 4 \sum_{k=0}^7 f(x_{k+\frac{1}{2}}) + 2 \sum_{k=1}^7 f(x_k) + f(b)] = 0.11157$$

$$(2) n=10, a=0, b=1, h=\frac{1}{10}, f(x)=\frac{(1-e^{-x})^{\frac{1}{2}}}{x}$$

复化梯形公式为

$$T_{10} = \frac{h}{2} [f(a) + 2 \sum_{k=1}^9 f(x_k) + f(b)] = 1.39148$$

复化辛普森公式为

$$S_{10} = \frac{h}{6} [f(a) + 4 \sum_{k=0}^9 f(x_{k+\frac{1}{2}}) + 2 \sum_{k=1}^9 f(x_k) + f(b)] = 1.45471$$

$$(3) n=4, a=1, b=9, h=2, f(x)=\sqrt{x},$$

复化梯形公式为

$$T_4 = \frac{h}{2} [f(a) + 2 \sum_{k=1}^3 f(x_k) + f(b)] = 17.22774$$

复化辛普森公式为

$$S_4 = \frac{h}{6} [f(a) + 4 \sum_{k=0}^3 f(x_{k+\frac{1}{2}}) + 2 \sum_{k=1}^3 f(x_k) + f(b)] = 17.32222$$

$$(4)n=6, a=0, b=\frac{\pi}{6}, h=\frac{\pi}{36}, f(x)=\sqrt{4-\sin^2 \varphi}$$

复化梯形公式为

$$T_6 = \frac{h}{2} [f(a) + 2 \sum_{k=1}^5 f(x_k) + f(b)] = 1.03562$$

复化辛普森公式为

$$S_6 = \frac{h}{6} [f(a) + 4 \sum_{k=0}^5 f(x_{k+\frac{1}{2}}) + 2 \sum_{k=1}^5 f(x_k) + f(b)] = 1.03577$$

3. 直接验证柯特斯教材公式 (2.4) 具有 5 交代数精度。

证明:

柯特斯公式为

$$\int_a^b f(x) dx = \frac{b-a}{90} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

令 $f(x) = 1$, 则

$$\int_a^b f(x) dx = \frac{b-a}{90}$$

$$\frac{b-a}{90} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] = b-a$$

令 $f(x) = x$, 则

$$\int_a^b f(x) dx = \int_a^b x dx = \frac{1}{2} (b^2 - a^2)$$

$$\frac{b-a}{90} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] = \frac{1}{2} (b^2 - a^2)$$

令 $f(x) = x^2$, 则

$$\int_a^b f(x) dx = \int_a^b x^2 dx = \frac{1}{3} (b^3 - a^3)$$

$$\frac{b-a}{90} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] = \frac{1}{3} (b^3 - a^3)$$

令 $f(x) = x^3$, 则

$$\int_a^b f(x)dx = \int_a^b x^3 dx = \frac{1}{4}(b^4 - a^4)$$

$$\frac{b-a}{90}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] = \frac{1}{4}(b^4 - a^4)$$

令 $f(x) = x^4$ ，则

$$\int_a^b f(x)dx = \int_a^b x^4 dx = \frac{1}{5}(b^5 - a^5)$$

$$\frac{b-a}{90}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] = \frac{1}{5}(b^5 - a^5)$$

令 $f(x) = x^5$ ，则

$$\int_a^b f(x)dx = \int_a^b x^5 dx = \frac{1}{6}(b^6 - a^6)$$

$$\frac{b-a}{90}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] = \frac{1}{6}(b^6 - a^6)$$

令 $f(x) = x^6$ ，则

$$\int_0^h f(x)dx \neq \frac{b-a}{90}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

因此，该柯特斯公式具有 5 次代数精度。

4. 用辛普森公式求积分 $\int_0^1 e^{-x} dx$ 并估计误差。

解：

辛普森公式为

$$S = \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

此时，

$$a=0, b=1, f(x) = e^{-x},$$

从而有

$$S = \frac{1}{6}(1 + 4e^{-\frac{1}{2}} + e^{-1}) = 0.63233$$

误差为

$$\begin{aligned} |R(f)| &= \left| -\frac{b-a}{180} \left(\frac{b-a}{2}\right)^4 f^{(4)}(\eta) \right| \\ &\leq \frac{1}{180} \times \frac{1}{2^4} \times e^0 = 0.00035, \eta \in (0,1) \end{aligned}$$

5. 推导下列三种矩形求积公式：

$$\int_a^b f(x)dx = (b-a)f(a) + \frac{f'(\eta)}{2}(b-a)^2;$$

$$\int_a^b f(x)dx = (b-a)f(b) - \frac{f'(\eta)}{2}(b-a)^2;$$

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(\eta)}{24}(b-a)^3;$$

证明:

$$(1) \because f(x) = f(a) + f'(\eta)(x-a), \eta \in (a, b)$$

两边同时在 $[a, b]$ 上积分, 得

$$\int_a^b f(x)dx = (b-a)f(a) + f'(\eta) \int_a^b (x-a)dx$$

即

$$\int_a^b f(x)dx = (b-a)f(a) + \frac{f'(\eta)}{2}(b-a)^2$$

$$(2) \because f(x) = f(b) - f'(\eta)(b-x), \eta \in (a, b)$$

两边同时在 $[a, b]$ 上积分, 得

$$\int_a^b f(x)dx = (b-a)f(b) - f'(\eta) \int_a^b (b-x)dx$$

即

$$\int_a^b f(x)dx = (b-a)f(b) - \frac{f'(\eta)}{2}(b-a)^2$$

$$(3) \because f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{f''(\eta)}{2}\left(x - \frac{a+b}{2}\right)^2, \eta \in (a, b)$$

两连边同时在 $[a, b]$ 上积分, 得

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right) \int_a^b \left(x - \frac{a+b}{2}\right)dx + \frac{f''(\eta)}{2} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

即

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(\eta)}{24}(b-a)^3;$$

6. 若用复化梯形公式计算积分 $I = \int_0^1 e^x dx$, 问区间 $[0, 1]$ 应分多少等分才能使截断误差不超

过 $\frac{1}{2} \times 10^{-5}$? 若改用复化辛普森公式, 要达到同样精度区间 $[0, 1]$ 应分多少等分?

解:

采用复化梯形公式时, 余项为

$$R_n(f) = -\frac{b-a}{12} h^2 f''(\eta), \eta \in (a, b)$$

$$\text{又} \because I = \int_0^1 e^x dx$$

$$\text{故 } f(x) = e^x, f''(x) = e^x, a = 0, b = 1.$$

$$\therefore |R_n(f)| = \frac{1}{12} h^2 |f''(\eta)| \leq \frac{e}{12} h^2$$

$$\text{若 } |R_n(f)| \leq \frac{1}{2} \times 10^{-5}, \text{ 则}$$

$$h^2 \leq \frac{6}{e} \times 10^{-5}$$

当对区间 [0,1] 进行等分时,

$$h = \frac{1}{n},$$

故有

$$n \geq \sqrt{\frac{e}{6} \times 10^{-5}} = 212.85$$

因此, 将区间 213 等分时可以满足误差要求
采用复化辛普森公式时, 余项为

$$R_n(f) = -\frac{b-a}{180} \left(\frac{h}{2}\right)^4 f^{(4)}(\eta), \eta \in (a, b)$$

$$\text{又} \because f(x) = e^x,$$

$$\therefore f^{(4)}(x) = e^x,$$

$$\therefore |R_n(f)| = -\frac{1}{2880} h^4 |f^{(4)}(\eta)| \leq \frac{e}{2880} h^4$$

$$\text{若 } |R_n(f)| \leq \frac{1}{2} \times 10^{-5}, \text{ 则}$$

$$h^4 \leq \frac{1440}{e} \times 10^{-5}$$

当对区间 [0,1] 进行等分时

$$n = \frac{1}{h}$$

故有

$$n \geq \left(\frac{1440}{e} \times 10^{-5}\right)^{\frac{1}{4}} = 3.71$$

因此, 将区间 8 等分时可以满足误差要求。

7. 如果 $f''(x) > 0$, 证明用梯形公式计算积分 $I = \int_a^b f(x) dx$ 所得结果比准确值 I 大, 并说明其几何意义。

解：采用梯形公式计算积分时，余项为

$$R_T = -\frac{f''(\eta)}{12}(b-a)^3, \eta \in [a, b]$$

又 $\because f''(x) > 0$ 且 $b > a$

$$\therefore R_T < 0$$

$$\text{又} \because R_T = 1 - T$$

$$\therefore I < T$$

即计算值比准确值大。

其几何意义为， $f''(x) > 0$ 为下凸函数，梯形面积大于曲边梯形面积。

8. 用龙贝格求积方法计算下列积分，使误差不超过 10^{-5} 。

$$(1) \frac{2}{\sqrt{\pi}} \int_0^1 e^{-x} dx$$

$$(2) \int_0^{2\pi} x \sin x dx$$

$$(3) \int_0^3 x \sqrt{1+x^2} dx.$$

解：

$$(1) I = \frac{2}{\sqrt{\pi}} \int_0^1 e^{-x} dx$$

k	$T_0^{(k)}$	$T_1^{(k)}$	$T_2^{(k)}$	$T_3^{(k)}$
0	0.7717433			
1	0.7280699	0.7135121		
2	0.7169828	0.7132870	0.7132720	
3	0.7142002	0.7132726	0.7132717	0.7132717

因此 $I = 0.713727$

$$(2) I = \int_0^{2\pi} x \sin x dx$$

k	$T_0^{(k)}$	$T_1^{(k)}$
0	3.451313×10^{-6}	
1	8.628283×10^{-7}	$-4.446923 \times 10^{-21}$

因此 $I \approx 0$

$$(3) I = \int_0^3 x \sqrt{1+x^2} dx$$

k	$T_0^{(k)}$	$T_1^{(k)}$	$T_2^{(k)}$	$T_3^{(k)}$	$T_4^{(k)}$	$T_5^{(k)}$
0	14.2302495					
1	11.1713699	10.1517434				
2	10.4437969	10.2012725	10.2045744			
3	10.2663672	10.2072240	10.2076207	10.2076691		
4	10.2222702	10.2075712	10.2075943	10.2075939	10.2075936	
5	10.2112607	10.2075909	10.2075922	10.2075922	10.2075922	10.2075922

因此 $I \approx 10.2075922$

9. 用 $n=2,3$ 的高斯-勒让德公式计算积分

$$\int_1^3 e^x \sin x dx.$$

解:

$$I = \int_1^3 e^x \sin x dx.$$

$\because x \in [1,3]$, 令 $t = x - 2$, 则 $t \in [-1,1]$

用 $n=2$ 的高斯-勒让德公式计算积分

$$I \approx 0.5555556 \times [f(-0.7745967) + f(0.7745967)] + 0.8888889 \times f(0) \\ \approx 10.9484$$

用 $n=3$ 的高斯-勒让德公式计算积分

$$I \approx 0.3478548 \times [f(-0.8611363) + f(0.8611363)] \\ + 0.6521452 \times [f(-0.3399810) + f(0.3399810)] \\ \approx 10.95014$$

10 地球卫星轨道是一个椭圆，椭圆周长的计算公式是

$$S = a \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{c}{a}\right)^2 \sin^2 \theta} d\theta,$$

这是 a 是椭圆的半径轴, c 是地球中心与轨道中心(椭圆中心)的距离, 记 h 为近地点距离, H 为远地点距离, $R=6371$ (km) 为地球半径, 则

$$a = (2R + H + h)/2, c = (H - h)/2.$$

我国第一颗地球卫星近地点距离 $h=439$ (km), 远地点距离 $H=2384$ (km)。试求卫星轨道的周长。

解:

$$\because R = 6371, h = 439, H = 2384$$

从而有。

$$a=(2R+H+h)/2=7782.5$$

$$c=(H-h)/2=972.5$$

$$S=4a\int_0^{\frac{\pi}{2}}\sqrt{1-(\frac{c}{a})^2\sin^2\theta}d\theta$$

k	$T_0^{(k)}$	$T_1^{(k)}$	$T_2^{(k)}$
0	1.564640		
1	1.564646	1.564648	
2	1.564646	1.564646	1.564646

$$I\approx 1.564646$$

$$S\approx 48708(km)$$

即人造卫星轨道的周长为 48708km

11. 证明等式

$$n\sin\frac{\pi}{n}=\pi-\frac{\pi^3}{3!n^2}+\frac{\pi^5}{5!n^4}-\cdots$$

试依据 $n\sin(\frac{\pi}{n})(n=3,6,12)$ 的值，用外推算法求 π 的近似值。

解

$$\text{若 } f(n)=n\sin\frac{\pi}{n},$$

$$\text{又}\because \sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots$$

\therefore 此函数的泰勒展式为

$$\begin{aligned} f(n) &= n\sin\frac{\pi}{n} \\ &= n[\frac{\pi}{n}-\frac{1}{3!}(\frac{\pi}{n})^3+\frac{1}{5!}(\frac{\pi}{n})^5-\cdots] \\ &= \pi-\frac{\pi^3}{3!n^2}+\frac{\pi^5}{5!n^4}-\cdots \end{aligned}$$

$$T_n^{(k)}\approx \pi$$

$$\text{当 } n=3 \text{ 时, } n\sin\frac{\pi}{n}=2.598076$$

$$\text{当 } n=6 \text{ 时, } n\sin\frac{\pi}{n}=3$$

$$\text{当 } n=12 \text{ 时, } n\sin\frac{\pi}{n}=3.105829$$

由外推法可得

n	$T_0^{(n)}$	$T_1^{(n)}$	$T_2^{(n)}$
3	2.598076		
6	3.000000	3.133975	
9	3.105829	3.141105	3.141580

故 $\pi \approx 3.14158$

12. 用下列方法计算积分 $\int_1^3 \frac{dy}{y}$ ，并比较结果。

- (1)龙贝格方法；
- (2)三点及五点高斯公式；
- (3)将积分区间分为四等分，用复化两点高斯公式。

解

$$I = \int_1^3 \frac{dy}{y}$$

(1)采用龙贝格方法可得

k	$T_0^{(k)}$	$T_1^{(k)}$	$T_2^{(k)}$	$T_3^{(k)}$	$T_4^{(k)}$
0	1.333333				
1	1.166667	1.099259			
2	1.116667	1.100000	1.099259		
3	1.103211	1.098726	1.098641	1.098613	
4	1.099768	1.098620	1.098613	1.098613	1.098613

故有 $I \approx 1.098613$

(2)采用高斯公式时

$$I = \int_1^3 \frac{dy}{y}$$

此时 $y \in [1, 3]$,

令 $x = y - z$, 则 $x \in [-1, 1]$,

$$I = \int_{-1}^1 \frac{1}{x+2} dx,$$

$$f(x) = \frac{1}{x+2},$$

利用三点高斯公式，则

$$I = 0.5555556 \times [f(-0.7745967) + f(0.7745967)] + 0.8888889 \times f(0) \\ \approx 1.098039$$

利用五点高斯公式，则

$$\begin{aligned} I &\approx 0.2369239 \times [f(-0.9061798) + f(0.9061798)] \\ &+ 0.4786287 \times [f(-0.5384693) + f(0.5384693)] + 0.5688889 \times f(0) \\ &\approx 1.098609 \end{aligned}$$

(3)采用复化两点高斯公式

将区间[1,3]四等分，得

$$\begin{aligned} I &= I_1 + I_2 + I_3 + I_4 \\ &= \int_1^{1.5} \frac{dy}{y} + \int_{1.5}^2 \frac{dy}{y} + \int_2^{2.5} \frac{dy}{y} + \int_{2.5}^3 \frac{dy}{y} \end{aligned}$$

作变换 $y = \frac{x+5}{4}$ ，则

$$\begin{aligned} I_1 &= \int_{-1}^1 \frac{1}{x+5} dx, \\ f(x) &= \frac{1}{x+5}, \\ I_1 &\approx f(-0.5773503) + f(0.5773503) \approx 0.4054054 \end{aligned}$$

作变换 $y = \frac{x+7}{4}$ ，则

$$\begin{aligned} I_2 &= \int_{-1}^1 \frac{1}{x+7} dx, \\ f(x) &= \frac{1}{x+7}, \\ I_2 &\approx f(-0.5773503) + f(0.5773503) \approx 0.2876712 \end{aligned}$$

作变换 $y = \frac{x+9}{4}$ ，则

$$\begin{aligned} I_3 &= \int_{-1}^1 \frac{1}{x+9} dx, \\ f(x) &= \frac{1}{x+9}, \\ I_3 &\approx f(-0.5773503) + f(0.5773503) \approx 0.2231405 \end{aligned}$$

作变换 $y = \frac{x+11}{4}$ ，则

$$\begin{aligned} I_4 &= \int_{-1}^1 \frac{1}{x+11} dx, \\ f(x) &= \frac{1}{x+11}, \\ I_4 &\approx f(-0.5773503) + f(0.5773503) \approx 0.1823204 \end{aligned}$$

因此，有

$$I \approx 1.098538$$

13.用三点公式和积分公式求 $f(x) = \frac{1}{(1+x)^2}$ 在 $x=1.0, 1.1$ ，和 1.2 处的导数值，并估计误差。

$f(x)$ 的值由下表给出：

x	1.0	1.1	1.2
F(x)	0.2500	0.2268	0.2066

解：

$$f(x) = \frac{1}{(1+x)^2}$$

由带余项的三点求导公式可知

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_1) - f(x_2)] + \frac{h^2}{3} f'''(\xi)$$

$$f'(x_1) = \frac{1}{2h}[-f(x_0) + f(x_2)] - \frac{h^2}{6} f'''(\xi)$$

$$f'(x_2) = \frac{1}{2h}[f(x_0) - 4f(x_1) + 3f(x_2)] + \frac{h^2}{3} f'''(\xi)$$

$$\text{又} \because f(x_0) = 0.2500, f(x_1) = 0.2268, f(x_2) = 0.2066,$$

$$\therefore f'(x_0) \approx \frac{1}{2h}[-3f(x_0) + 4f(x_1) - f(x_2)] = 0.247$$

$$f'(x_1) \approx \frac{1}{2h}[-f(x_0) + f(x_2)] = -0.217$$

$$f'(x_2) = \frac{1}{2h}[f(x_0) - 4f(x_1) + 3f(x_2)] = -0.187$$

$$\text{又} \because f(x) = \frac{1}{(1+x)^2}$$

$$\therefore f'''(x) = \frac{-24}{(1+x)^5}$$

$$\text{又} \because x \in [1.0, 1.2]$$

$$\therefore |f'''(\xi)| \leq 0.75$$

故误差分别为

$$|R(x_0)| = \left| \frac{h^2}{3} f'''(\xi) \right| \leq 2.5 \times 10^{-3}$$

$$|R(x_1)| = \left| \frac{h^2}{6} f'''(\xi) \right| \leq 1.25 \times 10^{-3}$$

$$|R(x_2)| = \left| \frac{h^2}{3} f'''(\xi) \right| \leq 2.5 \times 10^{-3}$$

利用数值积分求导,

设 $\varphi(x) = f'(x)$

$$f(x_{k+1}) = f(x_k) + \int_{x_k}^{x_{k+1}} \varphi(x) dx$$

由梯形求积公式得

$$\int_{x_k}^{x_{k+1}} \varphi(x) dx = \frac{h}{2} [\varphi(x_k) + \varphi(x_{k+1})]$$

从而有

$$f(x_{k+1}) = f(x_k) + \frac{h}{2} [\varphi(x_k) + \varphi(x_{k+1})]$$

故

$$\varphi(x_0) + \varphi(x_1) = \frac{2}{h} [f(x_1) - f(x_0)]$$

$$\varphi(x_1) + \varphi(x_2) = \frac{2}{h} [f(x_2) - f(x_1)]$$

$$\text{又} \because f(x_{k+1}) = f(x_{k-1}) + \int_{x_{k-1}}^{x_{k+1}} \varphi(x) dx$$

$$\text{且} \int_{x_{k-1}}^{x_{k+1}} \varphi(x) dx = h[\varphi(x_{k-1}) + \varphi(x_{k+1})]$$

从而有

$$f(x_{k+1}) = f(x_{k-1}) + h[\varphi(x_{k-1}) + \varphi(x_{k+1})]$$

$$\text{故} \varphi(x_0) + \varphi(x_2) = \frac{1}{h} [f(x_2) - f(x_0)]$$

即

$$\begin{cases} \varphi(x_0) + \varphi(x_1) = -0.464 \\ \varphi(x_1) + \varphi(x_2) = -0.404 \\ \varphi(x_0) + \varphi(x_2) = -0.434 \end{cases}$$

解方程组可得

$$\begin{cases} \varphi(x_0) = -0.247 \\ \varphi(x_1) = -0.217 \\ \varphi(x_2) = -0.187 \end{cases}$$

第五章 Equation Chapter 1 Section 1数值 分析课后习题全解

第 5 章：解线性方程组的直接方法

1. 证明：由消元公式及 A 的对称性得

$$a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}a_{j1}}{a_{11}} = -\frac{a_{j1}}{a_{11}}a_{1i} = a_{ji}^{(2)}, i, j = 2, 3, \dots, n$$

故 A_2 对称

2. 证明：（1）因 A 对称正定，故

$$a_{ii} = (Ae_i, e_i) > 0, i=1, 2, \dots, n$$

其中 $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ 为第 i 个单位向量.

(2)由 A 的对称性及消元公式得

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} d \\ d \\ \vdots \\ d_n \end{bmatrix} \quad a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}a_{j1}}{a_{11}} =$$

$$a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i} = a_{ji}^{(2)}, \quad i, j=2, \dots, n$$

故 A_2 也对称.

$$\text{又} \quad \begin{bmatrix} a_{11} & a_1^T \\ 0 & A_2 \end{bmatrix} = L_1 A A L_1^T$$

$$\text{其中} \quad L_1 = \begin{bmatrix} 1 & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ & \cdot & \ddots & \\ -\frac{a_{n1}}{a_{11}} & \dots & & 1 \end{bmatrix}$$

显然 L_1 非奇异, 从而对任意的 $x \neq 0$, 有

$$L_1^T X \neq 0, (x, L_1 A L_1^T X) = (L_1^T x, A L_1^T X) > 0 \quad (\text{由 } A \text{ 的正定性})$$

故 $L_1 A L_1^T$ 正定.

$$\text{又 } L_1 A L_1^T = \begin{bmatrix} a_{11} & 0 \\ 0 & A_2 \end{bmatrix}, \text{ 而 } a_{11} > 0, \text{ 故 } A_2 \text{ 正定.}$$

3. 证明 由矩阵乘法简单运算即得证.

4. 解 设有分解

$$\begin{bmatrix} 4 & 2 & & \\ 3 & -2 & 1 & \\ & 2 & 5 & 3 \\ & & -1 & 6 \end{bmatrix} = \begin{bmatrix} \alpha_1 & & & \\ 3 & \alpha_2 & 1 & \\ & 2 & \alpha_3 & 3 \\ & & -1 & \alpha_4 \end{bmatrix} \begin{bmatrix} 1 & \beta_1 & & \\ & 1 & \beta_2 & \\ & & 1 & \beta_3 \\ & & & 1 \end{bmatrix}$$

由公式

$$\begin{cases} b_1 = a_1 c_1 = \alpha_1 \beta_1 \\ b_i = \alpha_i \beta_{i-1} + \alpha_i, i = 2, 3, \dots, n \\ c_i = \alpha_i \beta_i, i = 2, 3, \dots, n-1 \end{cases}$$

其中 b_i, a_i, c_i 分别是系数矩阵的主对角线元素及下边和上边的次对角线元素. 故有

$$\begin{cases} \alpha_1 = 4, \beta_1 = \frac{1}{2} \\ \alpha_2 = -\frac{7}{2}, \beta_2 = -\frac{2}{7} \\ \alpha_3 = \frac{39}{7}, \beta_3 = \frac{7}{13} \\ \alpha_4 = \frac{85}{13} \end{cases}$$

从而有

$$\begin{bmatrix} 4 & 2 & & \\ 3 & -2 & 1 & \\ & 2 & 5 & 3 \\ & & -1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & & & \\ 3 & -\frac{7}{2} & & \\ & 2 & \frac{39}{7} & \\ & & -1 & \frac{85}{13} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & & \\ & 1 & -\frac{2}{7} & \\ & & 1 & \frac{7}{13} \\ & & & 1 \end{bmatrix}$$

$$\text{故 } y_1 = \frac{6}{4} = \frac{3}{2}, \quad y_2 = \frac{2 - 3y_1}{-\frac{2}{7}} = \frac{5}{7}$$

$$y_3 = \frac{10 - 2y_2}{\frac{39}{7}} = \frac{20}{13}, \quad y_4 = \frac{5 + y_3}{\frac{85}{13}} = 1$$

$$\text{故 } x_4 = 1, x_3 = \frac{20}{13} - \frac{7}{13}x_4 = 1, x_2 = \frac{5}{7} + \frac{2}{7}x_3 = 1, x_1 = \frac{3}{2} - \frac{1}{2}x_2 = 1$$

5. 解 (1) 设 U 为上三角阵

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

因 $u_{nn}x_n = d_n$, 故 $x_n = \frac{d_n}{u_{nn}}$.

$$\text{因 } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 0 & -3 & & \\ & \vdots & 0 & -2 & 1 & 7 & \\ & & 1 & -1 & 0 & 1 & \end{bmatrix} u_{ii}x_i + \sum_{j=i+1}^n u_{ij}x_j = d_i, \text{ 故}$$

$$x_i = \frac{d_i - \sum_{j=i+1}^n u_{ij}x_j}{u_{ii}}, i=n-1, n-2, \cdots, 1$$

当 U 为下三角阵时

$$\begin{bmatrix} u_{11} & & & \\ u_{21} & u_{22} & & \\ \vdots & \vdots & \ddots & \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$\text{得, } x_1 = \frac{d_1}{u_{11}}, \quad x_i = \frac{d_i - \sum_{j=1}^{i-1} u_{ij} x_j}{u_{ii}}, i=2,3,\dots,n.$$

(2)除法次数为 n ,乘法次数为

$$1+2+\dots+(n-1)=n(n-1)/2$$

故总的乘法次数为 $n+n(n-1)/2=n(n+1)/2$.

(3)设 U 为上三角阵, $U^{-1}=S$, 侧 S 也是上三角阵. 由

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ & s_{22} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\text{得} \quad s_{ii} = \frac{1}{u_{ii}}, \quad i=1,2,\dots,n$$

$$s_{ij} = -\frac{\sum_{k=i+1}^j u_{ik} s_{kj}}{u_{ii}}, j=i+1, i+2, \dots, n; i=n-1, n-2, \dots, 1$$

当 U 为下三角阵时, 由

$$\begin{bmatrix} d_{11} & & & \\ d_{21} & d_{22} & & \\ \vdots & \vdots & \ddots & \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} s_{11} & & & \\ s_{21} & s_{22} & & \\ \vdots & \vdots & \ddots & \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\text{得} \quad s_{ii} = \frac{1}{u_{ii}}, i=1,2,\dots,n$$

$$s_{ij} = -\frac{\sum_{k=1}^{i-1} u_{ik} s_{kj}}{u_{ii}}, i=2,3,\dots,n; j=1,2,\dots,i-1$$

6. 证明 (1)因 A 是对称正定阵,故存在唯一的分解 $A=LL^T$, 其中 L 是具有正对角元素的下三角阵. 从而

$$A^{-1}=(LL^T)^{-1}=(L^T)^{-1}L^{-1}=(L^{-1})^T L^{-1}$$

$$(A^{-1})^T = \left[(L^{-1})^T L^{-1} \right]^T = (L^{-1})^T L^{-1} = A^{-1}$$

故 A^{-1} 是对称矩阵.

又 L^{-1} 非奇异, 故对任意的 $x \neq 0$, 有 $L^{-1}x \neq 0$, 故

$$x^T A^{-1} x = x^T (L^{-1})^T L^{-1} x = (L^{-1}x)^T (L^{-1}x) > 0$$

故 A^{-1} 是对称正定矩阵, 即 A^{-1} 也对称正定.

(2) 由 A 对称正定, 故 A 的所有顺序主子式均不为零, 从而 A 有唯一的

Doolittle 分解 $A = \bar{L} U$. 又

$$U = \begin{bmatrix} u_{11} & & & \\ & u_{22} & & \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \dots & \frac{u_{1n}}{u_{11}} \\ & 1 & \dots & \frac{u_{2n}}{u_{22}} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} = D U_0$$

其中 D 为对三角阵, U_0 为单位上三角阵, 于是

$$A = U \bar{L} = D \bar{L} U_0$$

又

$$A = A^T = U_0^T D \bar{L}^T$$

由分解的唯一性即得

$$U_0^T = \bar{L}$$

从而有

$$A = D \bar{L} \bar{L}^T$$

又由 A 的对称正定性知

$$d_1 = D_1 > 0, \quad d_i = \frac{D_i}{D_{i-1}} > 0 \quad (i=2,3,\dots,n)$$

故 $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} =$

$$\begin{bmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{bmatrix} \begin{bmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{bmatrix} = D^{\frac{1}{2}} D^{\frac{1}{2}}$$

$$\text{故 } A = \bar{L} D \bar{L}^T = \bar{L} D^{\frac{1}{2}} D^{\frac{1}{2}} \bar{L}^T = (\bar{L} D^{\frac{1}{2}})(\bar{L} D^{\frac{1}{2}})^T = LL^T$$

其中 $L = \bar{L} D^{\frac{1}{2}}$ 为三角元为正的下三角矩阵.

$$\begin{aligned} 7. \text{ 解 } [A|I] &= \begin{bmatrix} 2 & 1 & -3 & -1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 7 & 0 & 1 & 0 & 0 \\ -1 & 2 & 4 & -2 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \\ &\begin{bmatrix} 1 & 0 & -1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & -8 & 0 & 1 & 0 & -3 \\ 0 & 2 & 3 & 3 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & -11 & 1 & 0 & 0 & -2 \end{bmatrix} \rightarrow \\ &\begin{bmatrix} 1 & 0 & -1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & -8 & 0 & 1 & 0 & -3 \\ 0 & 0 & -3 & 19 & 0 & -2 & 1 & 7 \\ 0 & 0 & -4 & -3 & 1 & -1 & 0 & 1 \end{bmatrix} \rightarrow \\ &\begin{bmatrix} 1 & 0 & 0 & -\frac{4}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 1 & 0 & 11 & 0 & -1 & 1 & 4 \\ 0 & 0 & 1 & -\frac{19}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{7}{3} \\ 0 & 0 & 0 & -\frac{85}{3} & 1 & \frac{5}{3} & -\frac{4}{3} & -\frac{25}{3} \end{bmatrix} \rightarrow \\ &\begin{bmatrix} & & & \frac{4}{85} & \frac{10}{17} & -\frac{23}{85} & -\frac{16}{17} \\ 1 & 0 & 0 & 0 & \frac{33}{85} & -\frac{6}{17} & \frac{41}{85} & \frac{13}{17} \\ 0 & 1 & 0 & 0 & \frac{85}{85} & -\frac{1}{17} & \frac{3}{85} & -\frac{8}{17} \\ 0 & 0 & 1 & 0 & -\frac{19}{85} & \frac{15}{17} & -\frac{3}{85} & -\frac{8}{17} \\ 0 & 0 & 0 & 1 & -\frac{3}{85} & -\frac{1}{17} & \frac{4}{85} & \frac{5}{17} \end{bmatrix} \rightarrow \end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{4}{85} & \frac{10}{17} & -\frac{23}{85} & -\frac{16}{17} \\ \frac{33}{85} & -\frac{6}{17} & \frac{41}{85} & \frac{13}{17} \\ -\frac{19}{85} & \frac{5}{17} & -\frac{3}{85} & -\frac{8}{17} \\ -\frac{3}{85} & -\frac{1}{17} & \frac{4}{85} & \frac{5}{17} \end{bmatrix} = \begin{bmatrix} 0.0470589 & 0.5882353 & -0.2705882 & -0.9411765 \\ 0.3882353 & -0.3529412 & 0.4823529 & 0.7647059 \\ -0.2235294 & 0.2941176 & -0.0352941 & -0.4705882 \\ -0.0352941 & -0.0588235 & 0.0470589 & 0.2941176 \end{bmatrix}$$

8. 解 设有分解

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & & & & \\ -1 & \alpha_2 & & & \\ & -1 & \alpha_3 & & \\ & & -1 & \alpha_4 & \\ & & & -1 & \alpha_5 \end{bmatrix} \begin{bmatrix} \beta_1 & & & & \\ 1 & \beta_2 & & & \\ & 1 & \beta_3 & & \\ & & 1 & \beta_4 & \\ & & & 1 & \beta_5 \end{bmatrix}$$

由公式

$$\begin{cases} b_1 = \alpha_1, c_1 = \alpha_1 \beta_1 \\ b_i = \alpha_i \beta_{i-1} + \alpha_i, (i = 2, 3, 4, 5) \\ c_i = \alpha_i \beta_i, (i = 2, 3, 4) \end{cases}$$

其中 b_i , α_i , c_i 分别是系数矩阵的主角线元素及其下边和上边的次对角线元素, 则有

$$\alpha_1 = 2, \quad \alpha_2 = \frac{3}{2}, \quad \alpha_3 = \frac{4}{3}, \quad \alpha_4 = \frac{5}{4}, \quad \alpha_5 = \frac{6}{5}$$

$$\beta_1 = -\frac{1}{2}, \quad \beta_2 = -\frac{2}{3}, \quad \beta_3 = -\frac{3}{4}, \quad \beta_4 = -\frac{4}{5}$$

由

$$\begin{bmatrix} 2 & & & & \\ -1 & \frac{3}{2} & & & \\ & -1 & \frac{4}{3} & & \\ & & -1 & \frac{5}{4} & \\ & & & -1 & \frac{6}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{得 } y_1 = \frac{1}{2}, \quad y_2 = \frac{1}{3}, \quad y_3 = \frac{1}{4}, \quad y_4 = \frac{1}{5}, \quad y_5 = \frac{1}{6}$$

由

$$\begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{bmatrix}$$

$$\text{得 } x_5 = \frac{1}{6}, \quad x_4 = \frac{1}{3}, \quad x_3 = \frac{1}{2}, \quad x_2 = \frac{2}{3}, \quad x_1 = \frac{5}{6}$$

9. 解 设

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ & 1 & l_{32} \\ & & 1 \end{bmatrix}$$

由矩阵乘法得

$$d_1 = 2, \quad l_{21} = -\frac{1}{2}, \quad l_{31} = \frac{1}{2}$$

$$d_2 = -\frac{5}{2}, \quad l_{32} = -\frac{7}{5}$$

$$d_3 = \frac{27}{5}$$

由

$$\begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ \frac{1}{2} & -\frac{7}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

得 $y_1 = 4, y_2 = 7, y_3 = \frac{69}{5}$

由

$$\begin{bmatrix} 2 & & \\ & -\frac{5}{2} & \\ & & \frac{27}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ & 1 & -\frac{7}{5} \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ \frac{69}{5} \end{bmatrix}$$

得 $\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ & 1 & -\frac{7}{5} \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & & \\ & -\frac{5}{2} & \\ & & \frac{27}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ \frac{69}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{14}{5} \\ \frac{23}{9} \end{bmatrix}$

故 $x_3 = \frac{23}{9} = 2.555\ 555\ 6, x_2 = \frac{7}{9} = 0.777\ 777\ 8, x_1 = \frac{10}{9} = 1.111\ 111\ 1$

10. 解 A 中 $\Delta_2 = 0$, 故不能分解。但 $\det(A) = -10 \neq 0$, 故若将 A 中第一行与第三行交换, 则可以分解, 且分解唯一。

B 中, $\Delta_2 = \Delta_3 = 0$, 但它仍可以分解为

$$B = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & l_{32} - 2 \end{bmatrix}$$

其中 l_{32} 为一任意常数, 且 U 奇异, 故分解且分解不唯一,

对 C, $\Delta_i \neq 0, i=1,2,3$, 故 C 可分解且分解唯一。

$$C = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ & 1 & 3 \\ & & 1 \end{bmatrix}$$

11. 解

$$\|A\|_{\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| = 1.1$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| = 0.8$$

$$\|A\|_F = \left(\sum_{i,j=1}^n a_{ij}^2 \right)^{\frac{1}{2}} = \sqrt{0.71} = 0.842\ 615\ 0$$

$$A^T A = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.37 & 0.33 \\ 0.33 & 0.34 \end{bmatrix}$$

$$\lambda_{\max}(A^T A) = 0.685\ 340\ 7$$

故 $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} = 0.827\ 853\ 1$

12. 证明 (1) 有定义知

$$\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i| \leq \sum_{i=1}^n |x_i| =$$

$$\|x\|_1 \leq \sum_{i=1}^n \max_{1 \leq i \leq n} |x_i| = \sum_{i=1}^n \|x\|_{\infty} = n \|x\|_{\infty}$$

故 $\|x\|_{\infty} \leq \|x\|_1 \leq n \|x\|_{\infty}$

(2) 由范数定义, 有

$$\|A\|_2^2 = \lambda_{\max}(A^T A) \leq$$

$$\lambda_1(A^T A) + \lambda_2(A^T A) + \cdots + \lambda_n(A^T A) = \text{tr}(A^T A) =$$

$$\sum_{i=1}^n a_{i1}^2 + \sum_{i=1}^n a_{i2}^2 + \cdots + \sum_{i=1}^n a_{in}^2 =$$

$$\sum_{j=1}^n \sum_{i=1}^n a_{ij}^2 = \|A\|_F^2$$

$$\|A\|_2^2 = \lambda_{\max}(A^T A) \geq$$

$$\frac{1}{n} [\lambda_1(A^T A) + \lambda_2(A^T A) + \cdots + \lambda_n(A^T A)] =$$

$$\frac{1}{n} \|A\|_F^2$$

故
$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F$$

13. 证明 (1) 因 P 非奇异, 故对任意的 $x \neq 0$, 有 $P_x \neq 0$, 故 $\|X\|_p = \|P_x\| \geq 0$,
当且仅当 $x=0$ 时, 有 $\|x\|_p = \|P_x\| = 0$ 成立。

(2) 对任意 $\alpha \in R^1$, 有

$$\|\alpha x\|_p = \|P_{\alpha x}\| = |\alpha| \|P_x\| = |\alpha| \|X\|_p$$

$$(3) \|x+y\|_p = \|P_{(x+y)}\| = \|P_x + P_y\| \leq$$

$$\|P_x\| + \|P_y\| = \|x\|_p + \|y\|_p$$

故 $\|x\|_p$ 是 R^n 上的一种向量范数。

14. 证明 (1) 因 A 正定对称, 故当 $x=0$ 时, $\|x\|_A=0$, 而当 $x \neq 0$ 时, $\|x\|_A$
 $= (x^T A x)^{\frac{1}{2}} > 0$ 。

(2) 对任意实数 c , 有

$$\|cx\|_A = \sqrt{(cx)^T A (cx)} = |c| \sqrt{x^T A x} = |c| \|x\|_A$$

(3) 因 A 正定, 故有分解 $A=LL^T$, 则

$$\|x\|_A = (x^T A x)^{\frac{1}{2}} = (x^T L L^T x)^{\frac{1}{2}} = ((L^T x)^T (L^T x))^{\frac{1}{2}} = \|L^T x\|_2$$

故对任意向量 x 和 y , 总有

$$\|x+y\|_A = \|L^T(x+y)\|_2 = \|L^T x + L^T y\|_2 \leq$$

$$\|L^T x\|_2 + \|L^T y\|_2 = \|x\|_A + \|y\|_A$$

综上所述, $\|x\|_A = (x^T A x)^{\frac{1}{2}}$ 是一种向量范数。

15. 证明 因为

$$\|A\|_s = \max_{x \neq 0} \frac{\|Ax\|_s}{\|x\|_s}$$

由向量范数的等价性知, 存在常数 $C'_1, C'_2 > 0$, 使对任意 x , 有

$$C'_1 \|Ax\|_s \leq \|Ax\|_t \leq C'_2 \|Ax\|_s$$

$$C'_1 \|x\|_s \leq \|x\|_t \leq C'_2 \|x\|_s$$

故

$$\frac{C_1' \|x\|_s}{C_2' \|x\|_s} \leq \frac{\|Ax\|_t}{\|x\|_t} \leq \frac{C_2' \|Ax\|_s}{C_1' \|x\|_s}$$

令 $\frac{C_1'}{C_2'} = C_1, \frac{C_2'}{C_1'} = C_2$, 则有

$$C_1 \frac{\|Ax\|_s}{\|x\|_s} \leq \frac{\|Ax\|_t}{\|x\|_t} \leq C_2 \frac{\|Ax\|_s}{\|x\|_s}$$

$$C_1 \max_{x \neq 0} \frac{\|Ax\|_s}{\|x\|_s} \leq \max_{x \neq 0} \frac{\|Ax\|_t}{\|x\|_t} \leq C_2 \max_{x \neq 0} \frac{\|Ax\|_s}{\|x\|_s}$$

即

$$C_1 \|A\|_s \leq \|A\|_t \leq C_2 \|A\|_s$$

$$\|A^{-1}\|_\infty = \max_{x \neq 0} \frac{\|A^{-1}x\|_\infty}{\|x\|_\infty} \stackrel{A^{-1}x=y}{=} \max_{x \neq 0} \frac{\|y\|_\infty}{\|Ay\|_\infty} =$$

16. 证明 $\|A^{-1}\|_2^2 \|A\|_2^2 = \text{Cond}[(A)_2]^2$

$$\max_{y \neq 0} \frac{1}{\frac{\|Ay\|_\infty}{\|y\|_\infty}}$$

故

$$\frac{1}{\|A^{-1}\|_\infty} = \min_{y \neq 0} \frac{\|Ay\|_\infty}{\|y\|_\infty}$$

17. 证明 设 $\lambda \neq 0$, 则

$$\|A\|_\infty = \begin{cases} 3|\lambda|, & |\lambda| \geq \frac{2}{3} \\ 2, & |\lambda| \leq \frac{2}{3} \end{cases}$$

又

$$A^{-1} = \frac{1}{\lambda} \begin{bmatrix} 1 & -\lambda \\ -1 & 2\lambda \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \frac{2|\lambda|+1}{|\lambda|}$$

故

$$\text{Cond}(A)_\infty = \|A^{-1}\|_\infty \|A\|_\infty = \begin{cases} 6|\lambda|+3, & |\lambda| \geq \frac{2}{3} \\ 2(2+\frac{1}{|\lambda|}), & |\lambda| \leq \frac{2}{3} \end{cases}$$

从而当 $|\lambda| = \frac{2}{3}$ 时, 即 $|\lambda| = \pm \frac{2}{3}$ 时, $\text{Cond}(A)_\infty$ 有最小值, 且

$$\min \text{Cond}(A)_\infty = 7$$

18. 解 $A^{-1} = \begin{bmatrix} -9 & 8 \\ 9 & 9 \end{bmatrix}, \|A\|_\infty = 19, \|A^{-1}\|_\infty = 1$

$$\text{Cond}(A)_2 = \|A^{-1}\|_\infty \|A\|_\infty = 39601$$

$$A^T A = \begin{bmatrix} 19801 & 19602 \\ 19602 & 19405 \end{bmatrix}$$

故 $\text{Cond}(A)_2 = \|A^{-1}\|_2 \|A\|_2 =$

$$\sqrt{\frac{\lambda_{\max}(A^T A)}{\lambda_{\min}(A^T A)}} = 39205.9745$$

19. 证明 因 A 正交, 故 $A^T A = A A^T = I, A^{-1} = A^T$, 从而有

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho(I)} = 1$$

$$\|A^{-1}\|_2 = \|A^T\|_2 = \sqrt{\rho(A A^T)} = \sqrt{\rho(I)} = 1$$

故 $\text{Cond}(A)_2 = \|A^{-1}\|_2 \|A\|_2 = 1$

20. 证明 $\text{Cond}(AB) = \|(AB)^{-1}\| \|AB\| \leq \|A^{-1}\| \|B^{-1}\| \|A\| \|B\| =$

$$\|A^{-1}\| \|A\| \|B^{-1}\| \|B\| = \text{Cond}(A) \text{Cond}(B)$$

21. 证明 (1) $(A^T A)^T = A^T (A^T)^T = A^T A$

故 $A^T A$ 为对称矩阵。

又 A 非奇异, 故对任意向量 $x \neq 0$, 有 $Ax \neq 0$, 从而有

$$x^T A^T A x = (Ax)^T (Ax) > 0$$

即 $A^T A$ 为对称正定矩阵。

(2) $\text{Cond}(A^T A)_2 = \|(A^T A)^{-1}\|_2 \|A^T A\|_2 =$

$$\sqrt{\lambda_{\max}[(A^T A)^{-1}]^T (A^T A)^{-1}} \sqrt{\lambda_{\max}[(A^T A)^T (A^T A)]} =$$

$$\begin{aligned}
& \sqrt{\lambda_{\max} [((A^T A)^{-1})^2]} \sqrt{\lambda_{\max} [(A^T A)^2]} = \\
& \sqrt{\lambda_{\max}^2 (A^T A)^{-1}} \sqrt{\lambda_{\max}^2 (A^T A)} = \\
& [\sqrt{\lambda_{\max} (A^T A)^{-1}}]^2 [\sqrt{\lambda_{\max} (A^T A)}]^2 = \\
& \|A^{-1}\|_2^2 \|A\|_2^2 = [\text{Cond}(A)_2]^2
\end{aligned}$$

第六章课后习题解答

1.解: (a)因系数矩阵按行严格对角占优, 故雅可比法与高斯-塞德尔均收敛。

(b)雅可比法的迭代格式为

$$\begin{aligned}x_1^{(k+1)} &= -\frac{2}{5}x_2^{(k)} - \frac{1}{5}x_3^{(k)} - \frac{12}{5} \\x_2^{(k+1)} &= \frac{1}{4}x_1^{(k)} - \frac{1}{2}x_3^{(k)} + 5 \\x_3^{(k+1)} &= -\frac{1}{5}x_1^{(k)} + \frac{3}{10}x_2^{(k)} + \frac{3}{10}\end{aligned}$$

取 $x^{(0)} = (1, 1, 1)^T$, 迭代到17次达到精度要求

$$x^{(17)} = (-4.0000186, 2.9999915, 2.0000012)^T$$

高斯-塞德尔法的迭代格式为

$$\begin{aligned}x_1^{(k+1)} &= -\frac{2}{5}x_2^{(k)} - \frac{1}{5}x_3^{(k)} - \frac{12}{5} \\x_2^{(k+1)} &= \frac{1}{4}x_1^{(k+1)} - \frac{1}{2}x_3^{(k)} + 5 \\x_3^{(k+1)} &= -\frac{1}{5}x_1^{(k+1)} + \frac{3}{10}x_2^{(k+1)} + \frac{3}{10}\end{aligned}$$

取 $x^{(0)} = (1, 1, 1)^T$, 迭代到8次达到精度要求

$$x^{(8)} = (-4.0000186, 2.9999915, 2.0000012)^T$$

2: 解 (a) 雅可比法的迭代矩阵

$$B_J = D^{-1}(L + U) = \begin{bmatrix} 0 & -0.4 & -0.4 \\ 0.4 & 0 & -0.8 \\ 0.4 & -0.8 & 0 \end{bmatrix}$$

$$\|I - B_J\| = (I - 0.8)(I^2 + 0.8I - 0.32)$$

$\rho(B_J) = 1.0928203 > 1$, 故雅可比迭代法不收敛

高斯-塞德尔法迭代矩阵

$$B_S = (D - L)^{-1}U = \begin{bmatrix} 0 & -0.4 & -0.4 \\ 0 & 0.16 & -0.64 \\ 0 & 0.032 & 0.672 \end{bmatrix}$$

$$\rho(B_S) = \|B_S\|_\infty = 0.8 < 1$$

故高斯-塞德尔迭代法收敛。

(b) 雅可比法的迭代矩阵

$$B_J = D^{-1}(L + U) = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\|I - B_J\| = I^3, \quad \rho(B_J) = 0 < 1$$

故雅可比迭代法收敛。

高斯-塞德尔法的迭代矩阵

$$B_S = (D - L)^{-1}U = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\|I - B_S\| = I(I - 2)^2, \quad \rho(B_S) = 2 > 1$$

故高斯-塞德尔法不收敛。

3: 证明

必要条件: 由 $\lim_k A_k = A$, 知 $\lim_k a_{ij}^{(k)} = a_{ij}$, 从而有 $\|A_k - A\| \rightarrow 0$ ($k \rightarrow \infty$).

故对任意的 x , 有 $\|A_k x - Ax\| \leq \|A_k - A\| \|x\| \rightarrow 0$ ($k \rightarrow \infty$).

即 $A_k x \rightarrow Ax$, $\lim_k A_k x = Ax$.

充分条件: 对任意的 $x \in \mathbb{R}^n$, 有 $A_k x \rightarrow Ax$ ($k \rightarrow \infty$), 取 $x_i = (0, \dots, 0, 1, 0, \dots, 0)^T$

($i = 1, 2, \dots, n$)

$$A_k x_i = (a_{1i}^{(k)}, a_{2i}^{(k)}, \dots, a_{ni}^{(k)})^T \rightarrow Ax_i (k \rightarrow \infty)$$

$$Ax_i = (a_{1i}, a_{2i}, \dots, a_{ni})^T$$

$$\text{故 } a_{ji}^{(k)} \rightarrow a_{ji} (j = 1, 2, \dots, n; i = 1, 2, \dots, n)$$

$$\text{即 } A_k \rightarrow A, \lim_k A_k = A.$$

4. 解: 不一定, 因其谱半径 $\rho(B_j)$ 不一定小于 1。

对习题 2(a), A 对称, 又 $V_1 = 1 > 0, V_2 = 0.84 > 0, V_3 = |A| = 0.296 > 0$, 故 A 正定, 但其雅可比迭代法不收敛

5. 解答见例 6-4

6. 解:

SOR 迭代格式为

$$\begin{aligned} x_1^{(k+1)} &= x_1^{(k)} + w \left(-\frac{12}{5} - x_1^{(k)} - \frac{2}{5} x_2^{(k)} - \frac{1}{5} x_3^{(k)} \right) \\ x_2^{(k+1)} &= x_2^{(k)} + w \left(5 + \frac{1}{4} x_1^{(k+1)} - x_2^{(k)} - \frac{1}{2} x_3^{(k)} \right) \\ x_3^{(k+1)} &= x_3^{(k)} + w \left(\frac{3}{10} - \frac{1}{5} x_1^{(k+1)} + \frac{3}{10} x_2^{(k+1)} - x_3^{(k)} \right) \end{aligned}$$

取初始值 $x^0 = (1, 1, 1)^T$, 计算如表.

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	-2.6000000	3.5650000	1.8005500
2	-4.0274990	3.1400652	2.0228224
3	-4.0572814	2.9908481	2.0101219
4	-4.0042554	2.9935725	2.0000427
5	-3.9981193	2.9997612	1.9996013
6	-3.9996542	3.0002334	1.9999609
7	-4.0000424	3.0000314	2.0000122
8	-4.0000177	2.9999937	2.0000027

因 $\|x^{(8)} - x^{(7)}\|_y = 0.000377 < 10^{-4}$, 故取 $x^{(8)} = (-4.0000177, 2.9999937, 2.0000027)^T$.

由 $|m| < 1, |1 - w l(A)| < 1$ 得

$$0 < w < \frac{2}{l(A)}$$

故当 $0 < w < \frac{2}{b}$ 时, 更有 $0 < w < \frac{2}{l(A)}$, 从而有 $|m| < 1, l(B) < 1$, 迭代格式收敛.

8. 证明: 当 $-\frac{1}{2} < a < 1$ 时, 由

$$\det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = 1 - a^2 > 0, \det(A) = (1 - a)^2(1 + 2a) > 0$$

故 A 是正定的. 又雅可比法迭代矩阵

$$B_J = \begin{pmatrix} -a & -a \\ a & 0 \\ a & -a & 0 \end{pmatrix}$$

$$\det(I - B_J) = \begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix} = 1^3 - 3la^2 + 2a^3 = (1 - a)^2(1 + 2a)$$

故 $l(B_J) = |2a|$, 故当 $-\frac{1}{2} < a < \frac{1}{2}$ 时, 雅可比迭代法收敛.

9. 证明: G 相似与它的若当标准行 J , 即存在可逆阵 P , 使 $G = P^{-1}JP$

由于 G 的特征值全为零, 故 J 一定有如下形式

$$J = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \text{ 其中 } n_i = n$$

方程组 $x = Gx + g$ 等价于 $(I - G)x = g$, 由于 $l(G) = 0$, 故 $l(I - G) = 1 - l(G) = 1 > 0$,

从而 $I - G$ 非奇异, 即 $(I - G)x = g$ 有唯一解 x^* . 于是

$$x^* = Gx^* + g$$

与所述迭代格式相减, 有 $x^{(k+1)} - x^* = G(x^{(k)} - x^*)$

$$\text{故 } x^{(n)} - x^* = G^n(x^{(0)} - x^*)$$

$$\text{又 } G^n = P^{-1}J^nP = 0$$

$$\text{故 } x^{(n)} - x^* = 0, \text{ 即 } x^{(n)} = x^*$$

因此, 至多迭代 n 次即可收敛到方程组的解.

10. 证明: 因 A 严格对角占优, 故 $a_{ii} > \sum_{j \neq i} |a_{ij}|$ ($i = 1, 2, \dots, n$), 且 A 非奇异.

$$SOR \text{ 法的迭代矩阵 } L_w = (D - wL)^{-1}((1 - w)D + wU)$$

其中 $A = D - L - U$, 而 $D, -L, -U$ 分别为 A 的对角, 严格下三角与严格上三角. 只需证明 $0 < w < 1$ 时,

$$l(L_w) < 1 \text{ 即可.}$$

用反证法: 设 L_w 有一个特征值 l 满足 $|l| \geq 1$, 则有

$$\det(I - L_w) = 0$$

$$\text{从而有 } \det\{(D - wL)^{-1}[(D - wL) - \frac{1}{l}((1 - w)D + wU)]\} = 0$$

$$\det\{(D - wL)^{-1}\} \det[(1 - \frac{1}{l}(1 - w))D - wL - \frac{w}{l}U] = 0$$

因对 A 严格对角占优, 有 $\det(D - wL)^{-1} \neq 0$

$$\text{另 } C = (1 - \frac{1}{l}(1 - w))D - wL - \frac{w}{l}U, \text{ 则}$$

$$|C_{ii}| = |1 - \frac{1}{l}(1 - w)| + |a_{ii}| \geq [1 - \frac{1}{|l|}(1 - w)] + |a_{ii}| \geq w|a_{ii}| + w \sum_{j=i+1}^n |a_{ij}| \geq w \sum_{j=1}^{i-1} |a_{ij}| + \frac{w}{|l|} \sum_{j=i+1}^n |a_{ij}|$$

这表明 C 在 $0 < w < 1$ 时也严格对角占优, 故 $\det C \neq 0$. 这与 $\det(I - L_w) = 0$ 矛盾, 故假设不成立, 从而 $|l| < 1$,

即 $l(L_w) < 1$, SOR 迭代法收敛

第七章

1、用二分法求方程 $x^2 - x - 1 = 0$ 的正根,要求误差小于 0.05.

解 设 $f(x) = x^2 - x - 1$, $f(1) = -1 < 0$, $f(2) = 1 > 0$, 故 $[1, 2]$ 为 $f(x)$ 的有根区间. 又

$f'(x) = 2x - 1$, 故当 $0 < x < \frac{1}{2}$ 时, $f(x)$ 单增, 当 $x > \frac{1}{2}$ 时 $f(x)$ 单增. 而

$f(\frac{1}{2}) = -\frac{5}{4}$, $f(0) = -1$, 由单调性知 $f(x) = 0$ 的惟一正根 $x^* \in (1, 2)$. 根据二分法的误差估

计式(7.2)知要求误差小于 0.05, 只需 $\frac{1}{2^{k+1}} < 0.05$, 解得 $k + 1 > 5.322$, 故至少应二分 6 次. 具体计算结果见表 7-7.

表 7-7

k	a_k	b_k	x_k	$f(x_k)$ 的符号
0	1	2	1.5	-
1	1.5	2	1.75	+
2	1.5	1.75	1.625	+
3	1.5	1.625	1.5625	-
4	1.5625	1.625	1.59375	-
5	1.59375	1.625	1.609375	-

-

即 $x^* \approx x_5 = 1.609375$.

2、为求 $x^3 - x^2 - 1 = 0$ 在 $x_0 = 1.5$ 附近的一个根, 设将方程改写成下列等价形式, 并建立相应的迭代公式:

$$(1) \quad x = 1 + \frac{1}{x^2}, \text{迭代公式} \quad x_{k+1} = 1 + \frac{1}{x_k^2};$$

$$(2) \quad x^3 = 1 + x^2, \text{迭代公式} \quad x_{k+1} = (1 + x_k^2)^{\frac{1}{3}};$$

$$(3) \quad x^2 = \frac{1}{x-1}, \text{迭代公式} \quad x_{k+1} = \frac{1}{\sqrt{x_k-1}}.$$

试分析每种迭代公式的收敛性, 并选取一种公式求出具有四位有效数字的近似根.

解 取 $x_0 = 1.5$ 的邻域 $[1.3, 1.6]$ 来考察.

(1) 当 $x \in [1.3, 1.6]$ 时, $\varphi(x) = 1 + \frac{1}{x^2} \in [1.3, 1.6], |\varphi'(x)| = \left| -\frac{2}{x^3} \right| \leq \frac{2}{1.3^3} = L < 1$, 故迭代公式

$x_{k+1} = 1 + \frac{1}{x_k^2}$ 在 $[1.3, 1.6]$ 上整体收敛.

(2) 当 $x \in [1.3, 1.6]$ 时

$$\varphi(x) = (1 + x^2)^{1/3} \in [1.3, 1.6]$$

$$|\varphi'(x)| = \frac{2}{3} \left| \frac{x}{(1 + x^2)^{5/3}} \right| < \frac{2}{3} \frac{1.6}{(1 + 1.3^2)^{5/3}} \leq L = 0.522 < 1$$

故 $x_{k+1} = (1 + x_k^2)^{1/3}$ 在 $[1.3, 1.6]$ 上整体收敛.

(3) $\varphi(x) = \frac{1}{\sqrt{x-1}}, |\varphi'(x)| = \left| \frac{-1}{2(x-1)^{3/2}} \right| > \frac{1}{2(1.6-1)} > 1$ 故 $x_{k+1} = \frac{1}{\sqrt{x_k-1}}$ 发散.

由于(2)的 L 叫小, 故取(2)中迭代式计算. 要求结果具有四位有效数字, 只需

$$|x_k - x^*| \leq \frac{L}{1-L} |x_k - x_{k-1}| < \frac{1}{2} \times 10^{-3}$$

即

$$|x_k - x_{k-1}| < \frac{1-L}{L} \times \frac{1}{2} \times 10^{-3} < 0.5 \times 10^{-3}$$

取 $x_0 = 1.5$ 计算结果见表 7-8.

表 7-8

k		k	
1	1.481248034	4	1.467047973
2	1.472705730	5	1.466243010
3	1.468817314	6	1.465876820

由于 $|x_6 - x_5| < \frac{1}{2} \times 10^{-3}$, 故可取 $x^* \approx x_6 = 1.466$.

3、比较求 $e^x + 10x - 2 = 0$ 的根到三位小数所需的计算量:

(1) 在区间 $[0, 1]$ 内用二分法;

(2) 用迭代法 $x_{k+1} = \frac{2 - e^{x_k}}{10}$, 取初值 $x_0 = 0$. x_k

解 (1) 因 $x^* \in [0, 1], f(0) < 0, f(1) > 0$, 故 $0 < x^* < 1$, 用二分法计算结果见表 7-9.

表 7-9

k	a_k	b_k	x_k	$f(x_k)$ 的符号	$\frac{1}{2^{k+1}}$
0	0	1	0.5	+	0.5
1	0	0.5	0.25	+	0.25
2	0	0.25	0.125	+	0.125
3	0	0.125	0.0625	-	0.0625
4	0.0625	0.125	0.09375	+	0.03125
5	0.0625	0.09375	0.078125	-	0.015625
6	0.0778125	0.09375	0.0859375	-	0.0078125
7	0.0859375	0.09375	0.08984375	-	0.00390625
8	0.08984375	0.09375	0.091796875	+	0.001953125
9	0.08984375	0.091796875	0.090820312	+	0.000976562
10	0.08984375	0.090820312	0.090332031	-	0.000488281
11	0.090332031	0.090820312	0.090576171	+	0.00024414
12	0.090332031	0.090576171	0.090454101	-	0.00012207
13	0.090454101	0.090576171	0.090515136	-	0.000061035
14	0.090515136	0.090576171	0.090545653	+	0.000030517

此时 $|x_{14} - x^*| \leq \frac{1}{2^{15}} = 0.000030517 < \frac{1}{2} \times 10^{-4}$, $x^* \approx x_{14}$ 具有三位有效数字.

(2)当 $x \in [0, 0.5]$ 时, $\varphi(x) \in [0, 0.5], |\varphi'(x)| = \frac{1}{10} |-e^x| \leq L = 0.825$, 故迭代式

$x_{k+1} = \frac{1}{10}(2 - e^{x_k})$ 在 $[0, 0.5]$ 上整体收敛. 取 $x_0 = 0$, 迭代计算结果如表 7-10 所示.

表 7-10

k	x_k	k	x_k
1	0.1	4	0.090512616
2	0.089482908	5	0.090526468
3	0.090639135	6	0.090524951

此时 $|x_6 - x^*| \leq \frac{L}{1-L} |x_6 - x_5| \leq 0.00000720 < \frac{1}{2} \times 10^{-4}$, 故 $x^* \approx x_6$ 精确到三位小数.

4、给定函数 $f(x)$, 设对一切 x , $f'(x)$ 存在且 $0 < m \leq f'(x) \leq M$, 证明对于范围

$0 < \lambda < \frac{2}{M}$ 内的任意定数 λ , 迭代过程 $x_{k+1} = x_k - \lambda f(x_k)$ 均收敛于 $f(x) = 0$ 的根 x^* .

证明 由于 $f'(x) > 0$, $f(x)$ 为单增函数, 故方程 $f(x) = 0$ 的根 x^* 是惟一的 (假定方程有根 x^*). 迭代函数 $\varphi(x) = x - \lambda f(x)$, $|\varphi'(x)| = |1 - \lambda f'(x)|$, 由 $0 < m \leq f'(x) \leq M$ 及

$0 < \lambda < \frac{2}{M}$ 得, $0 < \lambda m \leq \lambda f'(x) \leq \lambda M < 2$, $-1 < 1 - \lambda M \leq 1 - \lambda f'(x) \leq 1 - \lambda m < 1$, 故

$|\varphi'(x)| \leq L = \max\{|1 - \lambda m|, |1 - \lambda M|\} < 1$, 由此可得

$|x_k - x^*| \leq L |x_{k-1} - x^*| \leq \dots \leq L^k |x_0 - x^*| \rightarrow 0 (k \rightarrow \infty)$

即 $\lim_{k \rightarrow \infty} x_k = x^*$.

5、用斯蒂芬森迭代法计算第 2 题中(2)的近似根, 精确到 10^{-5} .

解 记第 2 题中(2)的迭代函数 $\varphi_2(x) = (1 + x^2)^{\frac{1}{2}}$, (3)的迭代函数为 $\varphi_3(x) = \frac{1}{\sqrt{x-1}}$, 利用迭代式(7.11), 计算结果见表 7-11.

表 7-11

k	加速 $\varphi_2(x)$ 的结果 x_k	k	加速 $\varphi_3(x)$ 的结果 x_k
0	1.5	0	1.5
1	1.465558485	1	1.467342286
2	1.465571233	2	1.465576085
3	1.465571232	3	1.465571232
		4	1.465571232

6、设 $\varphi(x) = x - p(x)f(x) - q(x)f^2(x)$, 试确定函数 $p(x)$ 和 $q(x)$, 使求解 $f(x) = 0$ 且以 $\varphi(x)$ 为迭代函数的迭代法至少三阶收敛.

解 要求 $x_{k+1} = \varphi(x_k)$ 三阶收敛到 $f(x) = 0$ 的根 x^* , 根据定理 7.4, 应有

$\varphi(x^*) = x^*$, $\varphi'(x^*) = 0$, $\varphi''(x^*) = 0$. 于是由

$$x^* = x^* - p(x^*)f(x^*) - q(x^*)f^2(x^*) = x^*$$

$$\varphi'(x^*) = 1 - p(x^*)f'(x^*) = 0$$

$$\varphi''(x^*) = -2p'(x^*)f'(x^*) - p(x^*)f''(x^*) - 2q(x^*)[f'(x^*)]^2 = 0$$

得

$$p(x^*) = \frac{1}{f'(x^*)}, q(x^*) = \frac{1}{2} \frac{f''(x^*)}{[f'(x^*)]^3}$$

故取

$$p(x) = \frac{1}{f'(x)}, q(x) = \frac{1}{2} \frac{f''(x)}{[f'(x)]^3}$$

即迭代至少三阶收敛.

7、用下列方法求 $f(x) = x^3 - 3x - 1 = 0$ 在 $x_0 = 2$ 附近的根. 根的准确值

$x^* = 1.87938524\dots$, 要求计算结果准确到四位有效数字.

(1) 用牛顿法;

(2) 用弦截法, 取 $x_0 = 2, x_1 = 1.9$;

(3) 用抛物线法, 取 $x_0 = 1, x_1 = 3, x_2 = 2$.

解 $f(1) < 0, f(2) > 0, f(x) = 3x^2 - 3 = 3(x^2 - 1) \geq 0, f''(x) = 6x > 0, \forall x \in [1, 2]$.

(1) 取 $x_0 = 2$, 用牛顿迭代法

$$x_{k+1} = x_k - \frac{x_k^3 - 3x_k - 1}{3x_k^2 - 3} = \frac{2x_k^3 + 1}{3(x_k^2 - 1)}$$

计算得 $x_1 = 1.888888889, x_2 = 1.879451567, |x_2 - x^*| < \frac{1}{2} \times 10^{-3}$, 故

$$x^* \approx x_2 = 1.879451567$$

(2) 取 $x_2 = 2, x_1 = 1.9$, 利用弦截法

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})f(x_k)}{f(x_k) - f(x_{k-1})}$$

得, $x_2 = 1.981093936, x_3 = 1.880840630, x_4 = 1.879489903, |x_4 - x^*| < \frac{1}{2} \times 10^{-3}$, 故取

$$x^* \approx x_4 = 1.879489903$$

(3) $x_0 = 1, x_1 = 3, x_2 = 2$. 抛物线法的迭代式为

$$x_{k+1} = x_k - \frac{2f(x_k)}{w + \text{sign}(w)\sqrt{w^2 - 4f(x_k)f[x_k, x_{k-1}, x_{k-2}]}}$$

$$w = f[x_k, x_{k-1}] + f[x_k, x_{k-1}, x_{k-2}](x_k - x_{k-1})$$

迭代结果为: $x_3 = 1.953967549, x_4 = 1.87801539, x_5 = 1.879386866$ 已达四位有效数字.

8、分别用二分法和牛顿迭代法求 $x - \tan x = 0$ 的最小正根.

解 显然 $x^* = 0$ 满足 $x - \tan x = 0$. 另外当 $|x|$ 较小时, $\tan x = x + \frac{1}{3}x^3 + \dots + \frac{x^{2k+1}}{2k+1} + \dots$, 故

当 $x \in (0, \frac{\pi}{2})$ 时, $\tan x > x$, 因此, 方程 $x - \tan x = 0$ 的最小正根应在 $(\frac{\pi}{2}, \frac{3\pi}{2})$ 内.

记 $f(x) = x - \tan x, x \in (\frac{\pi}{2}, \frac{3\pi}{2})$, 容易算得 $f(4) = 2.842... > 0, f(4.6) = -4.26... < 0$, 因此

$[4, 4.6]$ 是 $f(x) = 0$ 的有限区间.

对于二分法, 计算结果见表 7-12.

表 7-12

k	a_k	b_k	x_k	$f(x_k)$ 的符号
0	4.0	4.6	4.3	+
1	4.3	4.6	4.45	+
2	4.45	4.6	4.525	-
3	4.45	4.525	4.4875	+
4	4.4875	4.525	4.50625	-
5	4.4875	4.50625	4.496875	-
6	4.4875	4.496875	4.4921875	+
7	4.4921875	4.496875	4.49453125	-
8	4.4921875	4.49453125	4.493359375	+
9	4.493359375	4.49453125	4.493445313	-

此时 $|x_9 - x^*| < \frac{1}{2^{10}} = \frac{1}{1024} < 10^{-3}$.

若用牛顿迭代法求解, 由于 $f'(x) = -(\tan x)^2 < 0, f''(x) = -2 \tan x \frac{1}{\cos^2 x} < 0$, 故取

$x_0 = 4.6$, 迭代计算结果如表 7-13 所示.

表 7-13

k	x_k	k	x_k
1	4.545732122	4	4.493412197
2	4.506145588	5	4.493409458
3	4.49417163	6	4.493409458

所以 $x - \tan x = 0$ 的最小正根为 $x^* \approx 4.493409458$.

9、研究求 \sqrt{a} 的牛顿公式

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right), x_0 > 0$$

证明对一切 k 序列是递减的.

证法一 用数列的办法, 因 $x_0 > 0$ 由 $x_k = \frac{1}{2} \left(x_{k-1} + \frac{a}{x_{k-1}} \right)$ 知 $x_k > 0$, 且

$$x_k = \frac{1}{2} \left(\sqrt{x_{k-1}} + \sqrt{\frac{a}{x_{k-1}}} \right)^2 + \sqrt{a}, k = 1, 2, 3, \dots$$

. 又由

$$\frac{x_{k+1}}{x_k} = \frac{1}{2} + \frac{a}{2x_k^2} = 1, \forall k \geq 1$$

故 $x_{k+1} \leq x_k$, 即 $\{x_k\}_{k=1}^{\infty}$ 单减有下界 \sqrt{a} . 根据单调原理知, $\{x_k\}$ 有极限. 易证起极限为 \sqrt{a} .

证法二 设 $f(x) = x^2 - a (a > 0)$. 易知 $f(x) = 0$ 在 $[0, +\infty)$ 内有惟一实根 $x^* = \sqrt{a}$. 对 $f(x)$ 应用牛顿迭代法, 得

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right), k = 0, 1, 2, \dots$$

利用例 7-9 的结论知, 当 $x_0 > \sqrt{a}$ 时, $\{x_k\}_{k=0}^{\infty}$ 单减有下界 \sqrt{a} , 且 $\lim_{k \rightarrow \infty} x_k = \sqrt{a}$. 当 $x_0 \in (0, \sqrt{a})$ 时,

$$x_1 = \frac{1}{2} \left(x_0 + \frac{a}{x_0} \right) = \frac{1}{2} \left[\sqrt{x_0} - \frac{a}{\sqrt{x_0}} \right]^2 + \sqrt{a} > \sqrt{a}$$

此时, 从 x_1 起, $\{x_k\}_{k=1}^{\infty}$ 单减有下界 \sqrt{a} , 且极限为 \sqrt{a} .

10、对于 $f(x) = 0$ 的牛顿公式 $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, 证明

$$R_k = \frac{x_k - x_{k-1}}{(x_{k-1} - x_{k-2})^2}$$

收敛到 $-\frac{f''(x^*)}{2f'(x^*)}$, 这里 x^* 为 $f(x)=0$ 的根.

证明见例 7-10.

11、用牛顿迭代法和求重根的牛顿迭代法(7.15)和(7.16)(书中式(4.13),(4.14))计算方程

$$f(x) = (\sin x - \frac{x}{2})^2 = 0 \quad \text{的一个近似根, 准确到 } 10^{-5}, \text{ 初始值 } x_0 = \frac{\pi}{2}.$$

解 $f(x) = (\sin x - \frac{x}{2})^2$ 的根 x^* 为 2 重根, 即

$$f'(x) = 2(\sin x - \frac{x}{2})(\cos x - \frac{1}{2})$$

用牛顿法迭代公式为

$$x_{k+1} = x_k - \frac{(\sin x_k - \frac{x_k}{2})^2}{2(\sin x_k - \frac{x_k}{2})(\cos x_k - \frac{1}{2})} =$$

$$x_k - \frac{\sin x_k - \frac{x_k}{2}}{2\cos x_k - 1}, k = 0, 1, 3, \dots$$

令 $x_0 = \frac{\pi}{2}$, 则 $x_1 = 1.785398, x_2 = 1.844562, \dots$, 迭代到

$$x_{20} = 1.895494, |x^* - 1.89549| < 10^{-5}.$$

用求重根的迭代公式(7.15), 迭代迭代公式为

$$x_{k+1} = x_k - \frac{\sin x_k - \frac{x_k}{2}}{\cos x_k - \frac{1}{2}}, k = 0, 1, 2, \dots$$

取 $x_0 = \frac{\pi}{2}$, 则 $x_1 = 2.000000, x_2 = 1.900996, x_3 = 1.895512, x_4 = 1.895494, x_5 = 1.895494$.

四次迭代达到上面 x_{20} 的结果.

若用公式(7.16), 则有

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{[f'(x_k)]^2 - f(x_k)f''(x_k)}$$

将 $f(x), f'(x)$ 及 $f''(x) = 2(\cos x - \frac{1}{2})^2 - 2\sin x(\sin x - \frac{1}{2})$ 代入上述迭代公式,得

$$x_{k+1} = x_k - \frac{(\sin x_k - \frac{x_k}{2})(\cos x_k - \frac{1}{2})}{(\cos x_k - \frac{1}{2})^2 + \sin x_k(\sin x_k - \frac{x_k}{2})}$$

取 $x_0 = \frac{\pi}{2}$, 得 $x_1 = 1.801749, x_2 = 1.889630, x_3 = 1.895474, x_4 = 1.895494, x_5 = 1.895494$.

结果与公式(7.15)的相同.

12、应用牛顿迭代法于方程 $x^3 - a = 0$, 导出求立方根 $\sqrt[3]{a}$ 的迭代公式, 并讨论其收敛性.

解 设 $f(x) = x^3 - a, f'(x) = 3x^2, f''(x) = 6x$, 牛顿迭代公式为

$$x_{k+1} = x_k - \frac{x_k^3 - a}{3x_k^2} = \frac{2x_k^3 + a}{3x_k^2}, k = 0, 1, 2, \dots$$

当 $x > 0, f'(x) > 0, f''(x) > 0$; 当 $x < 0$ 时, $f'(x) > 0, f''(x) < 0$, 因此, 对于 $a > 0$, 当

$x_0 > \sqrt[3]{a}$ 时 $f(x_0)f''(x_0) > 0$, 根据例 7-9 的结论知, 牛顿序列 $\{x_k\}$ 收敛到 $\sqrt[3]{a}$. 当

$x_0 \in (0, \sqrt[3]{a})$ 时,

$$x_1 - \sqrt[3]{a} = \frac{2x_0^3 + a}{3x_0^2} - \sqrt[3]{a} = \frac{(\sqrt[3]{a} - x_0)^2}{3x_0^2}(\sqrt[3]{a} + 2x_0) > 0, x_1 > \sqrt[3]{a}$$

从 x_1 起, 牛顿序列 $\{x_k\}$ 收敛到 $\sqrt[3]{a}$.

对于 $a < 0$, 当 $x_0 < \sqrt[3]{a} < 0$ 时 $f(x_0)f''(x_0) > 0$, 由牛顿法产生的序列 $\{x_k\}$ 单增趋于 $\sqrt[3]{a}$. 当

$x_0 \in (\sqrt[3]{a}, 0)$ 时,

$$x_1 - \sqrt[3]{a} = \frac{(\sqrt[3]{a} - x_0)^2}{3x_0^2}(\sqrt[3]{a} + 2x_0) < 0, x_1 < \sqrt[3]{a}$$

之后迭代也收敛.

当 $a = 0$ 时, 迭代式变为

$$x_{k+1} = x_k - \frac{x_k^3}{3x_k^2} = \frac{2}{3}x_k$$

该迭代对任何 $x_0 \in R$ 均收敛, 但收敛速度是线性的.

13、应用牛顿法于方程 $f(x) = 1 - \frac{a}{x^2} = 0$, 导出求 \sqrt{a} 的迭代公式, 并用此公式求 $\sqrt{115}$ 的值.

解 $f(x) = 1 - \frac{a}{x^2}, f'(x) = \frac{2a}{x^3}, x \neq 0$, 所以牛顿迭代公式有

$$x_{k+1} = x_k - \frac{1 - \frac{a}{x_k^2}}{\frac{2a}{x_k^3}} = \frac{1}{2} x_k \left(3 - \frac{x_k^2}{a} \right), k = 0, 1, 2, \dots$$

易知 $f''(x) = \frac{6a}{x^4} < 0$. 故取 $x_0 \in (0, \sqrt{a})$ 时, 迭代收敛.

对于 $\sqrt{115}$, 取 $x_0 = 9$, 迭代计算, 得

$$x_1 = 10.33043478, x_2 = 10.70242553, x_3 = 10.7237414, \\ x_4 = 10.72380529, x_5 = 10.72380529$$

故 $\sqrt{115} \approx 10.72380529$.

14、应用牛顿法于方程 $f(x) = x^n - a = 0$ 和 $f(x) = 1 - \frac{a}{x^n} = 0$, 分别导出求 $\sqrt[n]{a}$ 的迭代公式, 并求

$$\lim_{k \rightarrow \infty} \frac{\sqrt[n]{a} - x_{k+1}}{(\sqrt[n]{a} - x_k)^2}$$

解 对于 $f(x) = x^n - a, f'(x) = nx^{n-1}$, 因此牛顿迭代法为

$$x_{k+1} = x_k - \frac{x_k^n - a}{nx_k^{n-1}} = \frac{1}{n} \left[(n-1)x_k + \frac{a}{x_k^{n-1}} \right], k = 0, 1, 2, \dots$$

根据定理 7.4 知

$$(\varphi^n(\sqrt[n]{a}) = \frac{n-1}{\sqrt[n]{a}}) \\ \lim_{k \rightarrow \infty} \frac{(\sqrt[n]{a} - x_{k+1})}{(\sqrt[n]{a} - x_k)^2} = -\frac{1}{2} \frac{n-1}{\sqrt[n]{a}}$$

对于 $f(x) = 1 - \frac{a}{x^n}, f'(x) = \frac{na}{x^{n+1}}$, 牛顿法公式为

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \frac{x_k}{n} \left[(n+1) - \frac{x_k^n}{a} \right], k = 0, 1, 2, \dots$$

根据定理 7.4 知

$$(\varphi^n(\sqrt[n]{a}) = -\frac{n+1}{\sqrt[n]{a}})$$

$$\lim_{k \rightarrow \infty} \frac{\sqrt[n]{a} - x_{k+1}}{(\sqrt[n]{a} - x_k)^2} = \frac{1}{2} \frac{n+1}{\sqrt[n]{a}}$$

15、证明迭代公式

$$x_{k+1} = \frac{x_k(x_k^2 + 3a)}{3x_k^2 + a}$$

是计算 \sqrt{a} 的三阶方法. 假定初值 x_0 充分靠近根 $x^* = \sqrt{a}$, 求

$$\lim_{k \rightarrow \infty} \frac{\sqrt{a} - x_{k+1}}{(\sqrt{a} - x_k)^2}$$

证明 记 $\varphi(x) = \frac{x(x^2 + 3a)}{3x^2 + a}$, 则迭代式为 $x_{k+1} = \varphi(x_k)$ 且 $\varphi(\sqrt{a}) = \sqrt{a}$.

由 $\varphi(x)$ 的定义, 有

$$(3x^2 + a)\varphi(x) = x(x^2 + 3a)$$

对上式两端连续求导三次, 得

$$\begin{aligned} 6x\varphi(x) + (3x^2 + a)\varphi'(x) &= 3x^2 + 3a \\ 6\varphi(x) + 12x\varphi'(x) + (3x^2 + a)\varphi''(x) &= 6x \\ 18\varphi'(x) + 18x\varphi''(x) + (3x^2 + a)\varphi'''(x) &= 6 \end{aligned}$$

代 $x = \sqrt{a}$ 依次入上三式, 并利用 $\varphi(\sqrt{a}) = \sqrt{a}$, 得

$$\varphi'(\sqrt{a}) = 0, \varphi''(\sqrt{a}) = 0, \varphi'''(\sqrt{a}) = \frac{3}{2a} \neq 0$$

所以由定理 7.4 知, 迭代公式是求 \sqrt{a} 的三阶方法且

$$\lim_{k \rightarrow \infty} \frac{\sqrt{a} - x_{k+1}}{(\sqrt{a} - x_k)^2} = \frac{1}{3!} \frac{3}{2a} = \frac{1}{4a}$$

16、用牛顿法解方程组

$$\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 1 \end{cases}$$

取 $(x^{(0)}, y^{(0)})^T = (1.6, 1.2)^T$.

解 记 $f_1(x, y) = x^2 + y^2 - 4, f_2(x, y) = x^2 - y^2 - 1$, 则

$$F'(x, y) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}, [F'(x, y)]^{-1} = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4y} & -\frac{1}{4y} \end{bmatrix}$$

牛顿迭代法为

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} - [F'(x^{(k)}, y^{(k)})]^{-1} \begin{pmatrix} f_1(x^{(k)}, y^{(k)}) \\ f_2(x^{(k)}, y^{(k)}) \end{pmatrix}$$

代入初值 $(x^{(0)}, y^{(0)})^T = (1.6, 1.2)^T$, 迭代计算, 得

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} 1.581250000 \\ 1.225000000 \end{pmatrix}, \begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} = \begin{pmatrix} 1.58113834 \\ 1.224744898 \end{pmatrix}$$

$$\begin{pmatrix} x^{(3)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} 1.581138830 \\ 1.224744871 \end{pmatrix}, \begin{pmatrix} x^{(4)} \\ y^{(4)} \end{pmatrix} = \begin{pmatrix} 1.581138830 \\ 1.224744871 \end{pmatrix}$$

第八章 常微分方程初值问题数值解法

1、解：欧拉法公式为

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + h(x_n^2 + 100y_n^2), n = 0, 1, 2$$

代 $y_0 = 0$ 入上式, 计算结果为

$$y(0.1) \approx y_1 = 0.0, y(0.2) \approx y_2 = 0.0010, y(0.3) \approx y_3 = 0.00501$$

2、解：改进的欧拉法为

$$y_{n+1} = y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

将 $f(x, y) = x^2 + x - y$ 代入上式, 得

$$y_{n+1} = (1 - h + \frac{h^2}{2})y_n + \frac{h}{2}[(1 - h)x_n(1 + x_n) + (1 + x_{n+1})x_{n+1}]$$

同理, 梯形法公式为

$$y_{n+1} = \frac{2-h}{2+h} y_n + \frac{h}{2+h} [x_n(1+x_n) + x_{n+1}(1+x_{n+1})]$$

将 $y_0 = 0, h = 0.1$ 代入上二式, , 计算结果见表 9—5

表 9—5

x_n	改进欧拉 y_n	$ y(x_n) - y_n $	梯形法 y_n	$ y(x_n) - y_n $
0. 1	0. 005500	$0.337418036 \times 10^{-3}$	0. 005238095	$0.755132781 \times 10^{-4}$
0. 2	0. 021927500		0. 021405896	
0. 3	0. 050144388	$0.658253078 \times 10^{-3}$	0. 049367239	$0.136648778 \times 10^{-3}$
0. 4	0. 090930671		0. 089903692	
0. 5	0. 144992257	$0.962608182 \times 10^{-3}$	0. 143722388	$0.185459653 \times 10^{-3}$
		$0.125071672 \times 10^{-2}$		$0.223738443 \times 10^{-3}$
		$0.152291668 \times 10^{-2}$		$0.253048087 \times 10^{-3}$

可见梯形方法比改进的欧拉法精确。

3、证明：梯形公式为

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

代 $f(x, y) = -y$ 入上式, 得

$$y_{n+1} = y_n + \frac{h}{2}[-y_n - y_{n+1}]$$

解得

$$y_{n+1} = (\frac{2-h}{2+h})y_n = (\frac{2-h}{2+h})^2 y_{n-1} = \dots = (\frac{2-h}{2+h})^{n+1} y_0$$

因为 $y_0 = 1$, 故

$$y_n = (\frac{2-h}{2+h})^n$$

对 $\forall x > 0$, 以 h 为步长经 n 步运算可求得 $y(x)$ 的近似值 y_n , 故

$$x = nh, n = \frac{x}{h}, \text{代入上式有}$$

$$y_n = (\frac{2-h}{2+h})^{\frac{x}{h}}$$

$$\lim_{h \rightarrow 0} y_n = \lim_{h \rightarrow 0} \left(\frac{2-h}{2+h} \right)^{\frac{x}{h}} = \lim_{h \rightarrow 0} \left(1 - \frac{2h}{2+h} \right)^{\frac{x}{h}} = \lim_{h \rightarrow 0} \left[\left(1 - \frac{2h}{2+h} \right)^{\frac{2+h}{2h}} \right]^{\frac{2h}{2+h} \frac{x}{h}} = e^{-x}$$

4、解：令 $y(x) = \int_0^x e^{t^2} dt$ ，则有初值问题

$$y' = e^{x^2}, y(0) = 0$$

对上述问题应用欧拉法，取 $h=0.5$ ，计算公式为

$$y_{n+1} = y_n + 0.5e^{x_n^2}, n = 0, 1, 2, 3$$

由 $y(0) = y_0 = 0$ ，得

$$y(0.5) \approx y_1 = 0.5, y(1.0) \approx y_2 = 1.142012708$$

$$y(1.5) \approx y_3 = 2.501153623, y(2.0) \approx y_4 = 7.245021541$$

5、解：四阶经典龙格-库塔方法计算公式见式 (9.7)。对于问题 (1)，

$$f(x, y) = x + y; \text{ 对于问题(2), } f(x, y) = \frac{3y}{1+x}。 \text{ 取 } h=0.2, y_0 = y(0) = 1,$$

分别计算两问题的近似解见表 9-6。

表 9-6

x_n	(1) 的解 y_n	(2) 的解 y_n
0.2	1.242800000	1.727548209
0.4	1.583635920	2.742951299
0.6	2.044212913	4.094181355
0.8	2.651041652	5.829210728
1.0	3.436502273	7.996012143

6、证明：根据定义 9.2，只要证明 $T_{n+1} = o(h^3)$ 即可。而

$$T_{n+1} = y(x+h) - y(x) - h\varphi(x, y, h)$$

$$\varphi(x, y, h) = \frac{1}{2} [f(x + th, y + thy'(x)) + f(x + (1-t)h, y + (1-t)hy'(x))]$$

因此只须将 $y(x+h)$ 和 $\varphi(x, y, h)$ 都在 x 处展开即可得到余项表达式:

$$f(x + th, y + thy'(x)) = f(x, y) + th \frac{\partial f}{\partial x}(x, y) + thy'(x) \frac{\partial f}{\partial x}(x, y) + o(h^2)$$

$$f(x + (1-t)h, y + (1-t)hy'(x)) = f(x, y) + (1-t)h \frac{\partial f}{\partial x}(x, y)$$

$$+ (1-t)hy'(x) \frac{\partial f}{\partial x}(x, y) + o(h^2)$$

所以

$$T_{n+1} = y(x) + hy'(x) + \frac{1}{2}h^2 y''(x) + \frac{1}{3!}h^3 y'''(\xi) - y(x) - \frac{1}{2}h[2f(x, y) + h \frac{\partial f}{\partial x}(x, y) + hy'(x) \frac{\partial f}{\partial x}(x, y) + o(h^2)] = o(h^3)$$

故对任意参数 t , 题中方法是二阶的。

7、解:

$$\begin{aligned}
T_{n+1} &= y(x_{n+1}) - y(x_n) - hf(x_n + \frac{h}{2}, y(x_n) + \frac{h}{2} y'(x_n)) = \\
&= y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{1}{3!} h^3 y'''(x_n) + o(h^4) - \\
&= y(x_n) - h \left\{ f(x_n, y_n) + \frac{h}{2} \frac{\partial f(x_n, y(x_n))}{\partial x} + \right. \\
&\quad \left. \frac{1}{2!} \left[\left(\frac{h}{2} \right)^2 \frac{\partial^2 f(x_n, y(x_n))}{\partial x^2} + \frac{h}{2} y'(x_n) \frac{\partial^2 f(x_n, y(x_n))}{\partial x \partial y} + \right. \right. \\
&\quad \left. \left. \left(\frac{h}{2} y'(x_n) \right)^2 \frac{\partial^2 f(x_n, y(x_n))}{\partial y^2} \right] + o(h^3) \right\} = \frac{h^3}{3!} y'''(x_n) - \\
&\quad \frac{h}{8} \left[\frac{\partial^2 f}{\partial x^2} + y'(x) \frac{\partial^2 f}{\partial x \partial y} + (y'(x))^2 \frac{\partial^2 f}{\partial y^2} \right]_{(x_n, y(x_n))} + o(h^4) = o(h^3)
\end{aligned}$$

因此，中点公式是二阶的。

对模型方程 $y' = \lambda y$ ($\text{Re}(\lambda) < 0$) 使用中点公式求解，得

$$y_{n+1} = \left[1 + \lambda h + \frac{1}{2} (\lambda h)^2 \right] y_n$$

易知，当 $\left| 1 + \lambda h + \frac{1}{2} (\lambda h)^2 \right| \leq 1$ 时，中点公式绝对稳定。特别当 λ 为实数且

$\lambda < 0$ 时，上不等式的解为

$$-2 \leq \lambda h \leq 0$$

8. 解： (1) 用欧拉法求解题中初值问题，当 $\lambda h = -100h$ 满足

$$\left| 1 + (-100h) \right| \leq 1$$

时绝对稳定，即当 $0 < h \leq 0.2$ 时欧拉法绝对稳定。

(2) 当 $\lambda h = -100h$ 满足不等

$$\left| 1 + \lambda h + \frac{1}{2} (\lambda h)^2 + \frac{1}{3!} (\lambda h)^3 + \frac{1}{4!} (\lambda h)^4 \right| \leq 1$$

时，四阶龙格-库塔法绝对稳定，也即当 λh 满足

$$-2.785 \leq \lambda h < 0, 0 < h \leq \frac{-2.785}{\lambda} = 0.02785 \text{ 时绝对稳定。}$$

(3) 对于梯形公式，当 $\lambda h = -100h \in (-\infty, 0)$ 时，绝对稳定，此条件对 $\forall h \in (0, +\infty)$ 都成立，即梯形法对 h 无限制。

9、解： 二阶阿达姆斯显式和隐式方法分别为

$$y_{n+2} = y_{n+1} + \frac{h}{2}(3f_{n+1} - f_n)$$

$$y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n)$$

将 $f = 1 - y$ 代入上二式，化简得

显式方法
$$y_{n+2} = (1 - \frac{3}{2}h)y_{n+1} + \frac{h}{2}y_n + h$$

隐式方法
$$y_{n+1} = \frac{2-h}{2+h}y_n + \frac{2h}{2+h}$$

取 $h = 0.2, y_0 = 0, y_1 = 0.181$ ，计算结果如表 9-7 所示

表 9-7

x_n	显式 y_n	$ y(x_n) - y_n $	隐式 y_n	$ y(x_n) - y_n $
0.4	0.3267	0.2979953×10^{-2}	0.32990909	0.229136×10^{-3}
0.6	0.44679		0.451743801	
0.8	0.545423	0.4398363×10^{-2}	0.551426746	0.555437×10^{-3}
1.0	0.6264751		0.63298552	
		0.5248035×10^{-2}		0.755710×10^{-3}
		0.5645458×10^{-2}		0.864961×10^{-3}

可见，隐式方法比显式方法精确。

10. 证明：根据局部截断误差的定义知

$$\begin{aligned}
T_{n+1} &= y(x_n + h) - \frac{1}{2}(y(x_n) + y(x_n - h)) - \\
&\frac{1}{4}h[4y'(x_n + h) - y'(x_n) + 3y'(x_n - h)] = \\
&y(x_n) + hy'(x_n) + \frac{1}{2}h^2y''(x_n) + \frac{1}{3!}h^3y'''(x_n) + o(h^4) - \frac{1}{2}y(x_n) - \\
&\frac{1}{2}[y(x_n) - hy'(x_n) + \frac{1}{2}h^2y''(x_n) - \frac{1}{3!}h^3y'''(x_n) + o(h^4)] - \\
&\frac{h}{4}[4(y'(x_n) + hy''(x_n) + \frac{1}{2}h^2y'''(x_n) + o(h^3)) - y'(x_n) + \\
&3(y'(x_n) - hy''(x_n) + \frac{1}{2}h^2y'''(x_n) + o(h^3))] = (1 - \frac{1}{2} - \frac{1}{2})y(x_n) + \\
&(\frac{1}{2} - \frac{1}{4} - 1 + \frac{3}{4})h^2y''(x_n) + (\frac{1}{6} + \frac{1}{12} - \frac{1}{2} - \frac{3}{8})h^3y'''(x_n) + o(h^4) = \\
&-\frac{5}{8}h^3y'''(x_n) + o(h^4)
\end{aligned}$$

故方法是二阶的，局部截断误差的主项为 $-\frac{5}{8}h^3y'''(x_n)$ 。

11、解 由局部截断误差的定义知

$$\begin{aligned}
T_{n+2} &= y(x_n + 2h) + (b - 1)y(x_n + h) - by(x_n) - \\
&\frac{h}{4} [(b + 3)y'(x_n + 2h) + (3b + 1)y'(x_n)] = \\
&y(x_n) + 2hy'(x_n) + \frac{1}{2}(2h)^2 y''(x_n) + \frac{1}{3!}(2h)^3 y'''(x_n) + \\
&\frac{1}{4!}(2h)^4 y^{(4)}(x_n) + o(h^5) + (b - 1) [y(x_n) + hy'(x_n) + \\
&\frac{1}{2} h^2 y''(x_n) + \frac{1}{3!} h^3 y'''(x_n) + \frac{1}{4!} h^4 y^{(4)}(x_n) + o(h^5)] - \\
&by(x_n) - \frac{h}{4} (b + 3) [y'(x_n) + 2hy''(x_n) + \frac{1}{2!}(2h)^2 y'''(x_n) + \\
&\frac{1}{3!}(2h)^3 y^{(4)}(x_n) + o(h^5)] - \frac{h}{4} (3b + 1)y'(x_n) = \\
&(1 + b - 1 - b)y(x_n) + \\
&[2 + b - 1 - \frac{1}{4}(b + 3) - \frac{1}{4}(3b + 1)]hy'(x_n) + \\
&[2 + \frac{1}{2}(b - 1) - \frac{1}{2}(b + 3)]h^2 y''(x_n) + \\
&[\frac{4}{3} + \frac{1}{6}(b - 1) - \frac{1}{2}(b + 3)]h^3 y'''(x_n) + \\
&[\frac{2}{3} + \frac{1}{24}(b - 1) - \frac{1}{3}(b + 3)]h^4 y^{(4)}(x_n) + o(h^5) = \\
&-\frac{1}{3}(b + 1)h^3 y'''(x_n) - (\frac{3}{8} - \frac{7}{24}b)h^4 y^{(4)}(x_n) + o(h^5)
\end{aligned}$$

所以当 $b \neq -1$ 时

$$T_{n+2} = -\frac{1}{3}(b + 1)h^3 y'''(x_n) + o(h^4)$$

方法为二阶；当 $b = -1$ 时

$$T_{n+1} = -(\frac{3}{8} - \frac{7}{24}b)h^4 y^{(4)}(x_n) + o(h^5)$$

方法为三阶。

12、解： 根据刚性比的定义，若方程组的矩阵 $A = \begin{bmatrix} -10 & 9 \\ 10 & -11 \end{bmatrix}$ 的特征值 λ_j 满足条件 $\text{Re}(\lambda_j) < 0 (j = 1, 2)$ ，则

$$s = \frac{\max_{1 \leq j \leq 2} |\text{Re}(\lambda_j)|}{\min_{1 \leq j \leq 2} |\text{Re}(\lambda_j)|}$$

称为刚性比，易知 A 的两个特征值为

$$\lambda_1 = -1, \lambda_2 = -20$$

所以刚性比 $s=20$ 。

当 $\lambda h \in [-2.78, 0)$ 时，数值稳定。因此当 $0 < h \leq \frac{-2.78}{-20} = 0.139$ 时才

能保证数值稳定。